Introduction to Computational Manifolds and Applications

Part 1 - Constructions

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Parametric Pseudo-Manifolds

Transition Maps

We will now study some "candidates" for the g maps of our transition maps.

First, we will consider projective transformations in $\mathbb{RP}^2$.

Next, we will review some simple conformal maps.

Both maps above do not fulfill all requirements for the role of the g maps. But, if we allow a slight change in the geometry of the p-domains, *simple* conformal maps can do the job.

Parametric Pseudo-Manifolds

Projective Transformations

Our goal now is to define a *projective transformation*, $T : \mathbb{RP}^2 \to \mathbb{RP}^2$, that maps $\overset{\circ}{Q}_{uw}$ onto $\overset{\circ}{Q}$.

Recall that a family, $(a_i)_{1 \leq i \leq n+2}$, of $n+2$ points of the projective space $\mathbb{RP}^n$ is a *projective frame* (or *basis*) of $\mathbb{RP}^n$ if there exists some basis $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+1})$ of $\mathbb{R}^{n+1}$ such that

$$a_i = [\boldsymbol{e}_i]_\sim \,, \quad \text{for } 1 \leq i \leq n+1$$

and

$$a_{n+2} = [\boldsymbol{e}_{n+2}]_\sim \,, \quad \text{where } \boldsymbol{e}_{n+2} = \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_n + \boldsymbol{e}_{n+1} .$$

Any basis with the above property is said to be *associated with the projective frame* $(a_i)_{1 \leq i \leq n+2}$.

Parametric Pseudo-Manifolds

Projective Transformations

For instance,

$$\begin{aligned} \boldsymbol{e}_1 &= (1,0,\ldots,0,0) \\ \boldsymbol{e}_2 &= (0,1,\ldots,0,0) \\ &\vdots \\ \boldsymbol{e}_n &= (0,0,\ldots,1,0) \\ \boldsymbol{e}_{n+1} &= (0,0,\ldots,0,1)\,, \end{aligned}$$

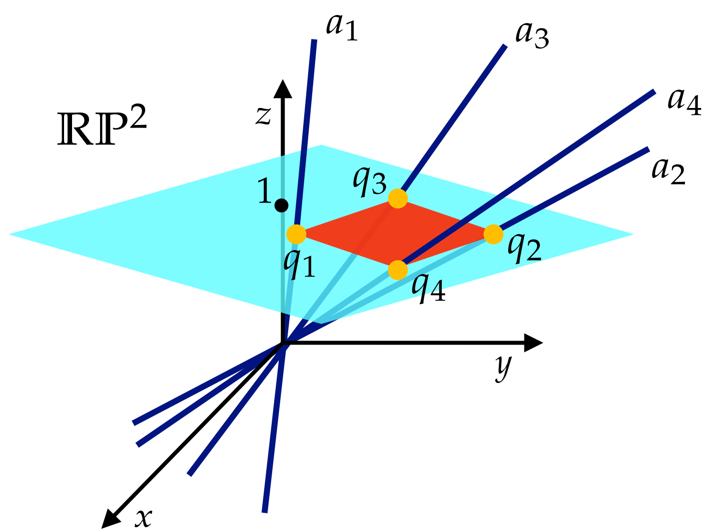
the canonical basis of $\mathbb{R}^{n+1}$, together with the vector $\boldsymbol{e}_{n+2} = \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_{n+1}$, defines a projective frame, $(a_1, \ldots, a_{n+2})$, of $\mathbb{RP}^n$ such that $a_i = [\boldsymbol{e}_i]_\sim$, for every $1 \leq i \leq n+2$.

We can view each a_i as a line in $\mathbb{R}^{n+1}$ passing through the origin in the direction of $\boldsymbol{e}_i$.

Parametric Pseudo-Manifolds

Projective Transformations

Consider $n = 2$.

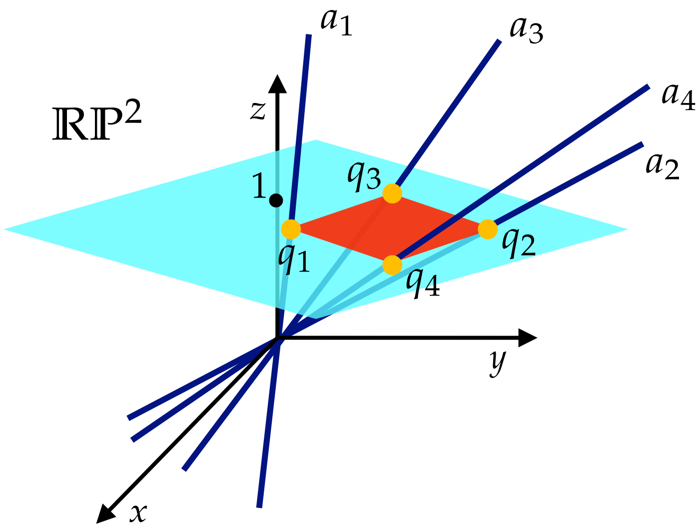


A projective frame in $\mathbb{RP}^2$ consists of four points, a_1, a_2, a_3, and a_4, which correspond to four lines through the origin of $\mathbb{R}^3$. The intersection of these lines and a plane in $\mathbb{R}^3$, e.g., $z = 1$, defines the vertices, q_1, q_2, q_3, and q_4, of a non-degenerate quadrilateral.

Parametric Pseudo-Manifolds

Projective Transformations

Consider $n = 2$.



Conversely, given a non-degenerate quadrilateral with vertices q_1, q_2, q_3, and q_4 in a plane in $\mathbb{R}^3$, e.g., $z = 1$, there is a projective frame consisting of the points a_1, a_2, a_3, and a_4, in $\mathbb{RP}^2$ such that q_i belongs to the line in $\mathbb{R}^3$ associated with a_i, for $i = 1, 2, 3, 4$.

Projective Transformations

Every bijective linear map, $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, induces a function,

$$\boldsymbol{P}(f) : \mathbb{RP}^n \to \mathbb{RP}^n \,,$$

called a *projective transformation*, defined as

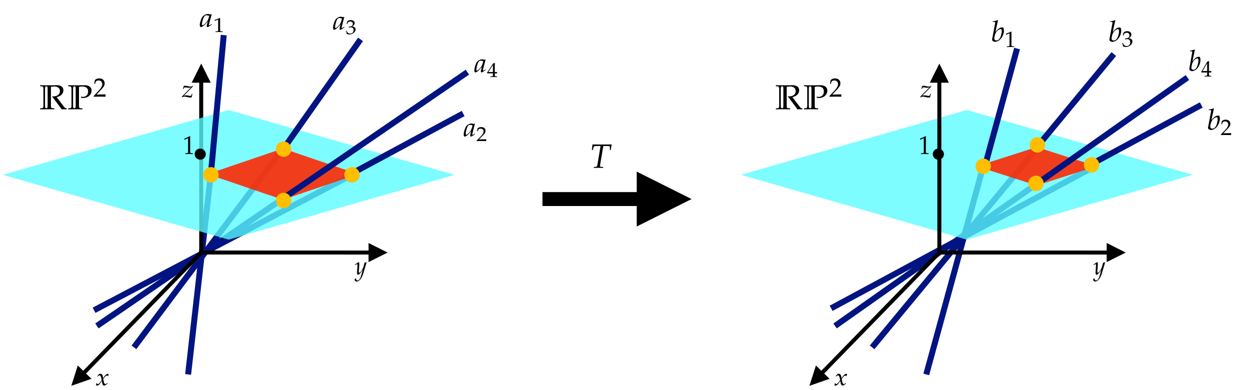
$$\boldsymbol{P}(f)([u]_\sim) = [f(u)]_\sim \,.$$

Parametric Pseudo-Manifolds

Projective Transformations

According to the Fundamental Theorem of Projective Geometry, if we are given any two projective frames, $(a_i)_{1 \le i \le n+2}$ and $(b_i)_{1 \le i \le n+2}$, of $\mathbb{RP}^n$, then there exists a *unique* projective transformation, $T : \mathbb{RP}^n \to \mathbb{RP}^n$, such that $T(a_i) = b_i$, for each $1 \le i \le n+2$.

An immediate consequence of the aforementioned theorem is that there exists a unique projective transformation between two non-degenerate quadrilaterals in the plane $z = 1$.



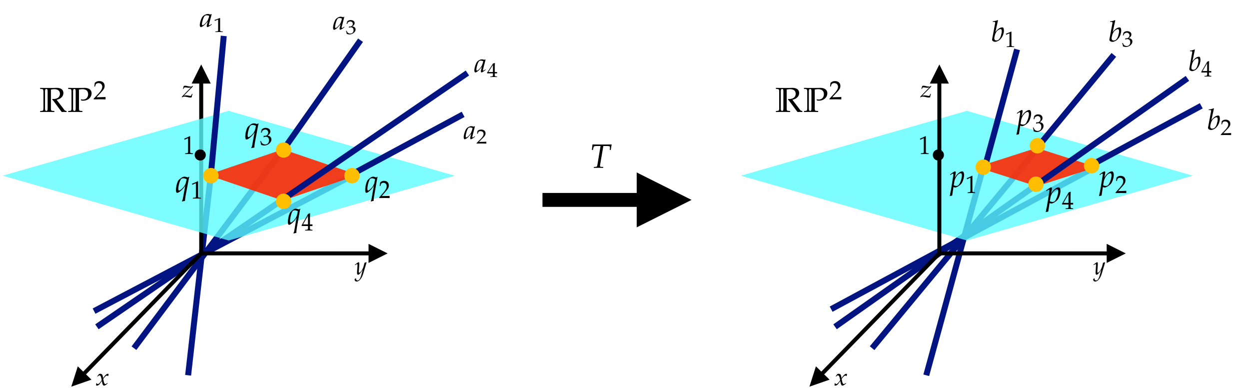
Parametric Pseudo-Manifolds

Projective Transformations

Given any two non-degenerate quadrilaterals,

$$Q_1 = [q_1, q_2, q_3, q_4] \qquad \text{and} \qquad Q_2 = [p_1, p_2, p_3, p_4] \, ,$$

