IMPA - Instituto de Matemática Pura e Aplicada, Rio de Janeiro, RJ, Brazil

Introduction to Computational Manifolds and Applications

Part 1 - Foundations

Prof. Jean Gallier

jean@cis.upenn.edu

Department of Computer and Information Science
University of Pennsylvania
Philadelphia, PA, USA

Parametric Pseudo-Manifolds

Sets of Gluing Data

Our definition of manifold is **not** constructive: it states what a manifold is by assuming that the space already exists. What if we are interested in “constructing" a manifold?

It turns out that a manifold can be built from what we call a set of gluing data.

The idea is to glue open sets in $\mathbb{E}^n$ in a controlled manner, and then embed them in $\mathbb{E}^d$.

André Weil introduced this gluing process to define abstract algebraic varieties from irreducible affine sets in a book published in 1946. However, as far as we know, Cindy Grimm and John Hughes were the first to give a constructive definition of manifold.

SIGGRAPH, 1995

Parametric Pseudo-Manifolds

Sets of Gluing Data

The pioneering work of Grimm and Hughes allows us to create smooth 2-manifolds (i.e., *smooth surfaces* equipped with an atlas) in $\mathbb{E}^3$ for the purposes of modeling and simulation.

In this lecture we will introduce a formal definition of sets of gluing data, which fixes a problem in the definition given by Grimm and Hughes, and includes a Hausdorff condition.

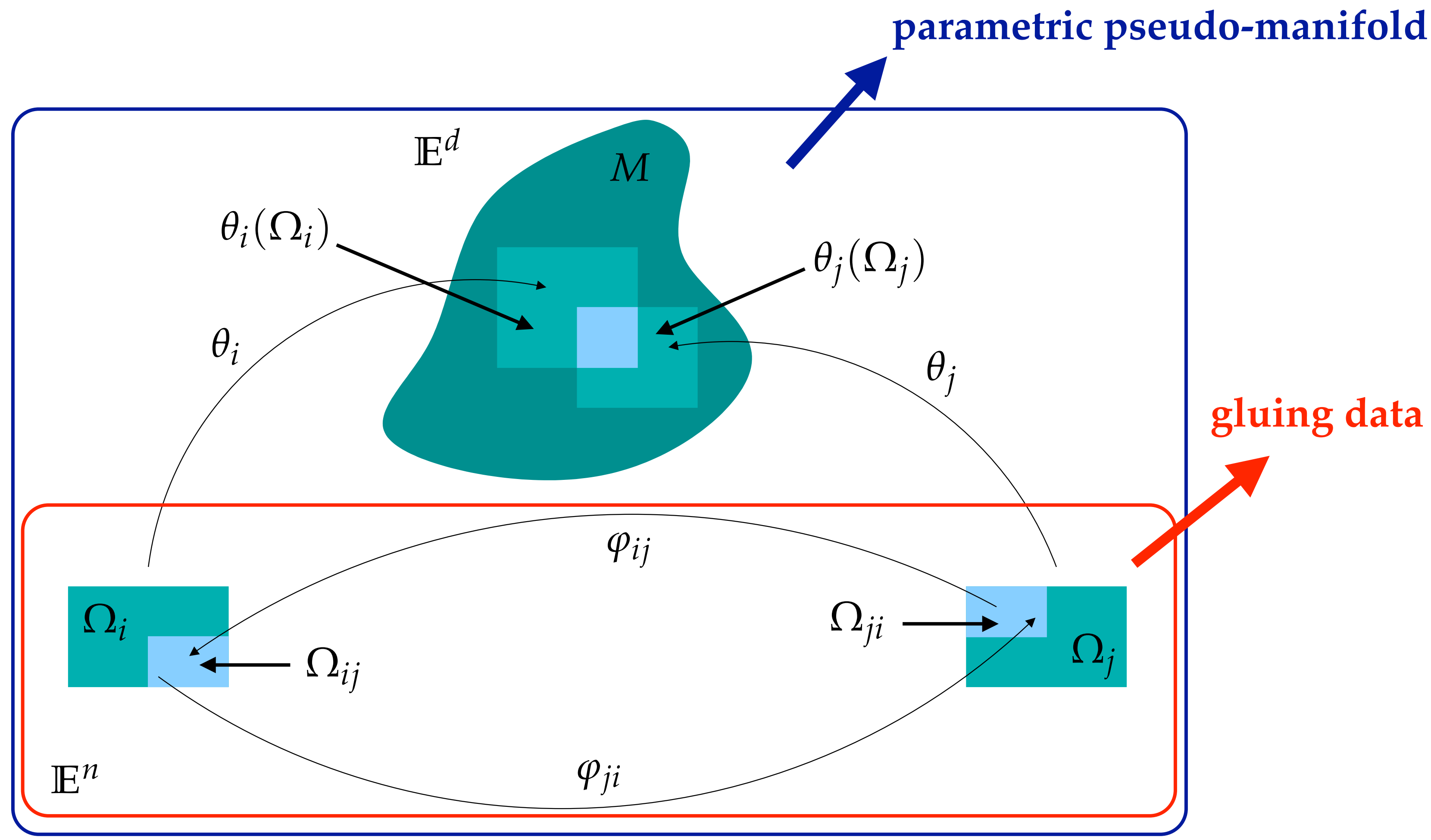
We also introduce the notion of parametric pseudo-manifolds.

A parametric pseudo-manifold (PPM) is a topological space defined from a set of gluing data.

Under certain conditions (which are often met in practice), PPM's are manifolds in $\mathbb{E}^m$.

Parametric Pseudo-Manifolds

Sets of Gluing Data



Parametric Pseudo-Manifolds

Sets of Gluing Data

Let I and K be (possibly infinite) countable sets such that I is nonempty.

Definition 7.1. Let n be an integer, with $n \geq 1$, and k be either an integer, with $k \geq 1$, or $k = \infty$.

A *set of gluing data* is a triple,

$$\mathcal{G} = \left((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K}\right),$$

satisfying the following properties:

Parametric Pseudo-Manifolds

Sets of Gluing Data

(1) For every $i \in I$, the set Ω_i is a nonempty open subset of $\mathbb{E}^n$ called *parametrization domain*, for short, *p-domain*, and any two distinct p-domains are pairwise disjoint, i.e.,

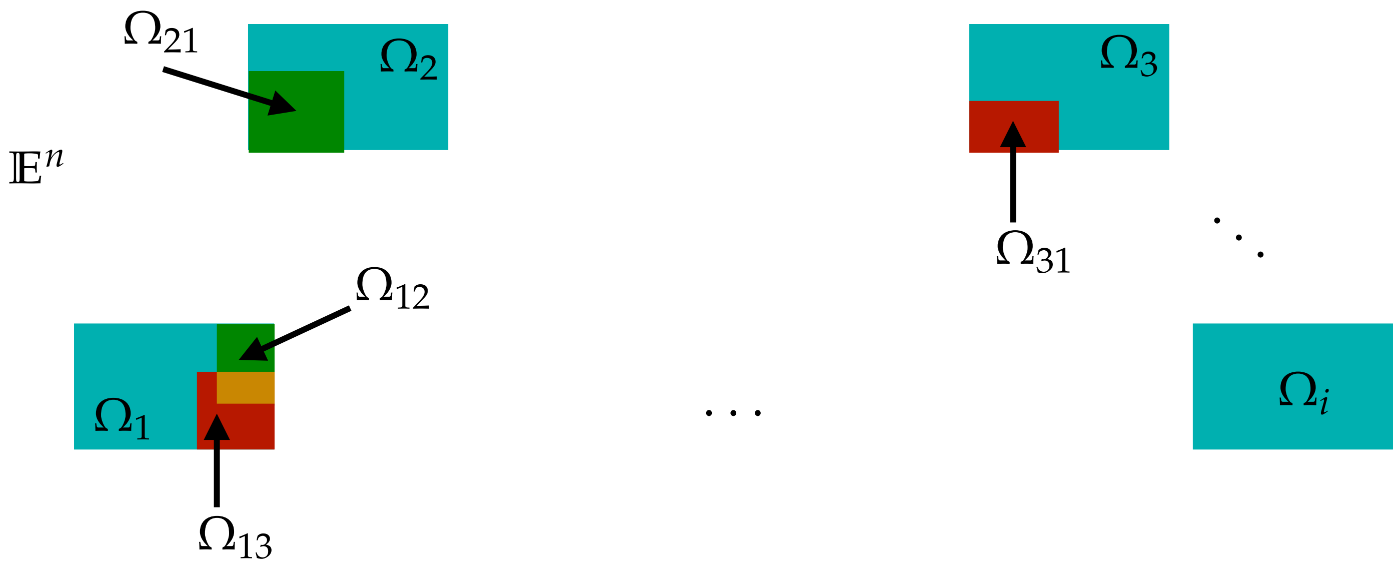
$$\Omega_i \cap \Omega_j = \emptyset,$$

for all $i \neq j$.

Parametric Pseudo-Manifolds

Sets of Gluing Data

(2) For every pair $(i, j) \in I \times I$, the set Ω_{ij} is an open subset of Ω_i. Furthermore, $\Omega_{ii} = \Omega_i$ and $\Omega_{ji} \neq \emptyset$ if and only if $\Omega_{ij} \neq \emptyset$. Each nonempty subset Ω_{ij} (with $i \neq j$) is called a *gluing domain*.



