IMPA - Instituto de Matemática Pura e Aplicada, Rio de Janeiro, RJ, Brazil

Introduction to Computational Manifolds and Applications

Part 1 - Foundations

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Manifolds

Quotient Topology

In some cases, the space M does not come with a topology in an obvious (or natural) way and the following slight variation of Definition 5.3 is more convenient in such a situation:

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Quotient Topology

Definition 6.1. Given a set, M, given some integer $n \geq 1$ and given some k such that k is either an integer, with $k \geq 1$, or $k = \infty$, a *C^k n-atlas* (or *n-atlas of class C^k*),

$$\mathcal{A} = \{(U_i, \varphi_i)\}_i ,$$

is a family of charts such that

(1) Each U_i is a subset of M and $\varphi_i : U_i \to \varphi_i(U_i)$ is a bijection onto an open subset, $\varphi_i(U_i) \subseteq \mathbb{E}^n$, for all i ;

(2) The U_i cover M, i.e.,

$$M = \bigcup_i U_i ;$$

(3) Whenever $U_i \cap U_j \neq \emptyset$, the sets $\varphi_i(U_i \cap U_j)$ and $\varphi_j(U_i \cap U_j)$ are open in $\mathbb{R}^n$ and the transition maps φ_{ji} and φ_{ij} are C^k-diffeomorphism.

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Then, the notion of a chart being compatible with an atlas and of two atlases being compatible is just as before and we get a new definition of a manifold, analogous to Definition 5.4.

We give M the topology in which the open sets are arbitrary unions of domains of charts, U_i.

More precisely, the U_i's of the maximal atlas defining the differentiable structure on M.

It is not difficult to verify that the axioms of a topology are verified and M is indeed a topological space with this topology.

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It can also be shown that when M is equipped with the above topology, then the maps

$$\varphi_i : U_i \to \varphi_i(U_i)$$

are homeomorphisms, so M is a manifold according to Definition 5.4.

We also require that under this topology, M is Hausdorff and second-countable. A sufficient condition (in fact, also necessary!) for being second-countable is that some atlas be countable.

A sufficient condition of M to be Hausdorff is that for all $p, q \in M$ with $p \neq q$, either $p, q \in U_i$ for some U_i or $p \in U_i$ and $q \in U_j$ for some disjoint U_i, U_j. Thus, we are back to the original notion of a manifold where it is assumed that M is already a topological space.

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Quotient Topology

If the underlying topological space of a manifold is compact, then M has some finite atlas.

Also, if $\mathcal{A}$ is some atlas for M and (U, φ) is a chart in $\mathcal{A}$, for any (nonempty) open subset, $V \subseteq U$, we get a chart, $(V, \varphi_{|V})$, and it is obvious that this chart is compatible with $\mathcal{A}$.

Thus, $(V, \varphi_{|V})$ is also a chart for M.

This observation shows that if U is any open subset of a C^k-manifold, say M, then U is also a C^k-manifold whose charts are the restrictions of charts on M to U.

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Quotient Topology

Interesting manifolds often occur as the result of a quotient construction.

For example, real projective spaces and Grassmannians are obtained this way.

In this situation, the natural topology on the quotient object is the quotient topology but, unfortunately, even if the original space is Hausdorff, the quotient topology may not be.

Therefore, it is useful to have criteria that insure that a quotient topology is Hausdorff. We will present two criteria. First, let us review the notion of quotient topology.

Quotient Topology

Definition 6.2. Given any topological space, X, and any set, Y, for any surjective function, $f : X \to Y$, we define the *quotient topology on Y determined by f* (also called the *identification topology on Y determined by f*), by requiring a subset, V, of Y to be open if $f^{-1}(V)$ is an open set in X. Given an equivalence relation R on a topological space X, if $\pi : X \to X/R$ is the projection sending every $x \in X$ to its equivalence class $[x]$ in X/R, the space X/R equipped with the quotient topology determined by π is called the *quotient space of X modulo R*. Thus a set, V, of equivalence classes in X/R is open iff $\pi^{-1}(V)$ is open in X, which is equivalent to the fact that $\bigcup_{[x]\in V}[x]$ is open in X.

