IMPA - Instituto de Matemática Pura e Aplicada, Rio de Janeiro, RJ, Brazil

Introduction to Computational Manifolds and Applications

Part 1 - Constructions

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Parametric Pseudo-Manifolds

More on Transition Maps

We've seen a class of complex functions that can play the role of the g maps in our transition functions. It is worth mentioning that *we still have to check assumption (5) for them*.

Recall that we had to *change* the geometry of the p-domains, so that we could define a C^k-diffeomorphism between $\overset{\circ}{Q}$ and the gluing domains, where k is a positive integer or $k = \infty$.

However, as we shall see in a coming lecture, this change in geometry imposes some difficulties for defining bump functions, shape functions, and parametrizations on the p-domains.

Now, we present an alternative choice for the g maps. This alternative also requires a change in the geometry of the p-domains. But, this change is more natural and less troublesome.

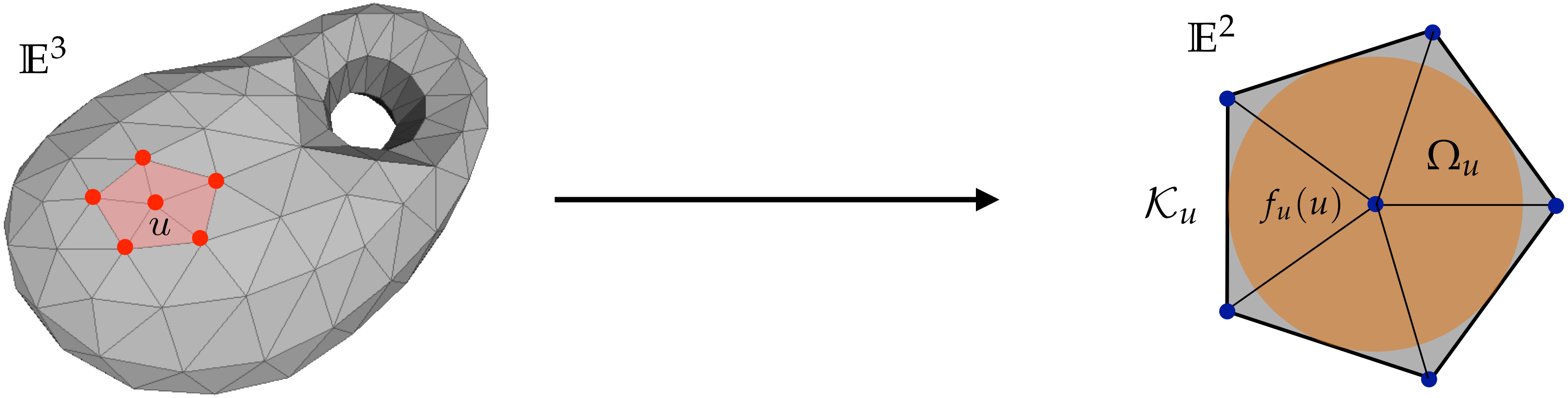
Parametric Pseudo-Manifolds

More on Transition Functions

The key idea is to consider the p-domain as an open disk in the underlying space of $\mathcal{K}_u$.

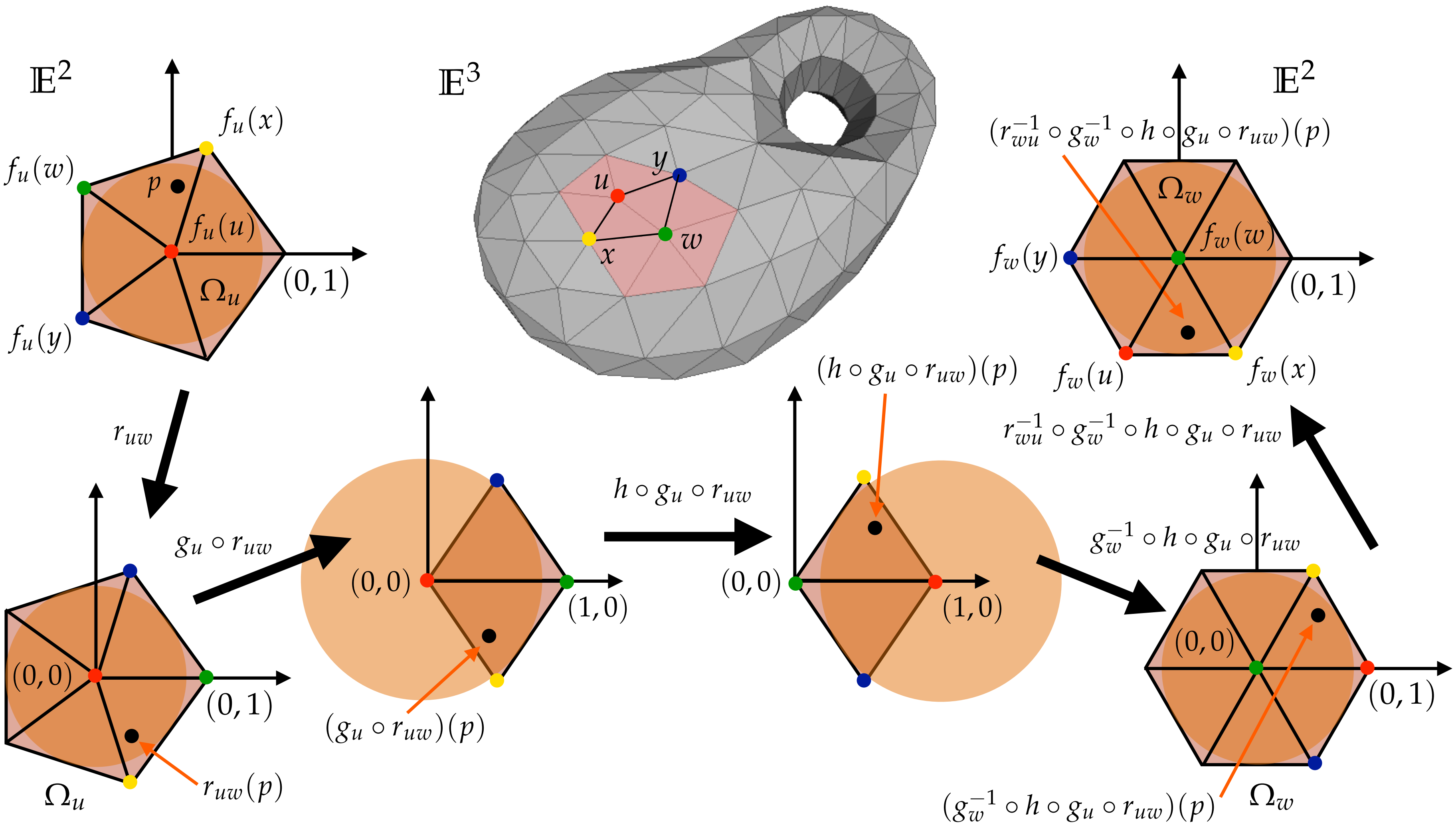
More specifically, Ω_u is the interior of the circle, C_u, inscribed in $|\mathcal{K}_u|$:

$$\Omega_u = \left\{ (x, y) \in \mathbb{E}^2 \mid x^2 + y^2 < \left(\cos\left(\frac{\pi}{n_u} \right) \right)^2 \right\} .$$



Parametric Pseudo-Manifolds

More on Transition Functions



Parametric Pseudo-Manifolds

More on Transition Functions

Like we did before, let $g_u : \mathbb{E}^2 - \{(0,0)\} \to \mathbb{E}^2 - \{(0,0)\}$ be given by the composition

$$g_u(p) = (\Pi^{-1} \circ \Gamma_u \circ \Pi)(p)\,,$$

for every $p \in \mathbb{R}^2 - \{(0,0)\}$. However, $\Gamma_u : \mathbb{R}_+ \times \,]-\pi\,,\,\pi[\to \mathbb{R}_+ \times \,]-\pi\,,\,\pi[$ is given by

$$\Gamma_u(r,\theta) = \left(\frac{\cos(\pi/6)}{\cos(\pi/n_u)} \cdot r\,,\, \frac{n_u}{6} \cdot \theta\right)\,,$$

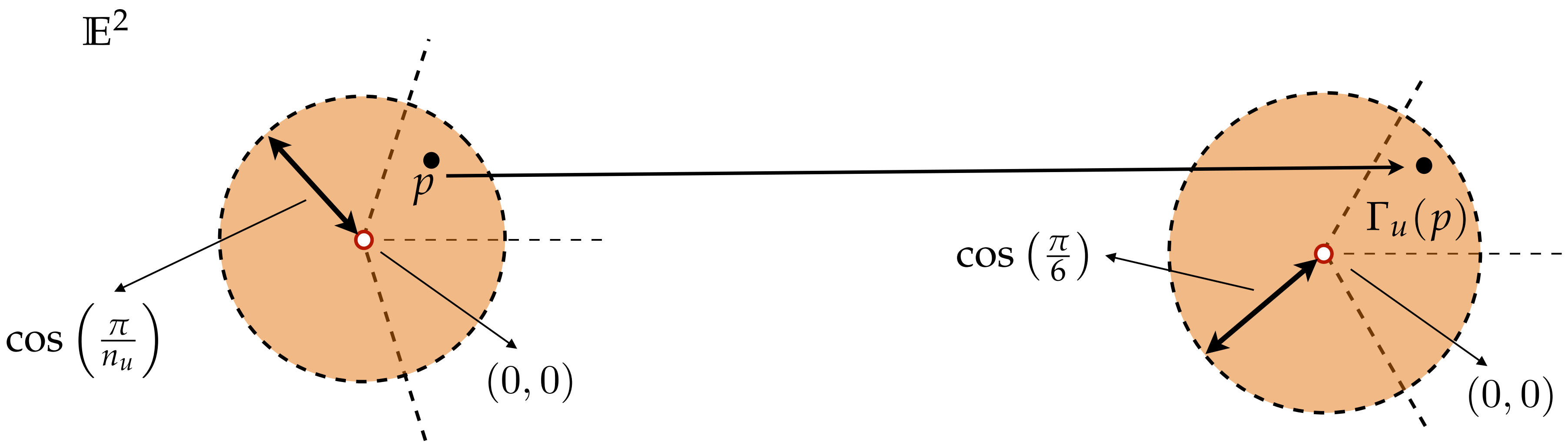
where $(r,\theta) = \Pi(p)$ are the polar coordinates of p.