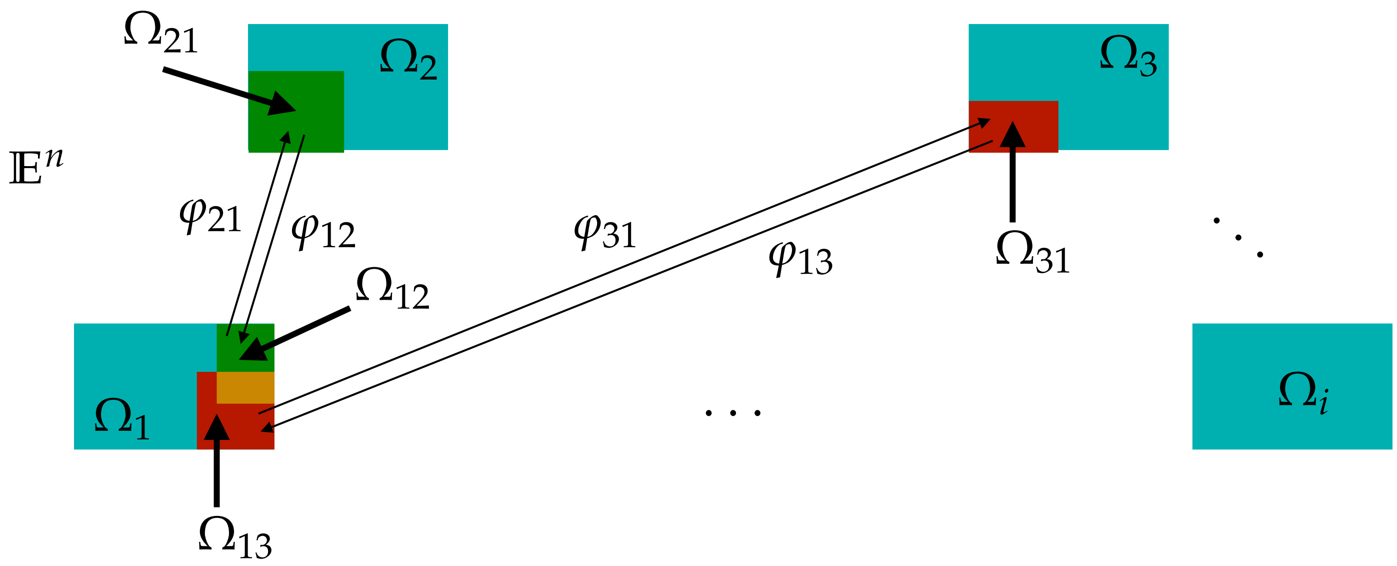
Parametric Pseudo-Manifolds

Sets of Gluing Data

(3) If we let

$$K = \{(i, j) \in I \times I \mid \Omega_{ij} \neq \emptyset\},$$

then $\varphi_{ji} \colon \Omega_{ij} \to \Omega_{ji}$ is a C^k bijection for every $(i, j) \in K$ called a *transition (or gluing) map*.



Parametric Pseudo-Manifolds

Sets of Gluing Data

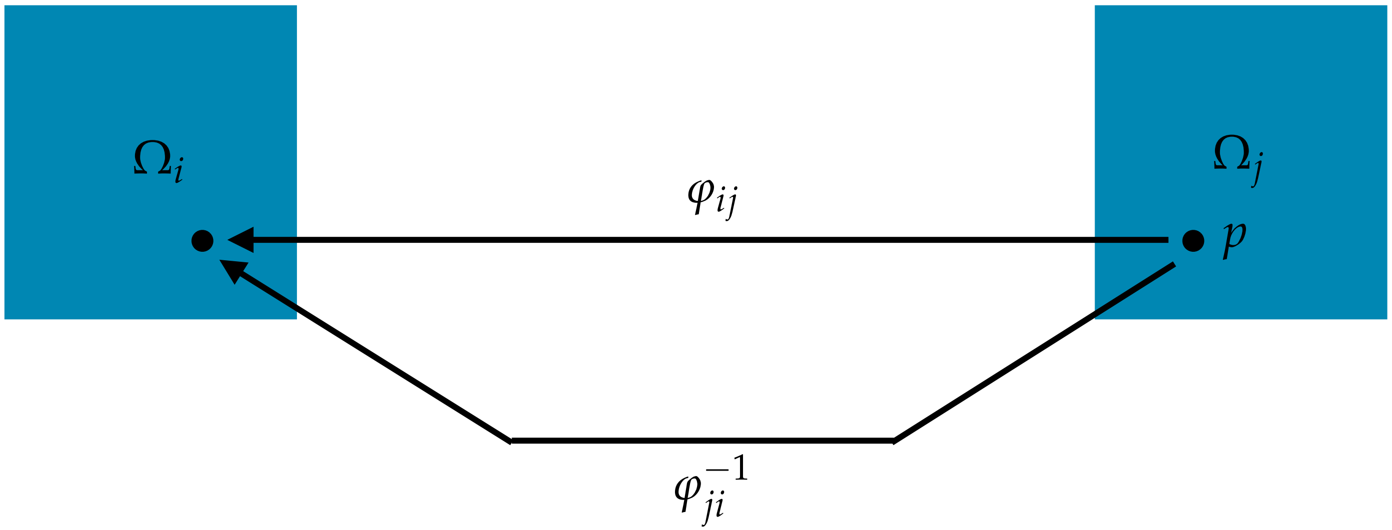
The transition functions must satisfy the following three conditions:

(a) $\varphi_{ii} = \mathrm{id}_{\Omega_i}$, for all $i \in I$,

Parametric Pseudo-Manifolds

Sets of Gluing Data

(b) $\varphi_{ij} = \varphi_{ji}^{-1}$, for all $(i, j) \in K$, and



Parametric Pseudo-Manifolds

Sets of Gluing Data

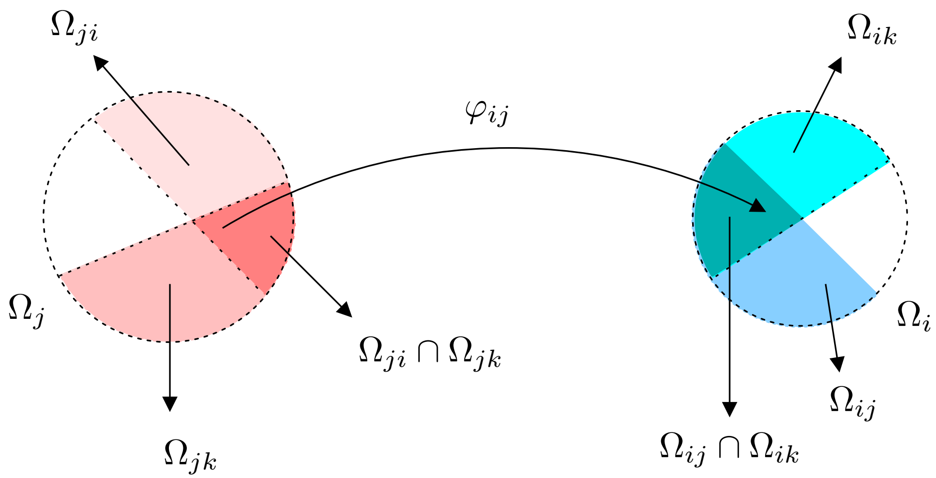
(c) For all i, j, k, if

$$\Omega_{ji} \cap \Omega_{jk} \neq \emptyset,$$

then

$$\varphi_{ij}(\Omega_{ji} \cap \Omega_{jk}) = \Omega_{ij} \cap \Omega_{ik} \qquad \text{and} \qquad \varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x),$$