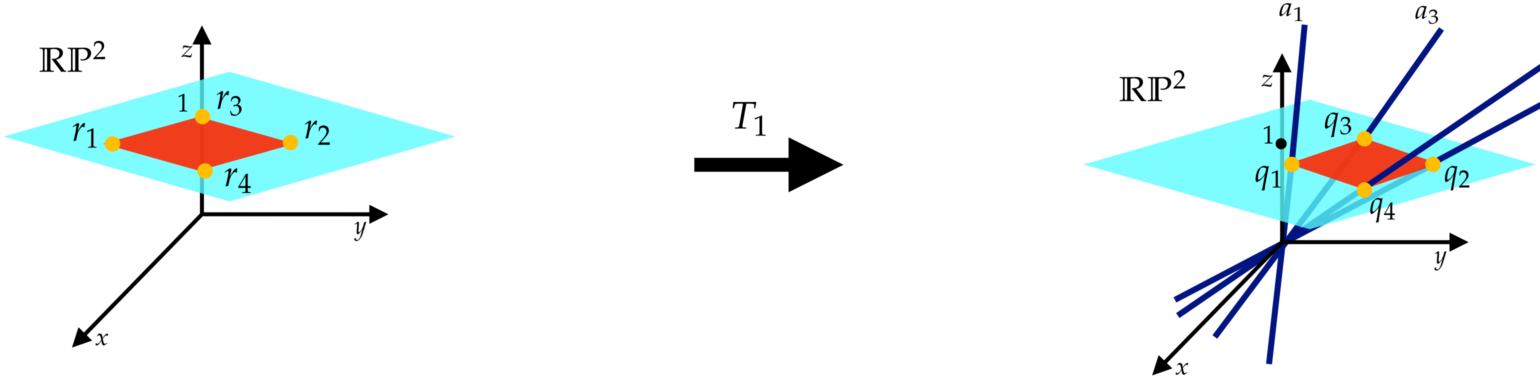
in the plane $z = 1$, the projective transformation, $T : \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$, that maps Q_1 to Q_2 can be computed in three steps as the composition of two projective transformations.



Parametric Pseudo-Manifolds

Projective Transformations

First, we compute the projective transformation, $T_1 : \mathbb{RP}^2 \to \mathbb{RP}^2$, that maps the square, $Q = [r_1, r_2, r_3, r_4]$, where $r_1 = (1,0,1)$, $r_2 = (0,1,1)$, $r_3 = (0,0,1)$, and $r_4 = (1,1,1)$ to the quadrilateral Q_1. In order to do so, we view T_1 as a linear map that takes r_i to a point in the line passing through the origin and q_i, for each $i = 1,2,3,4$.



Parametric Pseudo-Manifolds

Projective Transformations

Since (r_1, r_2, r_3, r_4) and (q_1, q_2, q_3, q_4) are non-degenerate quadrilaterals, we have that (r_1, r_2, r_3) and (q_1, q_2, q_3) are linearly independent. Furthermore, as points of the plane H of equation $z = 1$, they are also affinely independent. So, we can write r_4 and q_4 as

$$r_4 = r_1 + r_2 - r_3$$

and

$$q_4 = \lambda_1 q_1 + \lambda_2 q_2 + \lambda_3 q_3$$

for some unique scalars $\lambda_1, \lambda_2, \lambda_3$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

Parametric Pseudo-Manifolds

Projective Transformations

In fact, $\lambda_1, \lambda_2, \lambda_3$ are solutions of the system

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} x_4 \\ y_4 \\ 1 \end{pmatrix},$$

where $q_1 = (x_1, y_1, 1)$, $q_2 = (x_2, y_2, 1)$, $q_3 = (x_3, y_3, 1)$, $q_4 = (x_4, y_4, 1)$ are the coordinates of q_1, q_2, q_3, q_4 with respect to the basis (r_1, r_2, r_3). Furthermore, since (r_1, r_2, r_3, r_4) and (q_1, q_2, q_3, q_4) are non-degenerate quadrilaterals, we get $\lambda_i \neq 0$ for $i = 1, 2, 3$.

Parametric Pseudo-Manifolds

Projective Transformations

Let $a_1 = r_1$, $a_2 = r_2$, $a_3 = -r_3$, and let $b_1 = \lambda_1 q_1$, $b_2 = \lambda_2 q_2$, $b_3 = \lambda_3 q_3$, so that

$$r_4 = a_4 = a_1 + a_2 + a_3$$

and

$$q_4 = b_4 = b_1 + b_2 + b_3 \,.$$

Parametric Pseudo-Manifolds

Projective Transformations

Since r_1, r_2, r_3 are linearly independent, we know that there is a unique linear map,

$$f : \mathbb{R}^3 \to \mathbb{R}^3$$

such that

$$f(a_1) = b_1 , \quad f(a_2) = b_2 , \quad \text{and} \quad f(a_3) = b_3 ,$$

and by linearity,

$$f(r_4) = f(a_1 + a_2 + a_3) = f(a_1) + f(a_2) + f(a_3) = b_1 + b_2 + b_3 = q_4 .$$

Parametric Pseudo-Manifolds

Projective Transformations

With respect to the basis (r_1, r_2, r_3), we have

$$f(r_1) = b_1 , \qquad f(r_2) = b_2 \qquad \text{and} \qquad f(r_3) = -b_3 .$$

So, with respect to the basis (r_1, r_2, r_3), the associated matrix, A, of the map f is

$$A = \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 & -\lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & -\lambda_3 y_3 \\ \lambda_1 & \lambda_2 & -\lambda_3 \end{pmatrix} .$$

Parametric Pseudo-Manifolds

Projective Transformations

The change of basis matrix P from the canonical basis (e_1, e_2, e_3) to the basis (u_1, u_2, u_3) is

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

and its inverse is

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

Parametric Pseudo-Manifolds

Projective Transformations

If we assume that we pick the coordinates of q_1, q_2, q_3, q_4 with respect to the *canonical basis*, the matrix of our linear map with respect to the canonical basis is the unique matrix A' that maps each column u_1, u_2, and u_3 of the matriz P to the corresponding column of the matrix A representing v_1, v_2, and v_3 over the canonical basis, namely

$$A = \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix},$$

and this it must be given by

$$A' = A \cdot P^{-1} = AP .$$

Parametric Pseudo-Manifolds

Projective Transformations

That is,

$$A' = \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 & -\lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & -\lambda_3 y_3 \\ \lambda_1 & \lambda_2 & -\lambda_3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 x_1 + \lambda_3 x_3 & \lambda_2 x_2 + \lambda_3 x_3 & -\lambda_3 x_3 \\ \lambda_1 y_1 + \lambda_3 y_3 & \lambda_2 y_2 + \lambda_3 y_3 & -\lambda_3 y_3 \\ \lambda_1 + \lambda_3 & \lambda_2 + \lambda_3 & -\lambda_3 \end{pmatrix}.$$

Parametric Pseudo-Manifolds

Projective Transformations

Since

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ x+y-1 \end{pmatrix},$$

if we want to represent the restriction of the projective transformation to the plane H (in the *canonical basis*), we can also apply the matrix A to the point in $\mathbb{R}^3$ of coordinates

$$\begin{pmatrix} x \\ y \\ x+y-1 \end{pmatrix}.$$

Parametric Pseudo-Manifolds

Projective Transformations

Thus, we can define $T_1 : \mathbb{RP}^2 \to \mathbb{RP}^2$ as $T_1(s) = A_1 \cdot s$, for every $s \in \mathbb{R}^3$, where

$$A_1 = \begin{pmatrix} \lambda_1 x_1 + \lambda_3 x_3 & \lambda_2 x_2 + \lambda_3 x_3 & -\lambda_3 \cdot x_3 \\ \lambda_1 y_1 + \lambda_3 y_3 & \lambda_2 y_2 + \lambda_3 y_3 & -\lambda_3 \cdot y_3 \\ \lambda_1 + \lambda_3 & \lambda_2 + \lambda_3 & -\lambda_3 \end{pmatrix},$$

and the coordinates of $s \in \mathbb{R}^3$ is given with respect to the *canonical basis*, (e_1, e_2, e_3).

Parametric Pseudo-Manifolds

Projective Transformations

So, if $s = (x, y, 1) \in Q$, then we get $t = T_1(s) = (x', y', 1)$ such that x' and y' are

$$x' = \frac{(\lambda_1 x_1 + \lambda_3 x_3)x + (\lambda_2 x_2 + \lambda_3 x_3)y - \lambda_3 x_3}{(\lambda_1 + \lambda_3)x + (\lambda_2 + \lambda_3)y - \lambda_3}$$

$$y' = \frac{(\lambda_1 y_1 + \lambda_3 y_3)x + (\lambda_2 y_2 + \lambda_3 y_3)y - \lambda_3 y_3}{(\lambda_1 + \lambda_3)x + (\lambda_2 + \lambda_3)y - \lambda_3} \, .$$

Parametric Pseudo-Manifolds

Projective Transformations

We can proceed in a similar manner to define the map $T_2 : \mathbb{RP}^2 \to \mathbb{RP}^2$ taking Q onto Q_2.

The second step consists of defining the map $T_2 : \mathbb{RP}^2 \to \mathbb{RP}^2$ taking Q onto Q_2. We can proceed as before, but using p_1, p_2, p_3, and p_4 instead of q_1, q_2, q_3, and q_4, respectively.

The third step consists of defining the map T. This is done by noticing that T_1 is a bijection, as A_1 is invertible. So, T_1^{-1} maps Q_1 onto Q, and hence we define the map T as

$$T(p) = (T_2 \circ T_1^{-1})(p) = A_2 \cdot A_1^{-1} \cdot p\,,$$

for every $p \in Q_1$, where A_2 is the matrix associated with the projective transformation T_2.

