IMPA - Instituto de Matemática Pura e Aplicada, Rio de Janeiro, RJ, Brazil

Introduction to Computational Manifolds and Applications

Part 1 - Constructions

Prof. Marcelo Ferreira Siqueira

mfsiqueira@dimap.ufrn.br

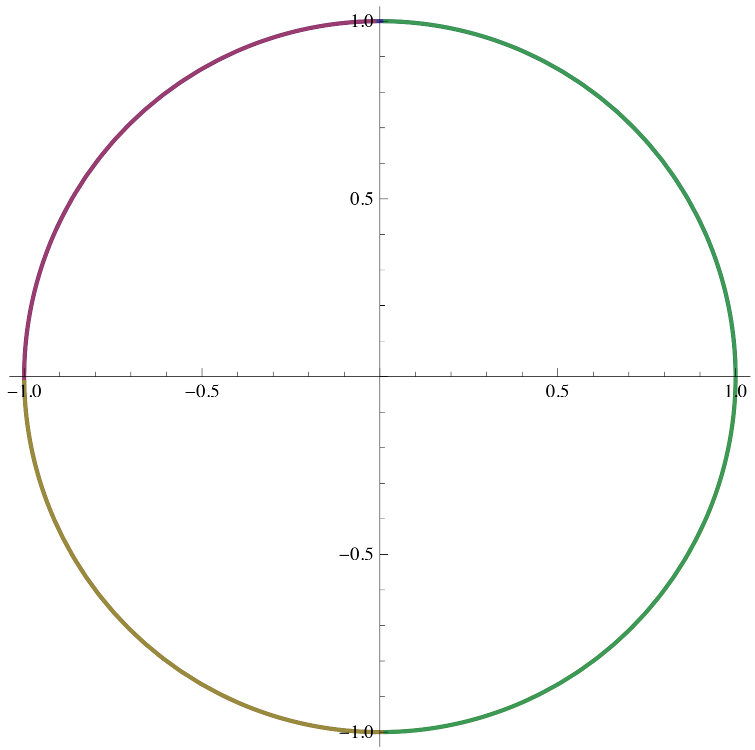
Departmento de Informática e Matemática Aplicada
Universidade Federal do Rio Grande do Norte
Natal, RN, Brazil

Parametric Pseudo-Manifolds

Reconstructing $\mathbf{S}^1$

Recall that

$$S^1 = \{(x, y) \in \mathbb{E}^2 \mid x^2 + y^2 = 1\} .$$



Parametric Pseudo-Manifolds

Reconstructing $\mathbf{S}^1$

S^1 is a one-dimensional parametric pseudo-manifold in $\mathbb{E}^2$.

To see why, let us define it as such.

We need to define a set of gluing data and a family of parametrizations.

The gluing data will define the topology of S^1, while the parametrizations the geometry.

We will start with the gluing data.

Parametric Pseudo-Manifolds

Reconstructing $\mathbf{S}^1$

We need to define

- the p-domains, $\{\Omega_i\}_{i \in I}$,
- the gluing domains, $\{\Omega_{ij}\}_{(i,j) \in I \times I}$, and
- the transition functions, $\{\varphi_{ij}\}_{(i,j) \in K}$.

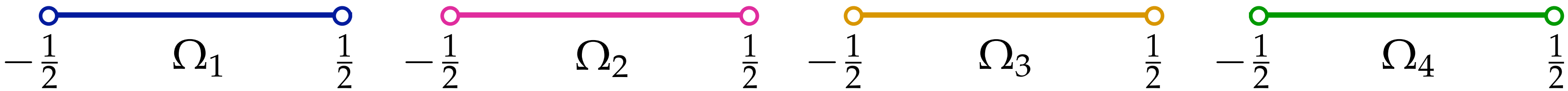
Parametric Pseudo-Manifolds

Reconstructing S^1

Let $I = \{1, 2, 3, 4\}$.

Define the p-domains Ω_1, Ω_2, Ω_3, and Ω_4 as distinct copies of the open interval,

$$\left] -\frac{1}{2}, \frac{1}{2} \right[\subset \mathbb{E} \, .$$



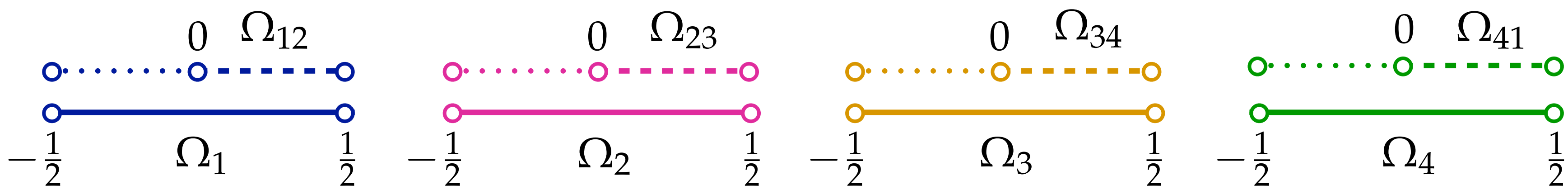
Parametric Pseudo-Manifolds

Reconstructing S^1

Define the gluing domains Ω_{12}, Ω_{23}, Ω_{34}, and Ω_{41} as the open interval

$$] \, 0, \frac{1}{2} \, [$$

contained in the p-domains Ω_1, Ω_2, Ω_3, and Ω_4, respectively.



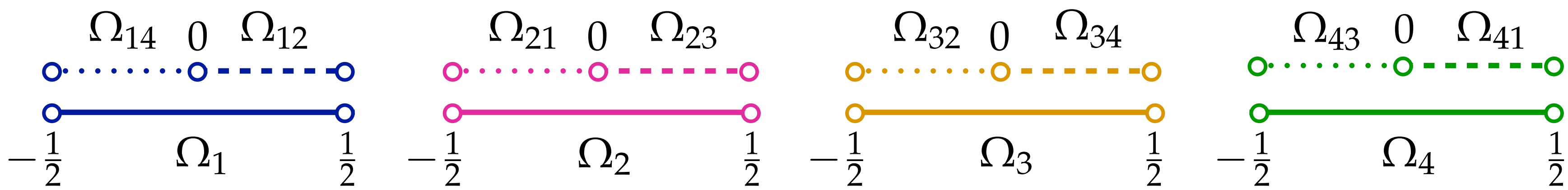
Parametric Pseudo-Manifolds

Reconstructing S^1

Similarly, define the gluing domains Ω_{21}, Ω_{32}, Ω_{43}, and Ω_{14} as the open interval

$$\left] -\frac{1}{2}, 0 \right[$$

contained in the p-domains Ω_2, Ω_3, Ω_4, and Ω_1, respectively.



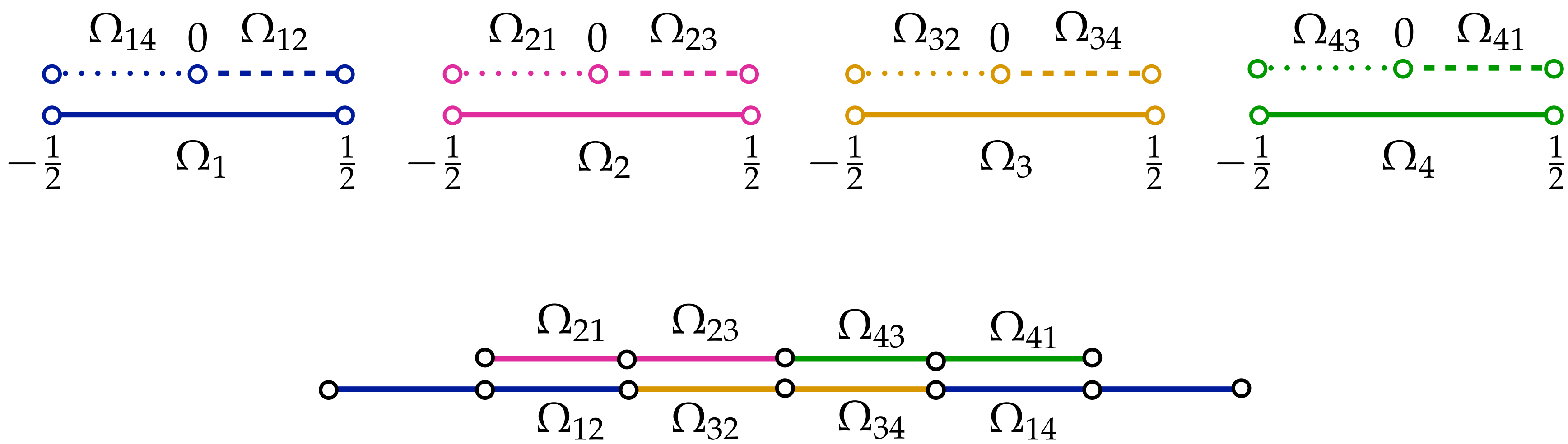
Finally, let $\Omega_{ii} = \Omega_i$, for $i = 1, 2, 3, 4$.