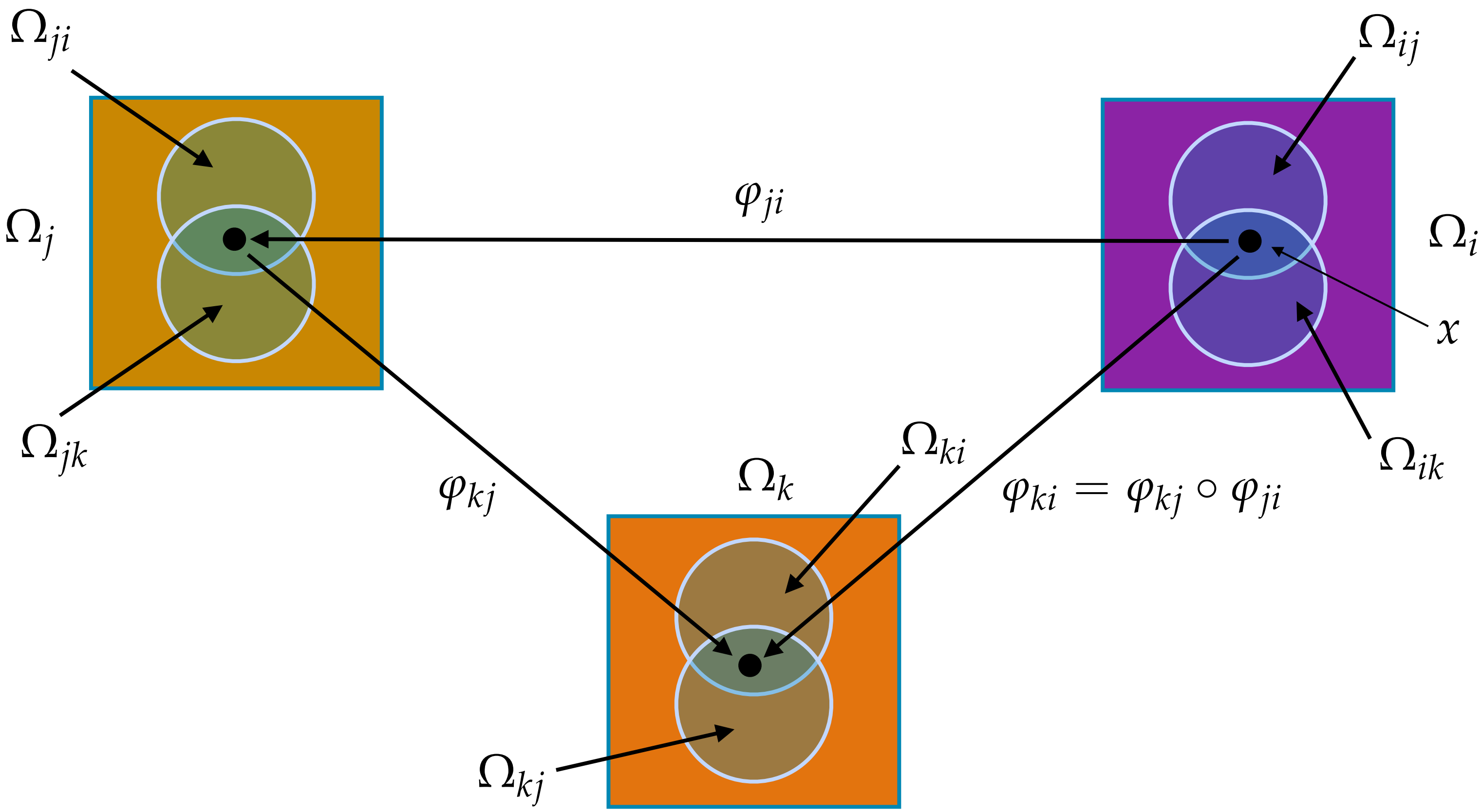
for all $x \in \Omega_{ij} \cap \Omega_{ik}$.



Parametric Pseudo-Manifolds

Sets of Gluing Data

$$\varphi_{ki}(x) = (\varphi_{kj} \circ \varphi_{ji})(x), \quad \text{for all } x \in (\Omega_{ij} \cap \Omega_{ik}).$$



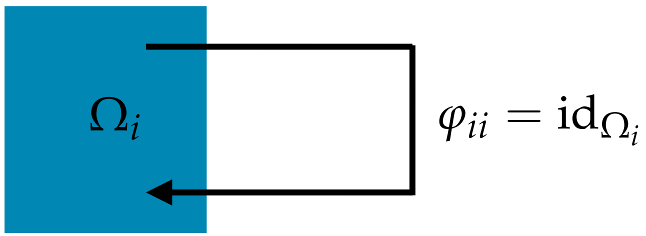
Parametric Pseudo-Manifolds

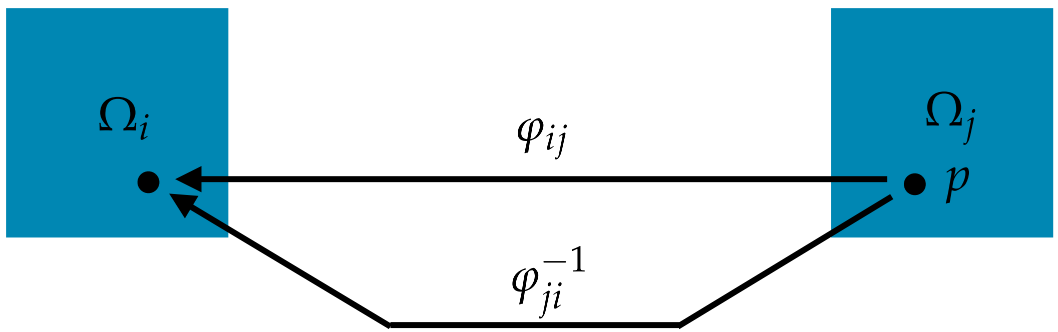
Sets of Gluing Data

The cocycle condition implies conditions (a) and (b):

(a) $\varphi_{ii} = \mathrm{id}_{\Omega_i}$, for all $i \in I$, and

(b) $\varphi_{ij} = \varphi_{ji}^{-1}$, for all $(i, j) \in K$.





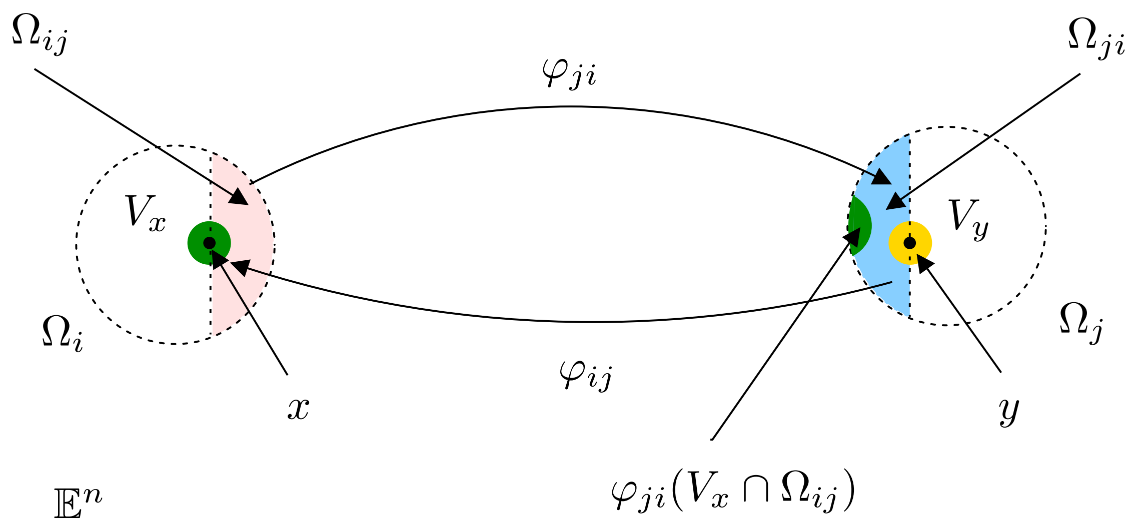
Parametric Pseudo-Manifolds

Sets of Gluing Data

(4) For every pair $(i, j) \in K$, with $i \neq j$, for every

$$x \in \partial(\Omega_{ij}) \cap \Omega_i \qquad \text{and} \qquad y \in \partial(\Omega_{ji}) \cap \Omega_j \,,$$

there are open balls, V_x and V_y, centered at x and y, so that no point of $V_y \cap \Omega_{ji}$ is the image of any point of $V_x \cap \Omega_{ij}$ by φ_{ji}.



Parametric Pseudo-Manifolds

Sets of Gluing Data

Given a set of gluing data, $\mathcal{G}$, can we build a manifold from it?

The answer is YES!

Indeed, such a manifold is built by a quotient construction.

Parametric Pseudo-Manifolds

Sets of Gluing Data

The idea is to form the disjoint union, $\coprod_{i\in I} \Omega_i$, of the Ω_i and then identify Ω_{ij} with Ω_{ji} using φ_{ji}.

Formally, we define a binary relation, $\sim$, on $\coprod_{i\in I} \Omega_i$ as follows: for all $x, y \in \coprod_{i\in I} \Omega_i$, we have

$$x \sim y \quad \text{iff} \quad (\exists (i,j) \in K)(x \in \Omega_{ij}, y \in \Omega_{ji}, y = \varphi_{ji}(x)).$$

We can prove that $\sim$ is an equivalence relation, which enables us to define the space

$$M_{\mathcal{G}} = \left(\coprod_{i\in I} \Omega_i \right) / \sim \; .$$

We can also prove that $M_{\mathcal{G}}$ is a Hausdorff and second-countable manifold.

Parametric Pseudo-Manifolds

Sets of Gluing Data

Sketching the proof:

For every $i \in I$, $\mathrm{in}_i \colon \Omega_i \to \coprod_{i \in I} \Omega_i$ is the *natural injection*.