Parametric Pseudo-Manifolds

More on Transition Functions

Function Γ_u maps $\Omega_u - \{(0,0)\}$ onto $\overset{\circ}{C} - \{(0,0)\}$, where

$$C = \left\{ (x,y) \in \mathbb{E}^2 \mid x^2 + y^2 \leq \left(\cos\left(\frac{\pi}{6}\right)\right)^2 \right\} .$$



Parametric Pseudo-Manifolds

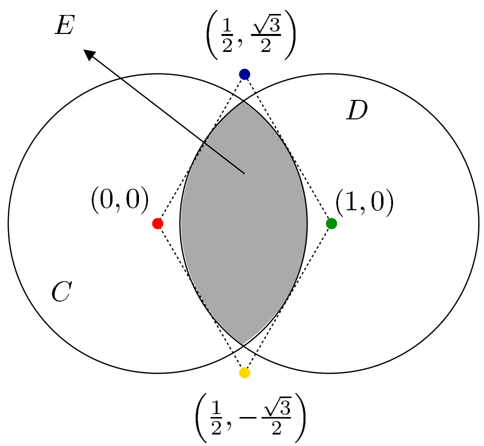
More on Transition Functions

For any $(u,w) \in I \times I$, the gluing domain Ω_{uw} is defined as the image, $(r_{uw}^{-1} \circ g_u^{-1})(\overset{\circ}{E})$, of the interior, $\overset{\circ}{E}$, of the *canonical lens*, E, under the composite function $r_{uw}^{-1} \circ g_u^{-1}$, where

$$E = C \cap D\,,$$

and

$$C = \{(x,y) \mid x^2 + y^2 \leq (\cos(\pi/6))^2\} \text{ and } D = \{(x,y) \mid (x-1)^2 + y^2 \leq (\cos(\pi/6))^2\}\,.$$

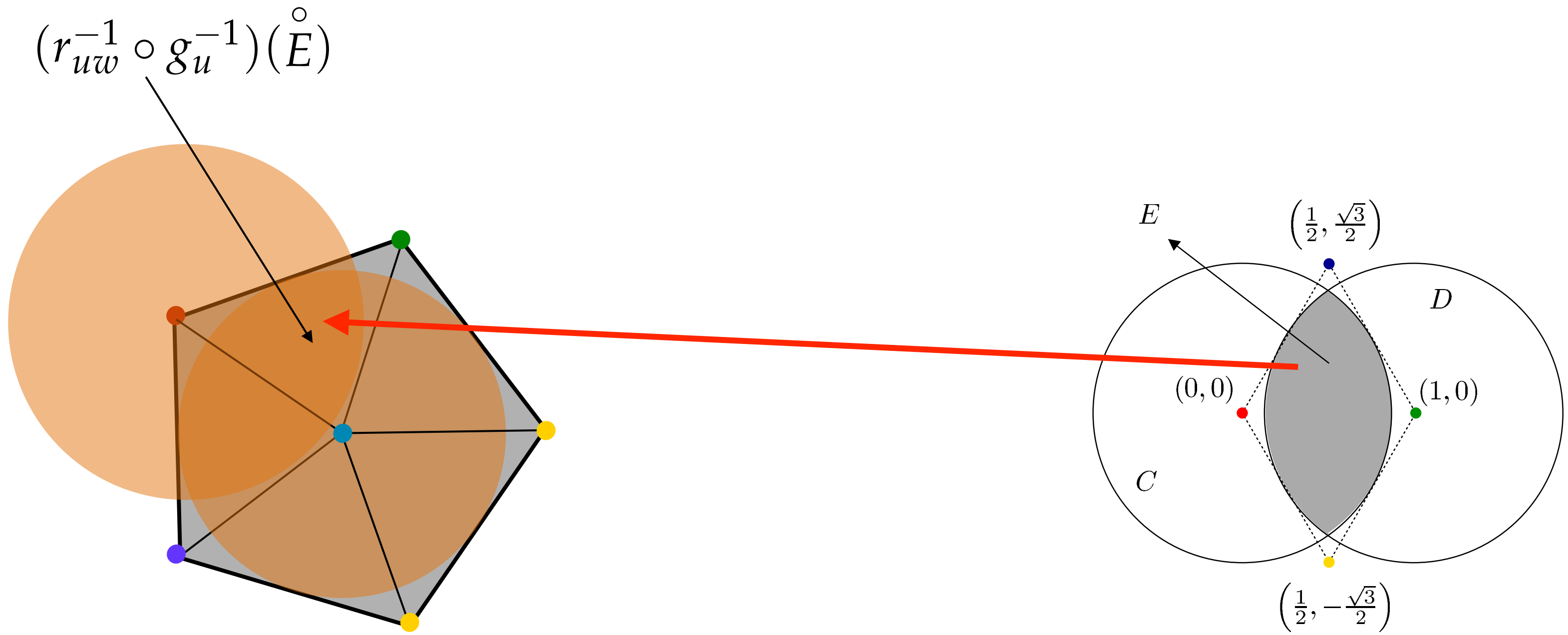


Parametric Pseudo-Manifolds

More on Transition Functions

So, for any $(u, w) \in I \times I$, the *gluing domain* Ω_{uw} is defined as

$$\Omega_{uw} = \begin{cases} \Omega_u & \text{if } u = w, \\ (r_{uw}^{-1} \circ g_u^{-1})(\mathring{E}) & \text{if } [u, w] \text{ is an edge of } \mathcal{K}, \\ \emptyset & \text{otherwise.} \end{cases}$$



Parametric Pseudo-Manifolds

More on Transition Functions

For any $(u, w) \in K$, the *transition map*,

$$\varphi_{wu} : \Omega_{uw} \to \Omega_{wu} ,$$

is such that, for every $p \in \Omega_{uw}$, we let

$$\varphi_{wu}(p) = \begin{cases} p & \text{if } u = w, \\ (r_{wu}^{-1} \circ g_w^{-1} \circ h \circ g_u \circ r_{uw})(p) & \text{otherwise.} \end{cases}$$

Parametric Pseudo-Manifolds

More on Transition Functions

It is now time for checking our assumptions regarding g_u:

(1) The g_u map is a C^k-diffeomorphism of $\mathbb{R}^2 - \{(0,0)\}$, for every $u \in I$

(2) The g_u map takes $r_{uw}(\Omega_{uw})$ onto $\overset{\circ}{E}$ for every $(u,w) \in K$.

(3) The g_u map satisfies $(g_u \circ r_{\frac{2\pi}{n_u}} \circ g_u^{-1})(q) = r_{\frac{\pi}{3}}(q)$, where $q \in g_u(\Omega_u)$.

(4) If $f_u(w)$ precedes $f_u(v)$ in a counterclockwise enumeration of the vertices of $\text{lk}(u, \mathcal{K})$, then $(g_u \circ r_{uw})(p) = (r_{\frac{\pi}{3}} \circ g_u \circ r_{uv})(p)$, for every point p in the gluing domain Ω_{uw}.

Parametric Pseudo-Manifolds

More on Transition Functions

We have **not** checked the following assumption:

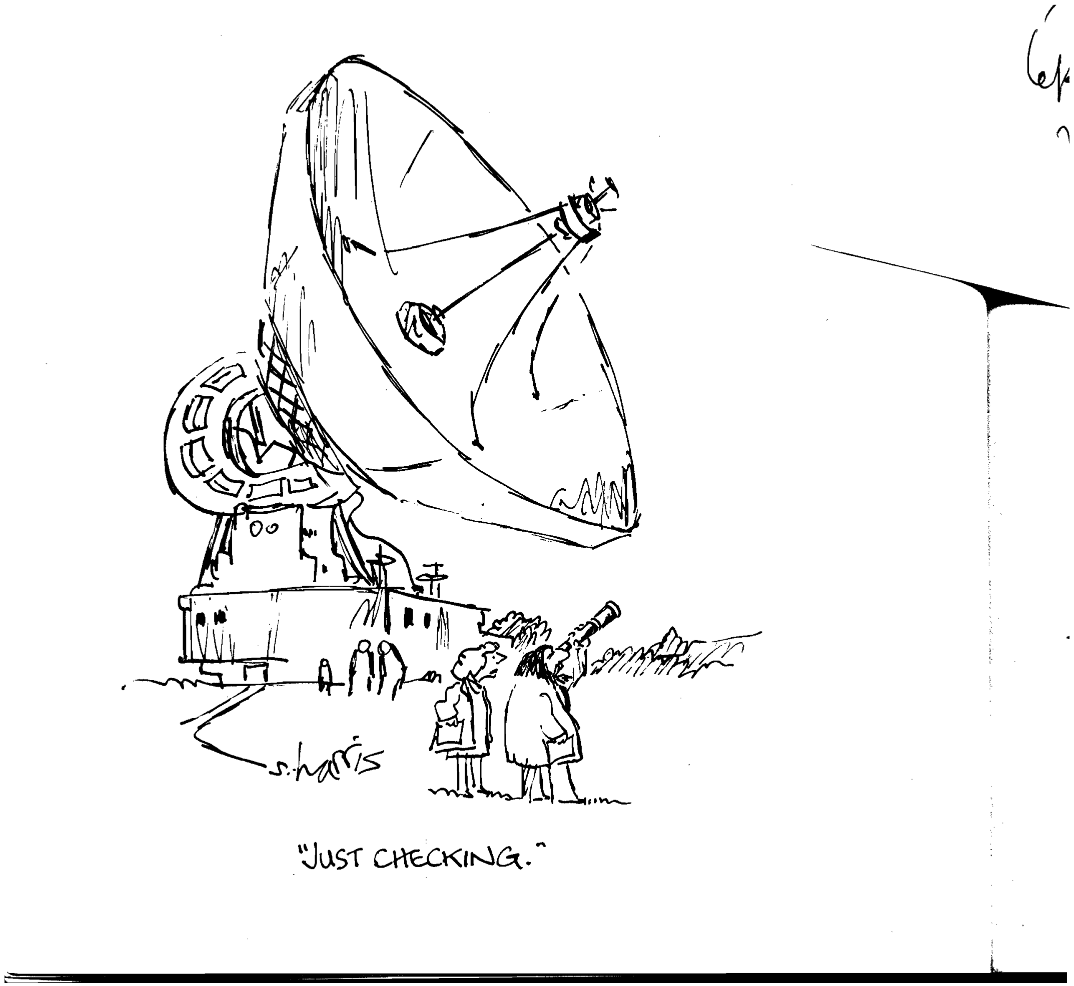
(5) For all u, v, w such that $[u, v, w]$ is a triangle of $\mathcal{K}$, if $\Omega_{wu} \cap \Omega_{wv} \neq \emptyset$ then

$$\varphi_{uw}(\Omega_{wu} \cap \Omega_{wv}) = \Omega_{uv} \cap \Omega_{uw} \, .$$

We will also explore that in a homework.

Parametric Pseudo-Manifolds

More on Transition Functions



Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

As far as we know, Cindy Grimm and John Hughes presented the first construction of parametric pseudo-manifolds from gluing data (see the Ph. D. thesis of Grimm, 1996).

Here, we will give an overview of this construction. We refer the audience to the aforementioned Ph. D. thesis and to Grimm and Hughes' SIGGRAPH 1995 paper for details.

Pointer to these references can be found on the course web page.

The construction of the gluing data is very intricate. So, reading the above references may be crucial for an in-depth understanding of their work and for implementation purposes.

Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

The input for the construction is any *polygonal mesh*. But, since we have not yet defined such meshes, we will restrict our attention to triangle meshes (i.e., simplicial surfaces).

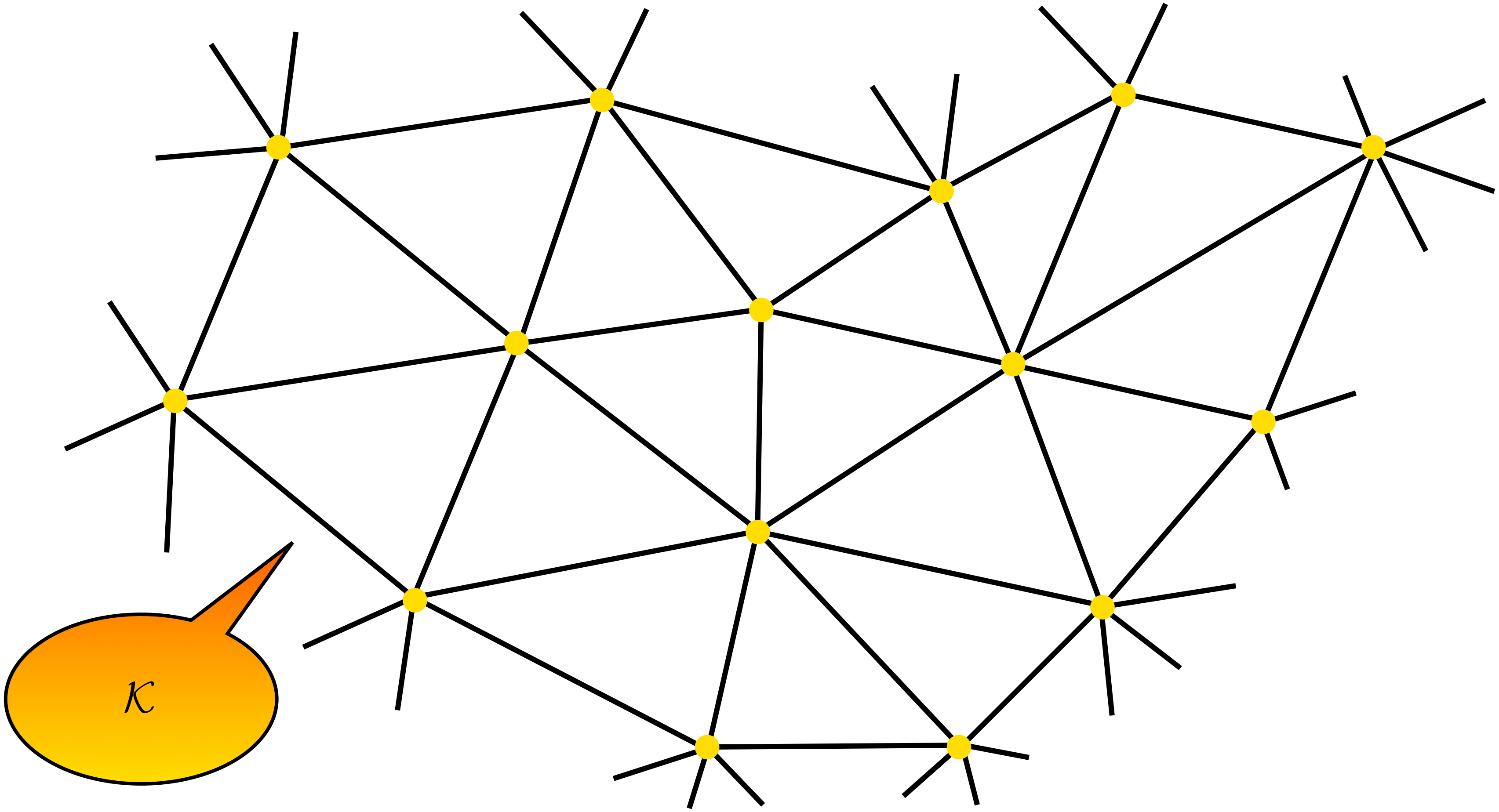
As usual, let us denote the given simplicial surface by $\mathcal{K}$.