Parametric Pseudo-Manifolds

Projective Transformations

Can the transformation T play the role of our g map in our transition functions?

The map T is definitely a C^∞-diffeomorphism of the plane (viewed as the plane $z = 1$ in $\mathbb{R}^3$).

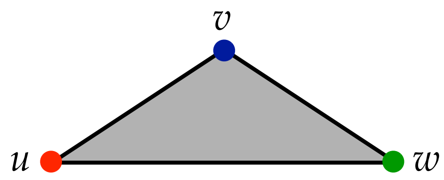
Furthermore, T maps $\mathring{Q}_{uw}$ onto $\mathring{Q}$, while T^{-1} maps $\mathring{Q}$ onto $\mathring{Q}_{uw}$.

However, the map T does not satisfies the cocycle condition.

Parametric Pseudo-Manifolds

Projective Transformations

To see why, consider a triangle, $\sigma = [u, v, w]$ of $\mathcal{K}$, such that $n_u = 5$, $n_v = 6$, and $n_w = 7$.



By construction,

$$r_{uw}(Q_{uw}) = r_{uv}(Q_{uv}) = \left[(0,0), \left(\cos\left(-\frac{2\pi}{5}\right), \sin\left(-\frac{2\pi}{5}\right)\right), (1,0), \left(\cos\left(\frac{2\pi}{5}\right), \sin\left(\frac{2\pi}{5}\right)\right)\right],$$

$$r_{vu}(Q_{vu}) = r_{vw}(Q_{vw}) = \left[(0,0), \left(\cos\left(-\frac{\pi}{3}\right), \sin\left(-\frac{\pi}{3}\right)\right), (1,0), \left(\cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right)\right)\right],$$

$$r_{wv}(Q_{wv}) = r_{wu}(Q_{wu}) = \left[(0,0), \left(\cos\left(-\frac{2\pi}{7}\right), \sin\left(-\frac{2\pi}{7}\right)\right), (1,0), \left(\cos\left(\frac{2\pi}{7}\right), \sin\left(\frac{2\pi}{7}\right)\right)\right].$$

Parametric Pseudo-Manifolds

Projective Transformations

We define

$$g_u : \mathbb{E}^2 \to \mathbb{E}^2, \quad g_v : \mathbb{E}^2 \to \mathbb{E}^2, \quad \text{and} \quad g_w : \mathbb{E}^2 \to \mathbb{E}^2$$

as the projective maps that takes $r_{uw}(Q_{uw})$, $r_{vu}(Q_{vu})$, and $r_{wv}(Q_{wv})$ onto Q, respectively, where

$$Q = \left[(0,0), \left(\cos\left(-\frac{\pi}{3}\right), \sin\left(-\frac{\pi}{3}\right)\right), (1,0), \left(\cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right)\right)\right] .$$

Parametric Pseudo-Manifolds

Projective Transformations

The matrices associated with the g_u and g_u^{-1} maps are:

$$\begin{pmatrix} 1.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.562777 & 0.000000 \\ 0.552786 & 0.000000 & 0.447214 \end{pmatrix}$$

and

$$\begin{pmatrix} 1.000000 & 0.000000 & 0.000000 \\ 0.000000 & 1.776900 & 0.000000 \\ -1.236070 & 0.000000 & 2.236070 \end{pmatrix},$$

respectively.

The matrices associated with the g_v and g_v^{-1} maps are the identity matrix.

Parametric Pseudo-Manifolds

Projective Transformations

The matrices associated with the g_w and g_w^{-1} maps are

$$\begin{pmatrix} 1.000000 & 0.000000 & 0.000000 \\ 0.000000 & 1.381260 & 0.000000 \\ -0.655971 & 0.000000 & 1.655970 \end{pmatrix}$$

and

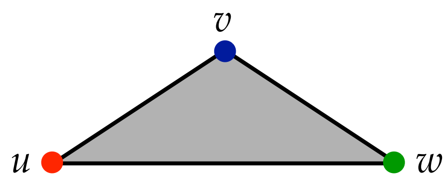
$$\begin{pmatrix} 1.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.723974 & 0.000000 \\ 0.396125 & 0.000000 & 0.603875 \end{pmatrix},$$

respectively.

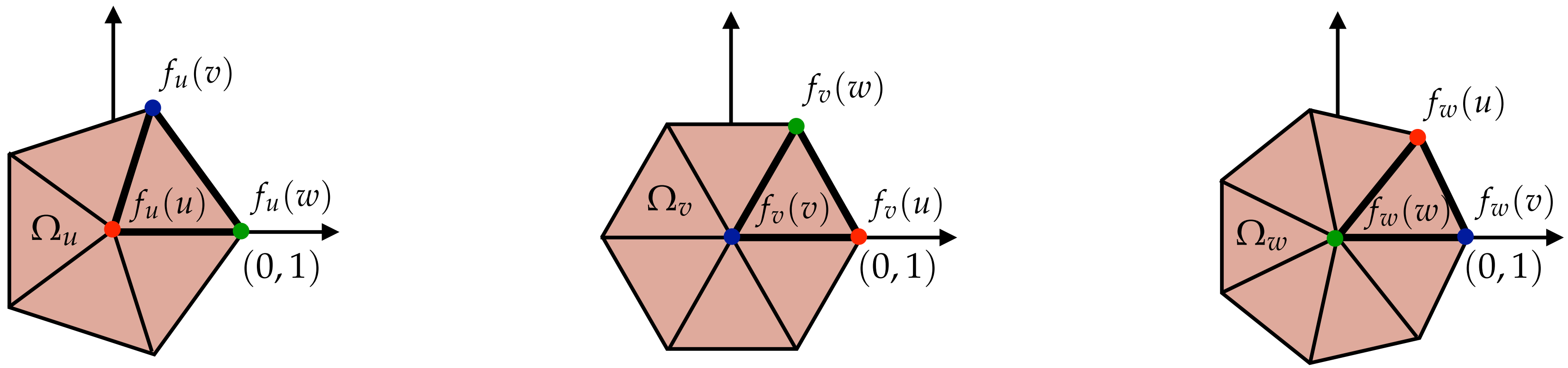
Parametric Pseudo-Manifolds

Projective Transformations

Suppose that w precedes v in a counterclockwise enumeration of the vertices in $\mathrm{lk}(u, \mathcal{K})$.



Suppose also that the p-domains are defined as below:



Parametric Pseudo-Manifolds

Projective Transformations

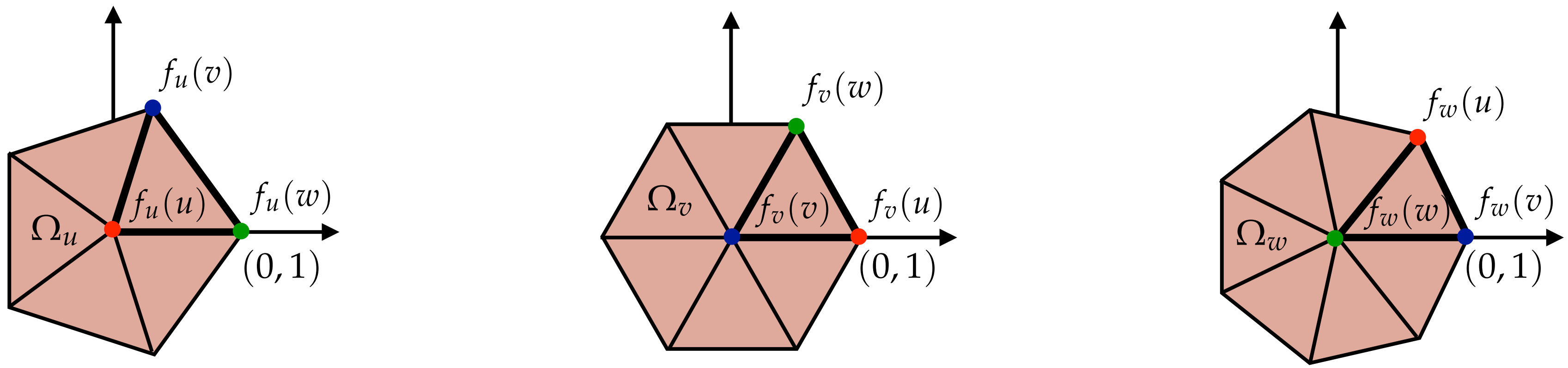
So,

$$\varphi_{vu}(x) = (g_v^{-1} \circ h \circ g_u \circ r_{-\frac{2\pi}{5}})(x), \quad \text{for all } x \in \Omega_{uv},$$

$$\varphi_{wu}(x) = (r_{\frac{2\pi}{7}} \circ g_w^{-1} \circ h \circ g_u)(x), \quad \text{for all } x \in \Omega_{uw},$$

and

$$\varphi_{vw}(x) = (r_{\frac{\pi}{3}} \circ g_v^{-1} \circ h \circ g_w)(x), \quad \text{for all } x \in \Omega_{wv}.$$