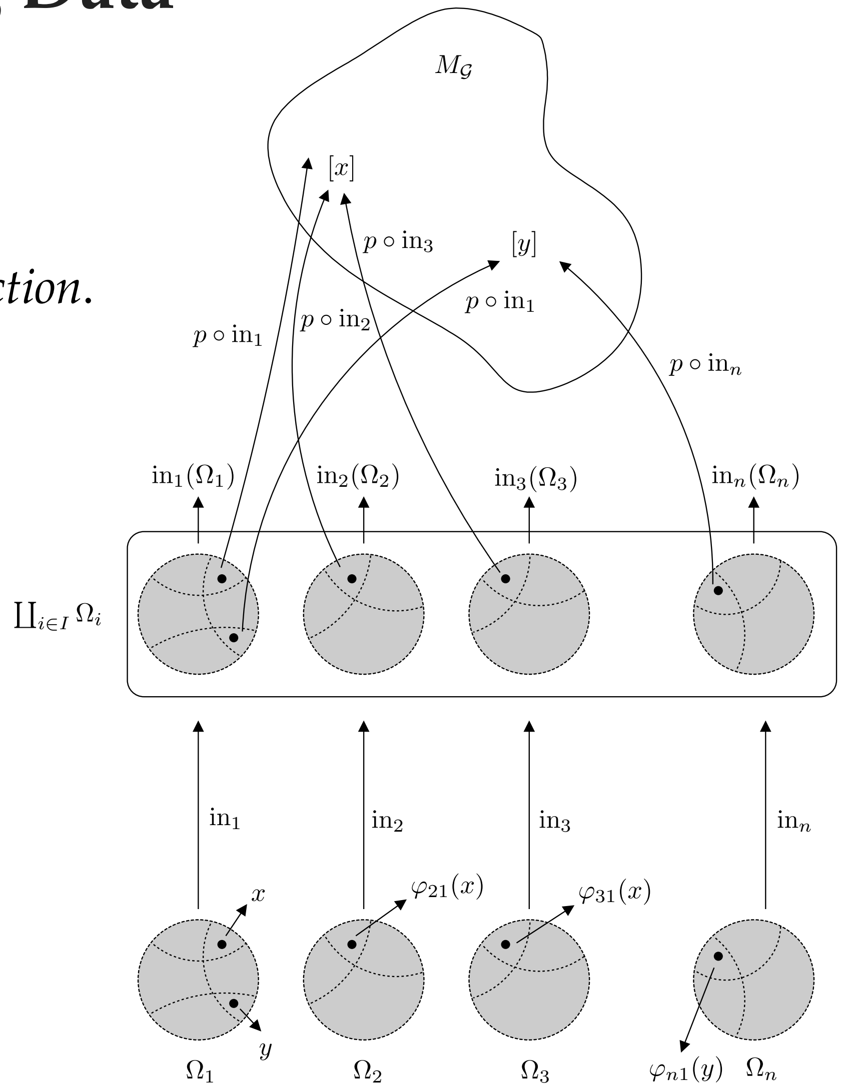
Let $p\colon \coprod_{i \in I} \Omega_i \to M_{\mathcal{G}}$ be the *quotient map*, with

$$p(x) = [x]\,.$$

For every $i \in I$, let $\tau_i = p \circ \mathrm{in}_i \colon \Omega_i \to M_{\mathcal{G}}$.

Let $U_i = \tau_i(\Omega_i)$ and $\varphi_i = \tau_i^{-1}$.

It is immediately verified that (U_i, φ_i) are charts and that this collection of charts forms a C^k atlas for $M_{\mathcal{G}}$.



Parametric Pseudo-Manifolds

Sets of Gluing Data

Sketching the proof:

We now prove that the topology of $M_{\mathcal{G}}$ is Hausdorff.

Pick $[x], [y] \in M_{\mathcal{G}}$ with $[x] \neq [y]$, for some $x \in \Omega_i$ and some $y \in \Omega_j$.

Either

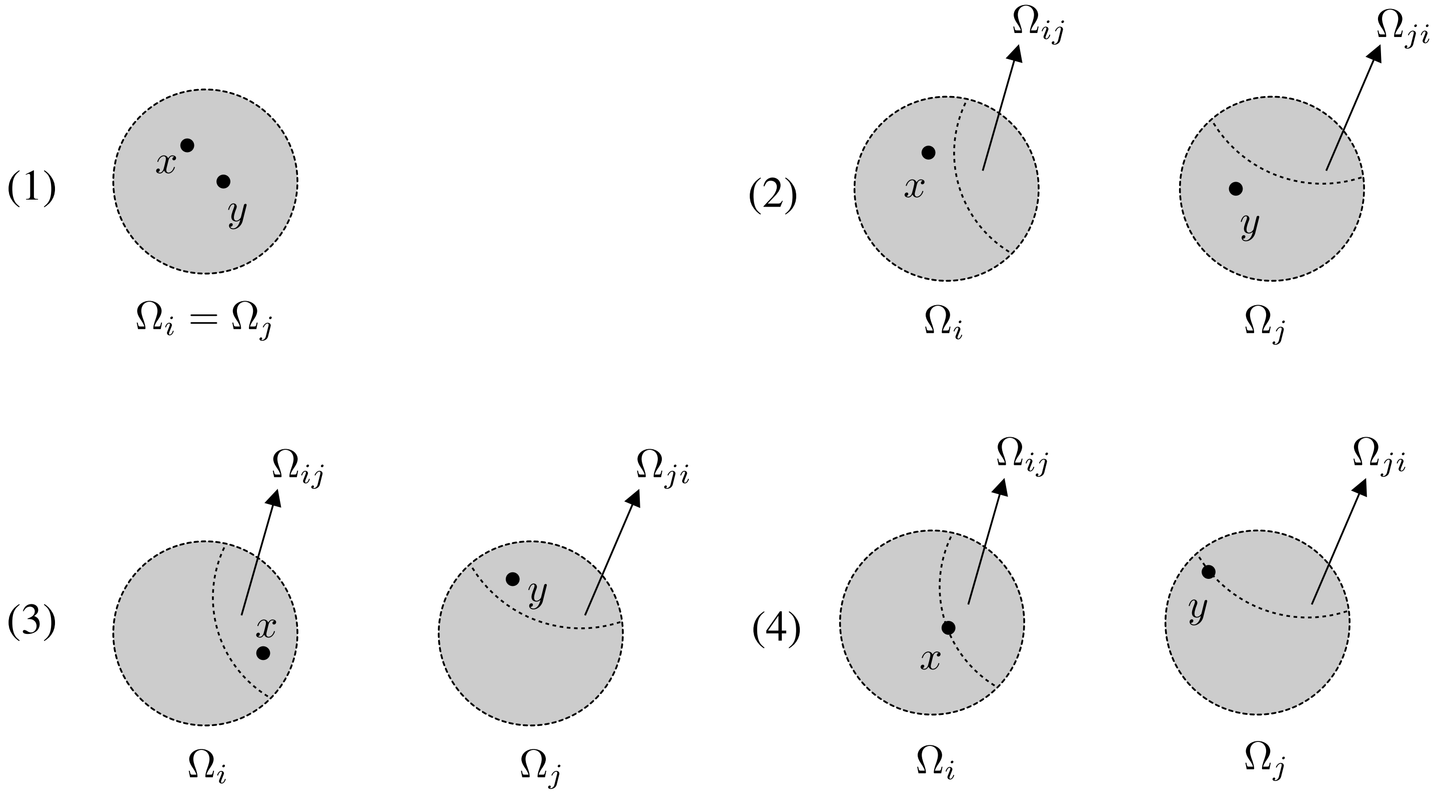
$$\tau_i(\Omega_i) \cap \tau_j(\Omega_j) = \emptyset \qquad \text{or} \qquad \tau_i(\Omega_i) \cap \tau_j(\Omega_j) \neq \emptyset\,.$$

In the former case, as τ_i and τ_j are homeomorphisms, $[x]$ and $[y]$ belong to the two disjoint open sets $\tau_i(\Omega_i)$ and $\tau_j(\Omega_j)$. In the latter case, we must consider four sub-cases:

Parametric Pseudo-Manifolds

Sets of Gluing Data

Sketching the proof:

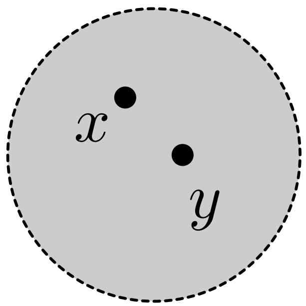


Parametric Pseudo-Manifolds

Sets of Gluing Data

Sketching the proof:

(1) If $i = j$ then x and y can be separated by disjoint opens, V_x and V_y, and as τ_i is a homeomorphism, $[x]$ and $[y]$ are separated by the disjoint open subsets $\tau_i(V_x)$ and $\tau_j(V_y)$.