It is immediately verified that Definition 6.2 defines topologies and that $f : X \to Y$ and $\pi : X \to X/R$ are continuous when Y and X/R are given these quotient topologies.

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Quotient Topology

One should be careful that if X and Y are topological spaces and $f : X \rightarrow Y$ is a continuous surjective map, Y *does not* necessarily have the quotient topology determined by f.

Indeed, it may not be true that a subset V of Y is open when $f^{-1}(V)$ is open.

However, this will be true in two important cases.

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Quotient Topology

Definition 6.3. A continuous map,

$$f : X \to Y,$$

is an *open map* (or simply *open*) if $f(U)$ is open in Y whenever U is open in X, and similarly, $f : X \to Y$, is a *closed map* (or simply *closed*) if $f(V)$ is closed in Y whenever V is closed in X.

Then, Y has the quotient topology induced by the continuous surjective map f if either f is open or f is closed. Indeed, if f is open, then assuming that $f^{-1}(V)$ is open in X, we have $f(f^{-1}(V)) = V$ open in Y. Now, since $f^{-1}(Y - B) = X - f^{-1}(B)$, for any subset, B, of Y, a subset, V, of Y is open in the quotient topology iff $f^{-1}(Y - V)$ is closed in X. As a result, if f is a closed map, then V is open in Y iff $f^{-1}(V)$ is open in X.

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Quotient Topology

Unfortunately, the Hausdorff separation property is not necessarily preserved under quotient.

Nevertheless, it is preserved in some special important cases.

Proposition 6.1. Let X and Y be topological spaces,

$$f : X \to Y$$

be a continuous surjective map, and assume that X is compact and that Y has the quotient topology determined by f. Then, the space Y is Hausdorff iff f is a closed map.

Quotient Topology

Proposition 6.2. Let

$$f : X \to Y$$

be a surjective continuous map between topological spaces. If f is an open map then Y is Hausdorff iff the set

$$\{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$$

is closed in $X \times X$.

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Quotient Topology

Given a topological space, X, and an equivalence relation, R, on X, we say that R is *open* if the projection map,

$$\pi : X \to X/R\,,$$

is an open map, where X/R is equipped with the quotient topology. Then, if R is an open equivalence relation on X, the topological space X/R is Hausdorff iff R is closed in $X \times X$.

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Quotient Topology

Proposition 6.3. If X is a topological space and R is an open equivalence relation on X, then for any basis,

$$\{B_\alpha\},$$

for the topology of X, the family

$$\{\pi(B_\alpha)\}$$

is a basis for the topology of X/R, where

$$\pi : X \to X/R$$

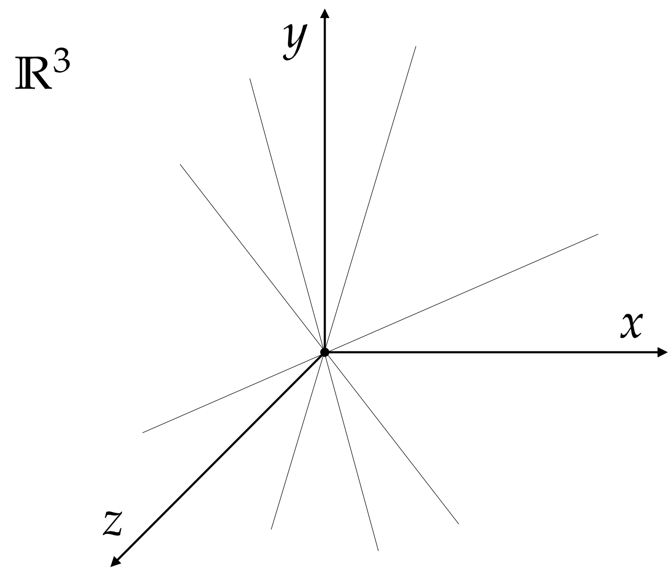
is the projection map. Consequently, if the space X is second-countable, then so is X/R.

