Introduction to Computational Manifolds and Applications

Part 1 - Foundations

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Parametric Pseudo-Manifolds

Sets of Gluing Data

Our definition of manifold is not constructive: it states what a manifold is by assuming that the space already exists. What if we are interested in “constructing” a manifold?

It turns out that a manifold can be built from what we call a set of gluing data.

The idea is to glue open sets in $\mathbb{E}^n$ in a controlled manner, and then embed them in $\mathbb{E}^d$.

André Weil introduced this gluing process to define abstract algebraic varieties from irreducible affine sets in a book published in 1946. However, as far as we know, Cindy Grimm and John Hughes were the first to give a constructive definition of manifold.
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Sets of Gluing Data

The pioneering work of Grimm and Hughes allows us to create smooth 2-manifolds (i.e., smooth surfaces equipped with an atlas) in $\mathbb{E}^3$ for the purposes of modeling and simulation.

In this lecture we will introduce a formal definition of sets of gluing data, which fixes a problem in the definition given by Grimm and Hughes, and includes a Hausdorff condition.

We also introduce the notion of parametric pseudo-manifolds.

A parametric pseudo-manifold (PPM) is a topological space defined from a set of gluing data.

Under certain conditions (which are often met in practice), PPM’s are manifolds in $\mathbb{E}^m$. 

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\[
\begin{align*}
\mathbb{E}^n & \quad \mathbb{E}^d \\
\Omega_i & \quad \Omega_{ij} \\
\varphi_{ij} & \\
\Omega_{ji} & \quad \Omega_j \\
\theta_i(\Omega_i) & \quad \theta_j(\Omega_j) \\
\end{align*}
\]
Sets of Gluing Data

Let $I$ and $K$ be (possibly infinite) countable sets such that $I$ is nonempty.

**Definition 7.1.** Let $n$ be an integer, with $n \geq 1$, and $k$ be either an integer, with $k \geq 1$, or $k = \infty$.

A *set of gluing data* is a triple,

$$\mathcal{G} = \left( (\Omega_i)_{i \in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\phi_{ji})_{(i,j) \in K} \right),$$

satisfying the following properties:
Sets of Gluing Data

(1) For every \( i \in I \), the set \( \Omega_i \) is a nonempty open subset of \( \mathbb{E}^n \) called *parametrization domain*, for short, *p-domain*, and any two distinct p-domains are pairwise disjoint, i.e.,

\[ \Omega_i \cap \Omega_j = \emptyset, \]

for all \( i \neq j \).
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(2) For every pair \((i, j) \in I \times I\), the set \(\Omega_{ij}\) is an open subset of \(\Omega_i\). Furthermore, \(\Omega_{ii} = \Omega_i\) and \(\Omega_{ji} \neq \emptyset\) if and only if \(\Omega_{ij} \neq \emptyset\). Each nonempty subset \(\Omega_{ij}\) (with \(i \neq j\)) is called a gluing domain.
Sets of Gluing Data

(3) If we let

\[ K = \{ (i, j) \in I \times I \mid \Omega_{ij} \neq \emptyset \}, \]

then \( \varphi_{ji} : \Omega_{ij} \rightarrow \Omega_{ji} \) is a \( C^k \) bijection for every \( (i, j) \in K \) called a transition (or gluing) map.
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The transition functions must satisfy the following three conditions:

(a) $\varphi_{ii} = \text{id}_{\Omega_i}$, for all $i \in I$,
(b) $\varphi_{ij} = \varphi_{ji}^{-1}$, for all $(i, j) \in K$, and
(c) For all $i, j, k$, if
\[ \Omega_{ji} \cap \Omega_{jk} \neq \emptyset, \]
then
\[ \varphi_{ij}(\Omega_{ji} \cap \Omega_{jk}) = \Omega_{ij} \cap \Omega_{ik} \quad \text{and} \quad \varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x), \]
for all $x \in \Omega_{ij} \cap \Omega_{ik}$. 
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\[ \varphi_{ki}(x) = (\varphi_{kj} \circ \varphi_{ji})(x), \quad \text{for all } x \in (\Omega_{ij} \cap \Omega_{ik}). \]
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The cocycle condition implies conditions (a) and (b):