Parametric Pseudo-Manifolds

Reconstructing $\mathbf{S}^1$

We let $\Omega_{13} = \Omega_{31} = \Omega_{24} = \Omega_{42} = \emptyset$.

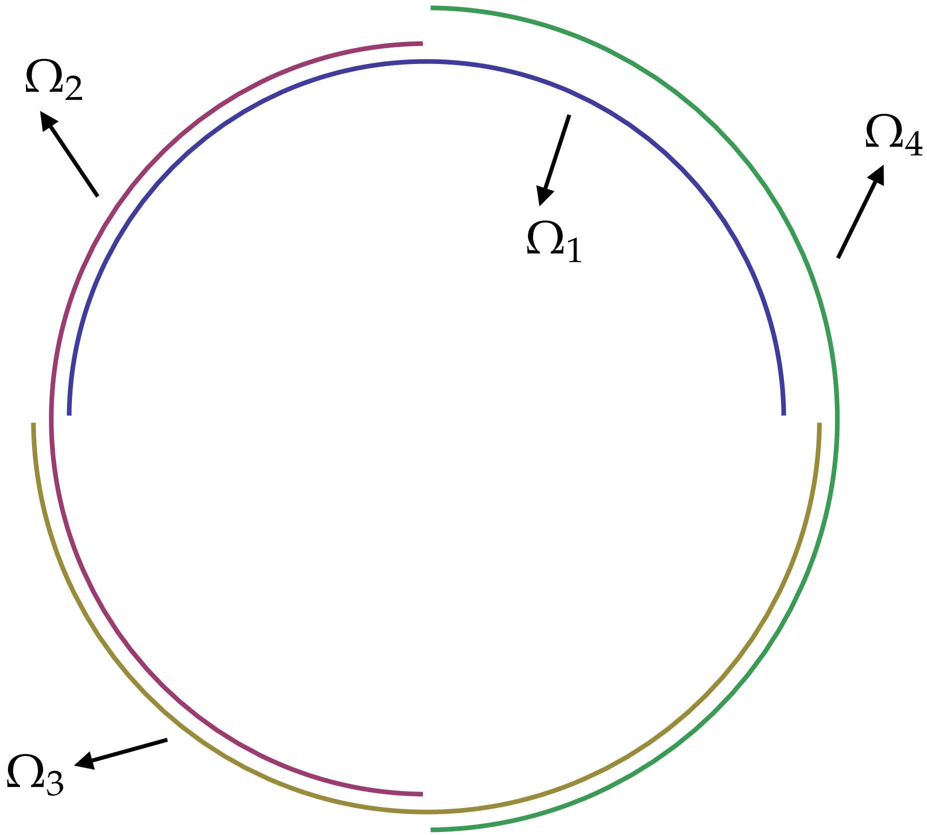
What does this "gluing" look like?



Parametric Pseudo-Manifolds

Reconstructing S^1

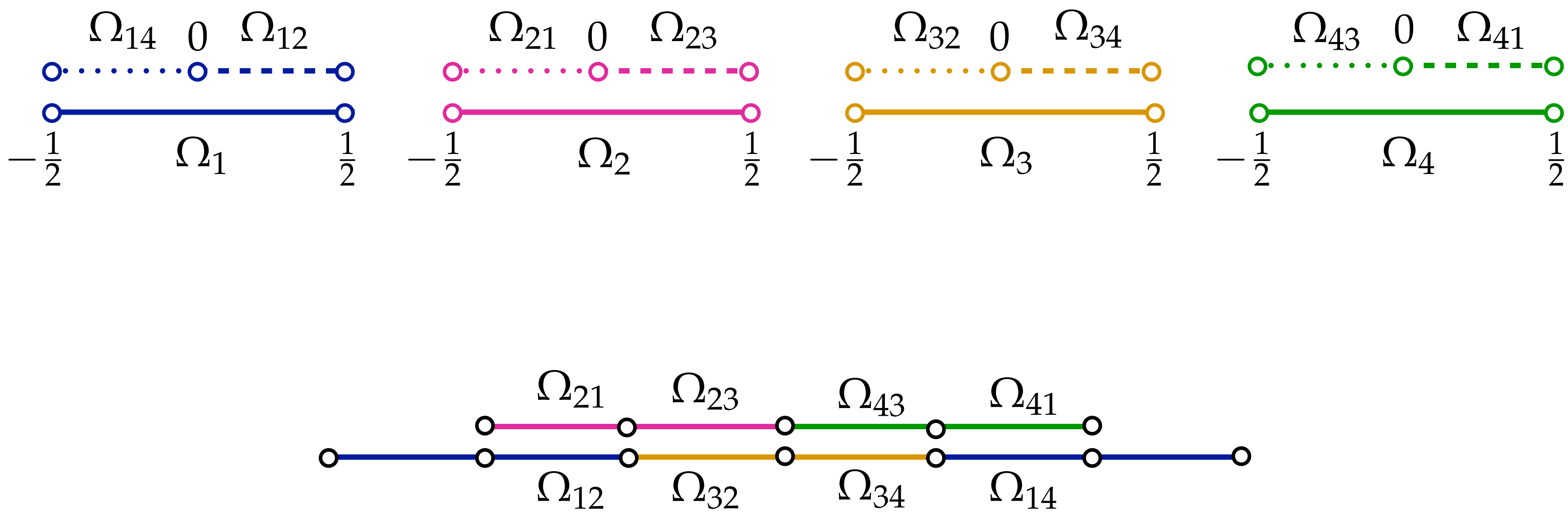
Our intuition is...



Parametric Pseudo-Manifolds

Reconstructing S^1

What transition functions can "realize" our intuition?



Parametric Pseudo-Manifolds

Reconstructing $\mathbb{S}^1$

Let $K = \{(1,1),(2,2),(3,3),(4,4),(1,2),(2,3),(3,4),(4,1),(1,4),(2,1),(3,2),(4,3)\}$.

For each $(i,j) \in K$ and for all $x \in \Omega_{ij}$, the transition map $\varphi_{ji} : \Omega_{ij} \to \Omega_{ji}$ is given by

$$\varphi_{ji}(x) = \begin{cases} x & \text{if } i = j, \\ x - \frac{1}{2} & \text{if } j = i+1 \text{ or } j = 1 \text{ and } i = 4, \\ x + \frac{1}{2} & \text{if } j = i-1 \text{ or } j = 4 \text{ and } i = 1. \end{cases}$$

Note that our transition maps are affine functions.

We claim that the triple $\mathcal{G} = \left((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K}\right)$ is a set of gluing data.

Parametric Pseudo-Manifolds

Reconstructing $\mathbb{S}^1$

Checking...

(1) For every $i \in I$, the set Ω_i is a nonempty open subset of $\mathbb{E}^n$ called parametrization domain, for short, p-domain, and any two distinct p-domains are pairwise disjoint, i.e.,

$$\Omega_i \cap \Omega_j = \emptyset\,,$$

for all $i \neq j$.

Our p-domains are (connected) open intervals of $\mathbb{E}$, which do not overlap (since they were assumed to be in distinct copies of $\mathbb{E}$). So, condition (1) of Definition 7.1 is satisfied.

Parametric Pseudo-Manifolds

Reconstructing $\mathbf{S}^1$

Checking...

(2) For every pair $(i, j) \in I \times I$, the set Ω_{ij} is an open subset of Ω_i. Furthermore, $\Omega_{ii} = \Omega_i$ and $\Omega_{ji} \neq \emptyset$ if and only if $\Omega_{ij} \neq \emptyset$. Each nonempty subset Ω_{ij} (with $i \neq j$) is called a gluing domain.

Our gluing domains are open intervals of $\mathbb{E}$. Furthermore, $\Omega_{ii} = \Omega_i$ and $\Omega_{ij} \neq \emptyset$ if and only if $\Omega_{ji} \neq \emptyset$, for $i, j = 1, 2, 3, 4$. So, condition (2) of Definition 7.1 is also satisfied.

Parametric Pseudo-Manifolds

Reconstructing $\mathbf{S}^1$

Checking...

(3) If we let

$$K = \{(i, j) \in I \times I \mid \Omega_{ij} \neq \emptyset\},$$

then

$$\varphi_{ji} \colon \Omega_{ij} \to \Omega_{ji}$$

is a C^k bijection for every $(i, j) \in K$ called a transition (or gluing) map.

Parametric Pseudo-Manifolds

Reconstructing $\mathbf{S}^1$

Checking...