Manifolds

Examples

Example 6.1. The projective space $\mathbb{RP}^n$

This is the space of all lines through the origin of $\mathbb{R}^{n+1}$:



Manifolds

Examples

To define an atlas on $\mathbb{RP}^n$ it is convenient to view $\mathbb{RP}^n$ as the set of equivalence classes of vectors in $\mathbb{R}^{n+1} - \{0\}$ (i.e., the nonzero vectors) modulo the equivalence relation,

$$u \sim v \quad \text{iff} \quad v = \lambda u, \quad \text{for some} \quad \lambda \neq 0 \in \mathbb{R}\,.$$

Given any $p = [x_1, \ldots, x_{n+1}] \in \mathbb{RP}^n$, we call $(x_1, \ldots, x_{n+1})$ the *homogeneous coordinates* of p.

It is customary to write $(x_1 : \cdots : x_{n+1})$ instead of $[x_1, \ldots, x_{n+1}]$.

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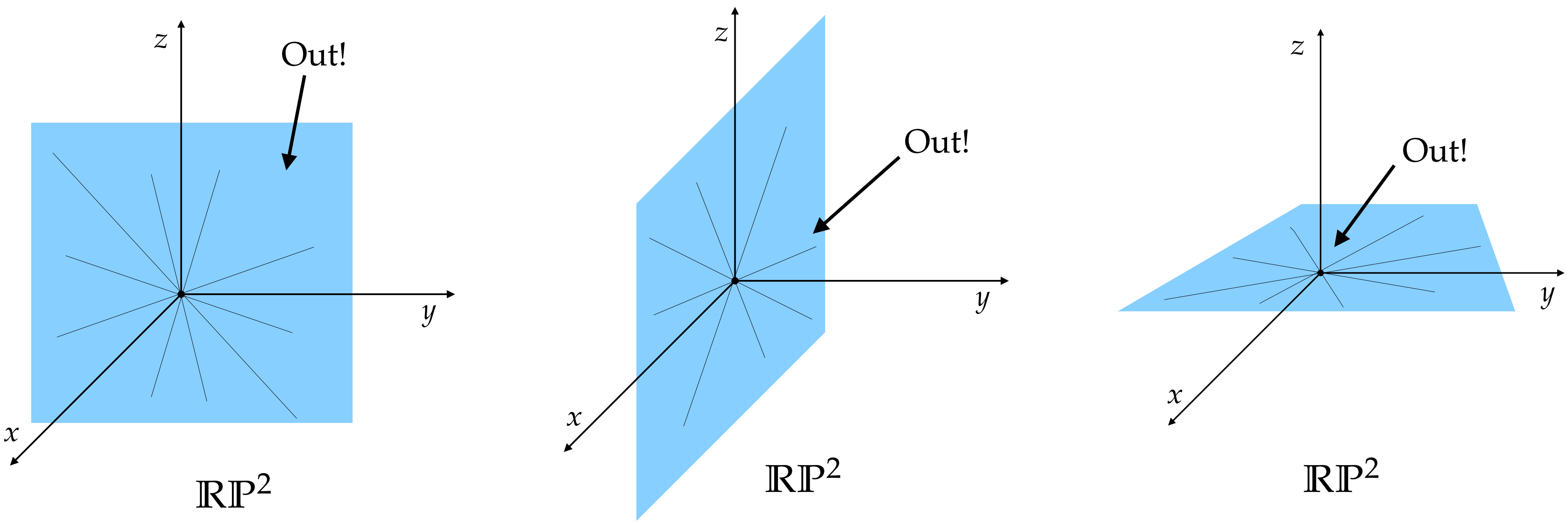
Examples

We define charts in the following way. For any i, with $1 \leq i \leq n+1$, let

$$U_i = \{(x_1 : \cdots : x_{n+1}) \in \mathbb{RP}^n \mid x_i \neq 0\} \, .$$



$U_1 = \{(x : y : z) \in \mathbb{RP}^2 \mid x \neq 0\}$ $U_2 = \{(x : y : z) \in \mathbb{RP}^2 \mid y \neq 0\}$ $U_3 = \{(x, y, z) \in \mathbb{RP}^2 \mid z \neq 0\}$

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Examples

Observe that U_i is well defined, because if

$$(y_1 : \cdots : y_{n+1}) = (x_1 : \cdots : x_{n+1}),$$

then there is some $\lambda \neq 0$ so that $y_j = \lambda \cdot x_j$, for all $j = 1, \ldots, n+1$.