(1) 

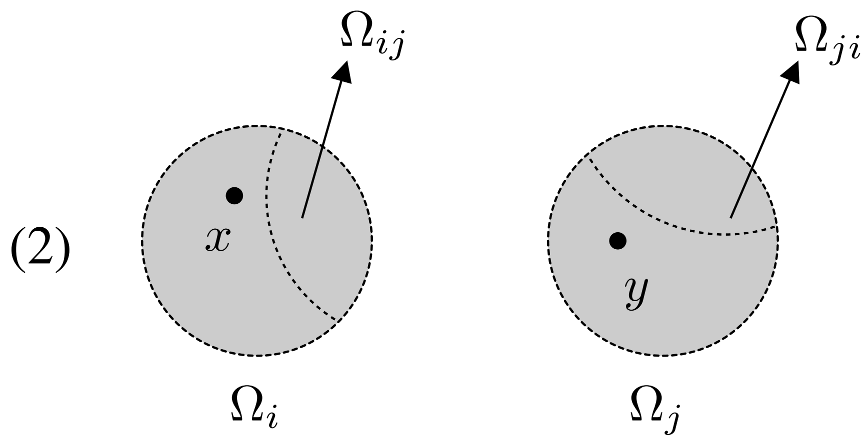
$\Omega_i = \Omega_j$

Parametric Pseudo-Manifolds

Sets of Gluing Data

Sketching the proof:

(2) If $i \neq j$, $x \in \Omega_i - \overline{\Omega_{ij}}$ and $y \in \Omega_j - \overline{\Omega_{ji}}$, then $\tau_i(\Omega_i - \overline{\Omega_{ij}})$ and $\tau_j(\Omega_j - \overline{\Omega_{ji}})$ are disjoint open subsets separating $[x]$ and $[y]$, where $\overline{\Omega_{ij}}$ and $\overline{\Omega_{ji}}$ are the closures of Ω_{ij} and Ω_{ji}, respectively.



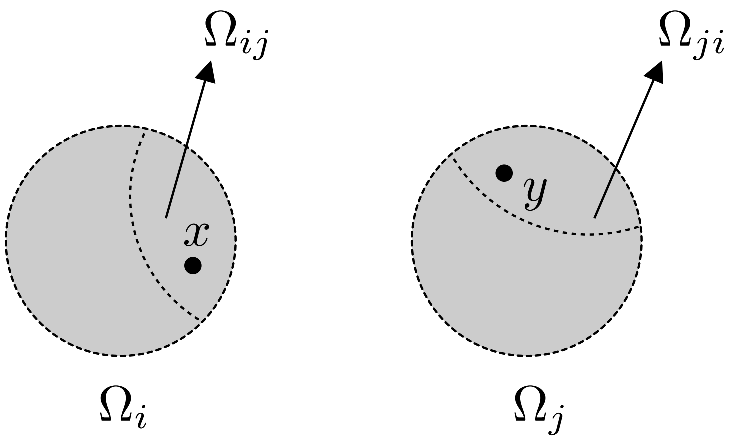
Parametric Pseudo-Manifolds

Sets of Gluing Data

Sketching the proof:

(3) If $i \neq j$, $x \in \Omega_{ij}$ and $y \in \Omega_{ji}$, as $[x] \neq [y]$ and $y \sim \varphi_{ij}(y)$, then $x \neq \varphi_{ij}(y)$. We can separate x and $\varphi_{ij}(y)$ by disjoint open subsets, V_x and V_y, and $[x]$ and $[y] = [\varphi_{ij}(y)]$ are separated by the disjoint open subsets $\tau_i(V_x)$ and $\tau_i(V_y)$.

(3)



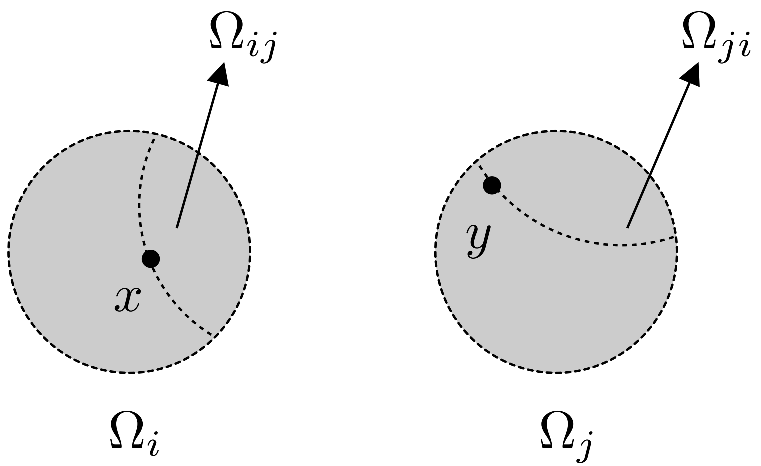
Parametric Pseudo-Manifolds

Sets of Gluing Data

Sketching the proof:

(4) If $i \neq j$, $x \in \partial(\Omega_{ij}) \cap \Omega_i$ and $y \in \partial(\Omega_{ji}) \cap \Omega_j$, then we use condition 4 of Definition 7.1. This condition yields two disjoint open subsets, V_x and V_y, with $x \in V_x$ and $y \in V_y$, such that no point of $V_x \cap \Omega_{ij}$ is equivalent to any point of $V_y \cap \Omega_{ji}$, and so $\tau_i(V_x)$ and $\tau_j(V_y)$ are disjoint open subsets separating $[x]$ and $[y]$.

(4)



Parametric Pseudo-Manifolds

Sets of Gluing Data

Sketching the proof:

So, the topology of $M_{\mathcal{G}}$ is Hausdorff and $M_{\mathcal{G}}$ is indeed a manifold.

$M_{\mathcal{G}}$ is also second-countable (WHY?).

Finally, it is trivial to verify that the transition maps of $M_{\mathcal{G}}$ are the original gluing functions,

$$\varphi_{ij}\,,$$

since

$$\varphi_i = \tau_i^{-1} \qquad \text{and} \qquad \varphi_{ji} = \varphi_j \circ \varphi_i^{-1}\,.$$

Parametric Pseudo-Manifolds

Sets of Gluing Data

Theorem 7.1. For every set of gluing data,

$$\mathcal{G} = \left((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K}\right),$$

there is an n-dimensional C^k manifold, $M_{\mathcal{G}}$, whose transition maps are the φ_{ji}'s.

Theorem 7.1 is nice, but...

- Our proof is not constructive;
- $M_{\mathcal{G}}$ is an *abstract* entity, which may not be orientable, compact, etc.

So, we know we *can* build a manifold from a set of gluing data, but that does not mean we know *how* to build a "concrete" manifold. For that, we need a formal notion of "concreteness".

Parametric Pseudo-Manifolds

Parametric Pseudo-Manifolds

The notion of "concreteness" is realized as *parametric pseudo-manifolds*:

Definition 7.2. Let n, d, and k be three integers with $d > n \geq 1$ and $k \geq 1$ or $k = \infty$. A *parametric C^k pseudo-manifold of dimension n in $\mathbb{E}^d$* (for short, *parametric pseudo-manifold* or PPM) is a pair,

$$\mathcal{M} = (\mathcal{G}, (\theta_i)_{i \in I}),$$

such that

$$\mathcal{G} = \left((\Omega_i)_{i \in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\varphi_{ji})_{(i,j) \in K}\right)$$

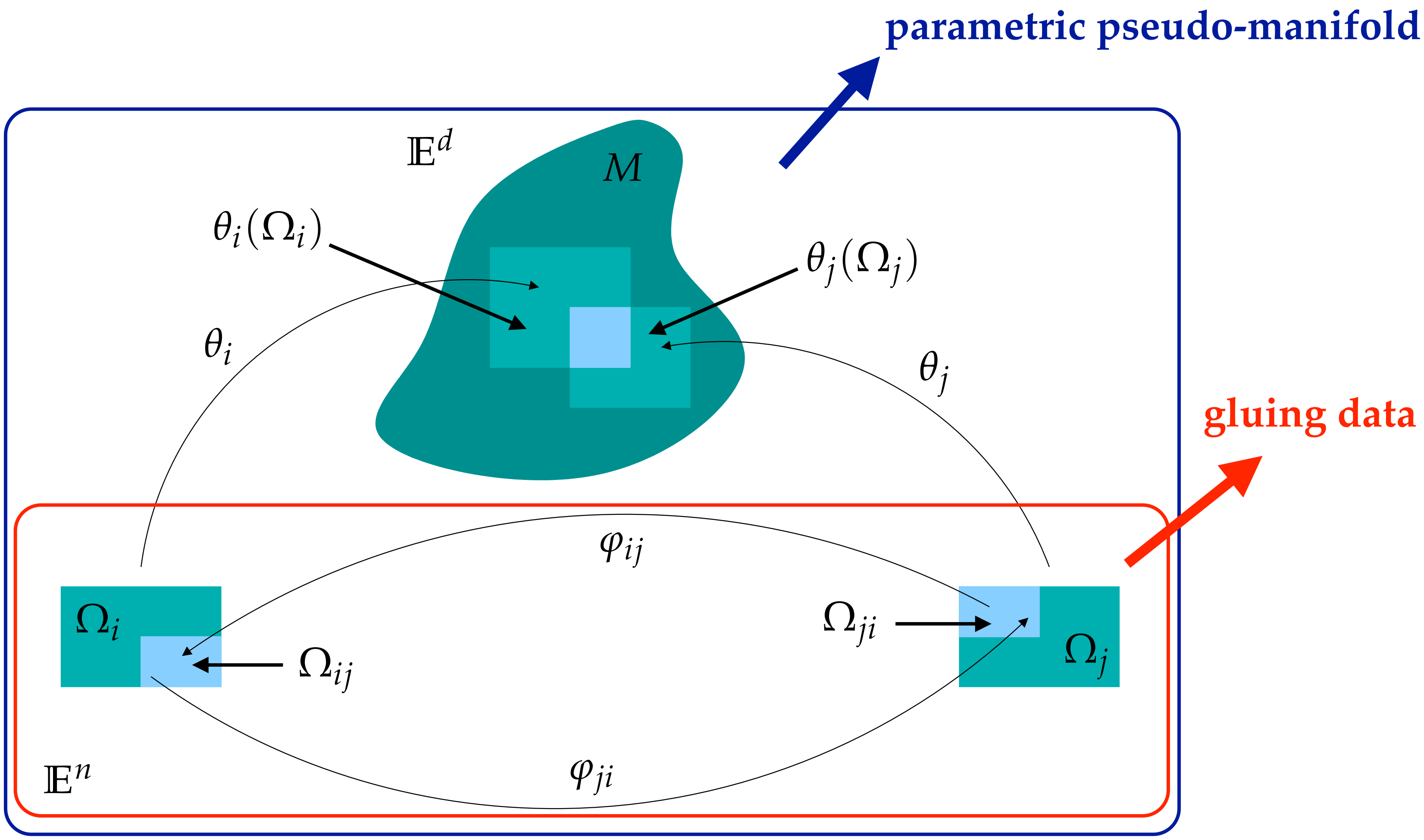
is a set of gluing data, for some finite set I, and each $\theta_i \colon \Omega_i \to \mathbb{E}^d$ is C^k and satisfies