Recall that, for each $(i, j) \in K$ and for all $x \in \Omega_{ij}$, we have

$$\varphi_{ji}(x) = \begin{cases} x & \text{if } i = j, \\ x - \frac{1}{2} & \text{if } j = i + 1 \text{ or } j = 1 \text{ and } i = 4, \\ x + \frac{1}{2} & \text{if } j = i - 1 \text{ or } j = 4 \text{ and } i = 1. \end{cases}$$

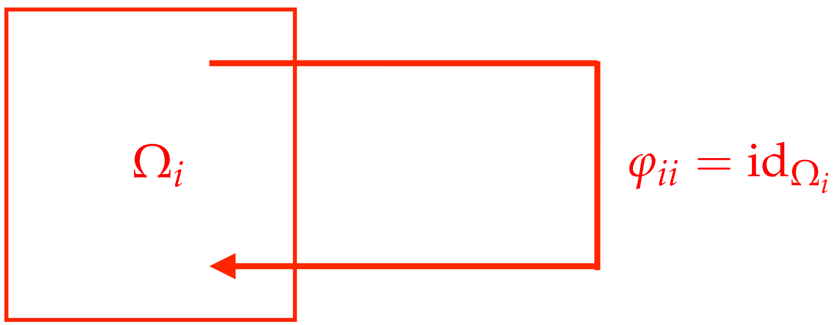
These maps are either the identity function or a "translation". In either case, they are C^∞ bijective functions. But, to satisfy condition (3), we still have to check three more cases.

Parametric Pseudo-Manifolds

Reconstructing S^1

Checking...

(a) $\varphi_{ii} = \text{id}_{\Omega_i}$, for all $i \in I$,



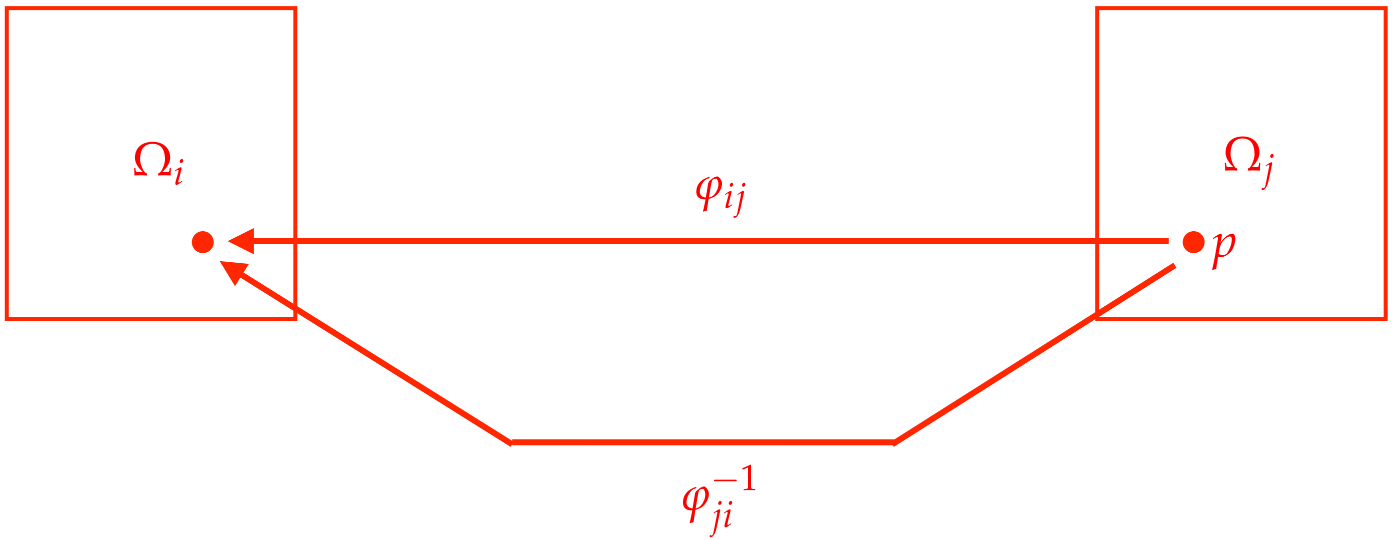
By definition, $\varphi_{ji}(x) = x$ whenever $i = j$. So, condition 3(a) is satisfied.

Parametric Pseudo-Manifolds

Reconstructing S^1

Checking...

(b) $\varphi_{ij} = \varphi_{ji}^{-1}$, for all $(i, j) \in K$, and



Parametric Pseudo-Manifolds

Reconstructing S^1

Checking...

If $i = j$ then condition 3(b) is trivially satisfied by our definition of φ_{ji}.

If $j = i + 1$ or $j = 1$ and $i = 4$ then $\varphi_{ji}(x) = x - (1/2)$. So, $\varphi_{ji}^{-1}(x) = x + (1/2)$.

If $j = i - 1$ or $j = 4$ and $i = 1$ then $\varphi_{ji}(x) = x + (1/2)$. So, $\varphi_{ji}^{-1}(x) = x - (1/2)$.

Thus,

$$\varphi_{ji}^{-1}(x) = \begin{cases} x & \text{if } i = j, \\ x - \frac{1}{2} & \text{if } j = i - 1 \text{ or } j = 4 \text{ and } i = 1, \\ x + \frac{1}{2} & \text{if } j = i + 1 \text{ or } j = 1 \text{ and } i = 4. \end{cases}$$

Parametric Pseudo-Manifolds

Reconstructing S^1

Checking...

But, since

$$\varphi_{ji}(x) = \begin{cases} x & \text{if } i = j, \\ x - \frac{1}{2} & \text{if } j = i + 1 \text{ or } j = 1 \text{ and } i = 4, \\ x + \frac{1}{2} & \text{if } j = i - 1 \text{ or } j = 4 \text{ and } i = 1, \end{cases}$$

we must have that

$$\varphi_{ij}(x) = \begin{cases} x & \text{if } j = i, \\ x - \frac{1}{2} & \text{if } j = i - 1 \text{ or } i = 1 \text{ and } j = 4, \\ x + \frac{1}{2} & \text{if } j = i + 1 \text{ or } i = 4 \text{ and } j = 1. \end{cases}$$

So,

$$\varphi_{ij}(x) = \varphi_{ji}^{-1}(x)\,.$$

Parametric Pseudo-Manifolds

Reconstructing $\mathbb{S}^1$

Checking...

(c) For all i, j, k, if

$$\Omega_{ji} \cap \Omega_{jk} \neq \emptyset ,$$

then

$$\varphi_{ij}(\Omega_{ji} \cap \Omega_{jk}) = \Omega_{ij} \cap \Omega_{ik} \qquad \text{and} \qquad \varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x) ,$$

for all $x \in \Omega_{ij} \cap \Omega_{ik}$.

Note that if $\Omega_{ji} \cap \Omega_{jk} \neq \emptyset$, then we get $i = k$. In other words, at most **two** gluing domains overlap at the same point of any p-domain. As a result condition 3(c) holds trivially.

Parametric Pseudo-Manifolds

Reconstructing S^1

Checking...

What about condition 4 of Definition 7.1 (the Hausdorff condition)?

(4) For every pair $(i, j) \in K$, with $i \neq j$, for every

$$x \in \partial(\Omega_{ij}) \cap \Omega_i \qquad \text{and} \qquad y \in \partial(\Omega_{ji}) \cap \Omega_j \,,$$

there are open balls, V_x and V_y, centered at x and y, so that no point of $V_y \cap \Omega_{ji}$ is the image of any point of $V_x \cap \Omega_{ij}$ by φ_{ji}.

Parametric Pseudo-Manifolds

Reconstructing S^1

Checking...

Let $j = i + 1$ or $j = 1$ and $i = 4$. So, if $x \in \partial(\Omega_{ij}) \cap \Omega_i$ and $y \in \partial(\Omega_{ji}) \cap \Omega_j$, then $x = y = 0$.

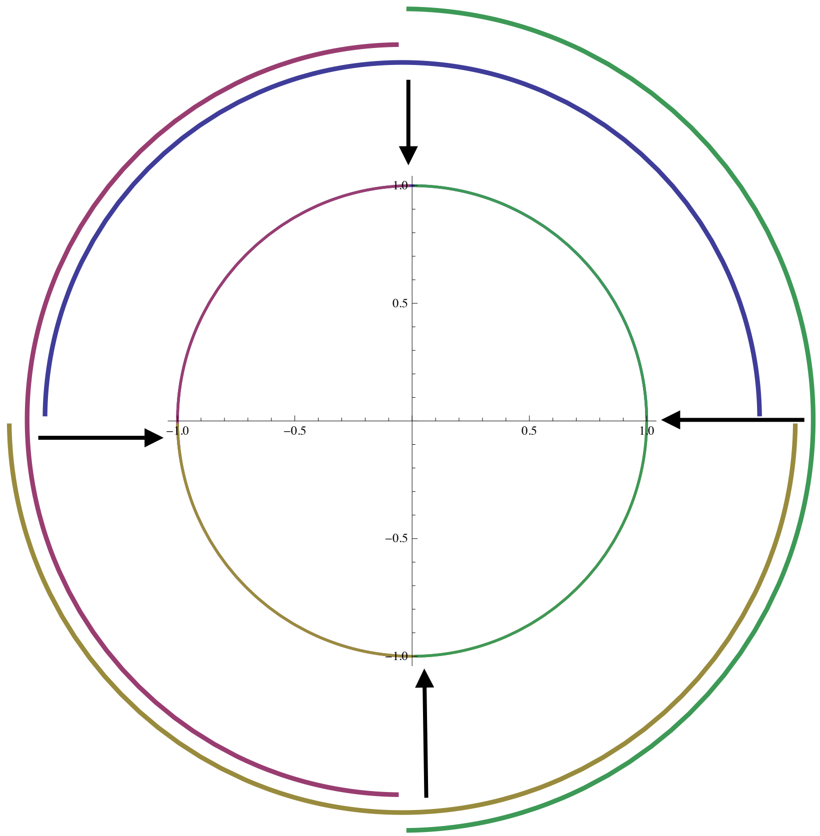
Thus, if we let $V_x = V_y =]-\epsilon, \epsilon[$, where $\epsilon < (1/4)$, then we have that $\varphi_{ji}(V_x) \cap V_y = \emptyset$.