We can define a homeomorphism, φ_i, of U_i onto $\mathbb{R}^n$, as follows:

$$\varphi_i(x_1 : \cdots : x_{n+1}) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i} \right),$$

where the i-th component is omitted.

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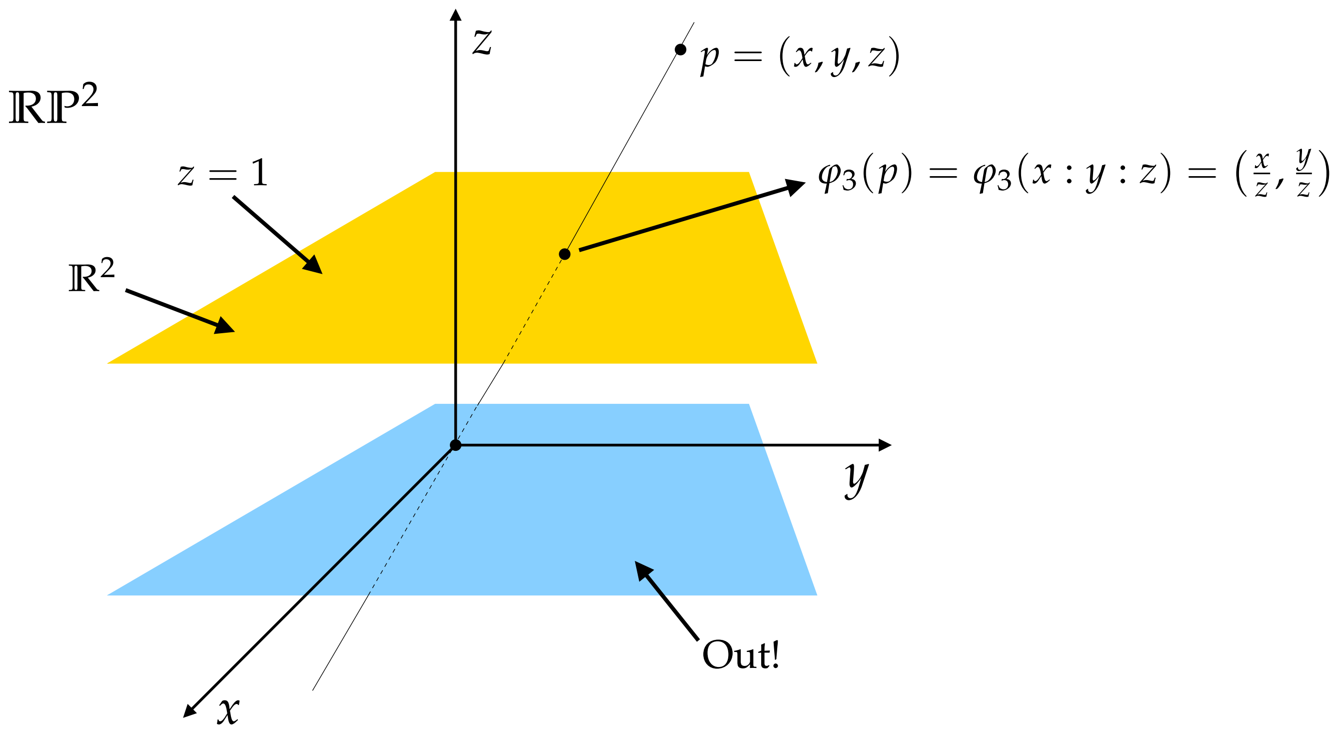
Examples



Again, it is clear that this map is well defined since it only involves ratios.

Manifolds

Examples

We can also define the maps, ψ_i, from $\mathbb{R}^n$ to $U_i \subseteq \mathbb{RP}^n$, given by

$$\psi_i(x_1, \ldots, x_n) = (x_1 : \cdots : x_{i-1} : 1 : x_i : \cdots : x_n)\,,$$

where the 1 goes in the i-th slot, for $i = 1, \ldots, n+1$.

We can easily check that φ_i and ψ_i are mutual inverses, so the φ_i are homeomorphisms.