The simplicial surface $\mathcal{K}$ is "refined" by one step of the *Catmull-Clark subdivision rule*, and then the *dual* of the resulting (cell) complex is considered for defining the gluing data.

The object resulting from the Catmull-Clark subdivision and its dual can be thought of as "graphs" with straight edges immersed in $\mathbb{E}^3$. Here, we will not define them in a formal way. Instead, we will illustrate how they are obtained using the subdivision rule.

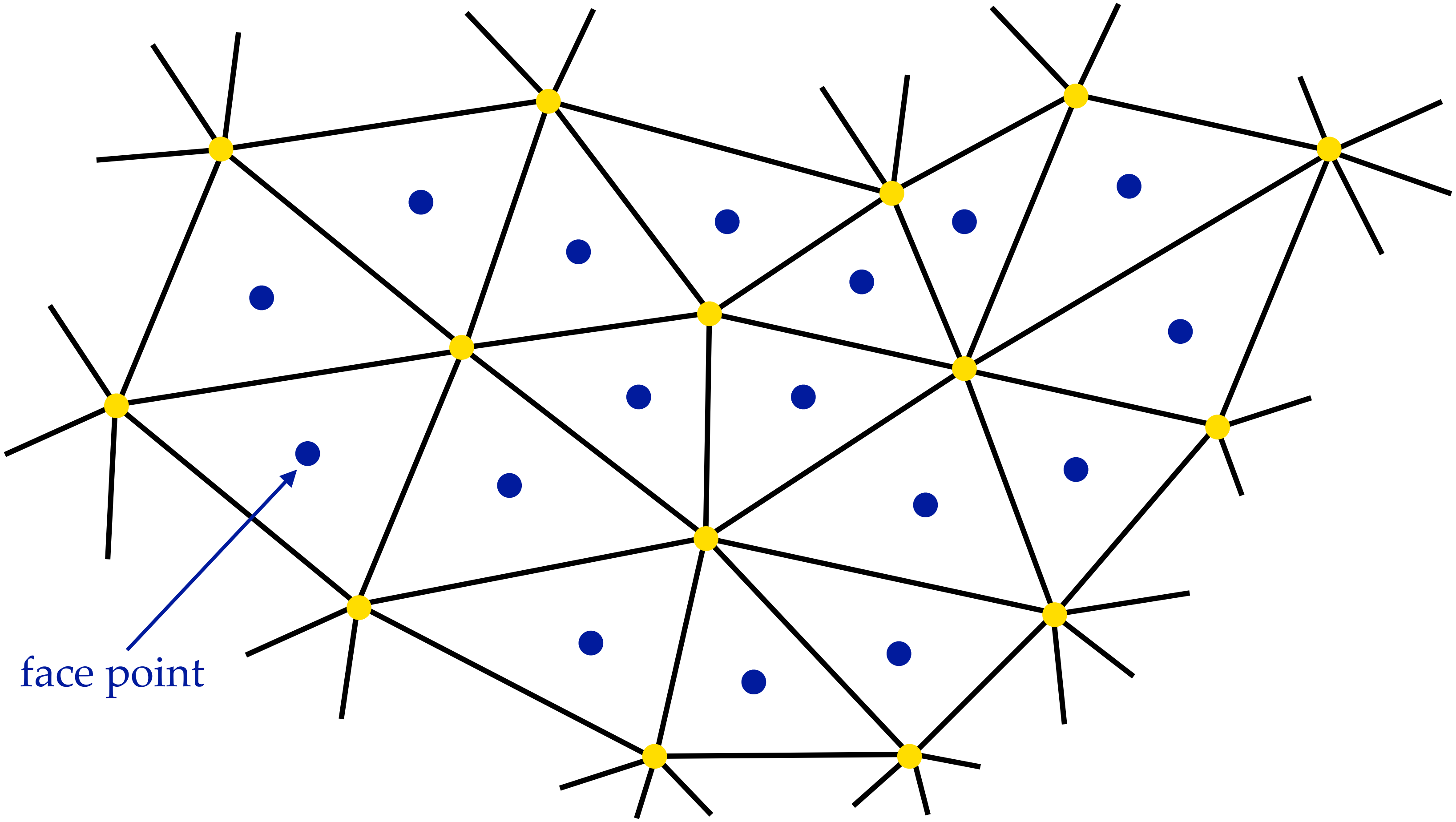
Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data



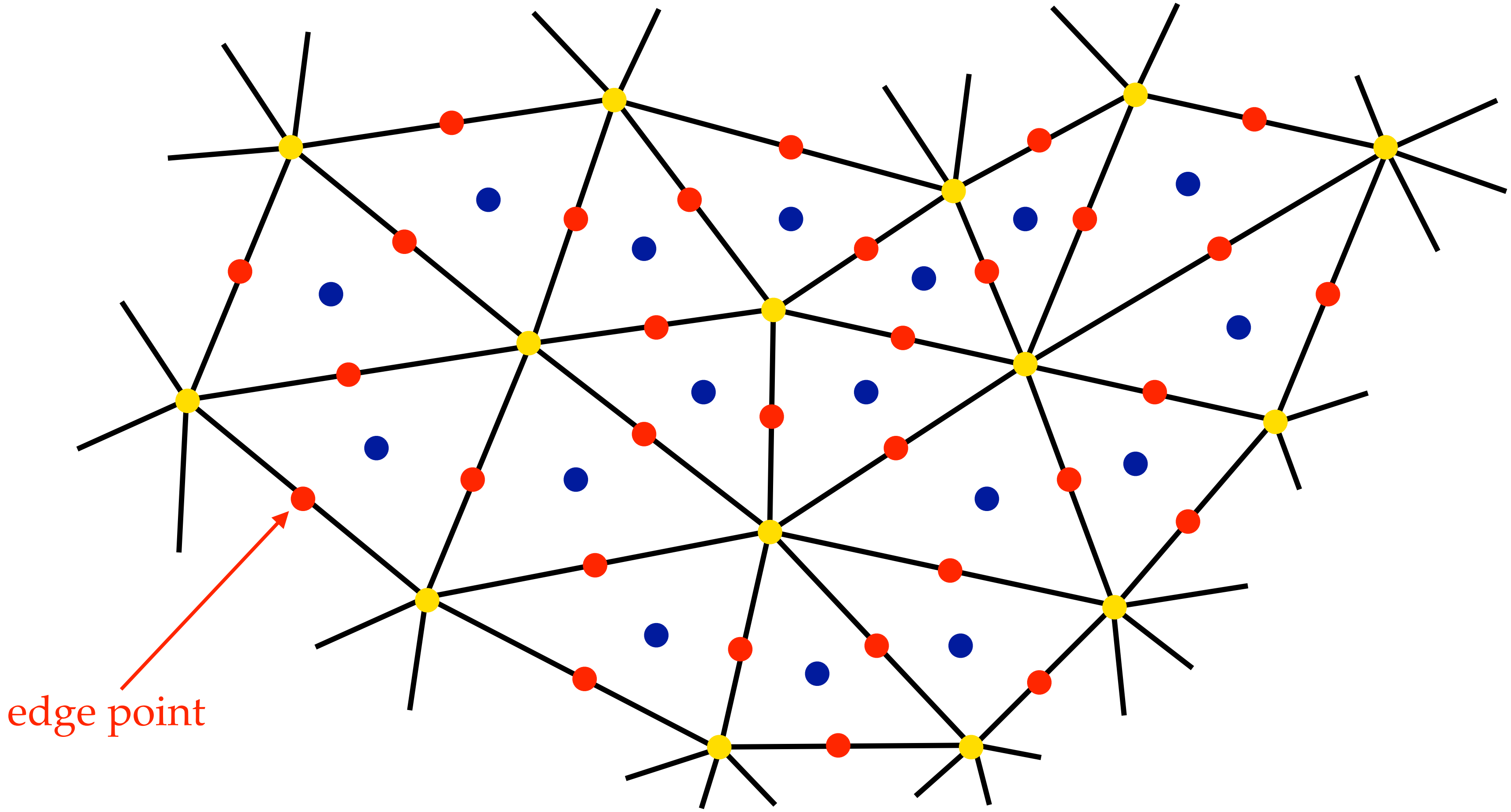
Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data



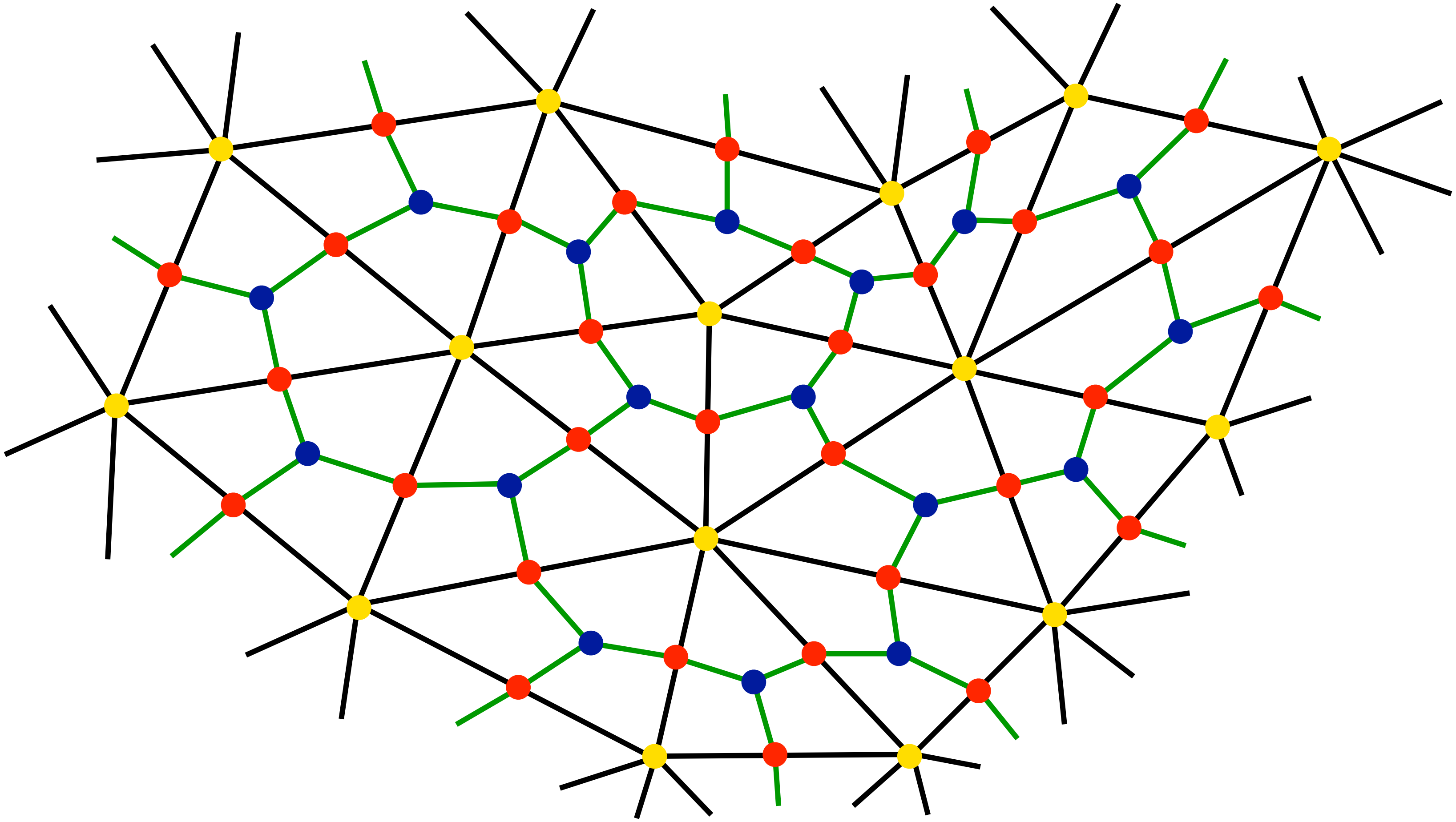
Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data