If $j = i - 1$ or $j = 4$ and $i = 1$, we can proceed in a similar manner. So, condition (4) holds.

Parametric Pseudo-Manifolds

Reconstructing S^1

We defined a gluing data that captures the topology of S^1. Now, we need to define the geometry of S^1. For that, we must define parametrizations that take the Ω_i's to $\mathbb{E}^2$.



Parametric Pseudo-Manifolds

Reconstructing S^1

Recall that a parametric C^k pseudo-manifold of dimension n in $\mathbb{E}^d$ (PPM) is a pair,

$$\mathcal{M} = (\mathcal{G}, (\theta_i)_{i \in I}),$$

such that

$$\mathcal{G} = \left((\Omega_i)_{i \in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\varphi_{ji})_{(i,j) \in K}\right)$$

is a set of gluing data, for some finite set I, and each $\theta_i \colon \Omega_i \to \mathbb{E}^d$ is C^k and satisfies

(C) For all $(i, j) \in K$, we have

$$\theta_i = \theta_j \circ \varphi_{ji}.$$

The key is to define parametrizations that respect condition (C).

In the case of S^1, this is a particularly easy job!

Parametric Pseudo-Manifolds

Reconstructing $\mathbf{S}^1$

Indeed, let

$$\theta_1(x) = \big(\cos((x+0.5)\cdot\pi)\,,\,\sin((x+0.5)\cdot\pi)\big),$$
$$\theta_2(x) = \big(\cos((x+1.0)\cdot\pi)\,,\,\sin((x+1.0)\cdot\pi)\big),$$
$$\theta_3(x) = \big(\cos((x+1.5)\cdot\pi)\,,\,\sin((x+1.5)\cdot\pi)\big),$$
$$\theta_4(x) = \big(\cos((x+2.0)\cdot\pi)\,,\,\sin((x+2.0)\cdot\pi)\big),$$

for all

$$x \in \left]-\frac{1}{2},\frac{1}{2}\right[.$$

Parametric Pseudo-Manifolds

Reconstructing $\mathbf{S}^1$

Let x be any point in $\Omega_{12} =]\,0, 1/2\,[$. Then,

$$\begin{aligned}
\theta_2 \circ \varphi_{21}(x) &= \big(\cos(\varphi_{21}(x) + 1.0) \cdot \pi\,,\, \sin(\varphi_{21}(x) + 1.0) \cdot \pi\big) \\
&= \big(\cos((x - 0.5) + 1.0) \cdot \pi\,,\, \sin((x - 0.5) + 1.0) \cdot \pi\big) \\
&= \big(\cos(x + 0.5) \cdot \pi\,,\, \sin(x + 0.5) \cdot \pi\big) \\
&= \theta_1(x)\,.
\end{aligned}$$

We can proceed in a similar way to show that $\theta_i = \theta_j \circ \varphi_{ji}$, for $i, j = 1, 2, 3, 4$.

So, S^1 is in fact a one-dimensional parametric pseudo-manifold in $\mathbb{E}^2$.

Parametric Pseudo-Manifolds

Reconstructing $\mathbf{S}^1$

It turns out that θ_1, θ_2, θ_3 and θ_4 are all injective and they also satisfy conditions

(C′) For all $(i, j) \in K$,

$$\theta_i(\Omega_i) \cap \theta_j(\Omega_j) = \theta_i(\Omega_{ij}) = \theta_j(\Omega_{ji}) \,.$$

(C″) For all $(i, j) \notin K$,

$$\theta_i(\Omega_i) \cap \theta_j(\Omega_j) = \emptyset \,.$$

So, our pseudo-manifold is actually a manifold, which comes with no surprise!

Parametric Pseudo-Manifolds

Reconstructing a curve homeomorphic to S^1

S^1 is certainly one of the easiest curves we could reconstruct with the gluing process.

This is because the topology and (mainly) the geometry of S^1 are quite simple.

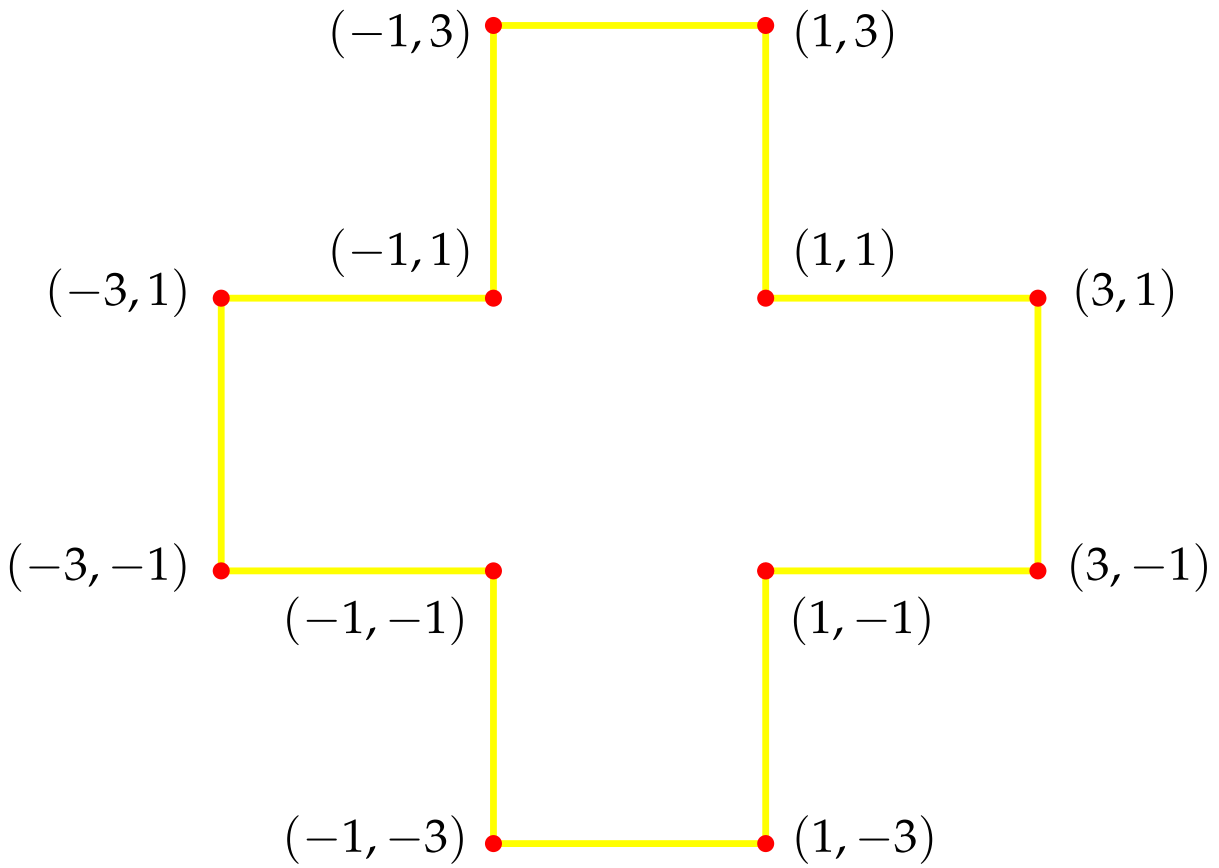
But, what if the curve has the same topology as S^1, but a more "challenging" shape?

In particular, what if we do not know any equation that captures the exact shape of the curve?

To illustrate this situation and how we can deal with it, let us consider a "sketch" of a shape.