(a) \( \varphi_{ii} = \text{id}_{\Omega_i} \), for all \( i \in I \), and

(b) \( \varphi_{ij} = \varphi_{ji}^{-1} \), for all \((i,j) \in K\).
(4) For every pair \((i, j) \in K\), with \(i \neq j\), for every

\[ x \in \partial(\Omega_{ij}) \cap \Omega_i \quad \text{and} \quad y \in \partial(\Omega_{ji}) \cap \Omega_j, \]

there are open balls, \(V_x\) and \(V_y\), centered at \(x\) and \(y\), so that no point of \(V_y \cap \Omega_{ji}\) is the image of any point of \(V_x \cap \Omega_{ij}\) by \(\varphi_{ji}\).
Sets of Gluing Data

Given a set of gluing data, $G$, can we build a manifold from it?

The answer is YES!

Indeed, such a manifold is built by a quotient construction.
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The idea is to form the disjoint union, $\bigsqcup_{i \in I} \Omega_i$, of the $\Omega_i$ and then identify $\Omega_{ij}$ with $\Omega_{ji}$ using $\varphi_{ji}$.

Formally, we define a binary relation, $\sim$, on $\bigsqcup_{i \in I} \Omega_i$ as follows: for all $x, y \in \bigsqcup_{i \in I} \Omega_i$, we have

$$x \sim y \quad \text{iff} \quad (\exists (i, j) \in K) (x \in \Omega_{ij}, y \in \Omega_{ji}, y = \varphi_{ji}(x)).$$

We can prove that $\sim$ is an equivalence relation, which enables us to define the space

$$M_G = \left( \bigsqcup_{i \in I} \Omega_i \right) / \sim.$$

We can also prove that $M_G$ is a Hausdorff and second-countable manifold.
Sets of Gluing Data

Sketching the proof:

For every $i \in I$, $\text{in}_i : \Omega_i \to \bigsqcup_{i \in I} \Omega_i$ is the natural injection.

Let $p : \bigsqcup_{i \in I} \Omega_i \to M_G$ be the quotient map, with

$$p(x) = [x].$$

For every $i \in I$, let $\tau_i = p \circ \text{in}_i : \Omega_i \to M_G$.

Let $U_i = \tau_i(\Omega_i)$ and $\varphi_i = \tau_i^{-1}$.

It is immediately verified that $(U_i, \varphi_i)$ are charts and that this collection of charts forms a $C^k$ atlas for $M_G$. 
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Sketching the proof:

We now prove that the topology of $M_G$ is Hausdorff.

Pick $[x], [y] \in M_G$ with $[x] \neq [y]$, for some $x \in \Omega_i$ and some $y \in \Omega_j$.

Either

$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \emptyset \quad \text{or} \quad \tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset.$$

In the former case, as $\tau_i$ and $\tau_j$ are homeomorphisms, $[x]$ and $[y]$ belong to the two disjoint open sets $\tau_i(\Omega_i)$ and $\tau_j(\Omega_j)$. In the latter case, we must consider four subcases:
Sketching the proof:

1. \( \Omega_i = \Omega_j \)

2. \( \Omega_i \)

3. \( \Omega_i \)

4. \( \Omega_j \)
Sets of Gluing Data

Sketching the proof:

(1) If \( i = j \) then \( x \) and \( y \) can be separated by disjoint opens, \( V_x \) and \( V_y \), and as \( \tau_i \) is a homeomorphism, \([x]\) and \([y]\) are separated by the disjoint open subsets \( \tau_i(V_x) \) and \( \tau_j(V_y) \).

\[
\Omega_i = \Omega_j
\]
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Sketching the proof:

(2) If \( i \neq j \), \( x \in \Omega_i - \overline{\Omega_{ij}} \) and \( y \in \Omega_j - \overline{\Omega_{ji}} \), then \( \tau_i(\Omega_i - \overline{\Omega_{ij}}) \) and \( \tau_j(\Omega_j - \overline{\Omega_{ji}}) \) are disjoint open subsets separating \([x]\) and \([y]\), where \( \overline{\Omega_{ij}} \) and \( \overline{\Omega_{ji}} \) are the closures of \( \Omega_{ij} \) and \( \Omega_{ji} \), respectively.
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Sketching the proof:

(3) If \( i \neq j \), \( x \in \Omega_{ij} \) and \( y \in \Omega_{ji} \), as \([x] \neq [y]\) and \( y \sim \varphi_{ij}(y) \), then \( x \neq \varphi_{ij}(y) \). We can separate \( x \) and \( \varphi_{ij}(y) \) by disjoint open subsets, \( V_x \) and \( V_y \), and \([x]\) and \([y] = [\varphi_{ij}(y)]\) are separated by the disjoint open subsets \( \tau_i(V_x) \) and \( \tau_i(V_y) \).
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Sketching the proof:

(4) If $i \neq j$, $x \in \partial(\Omega_{ij}) \cap \Omega_i$ and $y \in \partial(\Omega_{ji}) \cap \Omega_j$, then we use condition 4 of Definition 7.1. This condition yields two disjoint open subsets, $V_x$ and $V_y$, with $x \in V_x$ and $y \in V_y$, such that no point of $V_x \cap \Omega_{ij}$ is equivalent to any point of $V_y \cap \Omega_{ji}$, and so $\tau_i(V_x)$ and $\tau_j(V_y)$ are disjoint open subsets separating $[x]$ and $[y]$. 

![Diagram](image-url)
Sets of Gluing Data

Sketching the proof:

So, the topology of $M_G$ is Hausdorff and $M_G$ is indeed a manifold.

$M_G$ is also second-countable (WHY?).

Finally, it is trivial to verify that the transition maps of $M_G$ are the original gluing functions, 

$$
\varphi_{ij},
$$

since

$$
\varphi_i = \tau_i^{-1} \quad \text{and} \quad \varphi_{ji} = \varphi_j \circ \varphi_i^{-1}.
$$
Theorem 7.1. For every set of gluing data,
\[ G = (\{ \Omega_i \}_{i \in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\varphi_{ji})_{(i,j) \in K}), \]
there is an \( n \)-dimensional \( C^k \) manifold, \( M_G \), whose transition maps are the \( \varphi_{ji} \)'s.

Theorem 7.1 is nice, but...

- Our proof is not constructive;
- \( M_G \) is an abstract entity, which may not be orientable, compact, etc.

So, we know we can build a manifold from a set of gluing data, but that does not mean we know \textit{how} to build a "concrete" manifold. For that, we need a formal notion of "concreteness".
The notion of "concreteness" is realized as \textit{parametric pseudo-manifolds}:

\textbf{Definition 7.2.} Let \( n, d, \) and \( k \) be three integers with \( d > n \geq 1 \) and \( k \geq 1 \) or \( k = \infty \). A \textit{parametric} \( C^k \) \textit{pseudomanifold of dimension} \( n \) \textit{in} \( \mathbb{E}^d \) \textit{(for short, parametric pseudo-manifold or PPM)} is a pair,
\[
\mathcal{M} = (\mathcal{G}, (\theta_i)_{i \in I}),
\]
such that
\[
\mathcal{G} = ((\Omega_i)_{i \in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\varphi_{ji})_{(i,j) \in K})
\]
is a set of gluing data, for some finite set \( I \), and each \( \theta_i : \Omega_i \to \mathbb{E}^d \) is \( C^k \) and satisfies
\[(C) \text{ For all } (i,j) \in K, \text{ we have } \theta_i = \theta_j \circ \varphi_{ji}.\]
Parametric Pseudo-Manifolds

\[ \mathbb{E}^n \rightarrow \Omega_i \rightarrow M \rightarrow \mathbb{E}^d \rightarrow \theta_i(\Omega_i) \]

\[ \mathbb{E}^n \rightarrow \Omega_j \rightarrow M \rightarrow \mathbb{E}^d \rightarrow \theta_j(\Omega_j) \]

Gluing data

Parametric pseudo-manifold
As usual, we call \( \theta_i \) a *parametrization*.