Parametric Pseudo-Manifolds

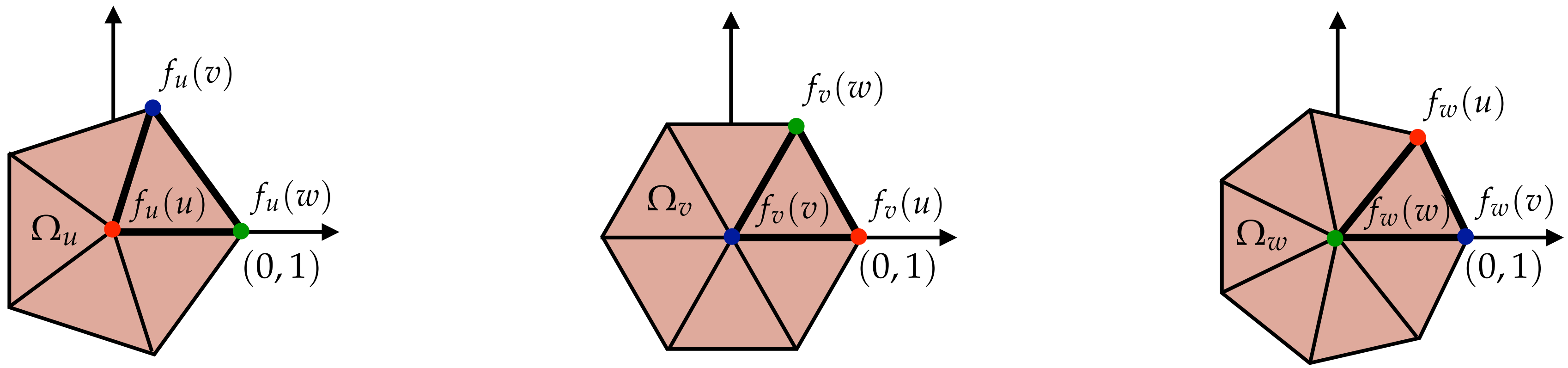
Projective Transformations

We can show that

$$\varphi_{uw}(\Omega_{wu} \cap \Omega_{wv}) = \Omega_{uv} \cap \Omega_{uw}\,.$$

So, the statement "*if* $\Omega_{wu} \cap \Omega_{wv} \neq \emptyset$ *then* $\varphi_{uw}(\Omega_{wu} \cap \Omega_{wv}) = \Omega_{uv} \cap \Omega_{uw}$" holds. But, it is **not** the case that $\varphi_{vu}(x) = (\varphi_{vw} \circ \varphi_{wu})(x)$, for all $x \in \Omega_{uw} \cap \Omega_{uv}$. For instance, pick

$$x = (0.5, 0.5) \in (\Omega_{uv} \cap \Omega_{uw})\,.$$



Parametric Pseudo-Manifolds

Projective Transformations

Indeed,

$$\varphi_{vu}(0.5, 0.5) = (0.207988, 0.227109)\,,$$

while

$$(\varphi_{vw} \circ \varphi_{wu})(0.5, 0.5) = (0.363339, 0.433479)\,.$$

It is worth noticing that map g_u is a C^∞-diffeomorphism of the plane. Furthermore, it maps $\mathring{Q}_{uv}$ onto $\mathring{Q}$, the canonical quadrilateral. But, the cocycle condition does not hold.

As a matter of fact, the map g_u does not satisfy $(g_u \circ r_{\frac{2\pi}{n_u}} \circ g_u^{-1})(x) = r_{\frac{\pi}{3}}$, for $q \in g_u(\Omega_u)$.

The map g_u does not satisfy $(g_u \circ r_{uw})(x) = (r_{\frac{\pi}{3}} \circ g_u \circ r_{uv})(x)$, for all $x \in \Omega_{uv}$ either.

Parametric Pseudo-Manifolds

Complex Functions as Mappings

We will now consider some elementary functions in one complex variable.

These functions can be viewed as mappings from one plane to the other.

So, we will investigate how they can play the role of the g map in our transition functions.

As we shall see, we will not succeed unless we change the geometry of the p-domains.

Parametric Pseudo-Manifolds

Complex Functions as Mappings

Let us recall a few elementary definitions...

A number of the form

$$z = x + i\,y\,,$$

where x and y are real numbers and i is a number such that $i^2 = -1$ is called a *complex number*. The number i is called the *imaginary unit*, and the numbers x and y are called the real part and the imaginary part of z, denoted by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively.

A complex number $z = x + i\,y$ is uniquely defined determined by an *ordered pair* of real numbers, (x, y). The first and second entries of the ordered pairs correspond to the real and imaginary parts of z. Conversely, $z = x + i\,y$ uniquely determines (x, y).

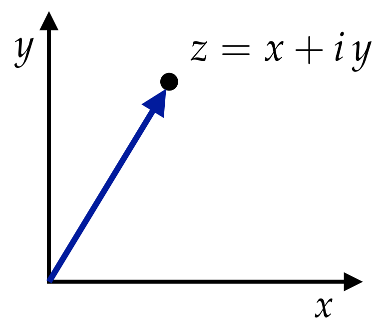
Parametric Pseudo-Manifolds

Complex Functions as Mappings

Since (x, y) can be interpreted as the components of a vector, a complex number

$$z = x + i\,y$$

can be viewed as a vector whose initial point is the origin and whose terminal point is (x, y).



The above coordinate plane is called the *complex plane* or simply the *z-plane*. The horizontal or x-axis is called the *real axis* and the vertical or y-axis is called the *imaginary axis*.

Parametric Pseudo-Manifolds

Complex Functions as Mappings

The *modulus* or *absolute value* of $z = x + i\,y$, denoted by $|z|$, is the real number

$$|z| = \sqrt{x^2 + y^2}\,.$$

A point (x, y) in rectangular coordinates has the polar description, (r, θ), where x, y, r, and θ are related by $x = r \cdot \cos(\theta)$ and $y = r \cdot \sin(\theta)$. Thus, a nonzero complex number,

$$z = x + i\,y\,,$$

can be written as

$$z = r \cdot \cos(\theta) + i\,r \cdot \sin(\theta) = r \cdot \big(\cos(\theta) + i\,\sin(\theta)\big)\,,$$

which is the *polar form* of the complex number z. The angle θ is the *argument*, $\arg(z)$, of z.

Complex Functions as Mappings

The polar form can be extremely convenient for certain operations on complex numbers.

If

$$z_1 = r_1 \cdot \big(\cos(\theta_1) + i\ \sin(\theta_1)\big) \quad \text{and} \quad z_2 = r_2 \cdot \big(\cos(\theta_2) + i\ \sin(\theta_2)\big)$$

are any two complex numbers, then the complex numbers $z_1 \cdot z_2$ and $\frac{z_1}{z_2}$ are equal to

$$z_1 \cdot z_2 = r_1 \cdot r_2 \cdot \big(\cos(\theta_1 + \theta_2) + i\ \sin(\theta_1 + \theta_2)\big)$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot \big(\cos(\theta_1 - \theta_2) + i\ \sin(\theta_1 - \theta_2)\big)\,.$$

Parametric Pseudo-Manifolds

Complex Functions as Mappings

Also, for any integer n and for any complex number $z = r \cdot \big(\cos(\theta) + i\ \sin(\theta)\big)$, we get

$$z^n = r^n \cdot \big(\cos(n \cdot \theta) + i\ \sin(n \cdot \theta)\big)\,,$$

the *n^{th} power*, z^n, of z. In particuar, when $z = \cos(\theta) + i\ \sin(\theta)$, we have $|z| = r = 1$ and

$$\big(\cos(n \cdot \theta) + i\ \sin(n \cdot \theta)\big)^n = \cos(n \cdot \theta) + i\ \sin(n \cdot \theta)\,.$$

Parametric Pseudo-Manifolds

Complex Functions as Mappings

If $z = x + i\,y$ is a complex number, then

$$e^z = e^{x+i\,y} = e^x \cdot \big(\cos(y) + i\,\sin(y)\big)$$

is the *exponential* of z. Note that e^z reduces to e^x when $y = 0$. Moreover, if $z = r \cdot \big(\cos(\theta) + i\,\sin(\theta)\big)$ is the polar form of the complex number z, then we have that $z = r \cdot e^{i\,\theta}$, as

$$e^{i\,\theta} = e^0 \cdot \cos(\theta) + i\,\sin(\theta) = \cos(\theta) + i\,\sin(\theta)\,.$$

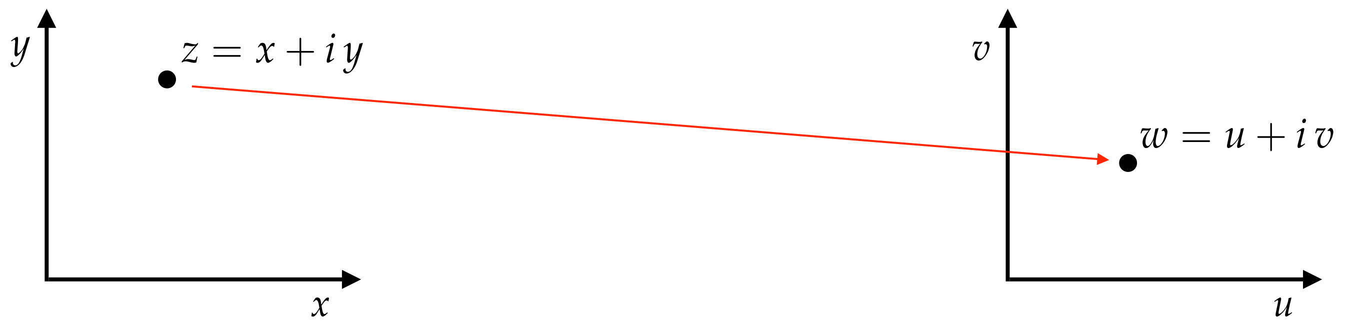
Parametric Pseudo-Manifolds

Complex Functions as Mappings

A function f defined on a set of complex numbers is called a function of a complex variable z or a complex function. The image w of z will be some complex number, $u + i\,v$, i.e.,

$$w = f(z) = u(x, y) + i\,v(x, y)\,,$$

where u and v are the imaginary parts of w and are real-valued functions. Obviously, *we cannot draw the graph of the complex function* $w = f(z)$ *with less than four axes*. However, we can interpret f as a mapping or transformation from the z-plane to the w-plane.