Parametric Pseudo-Manifolds

Reconstructing a curve homeomorphic to S^1

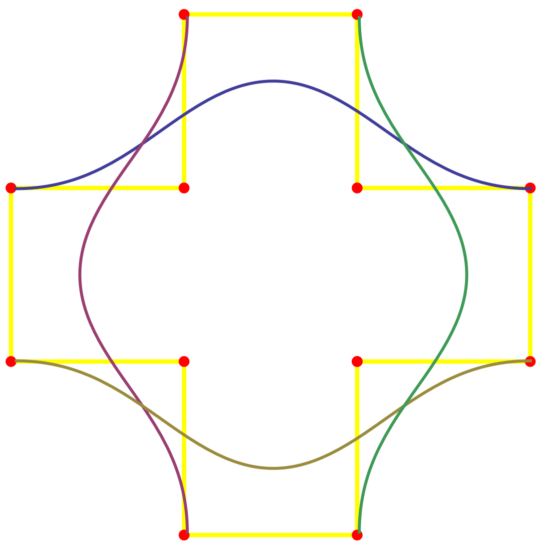


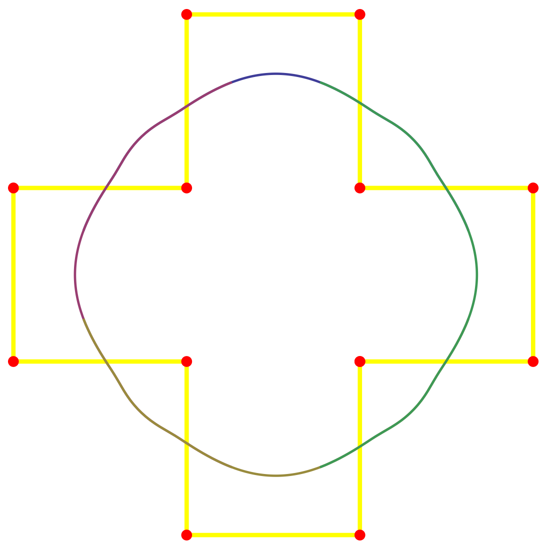
Parametric Pseudo-Manifolds

Reconstructing a curve homeomorphic to S^1

Let us assume that we do not know any equation that approximates the global shape, but that we know how to approximate it locally, using, for instance, arcs of Bézier curves.

In particular, assume that we can approximate the shape with four Bézier curves of degree 5.





Parametric Pseudo-Manifolds

Reconstructing a curve homeomorphic to $\mathbb{S}^1$

Recall that a Bézier curve, $C : [0,1] \to \mathbb{E}^3$, of degree 5 is expressed by the equation

$$C(t) = \sum_{i=0}^{5} \binom{5}{i} \cdot t^i \cdot (1-t)^{5-i} \cdot p_i \,,$$

where $(p_i)_{i=1}^{5}$ are the so-called control points.

Parametric Pseudo-Manifolds

Reconstructing a curve homeomorphic to S^1

In our example, the curves are

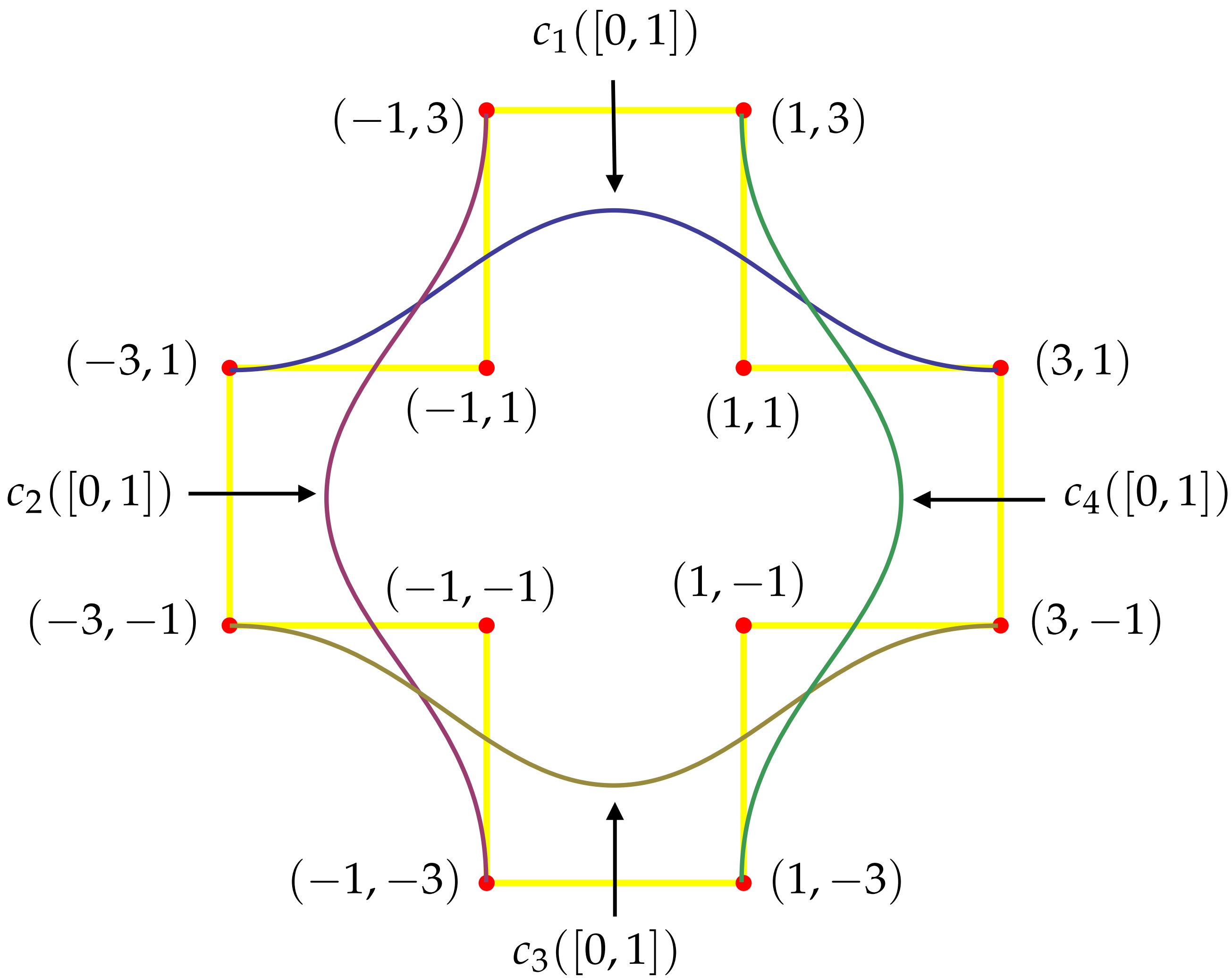
$$c_j(t) = \sum_{i=0}^{5} \binom{5}{i} \cdot t^i \cdot (1-t)^{5-i} \cdot p_i^{(j)} ,$$

for $j = 1, 2, 3, 4$, where the control points are the following:

$$\begin{array}{llllllllllll}
p_0^{(1)} = (3,1) & p_1^{(1)} = (1,1) & p_2^{(1)} = (1,3) & p_3^{(1)} = (-1,3) & p_4^{(1)} = (-1,1) & p_5^{(1)} = (-3,1) \\
p_0^{(2)} = (-1,3) & p_1^{(2)} = (-1,1) & p_2^{(2)} = (-3,1) & p_3^{(2)} = (-3,-1) & p_4^{(2)} = (-1,-1) & p_5^{(2)} = (-1,-3) \\
p_0^{(3)} = (-3,-1) & p_1^{(3)} = (-1,-1) & p_2^{(3)} = (-1,-3) & p_3^{(3)} = (1,-3) & p_4^{(3)} = (1,-3) & p_5^{(3)} = (3,-1) \\
p_0^{(4)} = (1,-3) & p_1^{(4)} = (1,-1) & p_2^{(4)} = (3,-1) & p_3^{(4)} = (3,1) & p_4^{(4)} = (1,1) & p_5^{(4)} = (1,3)
\end{array}$$

Parametric Pseudo-Manifolds

Reconstructing a curve homeomorphic to S^1



Parametric Pseudo-Manifolds

Reconstructing a curve homeomorphic to S^1

As we can see, the trace of the four Bézier curves do not match "exactly". But, before we worry about that, let us define a set of gluing data for the curve we want to build.

Since the topology is the same as before and since we also have four "pieces" of curve, we can basically re-use the same gluing data. We will make a slight modification only.

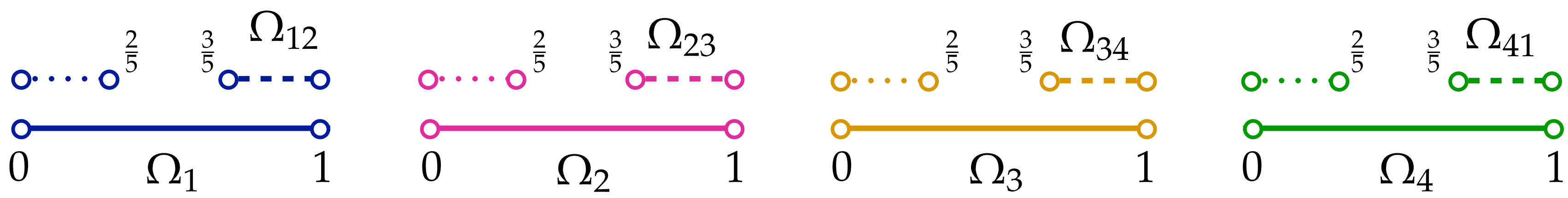
Our p-domains will be distinct copies of the open interval $]\,0\,,\,1\,[\, \subset \mathbb{E}$.

Our gluing domains will change too.