(C) For all $(i, j) \in K$, we have

$$\theta_i = \theta_j \circ \varphi_{ji}.$$

Manifolds

Parametric Pseudo-Manifolds



Parametric Pseudo-Manifolds

Parametric Pseudo-Manifolds

As usual, we call θ_i a *parametrization*.

The subset, $M \subset \mathbb{E}^d$, given by

$$M = \bigcup_{i \in I} \theta_i(\Omega_i)$$

is called the *image* of the parametric pseudo-manifold, $\mathcal{M}$.

Whenever $n = 2$ and $d = 3$, we say that $\mathcal{M}$ is a *parametric pseudo-surface* (or PPS, for short).

We also say that M, the image of the PPS $\mathcal{M}$, is a *pseudo-surface*.

Parametric Pseudo-Manifolds

Parametric Pseudo-Manifolds

Condition C of Definition 7.2,

(C) For all $(i, j) \in K$, we have

$$\theta_i = \theta_j \circ \varphi_{ji} \,,$$

obviously implies that

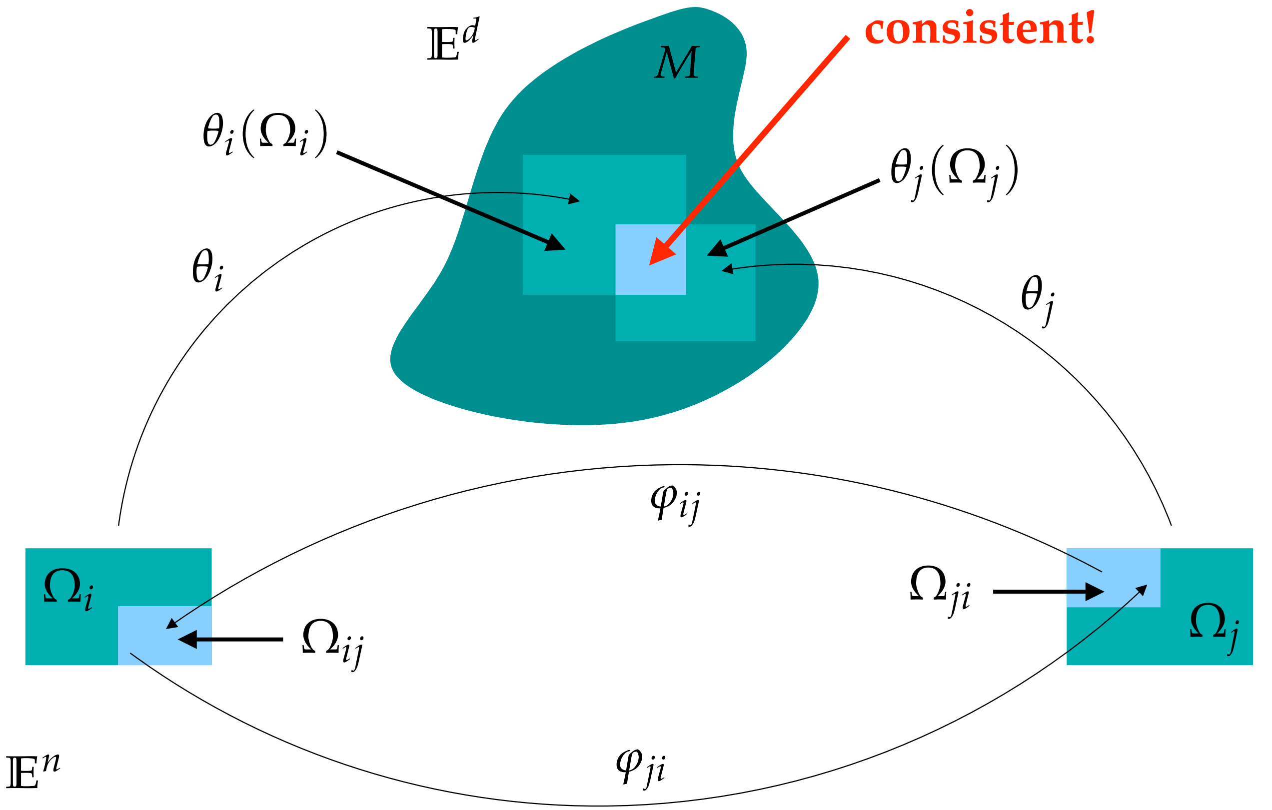
$$\theta_i(\Omega_{ij}) = \theta_j(\Omega_{ji}) \,,$$

for all $(i, j) \in K$. Consequently, θ_i and θ_j are consistent parametrizations of the overlap

$$\theta_i(\Omega_{ij}) = \theta_j(\Omega_{ji}) \,.$$

Parametric Pseudo-Manifolds

Parametric Pseudo-Manifolds



Parametric Pseudo-Manifolds

Parametric Pseudo-Manifolds

Thus, the set M, whatever it is, is covered by pieces, $U_i = \theta_i(\Omega_i)$, not necessarily open.

Each U_i is parametrized by θ_i, and each overlapping piece, $U_i \cap U_j$, is parametrized consistently.

The local structure of M is given by the θ_i's and its global structure is given by the gluing data.

Parametric Pseudo-Manifolds

Parametric Pseudo-Manifolds

We can equip M with an atlas if we require the θ_i's to be injective and to satisfy

(C′) For all $(i, j) \in K$,

$$\theta_i(\Omega_i) \cap \theta_j(\Omega_j) = \theta_i(\Omega_{ij}) = \theta_j(\Omega_{ji})\,.$$

(C″) For all $(i, j) \notin K$,

$$\theta_i(\Omega_i) \cap \theta_j(\Omega_j) = \emptyset\,.$$

Even if the θ_i's are not injective, properties C′ and C″ are still desirable since they ensure that $\theta_i(\Omega_i - \Omega_{ij})$ and $\theta_j(\Omega_j - \Omega_{ji})$ are uniquely parametrized. Unfortunately, properties C′ and C″ may be difficult to enforce in practice (at least for surface constructions).

Parametric Pseudo-Manifolds

Parametric Pseudo-Manifolds

Interestingly, regardless whether conditions C′ and C″ are satisfied, we can still show that M is the image in $\mathbb{E}^d$ of the abstract manifold, $M_{\mathcal{G}}$, as stated by Proposition 7.2:

Proposition 7.2. Let $\mathcal{M} = (\mathcal{G}, (\theta_i)_{i\in I})$ be a parametric C^k pseudo-manifold of dimension n in $\mathbb{E}^d$, where $\mathcal{G} = \big((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K}\big)$ is a set of gluing data, for some finite set I. Then, the parametrization maps, θ_i, induce a surjective map, $\Theta\colon M_{\mathcal{G}} \to M$, from the abstract manifold, $M_{\mathcal{G}}$, specified by $\mathcal{G}$ to the image, $M \subseteq \mathbb{E}^d$, of the parametric pseudo-manifold, $\mathcal{M}$, and the following property holds:

$$\theta_i = \Theta \circ \tau_i\,,$$

for every Ω_i, where $\tau_i\colon \Omega_i \to M_{\mathcal{G}}$ are the parametrization maps of the manifold $M_{\mathcal{G}}$. In particular, every manifold, $M \subset \mathbb{E}^d$, such that M is induced by $\mathcal{G}$ is the image of $M_{\mathcal{G}}$ by a map

$$\Theta\colon M_{\mathcal{G}} \to M\,.$$

Parametric Pseudo-Manifolds

The "Evil" Cocycle Condition

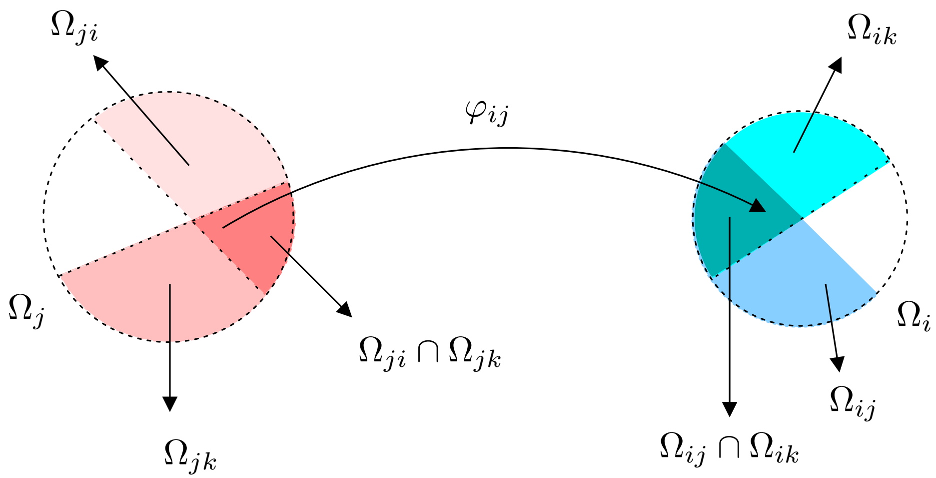
(c) For all i, j, k, if

$$\Omega_{ji} \cap \Omega_{jk} \neq \emptyset\,,$$

then

$$\varphi_{ij}(\Omega_{ji} \cap \Omega_{jk}) = \Omega_{ij} \cap \Omega_{ik} \qquad \text{and} \qquad \varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x)\,,$$