The subset, \( M \subset \mathbb{E}^d \), given by

\[
M = \bigcup_{i \in I} \theta_i(\Omega_i)
\]

is called the *image* of the parametric pseudo-manifold, \( M \).

Whenever \( n = 2 \) and \( d = 3 \), we say that \( M \) is a *parametric pseudo-surface* (or PPS, for short).

We also say that \( M \), the image of the PPS \( M \), is a *pseudo-surface*.
Parametric Pseudo-Manifolds

Condition C of Definition 7.2,

\[(C) \text{ For all } (i, j) \in K, \text{ we have } \theta_i = \theta_j \circ \varphi_{ji},\]

obviously implies that

\[\theta_i(\Omega_{ij}) = \theta_j(\Omega_{ji}),\]

for all \((i, j) \in K\). Consequently, \(\theta_i\) and \(\theta_j\) are consistent parametrizations of the overlap

\[\theta_i(\Omega_{ij}) = \theta_j(\Omega_{ji}).\]
Parametric Pseudo-Manifolds

\[ \mathbb{E}^d \]

\[ \mathbb{E}^n \]

\[ \Omega_i \]

\[ \Omega_j \]

\[ \varphi_{ij} \]

\[ \varphi_{ji} \]

\[ \theta_i(\Omega_i) \]

\[ \theta_j(\Omega_j) \]

\[ \mathcal{M} \]

consistent!
Parametric Pseudo-Manifolds

Thus, the set $M$, whatever it is, is covered by pieces, $U_i = \theta_i(\Omega_i)$, not necessarily open.

Each $U_i$ is parametrized by $\theta_i$, and each overlapping piece, $U_i \cap U_j$, is parametrized consistently.

The local structure of $M$ is given by the $\theta_i$’s and its global structure is given by the gluing data.
Parametric Pseudo-Manifolds

We can equip $M$ with an atlas if we require the $\theta_i$'s to be injective and to satisfy

(C') For all $(i, j) \in K$,
\[
\theta_i(\Omega_i) \cap \theta_j(\Omega_j) = \theta_i(\Omega_{ij}) = \theta_j(\Omega_{ji}).
\]

(C'') For all $(i, j) \notin K$,
\[
\theta_i(\Omega_i) \cap \theta_j(\Omega_j) = \emptyset.
\]

Even if the $\theta_i$'s are not injective, properties C’ and C” are still desirable since they ensure that $\theta_i(\Omega_i - \Omega_{ij})$ and $\theta_j(\Omega_j - \Omega_{ji})$ are uniquely parametrized. Unfortunately, properties C’ and C” may be difficult to enforce in practice (at least for surface constructions).
Interestingly, regardless whether conditions C’ and C” are satisfied, we can still show that $M$ is the image in $E^d$ of the abstract manifold, $M_G$, as stated by Proposition 7.2:

**Proposition 7.2.** Let $\mathcal{M} = (\mathcal{G}, (\theta_i)_{i \in I})$ be a parametric $C^k$ pseudo-manifold of dimension $n$ in $E^d$, where $\mathcal{G} = \left( (\Omega_i)_{i \in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\varphi_{ji})_{(i,j) \in K} \right)$ is a set of gluing data, for some finite set $I$. Then, the parametrization maps, $\theta_i$, induce a surjective map, $\Theta: M_G \to M$, from the abstract manifold, $M_G$, specified by $\mathcal{G}$ to the image, $M \subseteq E^d$, of the parametric pseudo-manifold, $\mathcal{M}$, and the following property holds:

$$\theta_i = \Theta \circ \tau_i,$$

for every $\Omega_i$, where $\tau_i: \Omega_i \to M_G$ are the parametrization maps of the manifold $M_G$. In particular, every manifold, $M \subseteq E^d$, such that $M$ is induced by $\mathcal{G}$ is the image of $M_G$ by a map

$$\Theta: M_G \to M.$$
Parametric Pseudo-Manifolds

The “Evil” Cocycle Condition

(c) For all $i, j, k$, if

$$\Omega_{ji} \cap \Omega_{jk} \neq \emptyset,$$

then

$$\varphi_{ij}(\Omega_{ji} \cap \Omega_{jk}) = \Omega_{ij} \cap \Omega_{ik} \quad \text{and} \quad \varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x),$$

for all $x \in \Omega_{ij} \cap \Omega_{ik}$.
Parametric Pseudo-Manifolds

The “Evil” Cocycle Condition

\[ \varphi_{ki}(x) = (\varphi_{kj} \circ \varphi_{ji})(x), \quad \text{for all } x \in (\Omega_{ij} \cap \Omega_{ik}). \]
The “Evil” Cocycle Condition

The statement

\[ \text{if } \Omega_{ji} \cap \Omega_{jk} \neq \emptyset \text{ then } \phi_{ij}(\Omega_{ji} \cap \Omega_{jk}) = \Omega_{ij} \cap \Omega_{ik} \]

is necessary for guaranteeing the transitivity of the equivalence relation \( \sim \).
Parametric Pseudo-Manifolds

The “Evil” Cocycle Condition

Consider the $p$-domains (i.e., open line intervals)

\[ \Omega_1 = ]0, 3[, \quad \Omega_2 = ]4, 5[, \quad \text{and} \quad \Omega_3 = ]6, 9[. \]

Consider the gluing domains

\[ \Omega_{12} = ]0, 1[, \quad \Omega_{13} = ]2, 3[, \quad \Omega_{21} = \Omega_{23} = ]4, 5[, \quad \Omega_{32} = ]8, 9[, \quad \Omega_{31} = ]6, 7[. \]
The “Evil” Cocycle Condition

Consider the transition maps:

\[ \varphi_{21}(x) = x + 4, \quad \varphi_{32}(x) = x + 4 \quad \text{and} \quad \varphi_{31}(x) = x + 4. \]
The “Evil” Cocycle Condition

Obviously,

\[(\varphi_{32} \circ \varphi_{21})(x) = x + 8, \quad \text{for all } x \in \Omega_{12}.\]

\[\varphi_{21}(0.5) = 4.5 \quad \text{and} \quad \varphi_{32}(4.5) = 8.5 \quad \implies \quad 0.5 \sim 4.5 \quad \text{and} \quad 4.5 \sim 8.5\]

So, if \(\sim\) were transitive, then we would have \(0.5 \sim 8.5\). But...
The “Evil” Cocycle Condition

it turns out that $\varphi_{31}$ is undefined at 0.5.

So, $0.5 \not\sim 8.5$.

The reason is that $\varphi_{31}$ and $\varphi_{32} \circ \varphi_{21}$ have disjoint domains.
The “Evil” Cocycle Condition

The reason they have disjoint domains is that condition "c" is not satisfied:

$$\text{if } \Omega_{21} \cap \Omega_{23} \neq \emptyset \text{ then } \phi_{12}(\Omega_{21} \cap \Omega_{23}) = \Omega_{12} \cap \Omega_{13}.$$ 

Indeed

$$\Omega_{21} \cap \Omega_{23} = \Omega_2 = ]4, 5[ \neq \emptyset,$$

but

$$\phi_{12}(\Omega_{21} \cap \Omega_{23}) = ]0, 1[ \neq \emptyset = \Omega_{12} \cap \Omega_{13}.$$