Parametric Pseudo-Manifolds

Complex Functions as Mappings

For the function

$$f(z) = z^2,$$

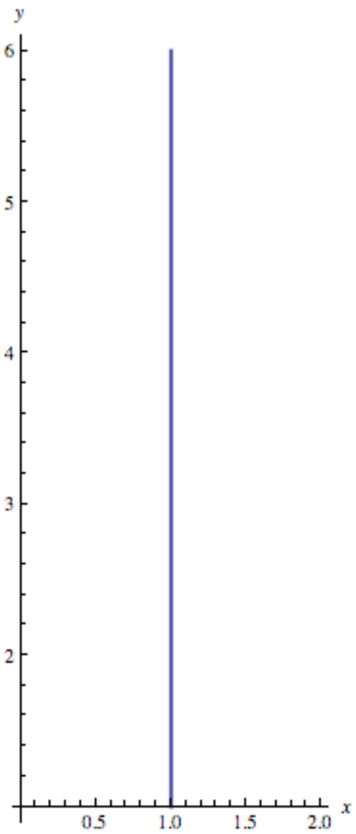
the image of the line $\text{Re}(z) = 1$ is a curve. Indeed, if we write z as $= x + i\,y$, then

$$z^2 = (x^2 - y^2) + i\,2xy \Longrightarrow f(z) = u(x, y) + i\,v(x, y),$$

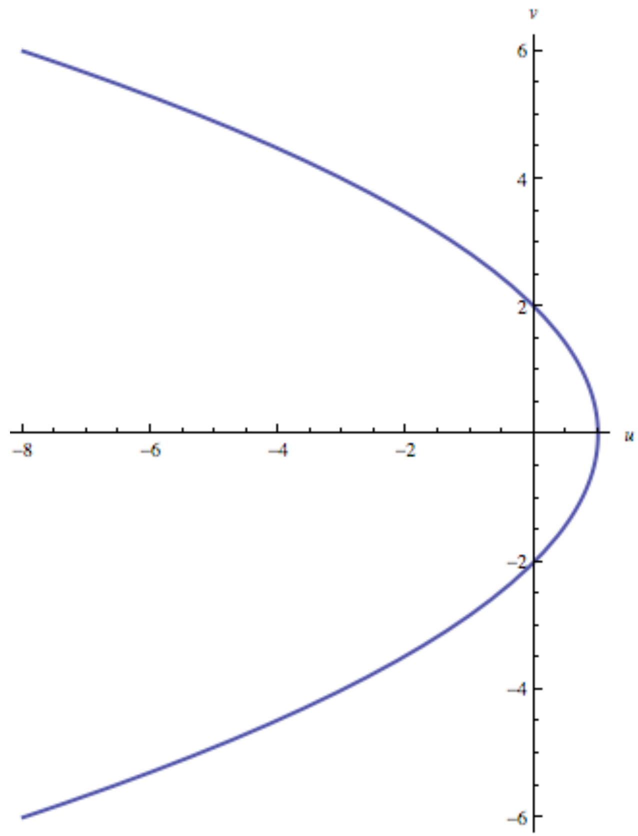
with $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Since $\text{Re}(z) = 1$, substituting $x = 1$ into u and v, we get $u = 1 - y^2$ and $v = 2y$. These parametric equations of a curve in the w-plane.

Parametric Pseudo-Manifolds

Complex Functions as Mappings



$\text{Re}(z) = 1$



$f(\text{Re}(z))$

Parametric Pseudo-Manifolds

Complex Functions as Mappings

In general, if $z(t) = x(t) + i\,y(t)$, with $a \leq t \leq b$, describes a curve C is the z-plane, then $w = f(z(t))$ is a parametric representation of the corresponding curve, C', in the w-plane.

Now, let us see some elementary maps.

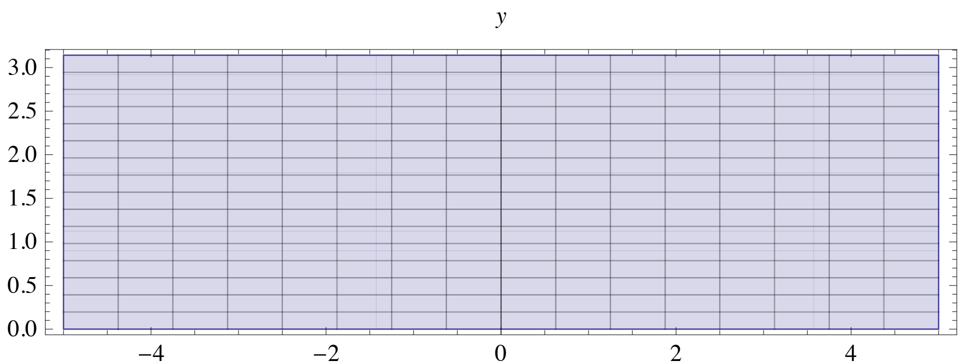
The mapping $f(z) = e^z$:

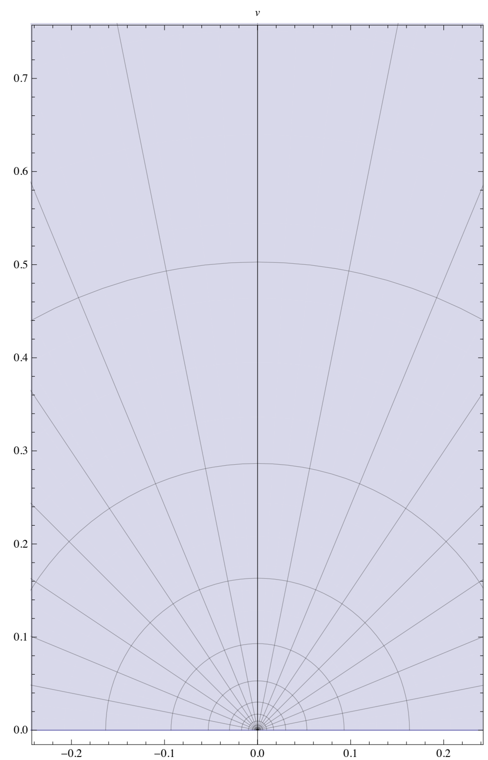
Recall that if $z = x + i\,y$ then $f(z) = e^z = e^x \cdot \big(\cos(y) + i\,\sin(y)\big)$.

Parametric Pseudo-Manifolds

Complex Functions as Mappings

A vertical line segment $x = a$ in the upper half of the z-plane can be described by the curve $z(t) = a + i\,t$, for $0 \leq t \leq \pi$. So, we get $f(z(t)) = e^{a} \cdot e^{i\,t}$. This means that the image of the line segment $z(t)$ is a semi-circle with center at $w = a$ and with radius $r = e^{a}$.

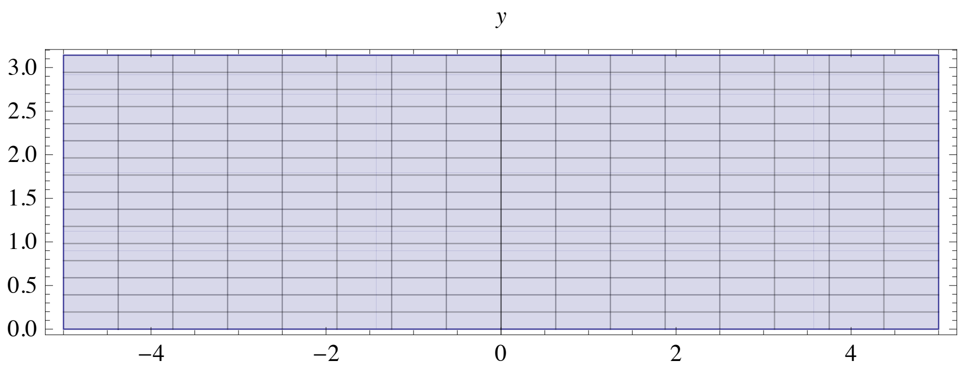


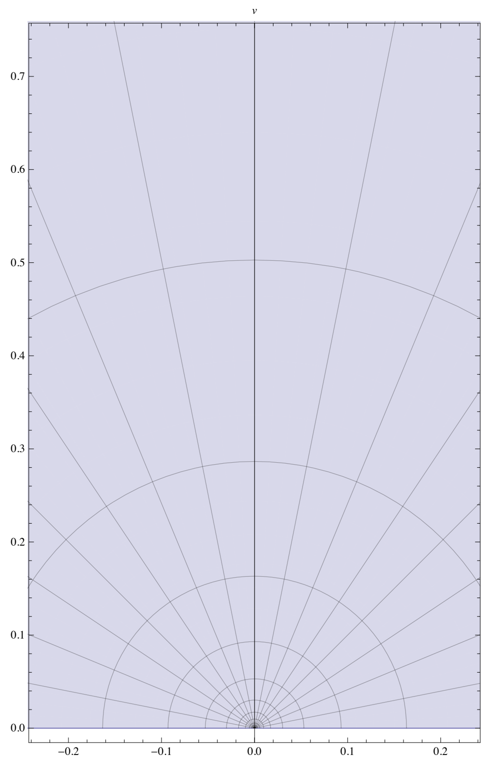


Parametric Pseudo-Manifolds

Complex Functions as Mappings

Similarly, a horizontal line $y = b$ can be parametrized by $z(t) = t + i\,b$, with $-\infty < t < \infty$, and so $f(z(t)) = e^t \cdot e^{i\,b}$. Since $\arg(w) = b$ and $|w| = e^t$, the image is a ray emanating from the origin. Because $0 \leq \arg(z) \leq \pi$, the image of the entire horizontal strip, $\{x + i\,y \mid -\infty \leq x \leq \infty \text{ and } 0 \leq y \leq \pi\}$, is the upper half-plane $v \geq 0$.





Parametric Pseudo-Manifolds

Complex Functions as Mappings

Unlike the real function e^x, the complex function $f(z) = e^z$ is **periodic** with the complex period $i\,2\pi$. Indeed, since $e^{i\,2\pi} = \cos(2\pi) + i\,\sin(2\pi) = 1$, we must have that

$$e^{z+i\,2\pi} = e^z \cdot e^{i\,2\pi} = e^z\,,$$

for all z. So,

$$f(z + i\,2\pi) = f(z)\,.$$

Parametric Pseudo-Manifolds

Complex Functions as Mappings

The elementary function $f(z) = z + z_0$ may be interpreted as a translation in the z-plane.