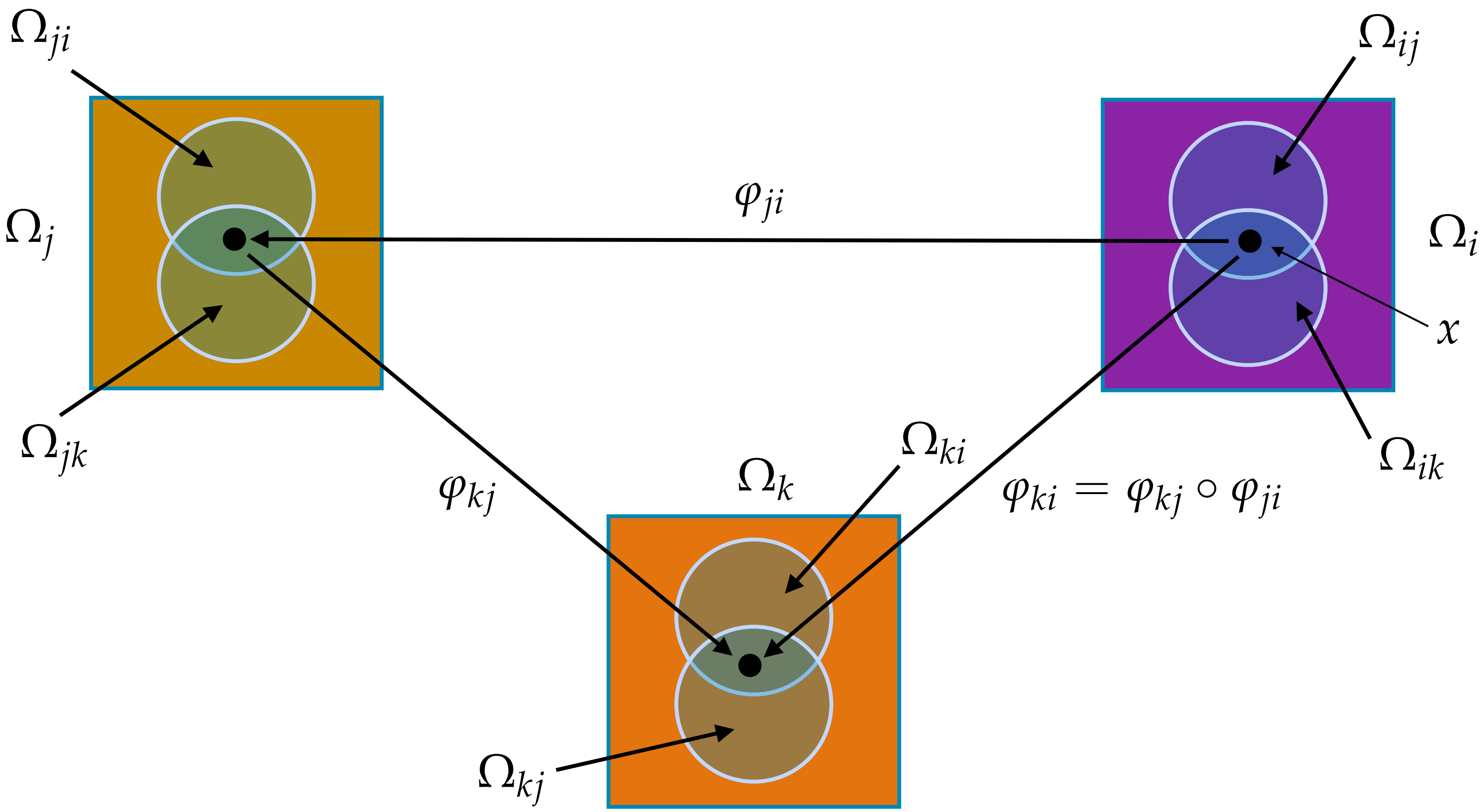
for all $x \in \Omega_{ij} \cap \Omega_{ik}$.



Parametric Pseudo-Manifolds

The "Evil" Cocycle Condition

$$\varphi_{ki}(x) = (\varphi_{kj} \circ \varphi_{ji})(x), \quad \text{for all } x \in (\Omega_{ij} \cap \Omega_{ik}).$$



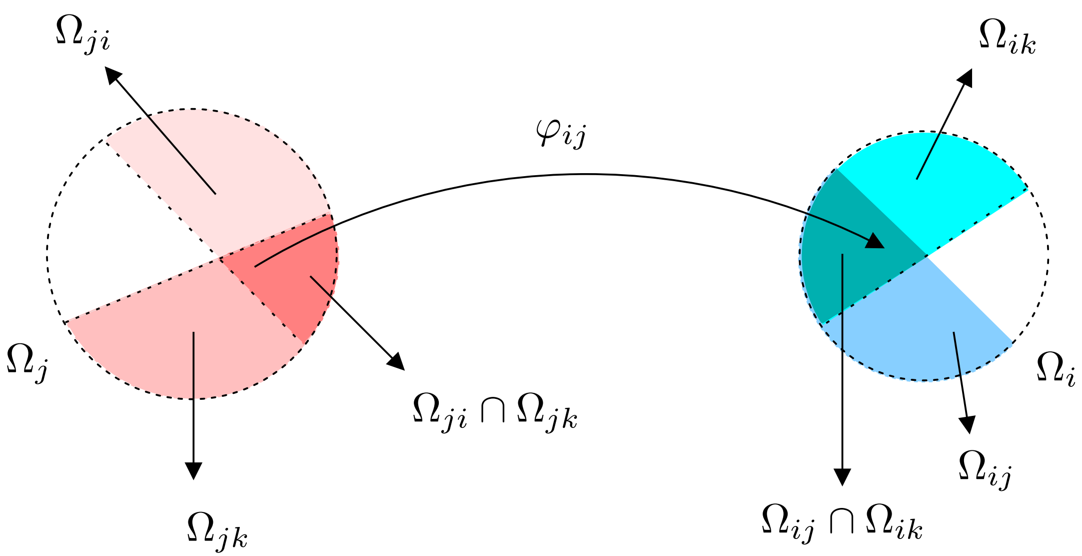
Parametric Pseudo-Manifolds

The "Evil" Cocycle Condition

The statement

$$\text{if } \Omega_{ji} \cap \Omega_{jk} \neq \emptyset \text{ then } \varphi_{ij}(\Omega_{ji} \cap \Omega_{jk}) = \Omega_{ij} \cap \Omega_{ik}$$

is necessary for guaranteeing the transitivity of the equivalence relation $\sim$.