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Examples

On the overlap, $U_i \cap U_j$, (where $i \neq j$), as $x_j \neq 0$, we have

$$(\varphi_j \circ \varphi_i^{-1})(x_1, \ldots, x_n) = \left(\frac{x_1}{x_j}, \ldots, \frac{x_{i-1}}{x_j}, \frac{1}{x_j}, \frac{x_i}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_n}{x_j} \right).$$

We assumed that $i < j$; the case $j < i$ is similar.

This is clearly a smooth function from $\varphi_i(U_i \cap U_j)$ to $\varphi_j(U_i \cap U_j)$.

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Examples

As the U_i cover $\mathbb{RP}^n$, we conclude that the (U_i, φ_i) are $n+1$ charts making a smooth atlas for $\mathbb{RP}^n$.

Intuitively, the space $\mathbb{RP}^n$ is obtained by gluing the open subsets U_i on their overlaps.

Even for $n = 2$, this is not easy to visualize!

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Examples

However, we can use the fact that $\mathbb{RP}^2$ is homeomorphic to the quotient of S^2 by the equivalence relation where antipodal points are identified. Then, there is a function, $\mathcal{H} : \mathbb{R}^3 \to \mathbb{R}^4$, given by

$$(x, y, z) \mapsto (xy, yz, xz, x^2 - y^2)$$

from Hilbert and Cohn-Vossen that allows us to concretely realize the projective plane in $\mathbb{R}^4$ as an embedded manifold. It can be shown that when it is restricted to S^2 (in $\mathbb{R}^3$), we have $\mathcal{H}(x, y, z) = \mathcal{H}(x', y', z')$ iff $(x', y', z') = (x, y, z)$ or $(x', y', z') = (-x, -y, -z)$; that is, the inverse image of every point in $\mathcal{H}(S^2)$ consists of two antipodal points.

The map $\mathcal{H}$ induces an injective map from the quotient space, i.e., $\mathbb{RP}^2$, onto $\mathcal{H}(S^2)$, and it is actually a homeomorphism. So, we can conclude that $\mathbb{RP}^2$ is a topological space in $\mathbb{R}^4$.

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Examples

We can do even better!

Indeed, the following three maps from $\mathbb{R}^2$ to $\mathbb{R}^4$,

$$
\begin{aligned}
\psi_1(u,v) &= \left(\frac{uv}{u^2+v^2+1}, \frac{v}{u^2+v^2+1}, \frac{u}{u^2+v^2+1}, \frac{u^2-v^2}{u^2+v^2+1}\right), \\
\psi_2(u,v) &= \left(\frac{u}{u^2+v^2+1}, \frac{v}{u^2+v^2+1}, \frac{uv}{u^2+v^2+1}, \frac{u^2-1}{u^2+v^2+1}\right), \\
\psi_3(u,v) &= \left(\frac{u}{u^2+v^2+1}, \frac{uv}{u^2+v^2+1}, \frac{v}{u^2+v^2+1}, \frac{1-u^2}{u^2+v^2+1}\right),
\end{aligned}
$$

are smooth parametrizations that make $\mathbb{RP}^2$ a smooth 2-manifold in $\mathbb{R}^4$.

This is an example of a surface that cannot be embedded in $\mathbb{R}^3$ without self-intersection.

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Examples

Although "normal" human beings cannot visualize that surface in $\mathbb{R}^4$, we can visualize projections of the surface onto $\mathbb{R}^3$. We do it by using orthogonal projection along an axis.

Remarkably, using the above projections, we only get two surfaces: the Steiner Roman and the Cross Cap surfaces. These surfaces were extensively investigated in the 1800's.