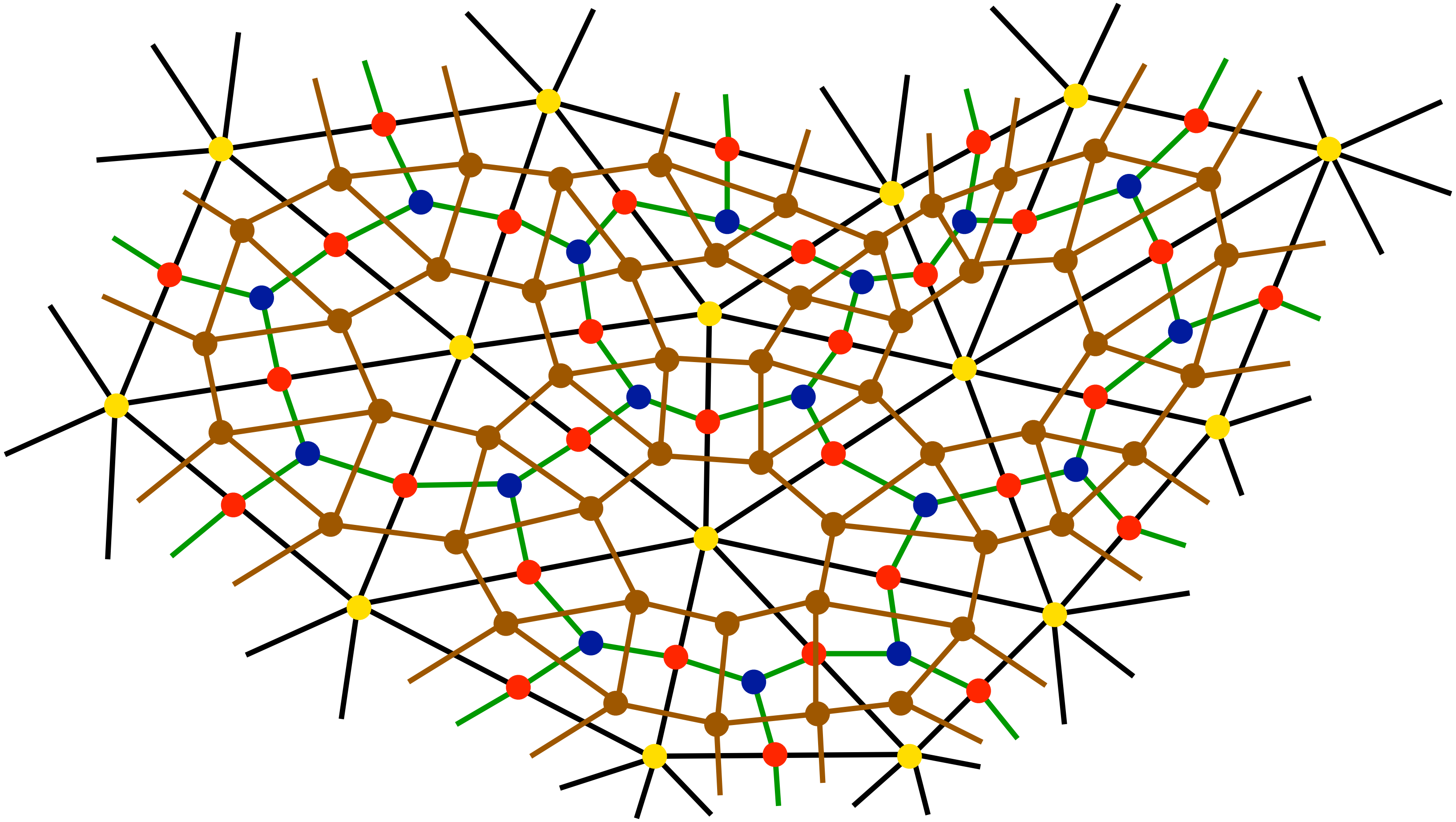
Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data



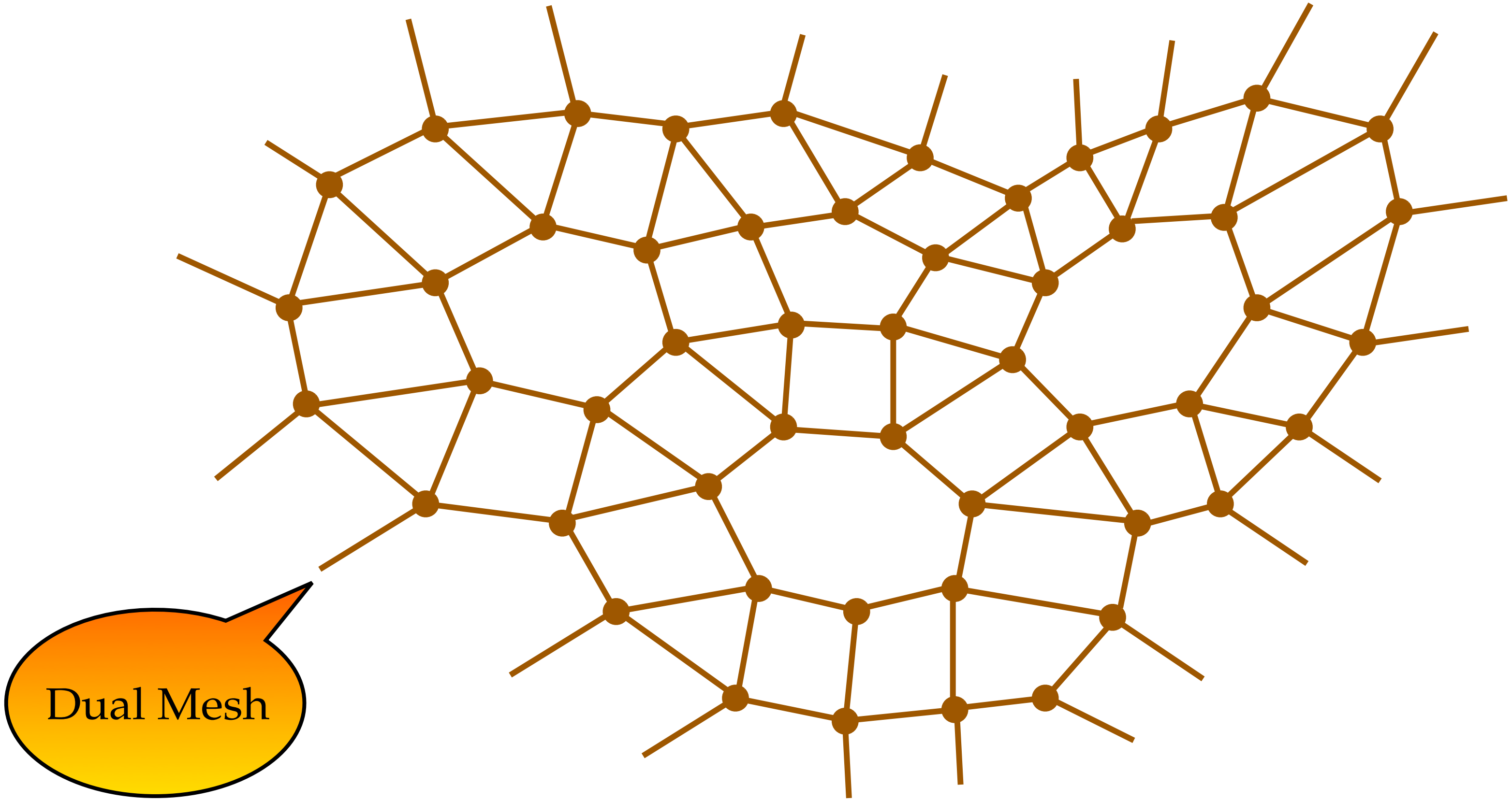
Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data