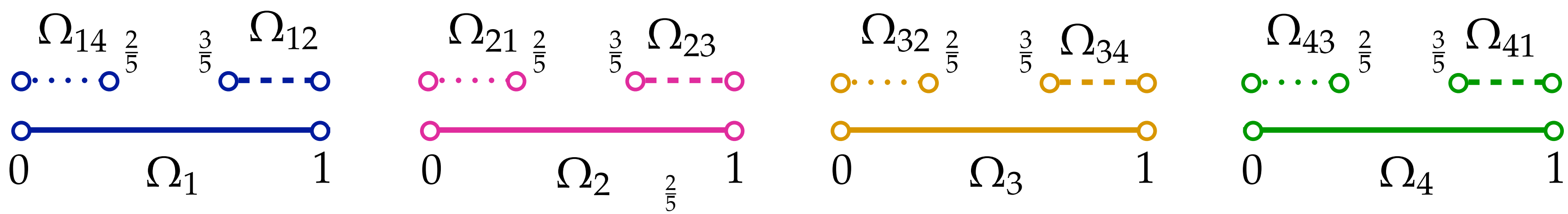
Parametric Pseudo-Manifolds

Reconstructing a curve homeomorphic to S^1

We will let Ω_{12}, Ω_{23}, Ω_{34}, and Ω_{41} be the subsets $\left]\frac{3}{5}, 1\right[$ of Ω_1, Ω_2, Ω_3, and Ω_4, respectively.



Similarly, we let Ω_{21}, Ω_{32}, Ω_{43}, and Ω_{14} be the subsets $\left]0, \frac{2}{5}\right[$ of Ω_2, Ω_3, Ω_4, and Ω_1, respectively.

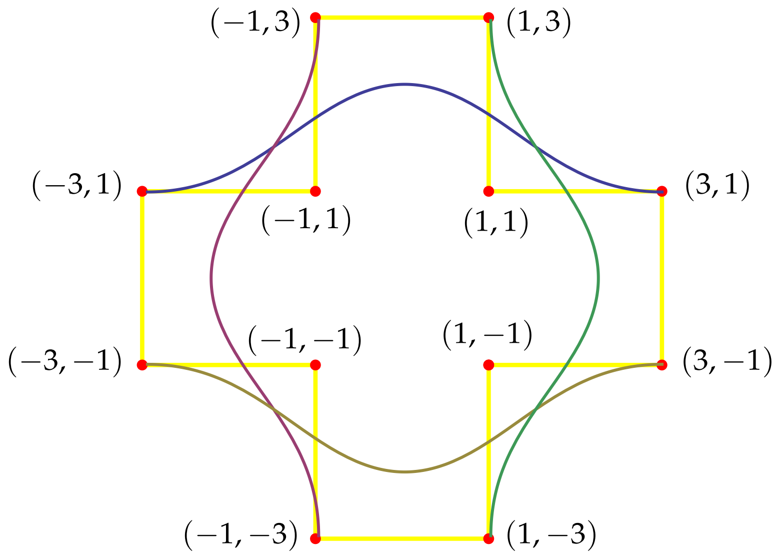


Parametric Pseudo-Manifolds

Reconstructing a curve homeomorphic to S^1

Finally, we let $\Omega_{13} = \Omega_{31} = \Omega_{24} = \Omega_{42} = \emptyset$, and $\Omega_{ii} = \Omega_i$, for $i = 1, 2, 3, 4$.

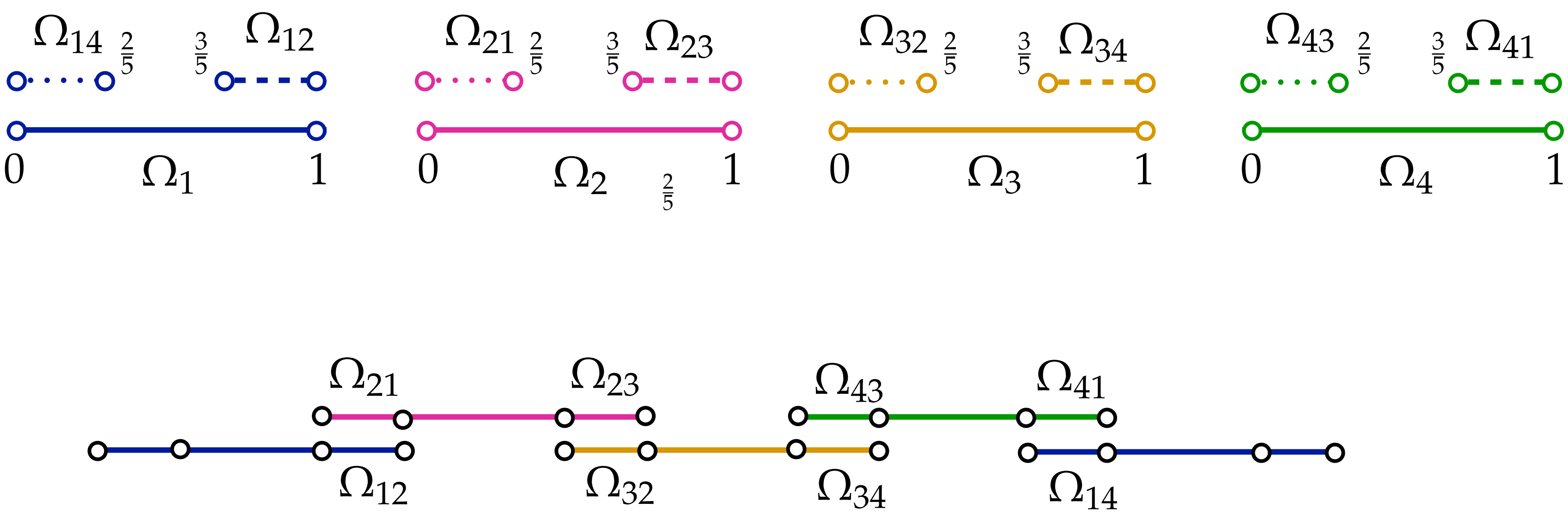
The intuition for defining the gluing domains comes from the fact that the Bézier curves overlaps in a (linear) region that corresponds to roughly $\frac{2}{5}$ of their parametric domain.



Parametric Pseudo-Manifolds

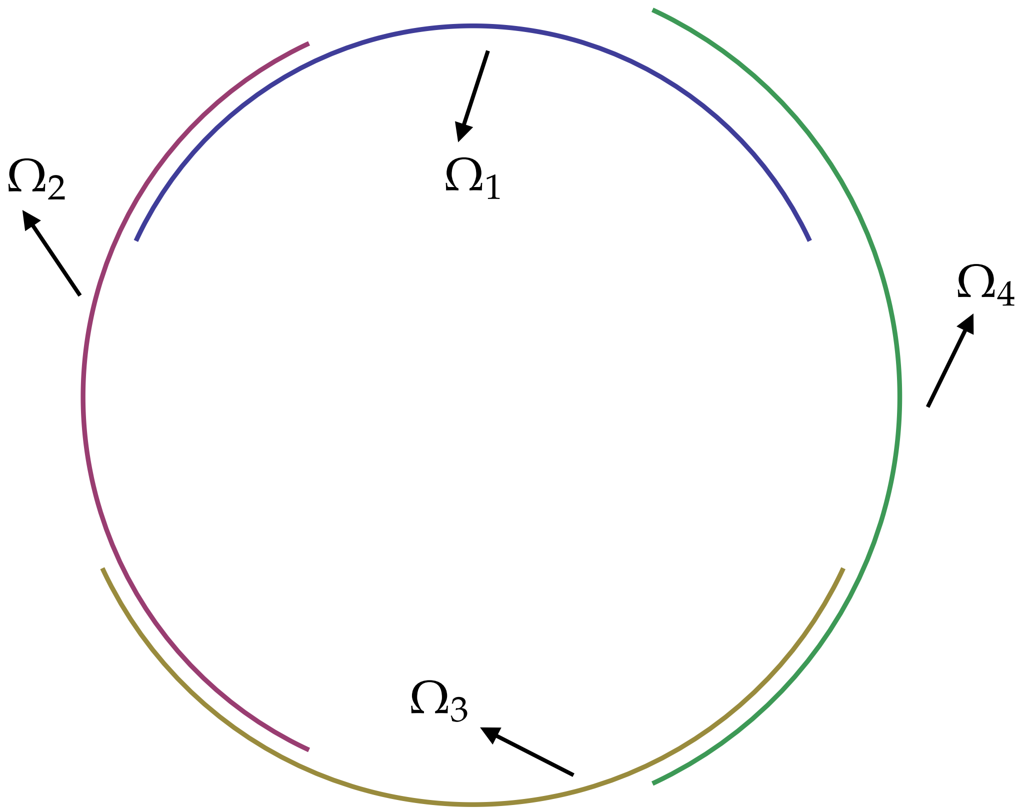
Reconstructing a curve homeomorphic to S^1

What does this "gluing" look like?



Parametric Pseudo-Manifolds

Reconstructing a curve homeomorphic to S^1



Parametric Pseudo-Manifolds

Reconstructing a curve homeomorphic to $\mathbf{S}^1$

We are now left with the transition functions.

Well, they are also affine maps (the identity and some translations).

Let $K = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,3), (3,4), (4,1), (1,4), (2,1), (3,2), (4,3)\}$.