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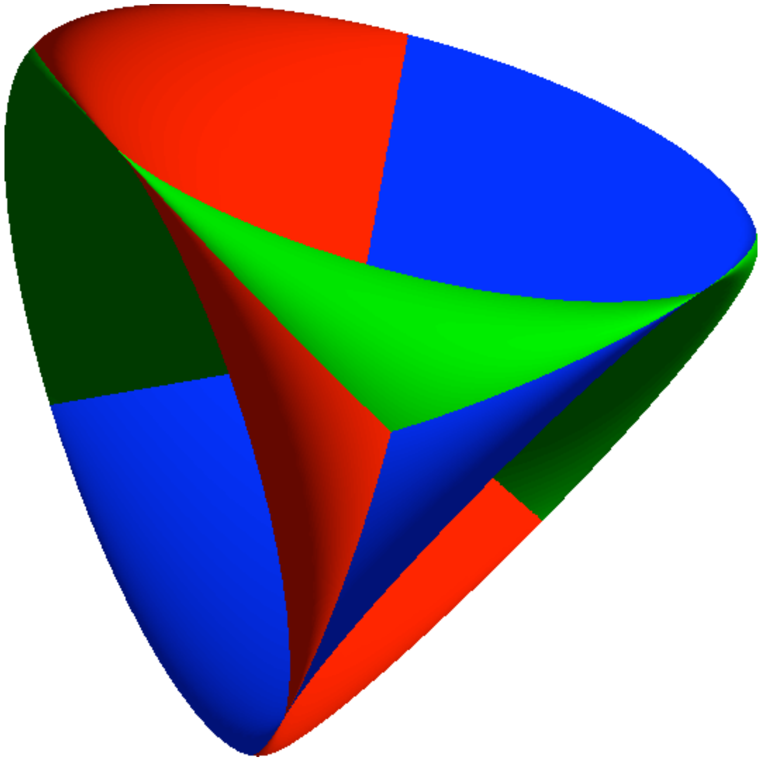
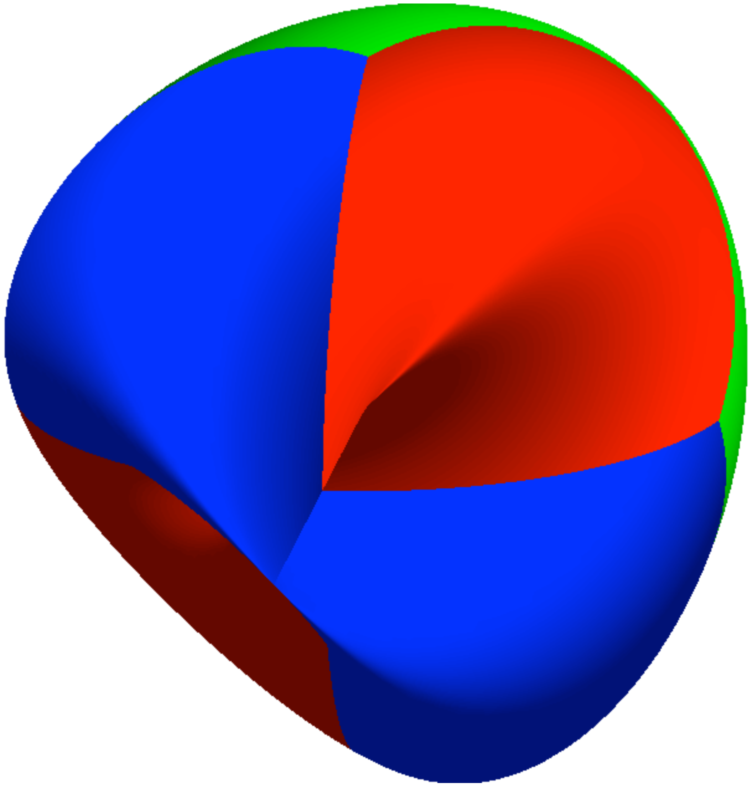
Examples





Manifolds

Examples

If we project the surface $\mathcal{H}(S^2)$ onto a hyperplane in $\mathbb{R}^4$ that is not orthogonal to one of the axes, we should see some "hybrid" of the Steiner Roman and the Cross Cap surfaces.

It is worth mentioning that the same map, $\mathcal{H}$, can be used to realize the Klein Bottle as a topological space in $\mathbb{R}^4$. For this, we must restrict $\mathcal{H}$ to the torus (check it if you like!).

It also worth mentioning that $\mathbb{RP}^3$ is homeomorphic to $\mathbf{SO}(3)$.