Parametric Pseudo-Manifolds

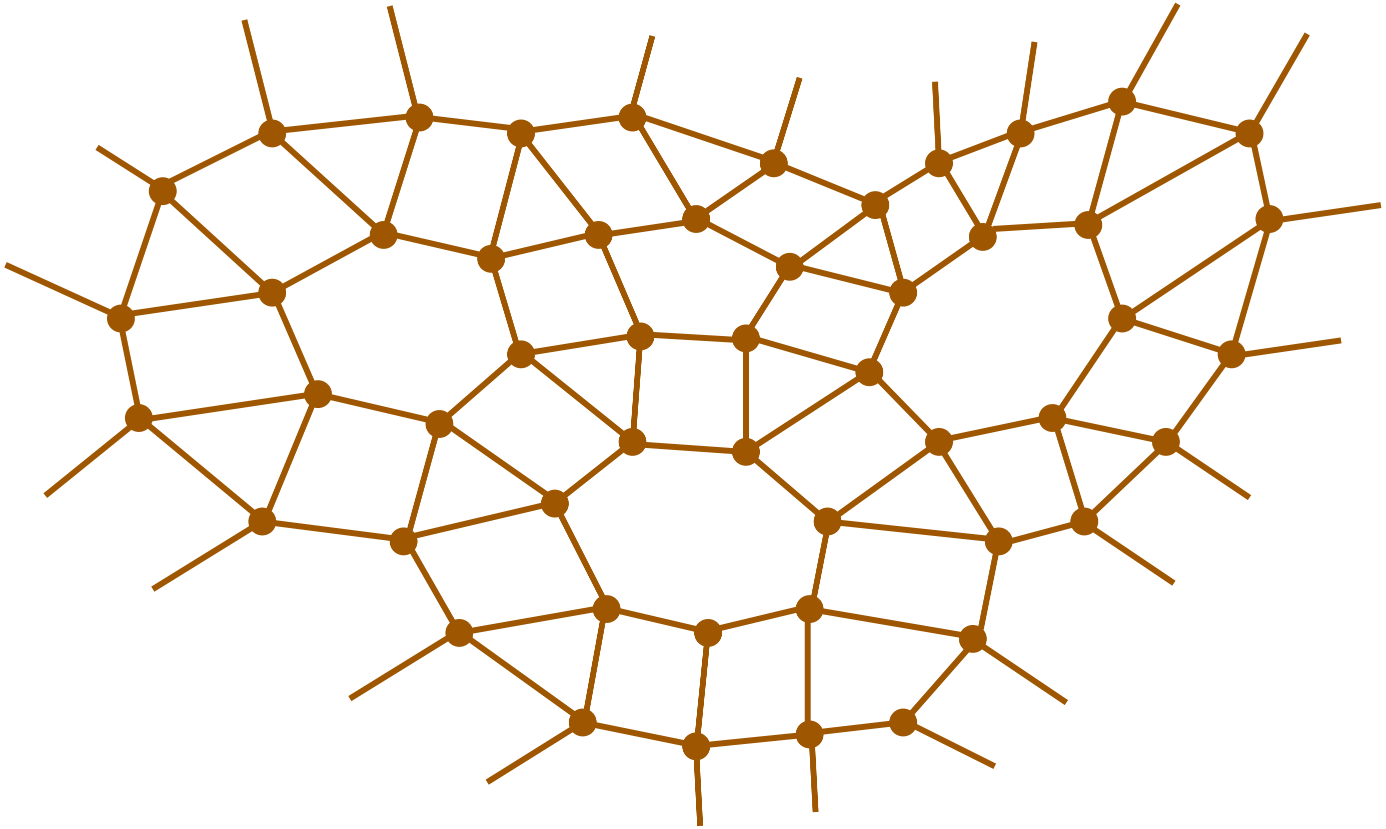
Grimm's Construction of Gluing Data



Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

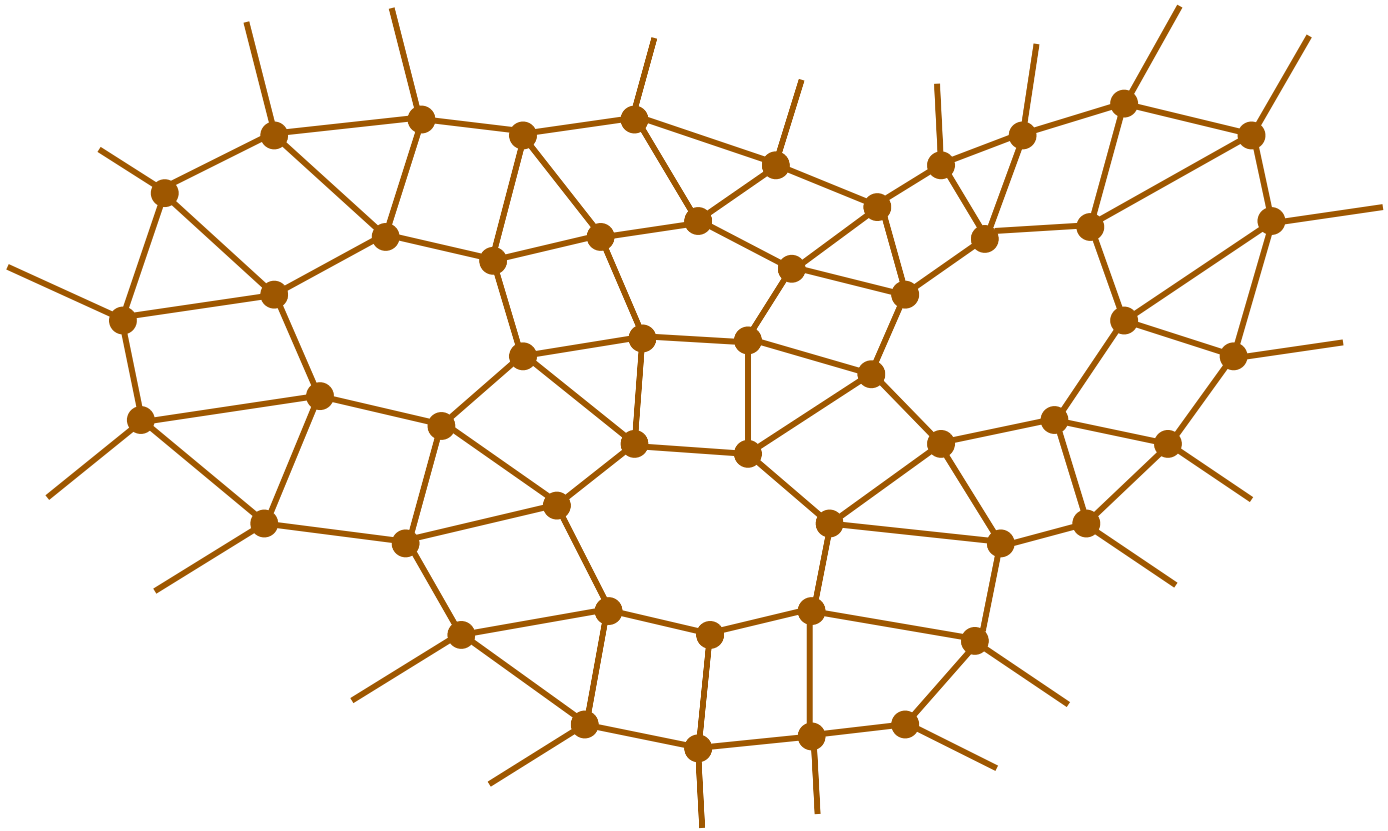
Let us denote the dual mesh by $\mathcal{K}'$.



Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

Note that all vertices of $\mathcal{K}'$ have degree four. This is the reason for defining $\mathcal{K}'$.



Parametric Pseudo-Manifolds

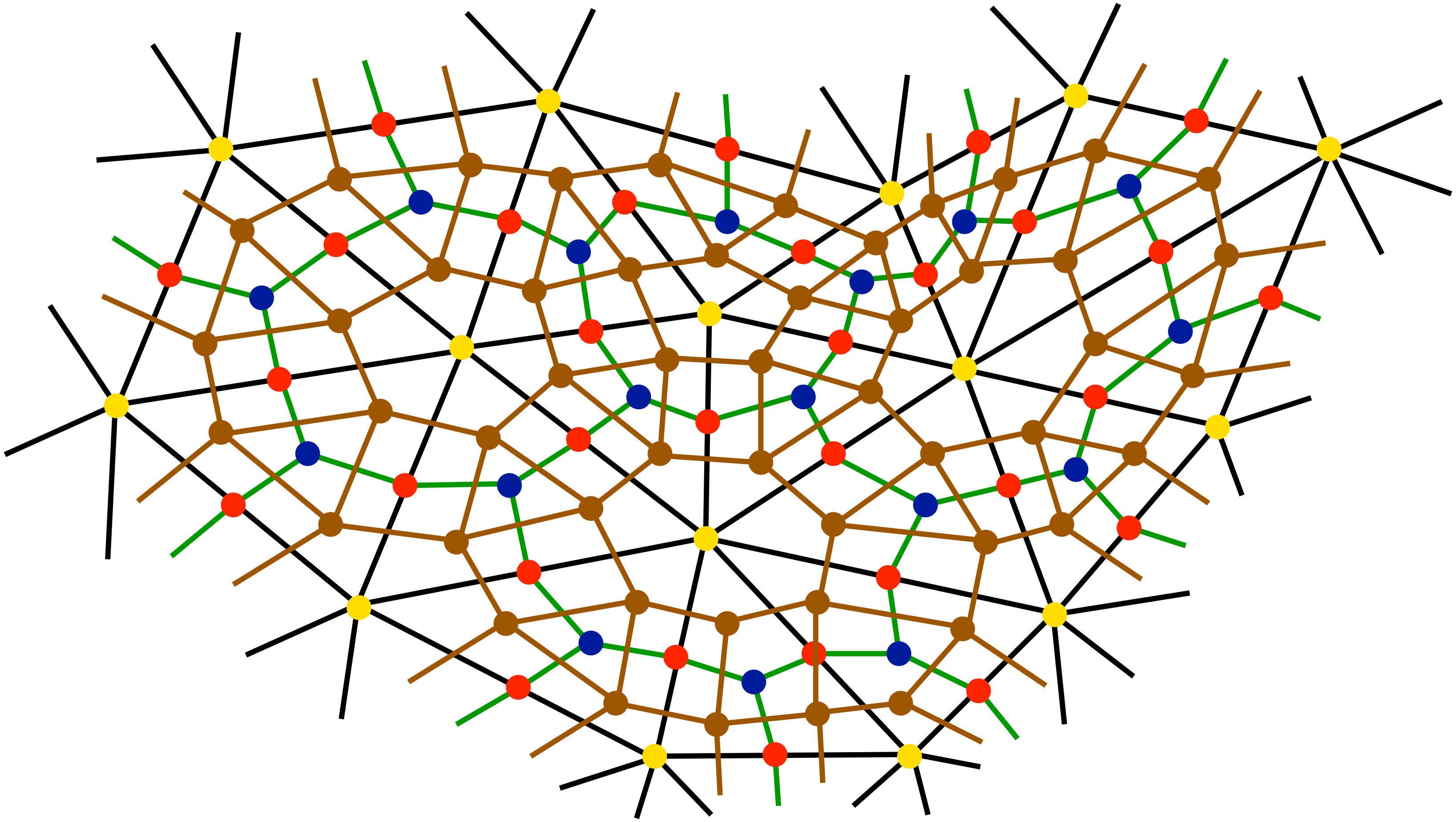
Grimm's Construction of Gluing Data

The gluing data defined by Grimm's constructions consists of one p-domain per each component of $\mathcal{K}'$; that is, we assign a p-domain with each dual mesh vertex, edge, and face.

Here, we view a "face" as a disk-like region bounded by a simple cycle of edges of $\mathcal{K}'$, which is the dual of a vertex of the graph obtained from the Catmull-Clark subdivision.

Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data



Parametric Pseudo-Manifolds

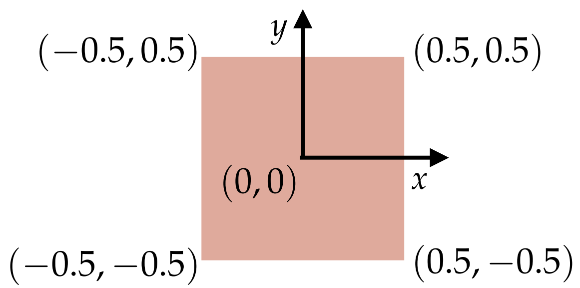
Grimm's Construction of Gluing Data

So, a face can be identified with a regular n-sided polygon in $\mathbb{E}^2$.

The p-domains associated with vertices, edges, and faces have distinct geometry. Furthermore, the geometry of the p-domains associated with edges (resp. faces) can also differ.

Let $\boldsymbol{V}$ be the set of vertices of $\mathcal{K}'$. Then, for each $v \in \boldsymbol{V}$, we define the p-domain Ω_v as

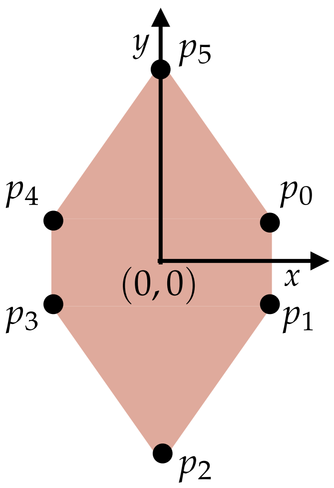
$$\Omega_v =]\,-0.5\,,\,0.5\,[^2\,.$$



Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

Let $\boldsymbol{E}$ be the set of edges of $\mathcal{K}'$. Then, for each $e \in \boldsymbol{E}$, we define the p-domain Ω_e as a *diamond-shaped* region that consists of the interior of an hexagon with vertices $p_0, \ldots, p_5$:



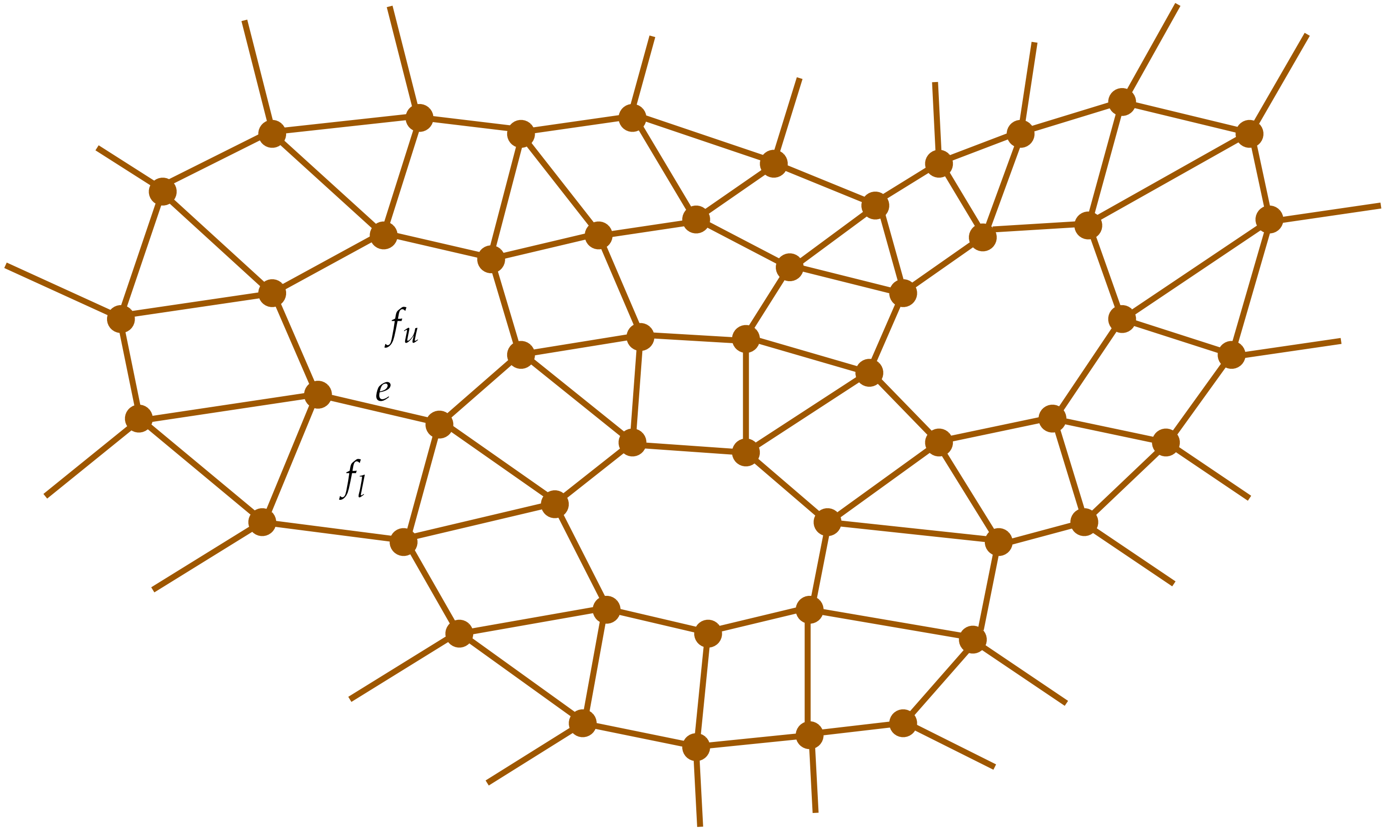
$$p_0 = (0.5 - h\,,\, h \cdot \cot(\pi/n_u))$$
$$p_1 = (0.5 - h\,,\, -h \cdot \cot(\pi/n_l))$$
$$p_2 = \left(0\,,\, -\frac{\cot(\pi/n_l)}{2}\right)$$
$$p_3 = (-0.5 + h\,,\, -h \cdot \cot(\pi/n_l))$$
$$p_4 = (-0.5 + h\,,\, h \cdot \cot(\pi/n_u))$$
$$p_5 = \left(0\,,\, \frac{\cot(\pi/n_u)}{2}\right)$$

So, the coordinates $p_0, \ldots, p_5$ depend on the parameters h, n_u, and n_l.

Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

By construction, each edge e of $\mathcal{K}'$ is incident with exactly two faces, say f_u and f_l, of K'.



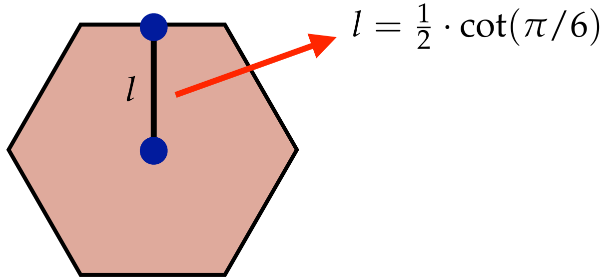
Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

We let n_u and n_l be the number of vertices of f_u and f_l, respectively.

Now, consider two regular polygons in $\mathbb{E}^2$, with n_u and n_l sides, respectively.

If the their sides have unit length, then the distance from the center of the polygon to the middle point of any edge of the polygon is equal to $\frac{1}{2} \cdot \cot(\pi/n_u)$ and $\frac{1}{2} \cdot \cot(\pi/n_l)$. This is why the second coordinate of p_2 and p_5 are given as $\frac{1}{2} \cdot \cot(\pi/n_u)$ and $-\frac{1}{2} \cdot \cot(\pi/n_l)$.



Parametric Pseudo-Manifolds

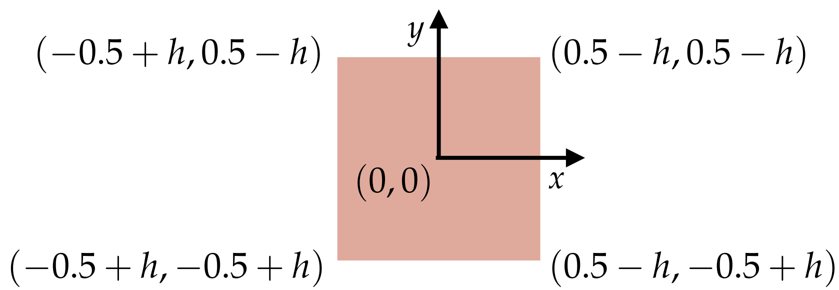
Grimm's Construction of Gluing Data

The parameter h is related to the transition functions and will be defined later.

Let $\boldsymbol{F}$ be the set of faces of $\mathcal{K}'$. Then, for each $f \in \boldsymbol{F}$, we define the p-domain Ω_f as the interior of a regular n-sided polygon centered at $(0,0)$ and whose sides are $1-2h$ units long.

The parameter n is the number of vertices of f.

$n = 4 \Longrightarrow$



Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

Just like before, gluing domains are determined by the adjacency relations of vertices, edges, and faces of $\mathcal{K}'$. In particular, p-domains associated with the same class of elements of $\mathcal{K}'$ (i.e., vertices, edges, and faces) are not identified by the gluing process.

So, if v_1 and v_2 are two distinct vertices in $\boldsymbol{V}$, then $\Omega_{v_1 v_2} = \Omega_{v_2 v_1} = \emptyset$. Likewise, if e_1 and e_2 are two distinct edges in $\boldsymbol{E}$ and if f_1 and f_2 are two distinct faces in $\boldsymbol{F}$, then we get

$$\Omega_{e_1 e_2} = \Omega_{e_2 e_1} = \Omega_{f_1 f_2} = \Omega_{f_2 f_1} = \emptyset\,.$$

Furthermore, if $v_1 = v_2$ then $\Omega_{v_1 v_2} = \Omega_{v_2 v_1} = \Omega_{v_1}$. The same is true for edges and faces.

Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

There are only three possibilities for nonempty gluing domains:

(1) The p-domain, Ω_v, associated with vertex $v \in \boldsymbol{V}$ is glued to Ω_e, the p-domain associated with edge $e \in \boldsymbol{E}$.

(2) The p-domain, Ω_v, associated with vertex $v \in \boldsymbol{V}$ is glued to Ω_f, the p-domain associated with face $f \in \boldsymbol{F}$.

(3) The p-domain, Ω_e, associated with edge $e \in \boldsymbol{E}$ is glued to Ω_f, the p-domain associated with face $f \in \boldsymbol{F}$.

Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

In particular, we have

(1) $\Omega_{ve} \neq \emptyset$ if and only if vertex v is a vertex of edge e.

(2) $\Omega_{vf} \neq \emptyset$ if and only if vertex v is a vertex of face f.

(3) $\Omega_{ef} \neq \emptyset$ if and only if edge e is an edge of face f.

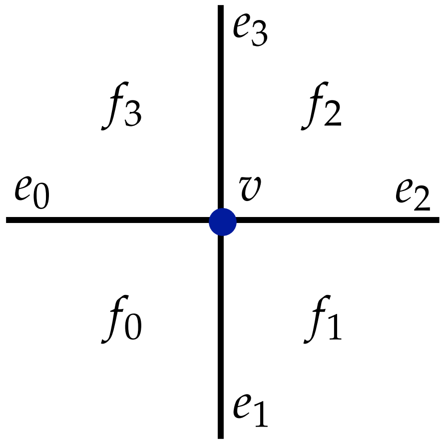
(4) $\Omega_{ev} \neq \emptyset$ if and only if edge e is incident with vertex v.

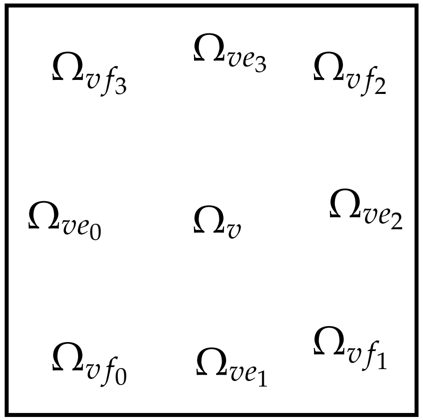
(5) $\Omega_{fv} \neq \emptyset$ if and only if face f is incident with vertex v.

(6) $\Omega_{fe} \neq \emptyset$ if and only if face f is incident with edge e.

Grimm's Construction of Gluing Data

For any given vertex $v \in \boldsymbol{V}$, there are exactly nine nonempty gluing domains in Ω_v:

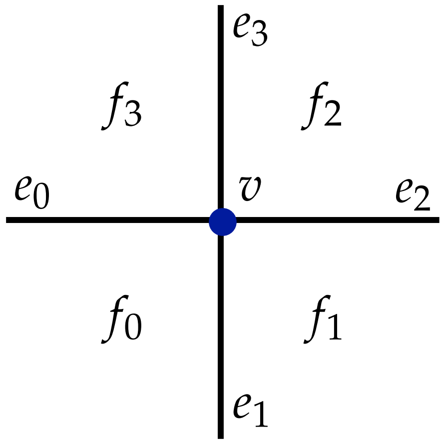


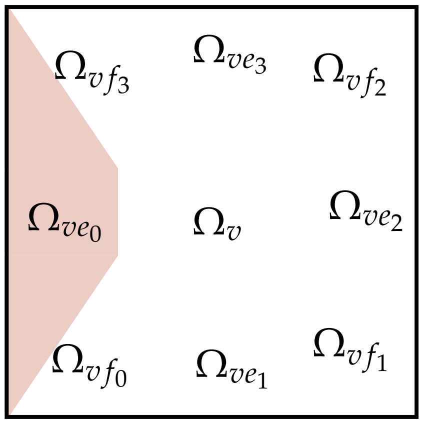


Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

The gluing domain Ω_{ve_i} corresponds to half a diamond-shaped region:

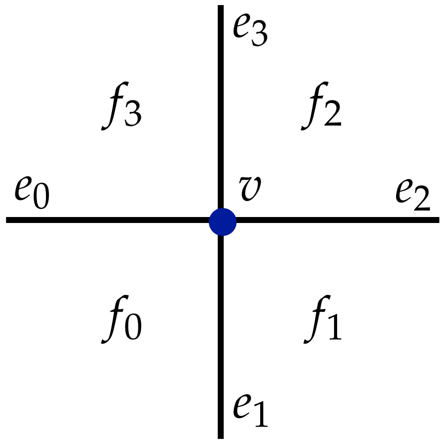


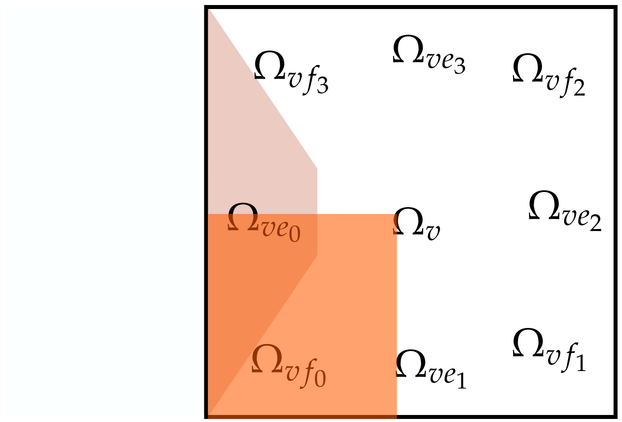


Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

The gluing domain Ω_{vf_i} corresponds to a non-degenerated quadrilateral:

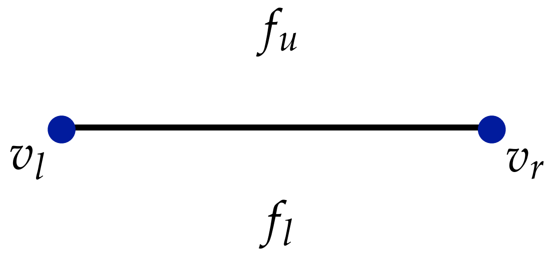


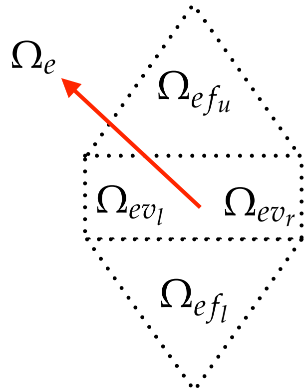


Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

For any given edge $e \in \boldsymbol{E}$, there are exactly five nonempty gluing domains in Ω_e:

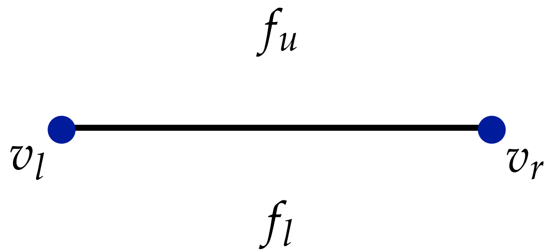


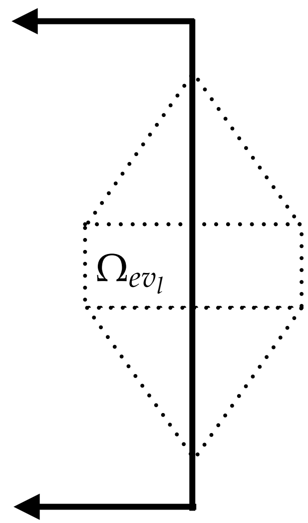


Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

The gluing domain Ω_{ev_l} corresponds to the set of points $(x, y) \in \Omega_e$ such that $x < 0$.

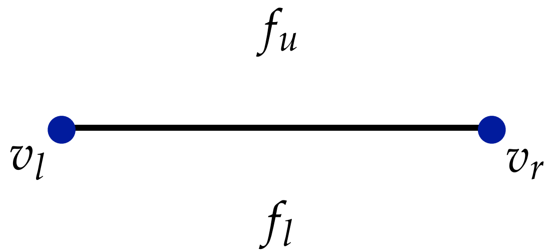


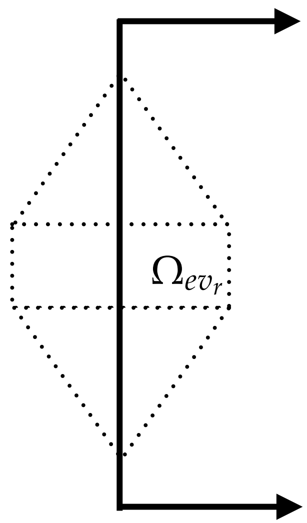


Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

The gluing domain Ω_{ev_r} corresponds to the set of points $(x, y) \in \Omega_e$ such that $x > 0$.



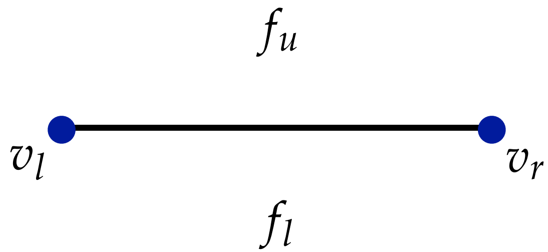


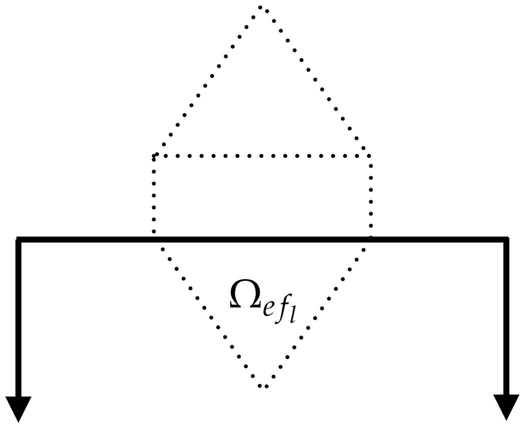
Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

The gluing domain Ω_{ef_l} corresponds to the set of points $(x, y) \in \Omega_e$ such that

$$y < -h \cdot \cot(\pi / n_l)\,.$$



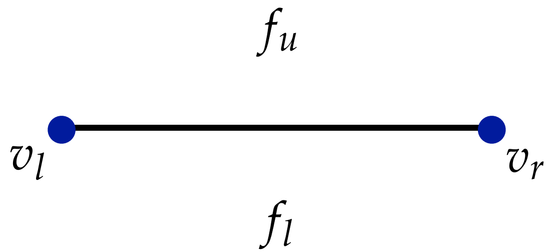


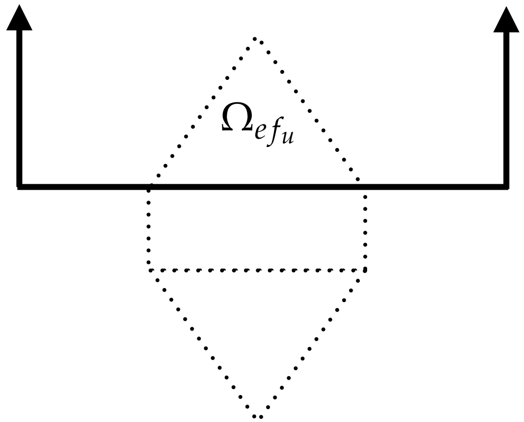
Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

The gluing domain Ω_{ef_r} corresponds to the set of points $(x, y) \in \Omega_e$ such that

$$y > h \cdot \cot(\pi / n_u)\,.$$

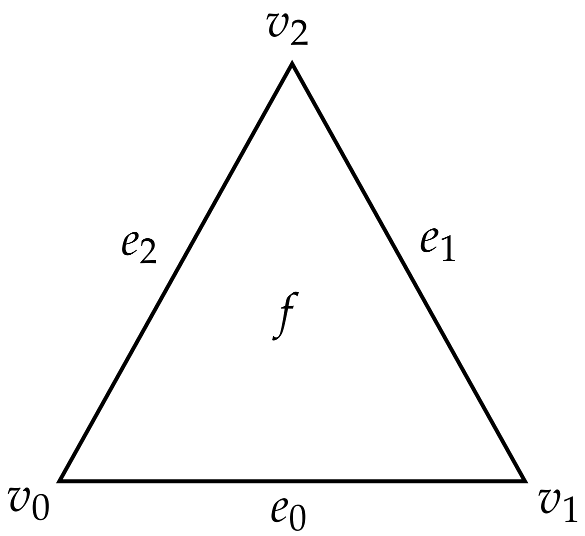


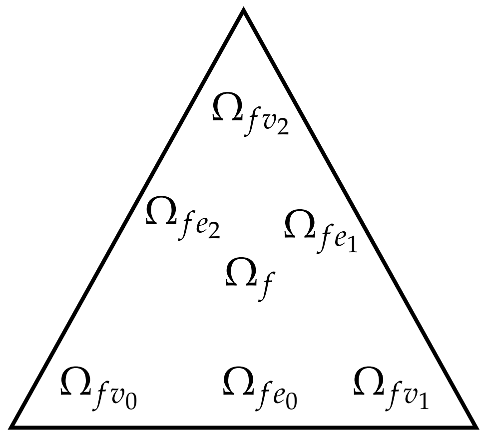


Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

For any given n-sided face $f \in \boldsymbol{F}$, there are exactly $2n+1$ nonempty gluing domains in Ω_f:

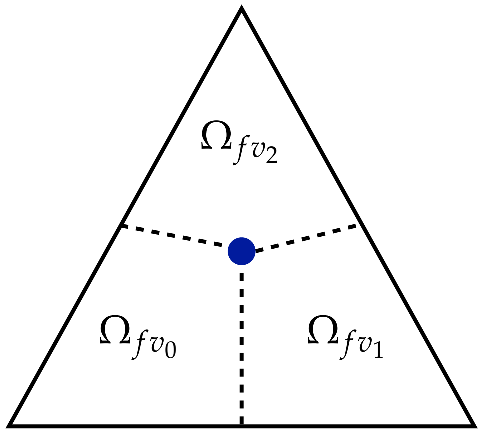




Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

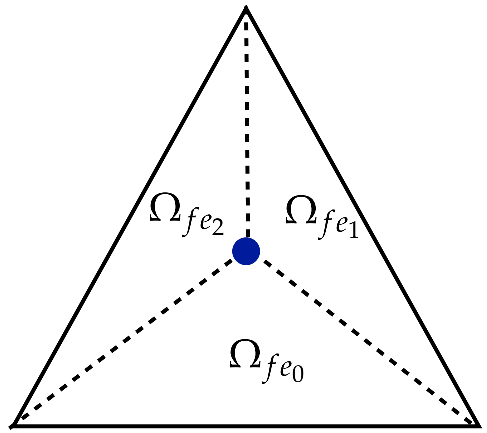
The gluing domains Ω_{fv_i} correspond to open quadrilaterals defined by connecting the center of Ω_f (i.e., the origin $(0,0)$) to the midpoint of the edges of the closure of Ω_f.



Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

The gluing domains Ω_{fe_i} correspond to open triangles defined by connecting the center of Ω_f (i.e., the origin $(0,0)$) to the vertices of the closure of Ω_f.



Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

There are six types of transition functions: (1) vertex-vertex, (2) edge-edge, (3) face-face, (4) vertex-edge, (5) vertex-face, and (6) edge-face. The first 3 functions are the identity.

Function (6) is an affine map (takes a triangle onto a triangle).

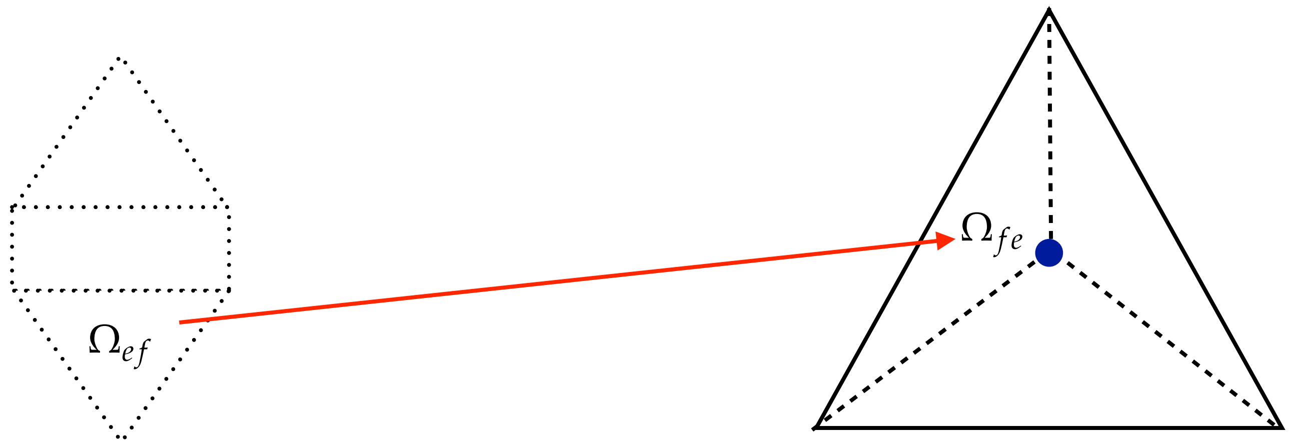
Function (5) is a projective map (takes a quadrilateral onto a quadrilateral).

Function (4) is defined as a weighted sum of two composite functions, each of which is the composition of an edge-face and vertex-face function. This function is bit complicated.

Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

The edge-face transition map, φ_{fe}:

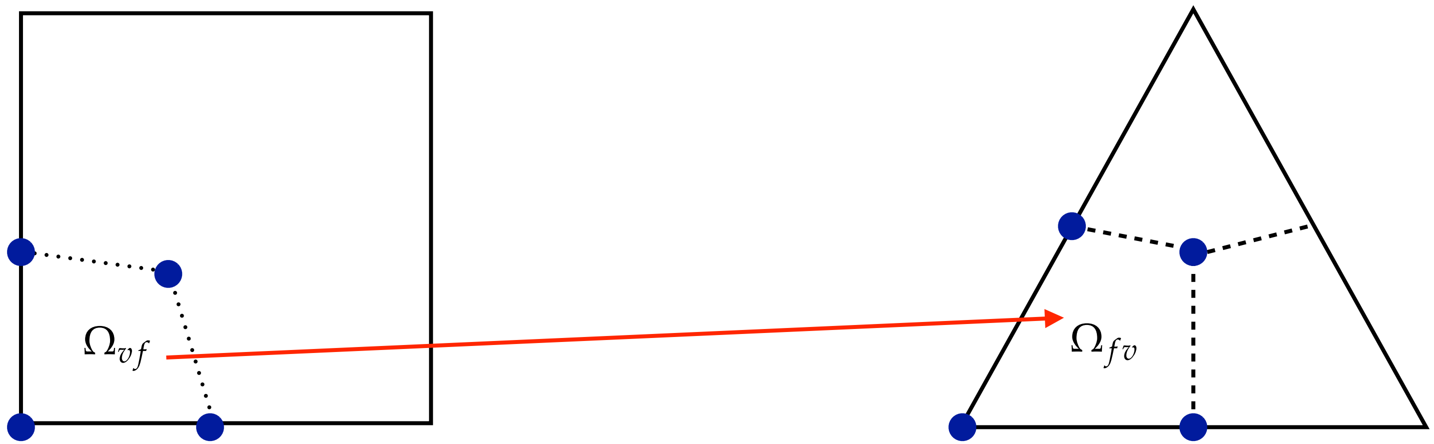


There is a unique affine transformation that takes the Ω_{ef} onto Ω_{fe} after a one-to-one correspondence between the vertices of the triangles corresponding to their closures is established.

Parametric Pseudo-Manifolds

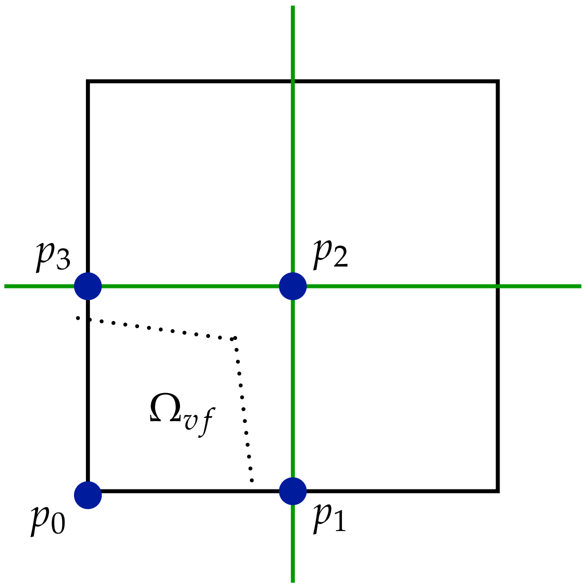
Grimm's Construction of Gluing Data

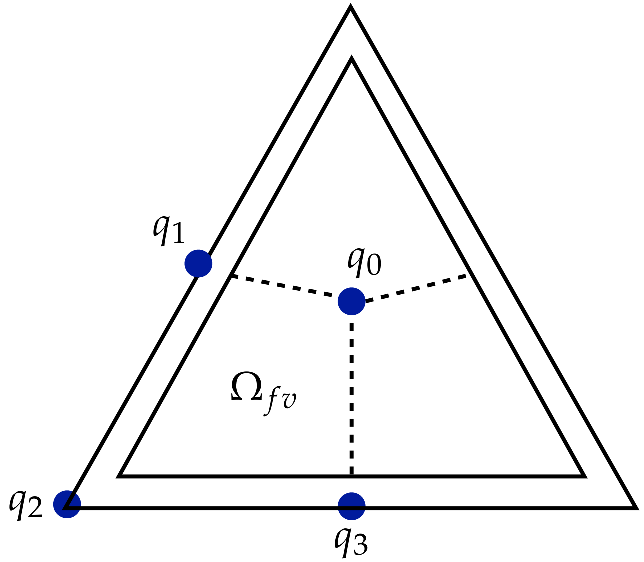
The vertex-face transition map, φ_{fv}:



Grimm's Construction of Gluing Data

To compute the projective transformation that takes Ω_{vf} onto Ω_{fv}, we consider two sets of points. The first contains the points p_0, p_1, p_2, p_3 that define the quadrant of Ω_v containing Ω_{vf}. The second contains the points q_0, q_1, q_2, and q_3, which define q quadrilateral in a regular, n-sided polygon centered at the origin and whose sides have length 1

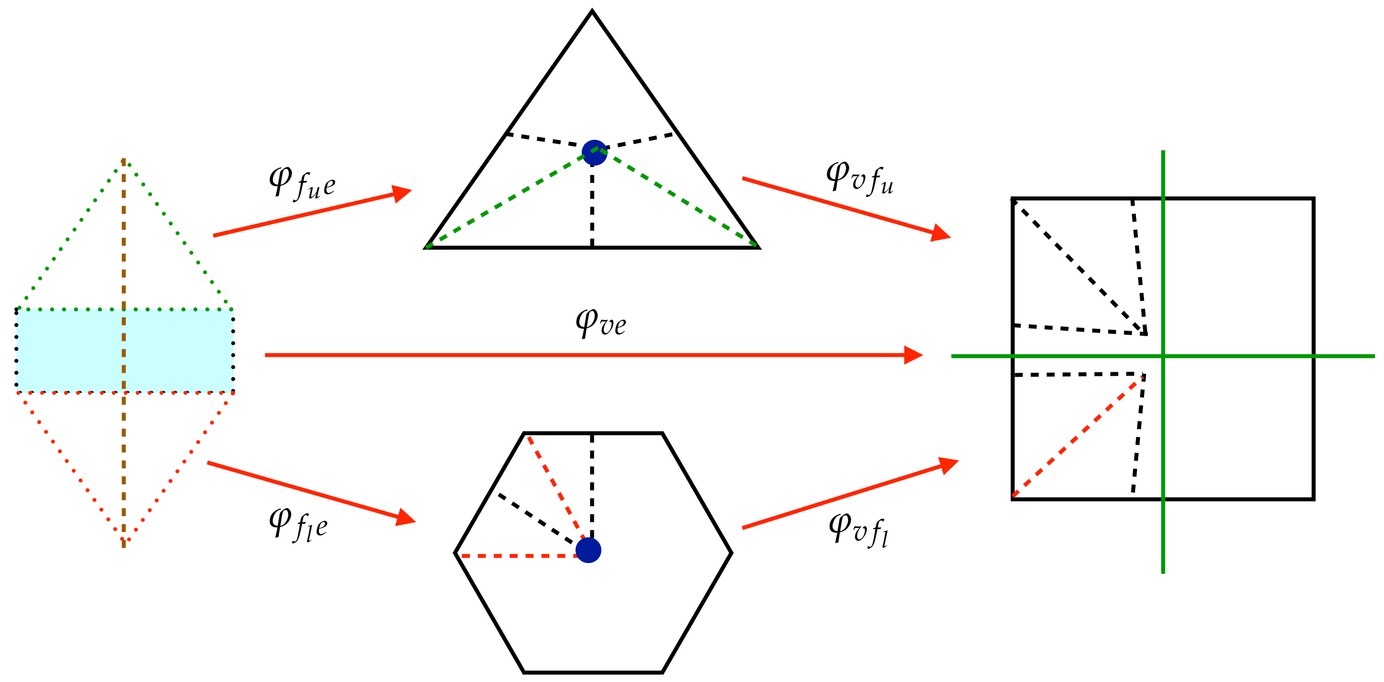




Grimm's Construction of Gluing Data

The domain Ω_{vf} is actually defined as $\varphi_{vf}(\Omega_{fv})$.

The vertex-edge transition map, φ_{ev}:

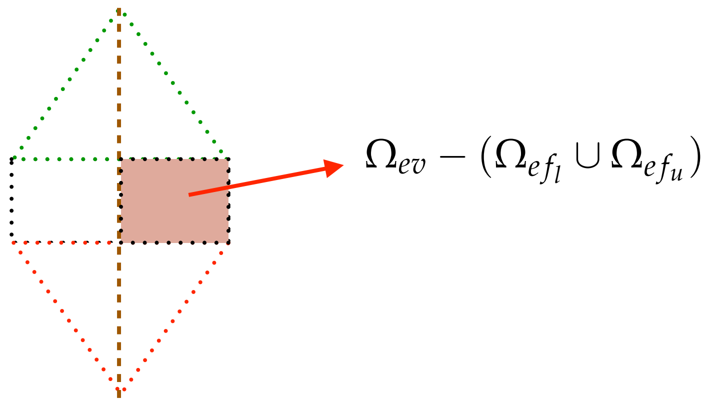


Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

How can we define φ_{ve} so that the cocycle condition holds?

Grimm did not define φ_{ve} in a direct manner. Instead, she forced φ_{ve} to be a weighted sum of two composite functions: $\varphi_{vf_l} \circ \varphi_{f_l e}$ and $\varphi_{vf_u} \circ \varphi_{f_u e}$. Since the domain of these functions are disjoint (in Ω_{ev}), she blended the functions along the region $\Omega_{ev} - (\Omega_{ef_l} \cup \Omega_{ef_u})$.



Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

The idea is to let $\varphi_{ve}(p) = (\varphi_{vf_l} \circ \varphi_{f_le})(p)$ if $p \in (\Omega_{ev} \cap \Omega_{ef_l})$, $\varphi_{ve}(p) = (\varphi_{vf_u} \circ \varphi_{f_ue})(p)$ if $p \in (\Omega_{ev} \cap \Omega_{ef_u})$, and $\varphi_{ve}(p)$ equal to some "average" value if $p \in (\Omega_{ev} - (\Omega_{ef_l} \cup \Omega_{ef_u}))$.

In particular,

$$\varphi_{ve}(p) = (1 - \beta(t)) \cdot (\varphi_{vf_l} \circ \varphi_{f_le})(p) + \beta(t) \cdot (\varphi_{vf_u} \circ \varphi_{f_ue})(p)\,,$$

where $\beta : \mathbb{R} \to [0,1]$ is a function satisfying the following properties:

- $\beta(t) = 0$ for $t < -h \cdot \cot(\pi/n_l)$.
- $\beta(t) = 1$ for $t > h \cdot \cot(\pi/n_u)$.

Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

- $\beta(t)$ is monotonically increasing.
- β is C^k for a given k (the desired continuity of the manifold).
- The derivative of β is bounded by the function

$$\beta'(t) = \begin{cases} \frac{h \cdot \cot(\pi/6)+t}{h \cdot \cot(\pi/6)} & \text{if } t \leq 0 \\ \frac{h \cdot \cot(\pi/6)-t}{h \cdot \cot(\pi/6)} & \text{if } t > 0 \end{cases} .$$

Grimm shows that for $k \geq 0$ the function φ_{ve} is invertible, one-to-one, and onto.

She also tells us how to build the function β.

Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

The choice of a value for the parameter h is related to the geometry of the resulting manifold.

Grimm computes this value by solving the equation

$$\varphi_{vf}(0.5 - h, -h \cdot \cot(\pi/6)) = \left(\frac{\delta_k}{2}, \frac{\delta_k}{2}\right),$$

where

$$\delta_k = \frac{1}{2 \cdot (2k+3)},$$

where k is the degree of continuity of β.

Parametric Pseudo-Manifolds

Grimm's Construction of Gluing Data

There are a few issues with this construction.

First, the number of p-domains is large compared to the number of p-domains in the approach we saw before.

Second, the definition of the map φ_{ve} is not elegant.

Third, the gluing regions are small (compared to the ones in other constructions), which may lead to visual artifacts in the resulting surfaces.