In turn, the elementary function $g(z) = e^{i\,\theta_0} \cdot z$ may be interpreted as a rotation through θ_0 degrees. Indeed, if we let z be the complex number $z = r \cdot e^{i\,\theta_0}$, then we get

$$w = g(z) = r \cdot e^{i\,(\theta+\theta_0)}\,.$$

Finally, if the complex mapping

$$h(z) = e^{i\,\theta_0} \cdot z + z_0$$

is applied to a region R that is centered at the origin, then the image region R' may be obtained by first rotating R through θ_0 degrees and then translating the center to z_0.

Parametric Pseudo-Manifolds

Complex Functions as Mappings

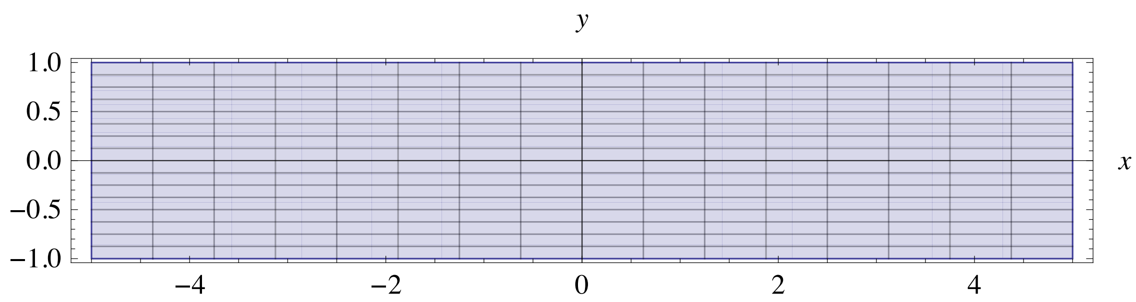
For instance,

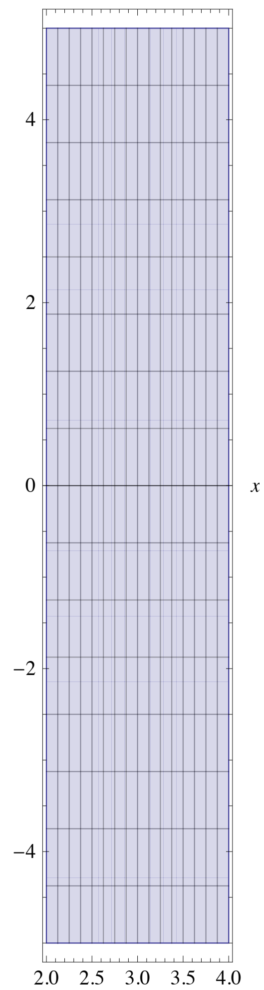
$$h(z) = i\,z + 3$$

maps the horizontal strip $-1 \leq y \leq 1$ onto the vertical strip $2 \leq x \leq 4$. Indeed, if the horizontal strip $-1 \leq x \leq 1$ is rotated through 90^{o} (i.e., $e^{i\,\pi/2} = i$), then the vertical $-1 \leq x \leq 1$ results. Finally, a translation of 3 units to the right yields the vertical strip $2 \leq x \leq 4$.

Parametric Pseudo-Manifolds

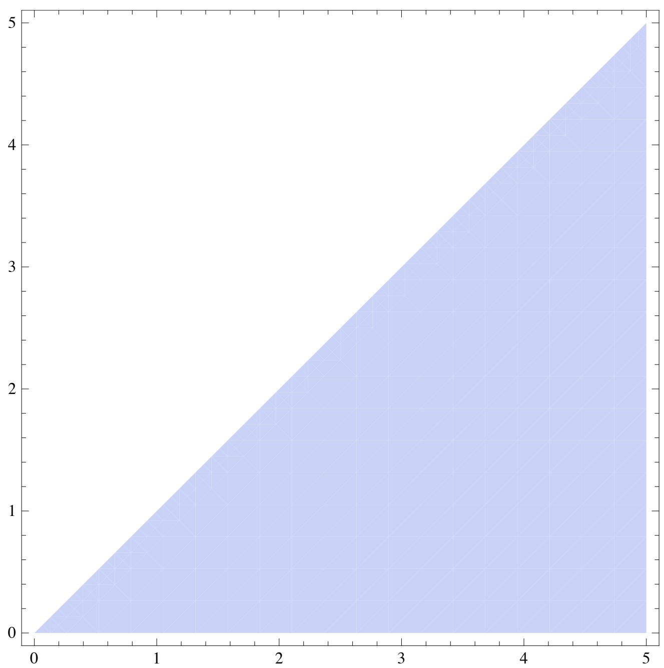
Complex Functions as Mappings

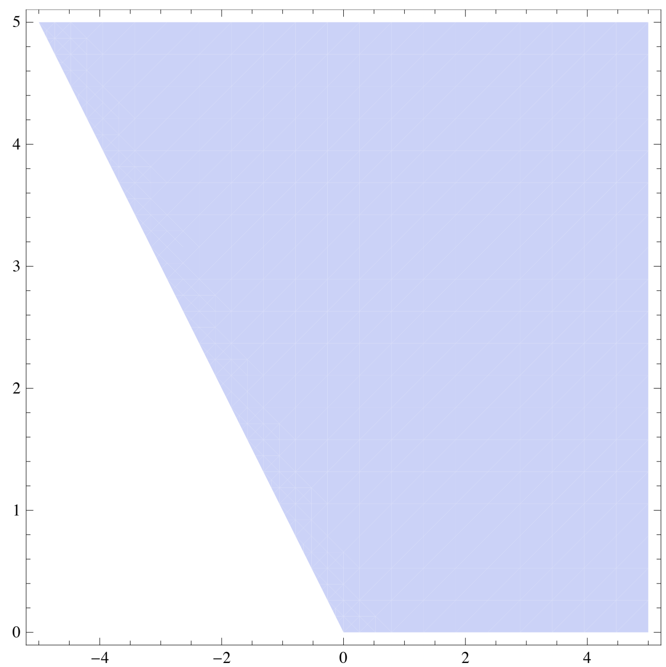




Complex Functions as Mappings

A complex function of the form $f(z) = z^{\alpha}$, where α is a fixed positive *real* number, is called a real power function. If $z = r \cdot e^{i\,\theta}$, then $w = f(z) = r^{\alpha} \cdot e^{i\,\alpha \cdot \theta}$. Since $0 \leq \arg(w) \leq \alpha \cdot \theta_0$, function f opens or contracts the wedge $0 \leq \arg(z) \leq \theta_0$ by a factor of α.

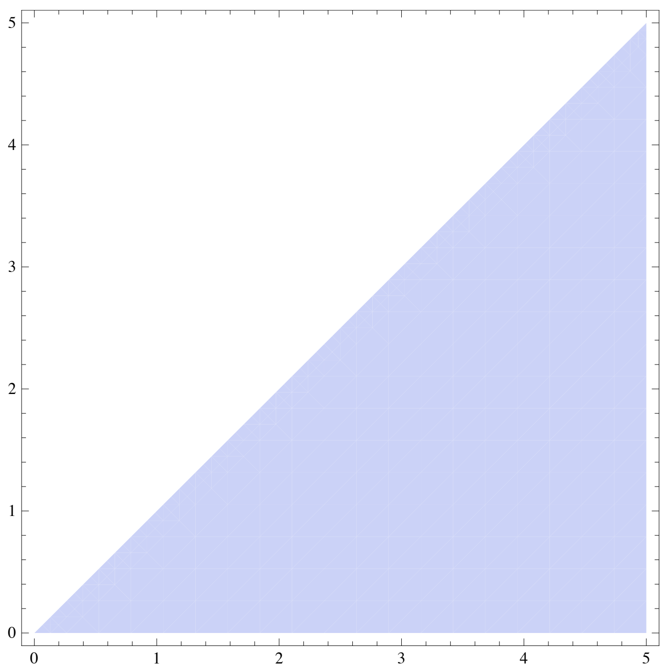


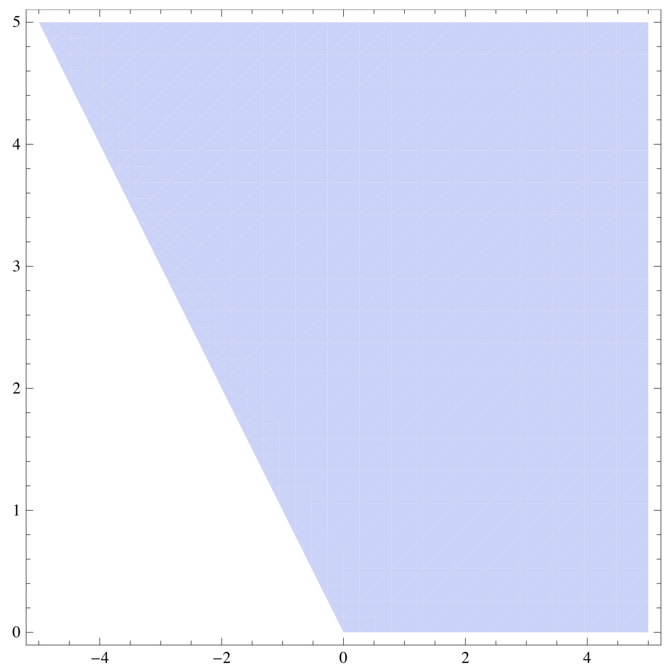


Parametric Pseudo-Manifolds

Complex Functions as Mappings

We can show that a circular arc with center at the origin is mapped by $f(z) = z^{\alpha}$ onto a similar circular arc, and that rays emanating from the origin are mapped by f to similar rays.