Manifolds

Examples

Example 6.2. Product Manifolds

The topological space, $M_1 \times M_2$, with the product topology (*the opens of* $M_1 \times M_2$ *are arbitrary unions of sets of the form* $U \times V$*, where* U *is open in* M_1 *and* V *is open in* M_2) can be given the structure of a C^k-manifold of dimension $n_1 + n_2$ by defining charts as follows:

For any two charts, (U_i, φ_i) on M_1 and (V_j, ψ_j) on M_2, we declare that the pair

$$(U_i \times V_j, \varphi_i \times \psi_j)$$

is a chart on $M_1 \times M_2$, where the map $\varphi_i \times \psi_j \colon U_i \times V_j \longrightarrow \mathbb{E}^{n_1+n_2}$ is defined so that

$$(\varphi_i \times \psi_j)\,(p, q) = (\varphi_i(p), \psi_j(q)), \quad \text{for all } (p, q) \in U_i \times V_j.$$

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Examples

The map $(\varphi_i \times \psi_j)$ is continuous and bijective. Furthermore, its inverse,

$$(\varphi_i \times \psi_j)^{-1} : \mathbb{E}^{n_1+n_2} \longrightarrow U_i \times V_j \, ,$$

where

$$(\varphi_i \times \psi_j)^{-1}(p,q) = (\varphi_i^{-1}(p), \psi_j^{-1}(q)), \quad \text{for all } (p,q) \in M_1 \times M_2 \, ,$$

is also a continuous map, which implies that $(\varphi_i \times \psi_j)$ is indeed a homeomorphism.

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Examples

If $(p, q) \in (U_i \times V_j) \cap (U_k \times V_l)$, then $p \in (U_i \cap U_k)$ and $q \in (V_j \cap V_l)$. So, on the overlap

$$(U_i \times V_j) \cap (U_k \times V_l) \, ,$$

we have

$$(\varphi_k \times \psi_l) \circ (\varphi_i \times \psi_j)^{-1} = (\varphi_k \times \psi_l)(\varphi_i^{-1}(p), \psi_j^{-1}(q)) = \Big((\varphi_k \circ \varphi_i^{-1})(p), (\psi_l \circ \psi_j^{-1})(q) \Big) \, .$$

This is clearly a smooth function from

$$(\varphi_i \times \psi_j) \left((U_i \times V_j) \cap (U_k \times V_l) \right)$$

to

$$(\varphi_k \times \psi_l) \left((U_i \times V_j) \cap (U_k \times V_l) \right) \, .$$

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Examples

Since the $(U_i \times V_j)$ form a cover for the space $M_1 \times M_2$, we conclude that the $\big((U_i \times V_j), (\varphi_i \times \psi_j)\big)$ form a smooth atlas for $M_1 \times M_2$. So, $M_1 \times M_2$ is a smooth $(n_1 + n_2)$-manifold.

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Examples

Example 6.3. Configuration spaces

Interesting classes of manifolds arise in motion planning for mobile robots.

The goal is to place several robots in motion, at the same time, in a such a way that collision is avoided. To model such a system, we assume that the location of each robot is a point in some topological space, X; for instance, the circle (i.e., S^1), $\mathbb{R}^2$, or $\mathbb{R}^3$.

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Examples

The *configuration space* of n distinct points on X, denoted by $\mathrm{Conf}^n(X)$, is the space

$$\mathrm{Conf}^n(X) = \left(\prod_1^n X\right) - \Delta\,,$$

where

$$\Delta = \{(x_1, \ldots, x_n) \mid x_i = x_j \text{ for some } i \neq j\}\,.$$

The set Δ, *pairwise diagonal*, represents those configurations of n points in X which experience a collision — this is the set of illegal configurations for the robots.

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Examples

The *unlabeled configuration space*, denoted by $\text{UConf}^n(X)$, is defined to be the quotient of $\text{Conf}^n(X)$ by the equivalence relation defined such that two tuples are equivalent iff one tuple is a permutation of the other. This space is given the quotient topology.

Configuration spaces of points on a manifold M are all (non-compact) manifolds of dimension $dim(\text{Conf}^n(X)) = n \cdot dim(M)$. The space $\text{Conf}^n(S^1)$ is homeomorphic to $(n-1)!$ distinct copies of $S^1 \times \mathbb{R}^{n-1}$, while $\text{UConf}^n(S^1)$ is a connected space.

It can also be shown that the configuration space of two points in $\mathbb{R}^2$, $\text{Conf}^n(\mathbb{R}^2)$, is homeomorphic to $\mathbb{R}^3 \times S^1$.

Configuration spaces and their applications to robot motion planning have been studied by Robert Ghrist (UPenn) and others.