For each $(i,j) \in K$ and for all $x \in \Omega_{ij}$, the transition map $\varphi_{ji} : \Omega_{ij} \to \Omega_{ji}$ is given by

$$\varphi_{ji}(x) = \begin{cases} x & \text{if } i = j, \\ x - \frac{3}{5} & \text{if } j = i + 1 \text{ or } j = 1 \text{ and } i = 4, \\ x + \frac{3}{5} & \text{if } j = i - 1 \text{ or } j = 4 \text{ and } i = 1. \end{cases}$$

Parametric Pseudo-Manifolds

Reconstructing a curve homeomorphic to S^1

We can show that we do have a set of gluing data, but the proof is similar to what we did before. So, it will be left as an easy problem for one of the following homeworks.

Let us now deal with a new challenge: our Bézier curves do not yield *consistent* parametrizations. So, they cannot be used as parametrizations. What should we do then?

We will resort to an approach that is often used in the gluing of 2-dimensional PPMs.

The idea is to create parametrizations by averaging the Bézier curves wherever their domains overlap (according to the gluing). For that, we will use the notion of partition of unity.

Parametric Pseudo-Manifolds

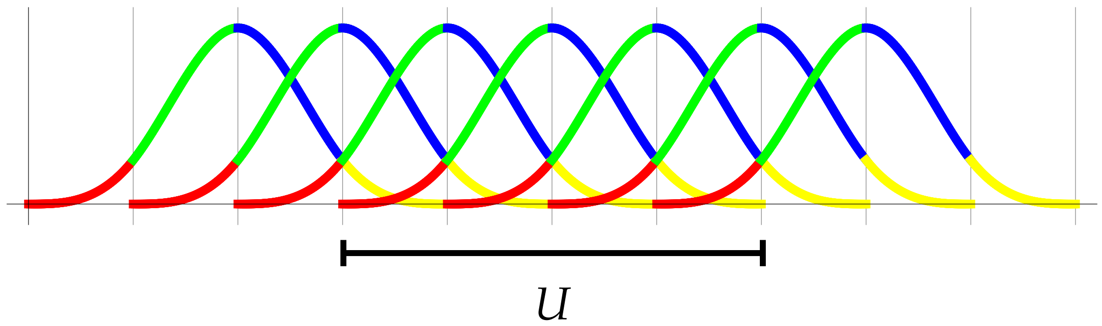
Reconstructing a curve homeomorphic to S^1

Definition 8.1. Given a subset U of $\mathbb{E}^n$, a *partition of unity* $\{\alpha_k\}_{k\in K}$ on U is a set of nonnegative compactly supported functions $\alpha_k : \mathbb{E}^n \to \mathbb{R}$ that add up to 1 at every point of U. More precisely, for each $k \in K$ and for each point $p \in U$, we have that

$$\alpha_k(p) \geq 0\,, \qquad \sum_{k\in K} \alpha_k(p) = 1\,, \qquad \text{and} \quad \{\text{supp}(\alpha_k)\}_{k\in K}$$

is a locally finite cover of U, where the support $\text{supp}(\alpha_k)$ of α_k is the closure of the point set

$$\{p \in U \mid \alpha_k(p) \neq 0\}\,.$$



Parametric Pseudo-Manifolds

Reconstructing a curve homeomorphic to S^1

We will (indirectly) define a partition of unity on each p-domain, Ω_1, Ω_2, Ω_3, and Ω_4.

First, we need a bump function:

For every $t \in \mathbb{R}$, we define

$$\xi : \mathbb{R} \to \mathbb{R}$$

as

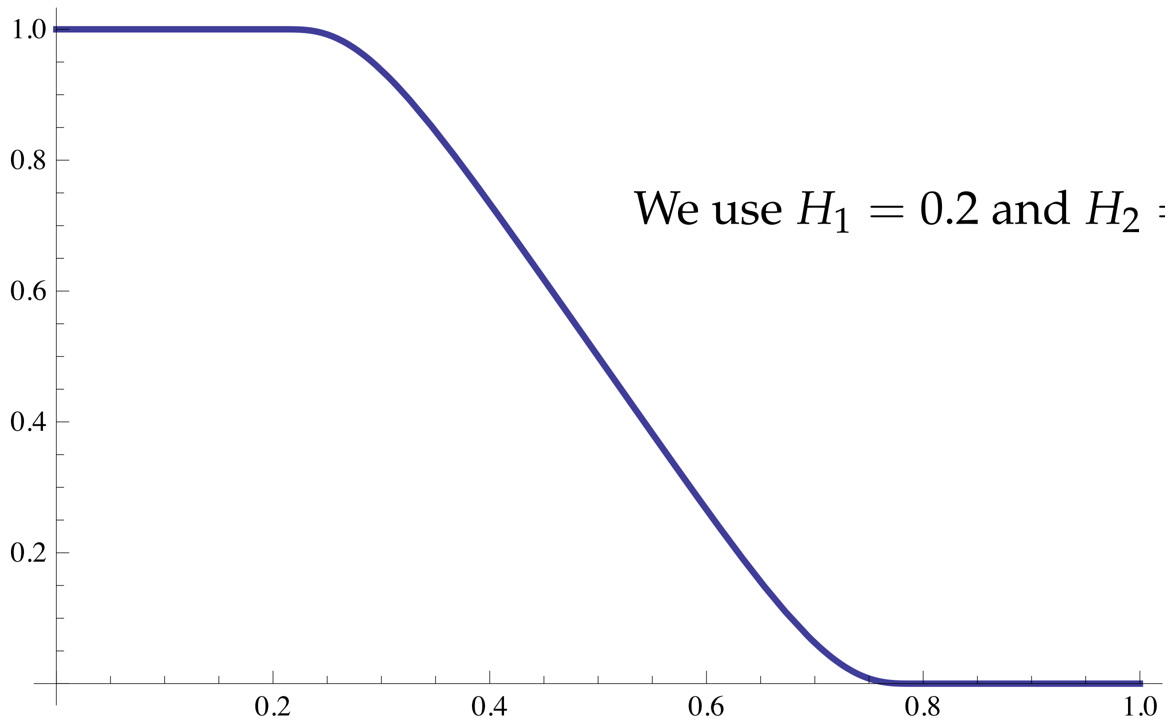
$$\xi(t) = \begin{cases} 1 & \text{if } t \leq H_1 \\ 0 & \text{if } t \geq H_2 \\ 1/(1 + e^{2 \cdot s}) & \text{otherwise} \end{cases}$$

where H_1, H_2 are constant, with $0 < H_1 < H_2 < 1$,

$$s = \left(\frac{1}{\sqrt{1-H}} \right) - \left(\frac{1}{\sqrt{H}} \right) \quad \text{and} \quad H = \left(\frac{t - H_1}{H_2 - H_1} \right) .$$

Parametric Pseudo-Manifolds

Reconstructing a curve homeomorphic to S^1



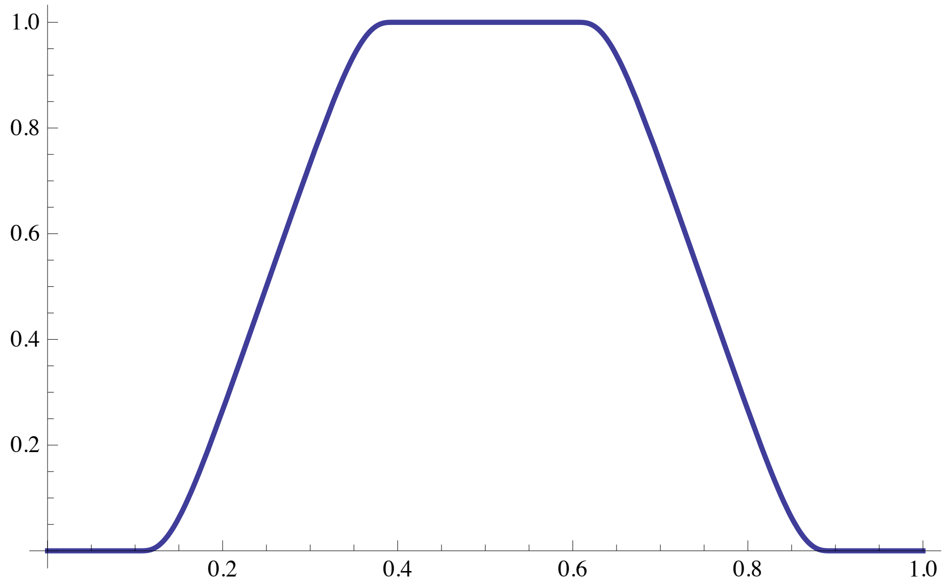
We use $H_1 = 0.2$ and $H_2 = 0.8$.