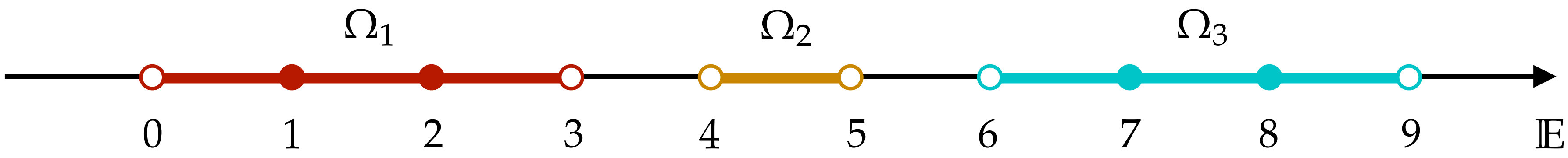


Parametric Pseudo-Manifolds

The "Evil" Cocycle Condition

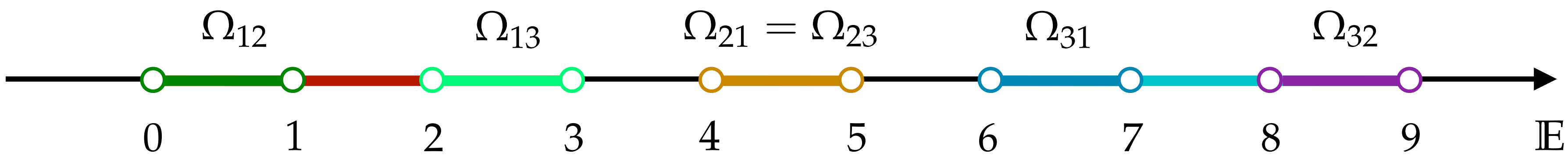
Consider the p-domains (i.e., open line intervals)

$$\Omega_1 =]\,0,3\,[\,, \quad \Omega_2 =]\,4,5\,[\,, \quad \text{and} \quad \Omega_3 =]\,6,9\,[\,.$$



Consider the gluing domains

$$\Omega_{12} =]\,0,1\,[\quad \Omega_{13} =]\,2,3\,[\,, \quad \Omega_{21} = \Omega_{23} =]\,4,5\,[\,, \quad \Omega_{32} =]\,8,9\,[\quad \Omega_{31} =]\,6,7\,[\,.$$

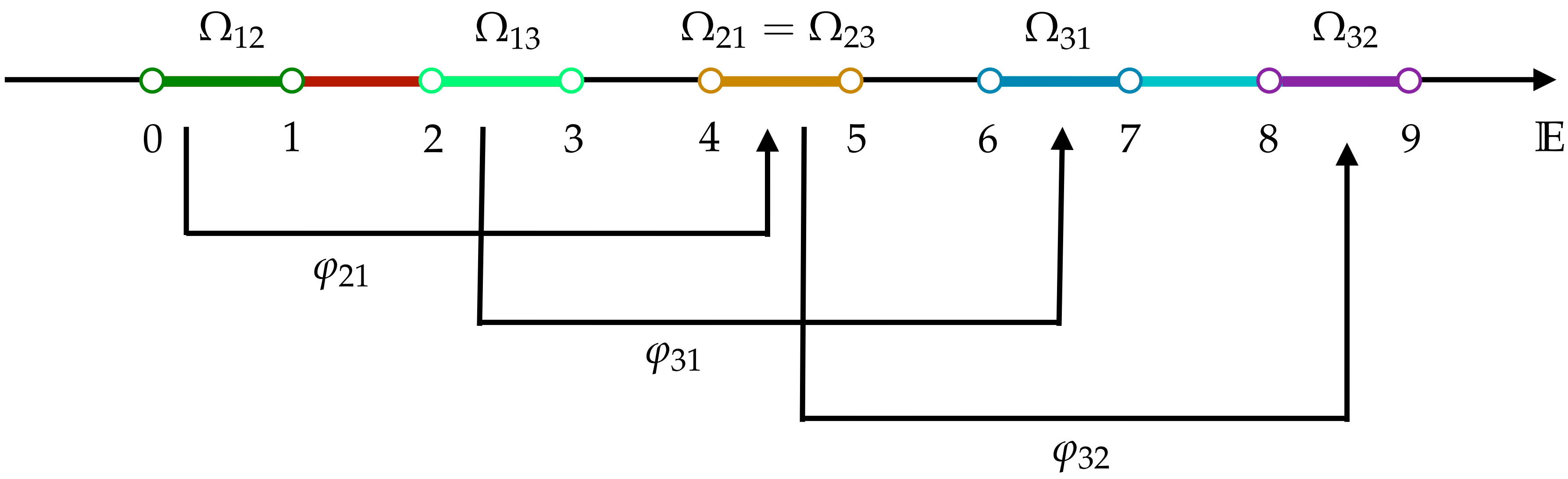


Parametric Pseudo-Manifolds

The "Evil" Cocycle Condition

Consider the transition maps:

$$\varphi_{21}(x) = x + 4\,, \quad \varphi_{32}(x) = x + 4 \quad \text{and} \quad \varphi_{31}(x) = x + 4\,.$$

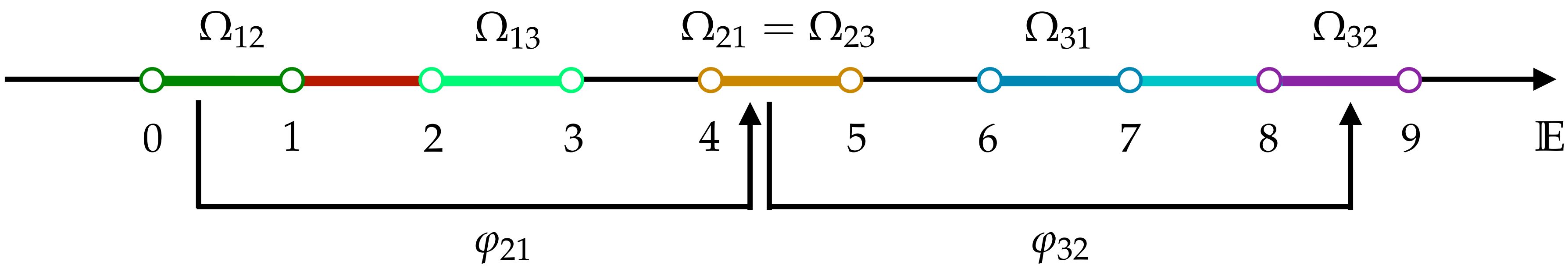


Parametric Pseudo-Manifolds

The "Evil" Cocycle Condition

Obviously,

$$(\varphi_{32} \circ \varphi_{21})(x) = x + 8, \qquad \text{for all } x \in \Omega_{12}.$$



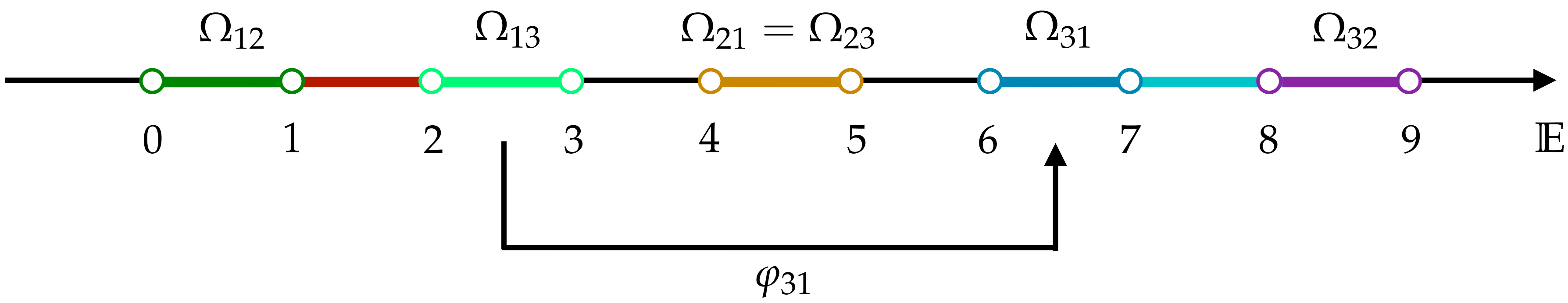
$$\varphi_{21}(0.5) = 4.5 \quad \text{and} \quad \varphi_{32}(4.5) = 8.5 \quad \Longrightarrow \quad 0.5 \sim 4.5 \quad \text{and} \quad 4.5 \sim 8.5$$

So, if $\sim$ were transitive, then we would have $0.5 \sim 8.5$. But...

Parametric Pseudo-Manifolds

The "Evil" Cocycle Condition

it turns out that φ_{31} is undefined at 0.5.



So, $0.5 \not\sim 8.5$.

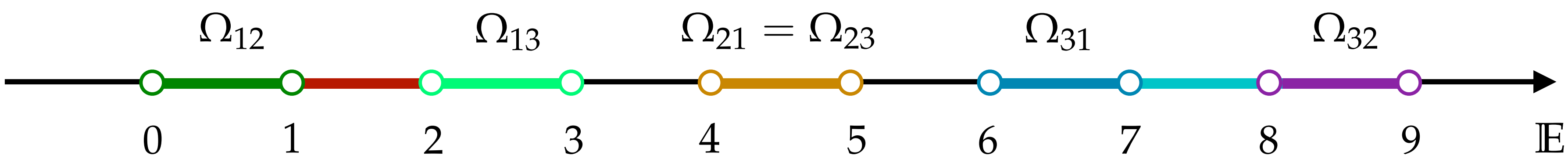
The reason is that φ_{31} and $\varphi_{32} \circ \varphi_{21}$ have disjoint domains.

Parametric Pseudo-Manifolds

The “Evil” Cocycle Condition

The reason they have disjoint domains is that condition "c" is not satisfied:

$$\text{if} \quad \Omega_{21} \cap \Omega_{23} \neq \emptyset \quad \text{then} \quad \varphi_{12}(\Omega_{21} \cap \Omega_{23}) = \Omega_{12} \cap \Omega_{13} \,.$$



Indeed

$$\Omega_{21} \cap \Omega_{23} = \Omega_2 = \left]\, 4, 5 \,\right[\neq \emptyset \,,$$

but

$$\varphi_{12}(\Omega_{21} \cap \Omega_{23}) = \left]\, 0, 1 \,\right[\neq \emptyset = \Omega_{12} \cap \Omega_{13} \,.$$