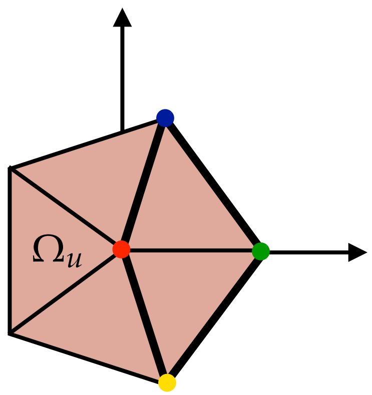




Parametric Pseudo-Manifolds

Complex Functions as Mappings

Now, let us consider a p-domain, Ω_u, where u is a vertex of $\mathcal{K}$ such that $n_u = 5$.



By definition,

$$r_{uv}(Q_{uv}) = \left[(0,0), \left(\cos\left(-\frac{2\pi}{5}\right), \sin\left(-\frac{2\pi}{5}\right)\right), (1,0), \left(\cos\left(\frac{2\pi}{5}\right), \sin\left(\frac{2\pi}{5}\right)\right)\right].$$

Parametric Pseudo-Manifolds

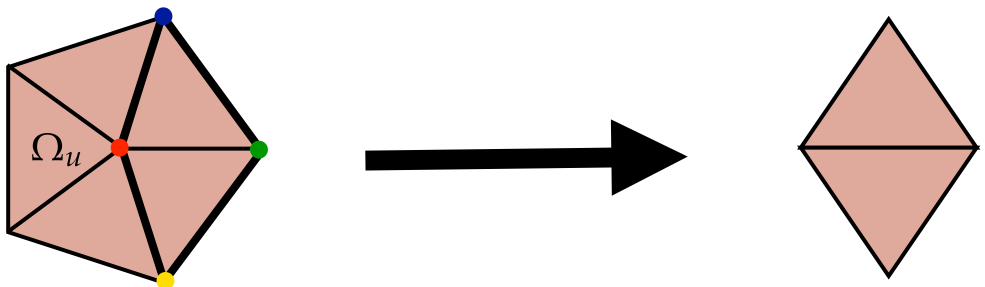
Complex Functions as Mappings

What is the image of $r_{uv}(Q_{uv})$ under the map $f(z) = z^{\alpha}$, where $\alpha = \frac{5}{6}$?

Note that

$$f(0 + i\,0) = 0\,, \quad f(1 + i\,0) = 1\,, \quad f\left(e^{i\,(-\frac{2\pi}{5})}\right) = e^{i\,(-\frac{\pi}{3})}, \quad \text{and} \quad f\left(e^{i\,\frac{2\pi}{5}}\right) = e^{i\,\frac{\pi}{3}}\,.$$

Is that the case that $f(r_{uv}(Q_{uv})) = Q$?

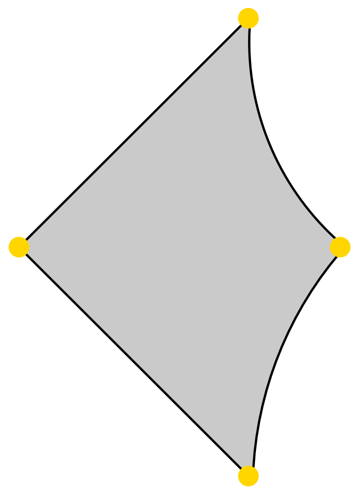


Parametric Pseudo-Manifolds

Complex Functions as Mappings

Unfortunately, NO!

The region $f(r_{uv}(Q_{uv}))$ will look like the picture below:



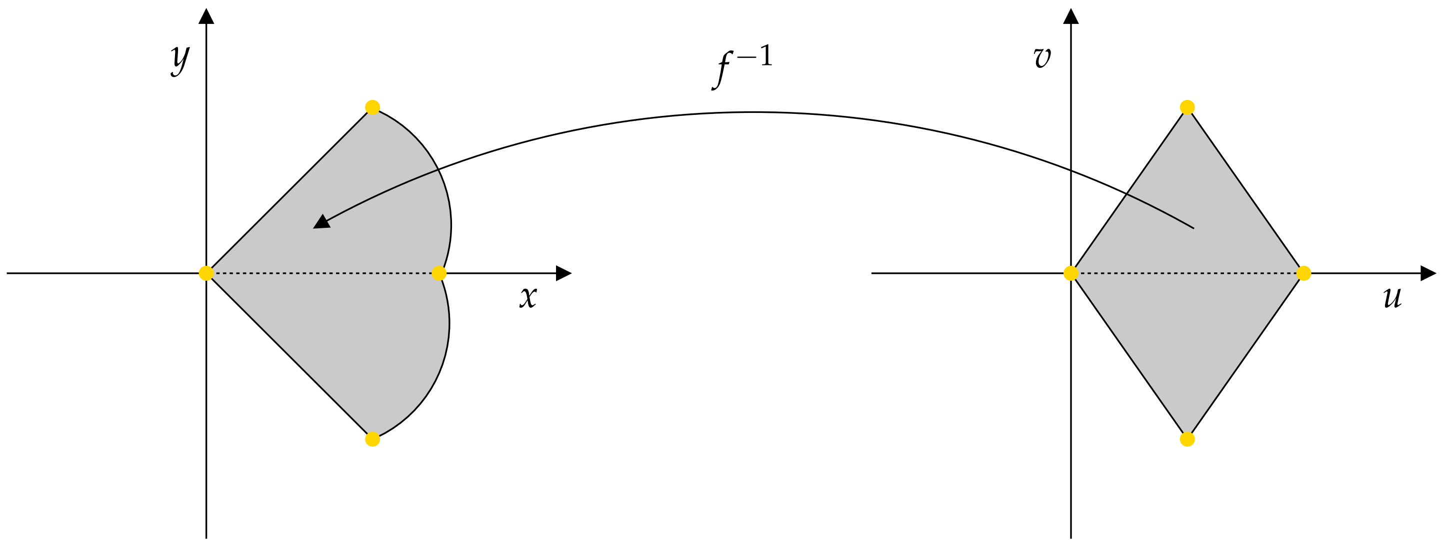
This is because $f(z) = z^{\alpha}$ scales the modulus of $z = r \cdot (\cos(\theta) + i\ \sin(\theta))$: r becomes r^{α}.

Parametric Pseudo-Manifolds

Complex Functions as Mappings

However, if we consider replacing our p-domains by "curved" p-domains, then we can make the f maps works in our favor. The idea is to let $r_{uv}(Q_{uv})$ be the image of Q under

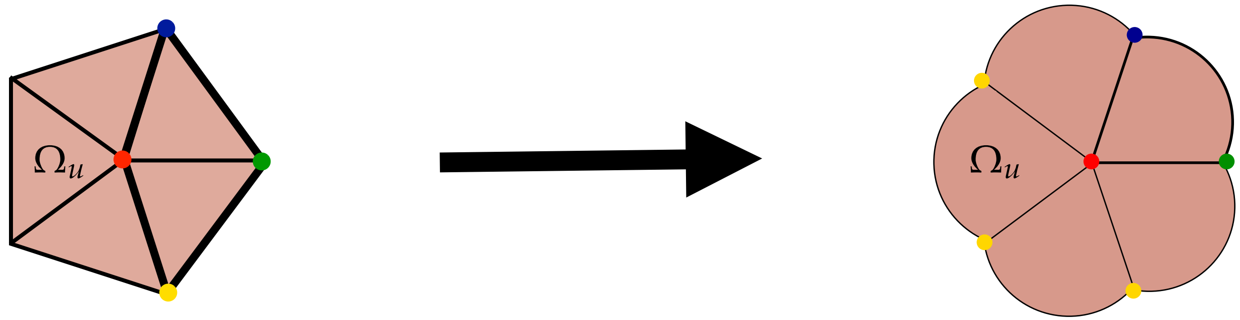
$$f^{-1}(w) = w^{\frac{6}{5}} = r^{\frac{6}{5}} \cdot \left(\cos\left(\frac{6}{5}\cdot\theta\right) + i\,\sin\left(\frac{6}{5}\cdot\theta\right)\right), \quad \text{for all } w \in Q.$$