Parametric Pseudo-Manifolds

Reconstructing a curve homeomorphic to S^1

Using ξ, we define the bump function $w : \mathbb{R} \to [0, 1]$ such that

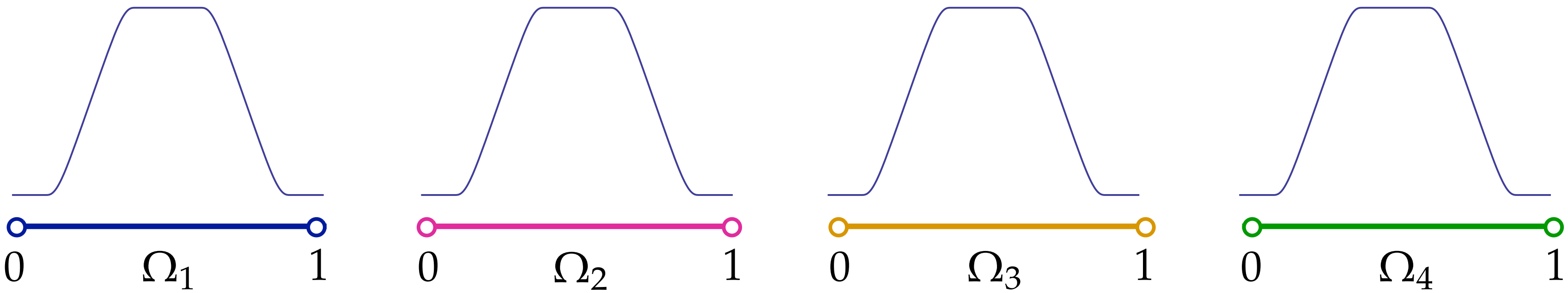
$$w(x) = \begin{cases} \xi(1-2x) & \text{if } x \leq 0.5, \\ \xi(2x-1) & \text{otherwise.} \end{cases}$$



Parametric Pseudo-Manifolds

Reconstructing a curve homeomorphic to S^1

Now, we are ready to define the parametrizations. The key idea is to assign a bump function, $w_i : \mathbb{R} \to [0, 1]$, with each p-domain, Ω_i, such that $w_i(x) = w(x)$, for every $x \in \Omega_i$.



Parametric Pseudo-Manifolds

Reconstructing a curve homeomorphic to $\mathbb{S}^1$

Finally, we assign a parametrization, $\theta_i : \Omega_i \to \mathbb{E}^2$, with each p-domain, Ω_i, such that

$$\theta_i(t) = \begin{cases} c_i(t) & \text{if } t \geq \frac{2}{5} \text{ and } t \leq \frac{3}{5}, \\ \dfrac{w_i(t) \cdot c_i(t) + w_j(\varphi_{ji}(t)) \cdot c_j(\varphi_{ji}(t))}{w_i(t) + w_j(\varphi_{ji}(t))} & \text{if } t > \frac{3}{5} \text{ and } t < 1 \text{ and } j = i+1 \text{ or } i = 4 \text{ and } j = 1, \\ \dfrac{w_i(t) \cdot c_i(t) + w_j(\varphi_{ji}(t)) \cdot c_j(\varphi_{ji}(t))}{w_i(t) + w_j(\varphi_{ji}(t))} & \text{if } t > 0 \text{ and } t < \frac{2}{5} \text{ and } j = i-1 \text{ or } i = 1 \text{ and } j = 4, \end{cases}$$

for $i = 1, 2, 3, 4$.

We also claim that $\theta_i(x) = \theta_j \circ \varphi_{ji}(x)$, for all $x \in \Omega_{ij}$ and for $i = 1, 2, 3, 4$.

Parametric Pseudo-Manifolds

Reconstructing a curve homeomorphic to $\mathbf{S}^1$

Let $j = i+1$ or $i = 4$ and $j = 1$ and $t \in]\,3/5\,,\,1\,[$. Then, we have $s = \varphi_{ji}(t) \in]\,0\,,\,2/5\,[$ and

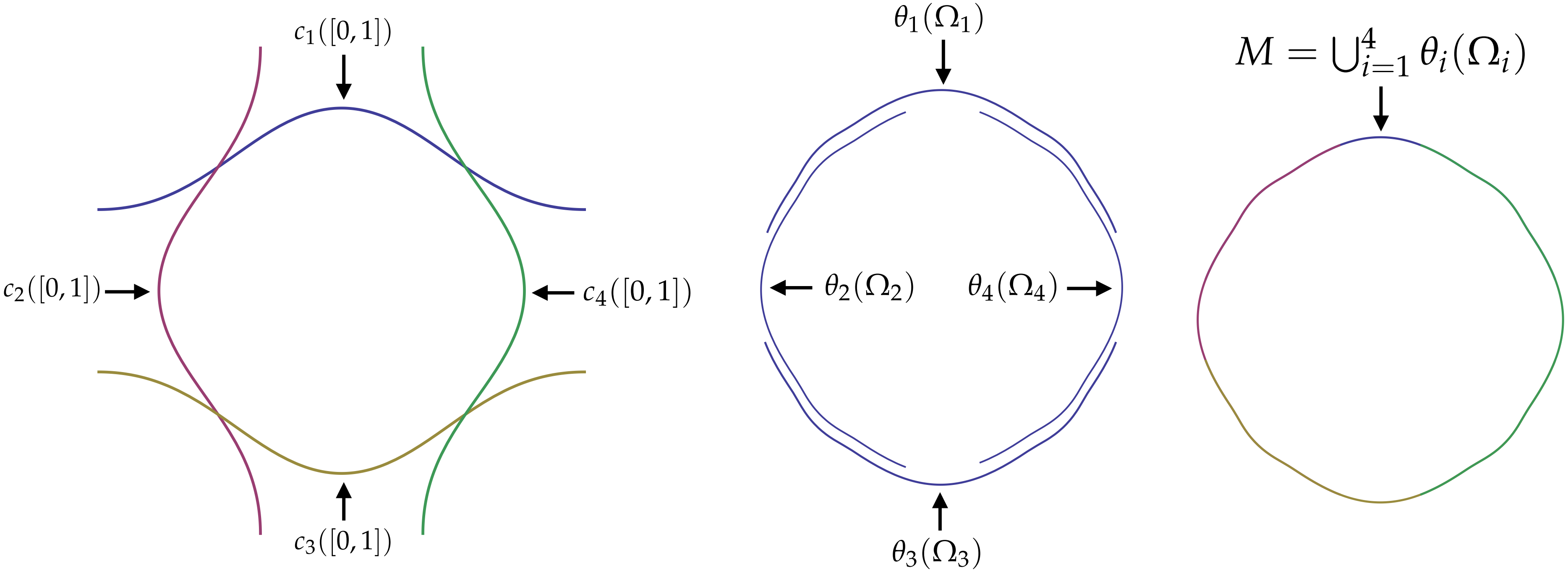
$$
\begin{aligned}
\theta_j \circ \varphi_{ji}(t) &= \theta_j(s) \\
&= \frac{w_j(s)\cdot c_j(s) + w_i(\varphi_{ij}(s))\cdot c_i(\varphi_{ij}(s))}{w_j(s) + w_i(\varphi_{ij}(s))} \\
&= \frac{w_j(s)\cdot c_j(s) + w_i(\varphi_{ji}^{-1}(s))\cdot c_i(\varphi_{ji}^{-1}(s))}{w_j(s) + w_i(\varphi_{ji}^{-1}(s))} \\
&= \frac{w_j(\varphi_{ji}(t))\cdot c_j(\varphi_{ji}(t)) + w_i(t)\cdot c_i(t)}{w_j(\varphi_{ji}(t)) + w_i(t)} \\
&= \frac{w_i(t)\cdot c_i(t) + w_j(\varphi_{ji}(t))\cdot c_j(\varphi_{ji}(t))}{w_i(t) + w_j(\varphi_{ji}(t))} \\
&= \theta_i(t)\,.
\end{aligned}
$$

Parametric Pseudo-Manifolds

Reconstructing a curve homeomorphic to S^1

If $j = i - 1$ or $i = 1$ and $j = 4$ and $t \in \,]\, 0\,,\, 2/5\, [$, then we can proceed in a similar manner.

So, our parametrizations are consistent.



Parametric Pseudo-Manifolds

Reconstructing a curve homeomorphic to S^1

Some important remarks:

- The partition of unity functions are "hidden" in the convex sum that defines the θ_i's. Indeed, if we denote the function associated with Ω_i by α_i, then we have that

$$\alpha_i(x) = \begin{cases} 1, & \text{if } x \geq \frac{2}{5} \text{ and } x \leq \frac{3}{5} \quad , \\ w_i(x)/(w_i(x) + w_j(\varphi_{ji}(x)), & \text{if } x > 0 \text{ and } x < \frac{2}{5}, j = i-1 \text{ or } j = 4 \text{ and } i = 1, \quad , \\ w_i(x)/(w_i(x) + w_j(\varphi_{ji}(x)), & \text{if } x > \frac{3}{5} \text{ and } x < 1, j = i+1 \text{ or } j = 1 \text{ and } i = 4, \quad . \end{cases}$$

- The bump functions (the w_i's), the transition maps (the φ_{ij}'s), and the Bézier curves (the c_i's) are all C^∞-functions. As a result the parametrizations (the θ_i's) are C^∞. In turn, these facts imply that the one-dimensional PPM we just built is C^∞.