Parametric Pseudo-Manifolds

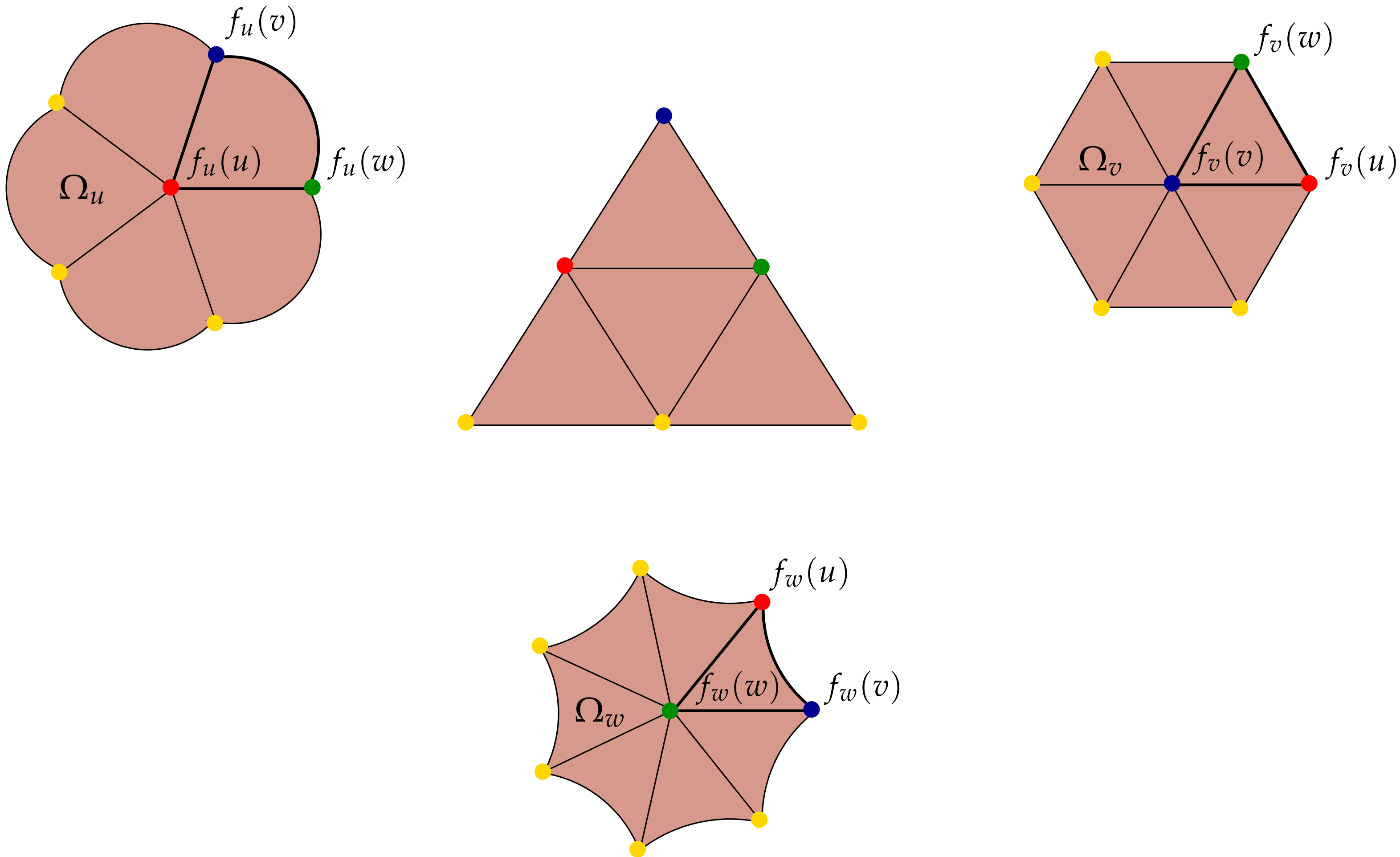
Complex Functions as Mappings

The picture below illustrates the shape of the p-domain Ω_u (left) obtained by applying f^{-1} to Q and then rotating $f^{-1}(Q)$ around the origin. The result is a "curved" p-domain (right).



Parametric Pseudo-Manifolds

Complex Functions as Mappings



Parametric Pseudo-Manifolds

Complex Functions as Mappings

So,

$$g_u(x,y) = (\Pi^{-1} \circ \Gamma_u \circ \Pi)(x,y)\,,$$

where

$$\Pi : \mathbb{E}^2 - \{(0,0)\} \rightarrow \ \mathbb{R}_+ \times \ ]-\pi\,,\,\pi\ [$$

is the map that converts Cartesian coordinates to polar coordinates, $\Pi(x,y) = (r,\theta)$, and

$$\Gamma_u : \mathbb{R}_+ \times \ ]-\pi\,,\,\pi\ [\ \rightarrow \ \mathbb{R}_+ \times \ ]-\pi\,,\,\pi\ [$$

is the map

$$\Gamma_u(r,\theta) = \left(r^{\frac{n_u}{6}}, \frac{n_u}{6} \cdot \theta \right)\,.$$

The map Π is a C^∞-diffeomorphism. So, working with polar coordinates is fine as well.

Parametric Pseudo-Manifolds

Complex Functions as Mappings

Note that the previous g maps are defined in $\mathbb{E}^2 - \{(0,0)\}$. The fact that $(0,0)$ does not belong to the domain of g is not a problem, as $(0,0)$ is not part of a gluing domain, except when the gluing domain is the p-domain itself. But, in this case, the transition map is defined as the identity map, rather than in terms of the g maps. So, we are safe!

Indeed, for every $(u,w) \in K$,

$$\varphi_{wu} : \Omega_{uw} \to \Omega_{wu} \, ,$$

where

$$\varphi_{wu}(x) = \begin{cases} x & \text{if } u = w, \\ (r_{wu}^{-1} \circ g_w^{-1} \circ h \circ g_u \circ r_{uw})(x) & \text{if } u \neq w, \end{cases}$$

for every $x \in \Omega_{uw}$.

Parametric Pseudo-Manifolds

Complex Functions as Mappings

Let q be a point in Q (the *canonical quadrilateral*). If (s, β) are the polar coordinates of q, then

$$
\begin{aligned}
(g_u \circ r_{\frac{2\pi}{n_u}} \circ g_u^{-1})(q) &= (\Pi^{-1} \circ \Gamma_u \circ \Pi \circ r_{\frac{2\pi}{n_u}} \circ \Pi^{-1} \circ \Gamma_u^{-1} \circ \Pi)(q) \\
&= (\Pi^{-1} \circ \Gamma_u \circ \Pi \circ r_{\frac{2\pi}{n_u}} \circ \Pi^{-1} \circ \Gamma_u^{-1})(s, \beta) \\
&= (\Pi^{-1} \circ \Gamma_u \circ \Pi \circ r_{\frac{2\pi}{n_u}} \circ \Pi^{-1}) \left(s^{\frac{6}{n_u}}, \frac{6}{n_u} \cdot \beta \right) \\
&= (\Pi^{-1} \circ \Gamma_u) \left(s^{\frac{6}{n_u}}, \frac{6}{n_u} \cdot \beta + \frac{2\pi}{n_u} \right) \\
&= \Pi^{-1} \left(\left(s^{\frac{6}{n_u}} \right)^{\frac{n_u}{6}}, \frac{n_u}{6} \cdot \left(\frac{6}{n_u} \cdot \beta + \frac{2\pi}{n_u} \right) \right) \\
&= \Pi^{-1} \left(s, \beta + \frac{\pi}{3} \right) \\
&= r_{\frac{\pi}{3}}(q) \, .
\end{aligned}
$$

Parametric Pseudo-Manifolds

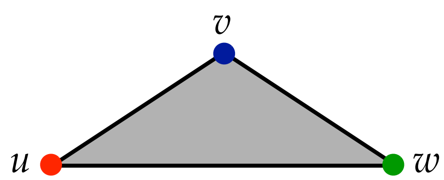
Complex Functions as Mappings

Let p be a point in $\Omega_u - \{(0,0)\}$.

If (t, α) are the polar coordinates of p and if $-\theta$ is the angle of rotation of r_{uw}, then

$$(t, \alpha - \theta) \qquad \text{and} \qquad \left(t, \alpha - \theta - \frac{2\pi}{n_u}\right)$$

are the polar coordinates of $r_{uw}(p)$ and $r_{uv}(p)$, respectively, as we assumed (in our example) that w precedes v in a counterclockwise enumeration of the vertices of $\text{lk}(u, \mathcal{K})$.



Parametric Pseudo-Manifolds

Complex Functions as Mappings

So,

$$\begin{aligned}(g_u \circ r_{uw})(p) &= (\Pi^{-1} \circ \Gamma_u \circ \Pi \circ r_{uw})(p) \\ &= (\Pi^{-1} \circ \Gamma_u)(t, \alpha - \theta) \\ &= (\Pi^{-1}) \left(t^{\frac{n_u}{6}}, \frac{n_u}{6} \cdot (\alpha - \theta) \right) .\end{aligned}$$

Parametric Pseudo-Manifolds

Complex Functions as Mappings

In turn,

$$
\begin{aligned}
(r_{\frac{\pi}{3}} \circ g_u \circ r_{uv})(p) &= (r_{\frac{\pi}{3}} \circ \Pi^{-1} \circ \Gamma_u \circ \Pi \circ r_{uv})(p) \\
&= (r_{\frac{\pi}{3}} \circ \Pi^{-1} \circ \Gamma_u) \left(t, \alpha - \theta - \frac{2\pi}{n_u} \right) \\
&= (r_{\frac{\pi}{3}} \circ \Pi^{-1}) \left(t^{\frac{n_u}{6}}, \frac{n_u}{6} \cdot \left(\alpha - \theta - \frac{2\pi}{n_u} \right) \right) \\
&= (r_{\frac{\pi}{3}} \circ \Pi^{-1}) \left(t^{\frac{n_u}{6}}, \frac{n_u}{6} \cdot (\alpha - \theta) - \frac{\pi}{3} \right) \\
&= \Pi^{-1} \left(t^{\frac{n_u}{6}}, \frac{n_u}{6} \cdot (\alpha - \theta) \right) \\
&= (g_u \circ r_{uw})(p) \,.
\end{aligned}
$$

Complex Functions as Mappings

So, the g_u map satisfies the following four conditions:

(1) The g_u map is a C^k-diffeomorphism of $\mathbb{E}^2 - \{(0,0)\}$, for every $u \in I$.

(2) The g_u map takes $r_{uw}(\Omega_{uw})$ onto $\overset{\circ}{Q}$, for every $(u,w) \in K$.

(3) The g_u map satisfies $(g_u \circ r_{\frac{2\pi}{n_u}} \circ g_u^{-1})(q) = r_{\frac{\pi}{3}}(q)$, where $q \in g_u(\Omega_u)$.

(4) If $f_u(w)$ precedes $f_u(v)$ in a counterclockwise enumeration of the vertices of $\text{lk}(u, \mathcal{K})$, then $(g_u \circ r_{uw})(p) = (r_{\frac{\pi}{3}} \circ g_u \circ r_{uv})(p)$, for every point p in the gluing domain Ω_{uw}.

Parametric Pseudo-Manifolds

Complex Functions as Mappings

We have **not** checked the following assumption:

(5) For all u, v, w such that $[u, v, w]$ is a triangle of $\mathcal{K}$, if $\Omega_{wu} \cap \Omega_{wv} \neq \emptyset$ then

$$\varphi_{uw}(\Omega_{wu} \cap \Omega_{wv}) = \Omega_{uv} \cap \Omega_{uw} \, .$$

We will also explore that in a homework.