Introduction to Computational Manifolds and Applications

Part 1 - Foundations

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Definition 4.1. Given any integers $N, m$, with $N \geq m \geq 1$, an $m$-dimensional smooth manifold in $\mathbb{E}^N$, for short a manifold, is a nonempty subset $M$ of $\mathbb{E}^N$ such that for every point $p \in M$, there are two open subsets, $\Omega \subseteq \mathbb{E}^m$ and $U \subseteq M$, with $p \in U$, and a smooth function,

$$\varphi : \Omega \to \mathbb{E}^N,$$

such that $\varphi$ is a homeomorphism between $\Omega$ and $U = \varphi(\Omega)$, and $(d\varphi)_{t_0}$ is injective, for

$$t_0 = \varphi^{-1}(p).$$
The function

$$\varphi : \Omega \to U$$

is called a (local) parametrization of $M$ at $p$. If $0_m \in \Omega$ and $\varphi(0_m) = p$, we say that $\varphi$ is centered at $p$. 
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Recall that

$$M \subseteq \mathbb{E}^N$$

is a topological space under the subspace topology, and $U$ is some open subset of $M$ in the subspace topology, which means that $U = M \cap W$ for some open subset $W$ of $\mathbb{E}^N$. 
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Since $\varphi : \Omega \to U$ is a homeomorphism, it has an inverse,

$$\varphi^{-1} : U \to \Omega,$$

that is also a homeomorphism, called a (local) chart.
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Since $\Omega \subseteq \mathbb{E}^m$, for every $p \in M$ and every parametrization $\varphi : \Omega \rightarrow U$ of $M$ at $p$, we have

$$\varphi^{-1}(p) = (z_1, \ldots, z_m),$$

for some $z_i \in \mathbb{E}$, and we call $z_1, \ldots, z_m$ the local coordinates of $p$ (with respect to $\varphi^{-1}$).
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We often refer to a manifold $M$ without explicitly specifying its dimension (the integer $m$).

Intuitively, a chart provides a "flattened" local map of a region on a manifold.
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Example 4.1.

Every open subset, $U$, of $\mathbb{E}^N$ is a manifold in a trivial way. Indeed, we can use the inclusion map, $\varphi : U \to \mathbb{E}^N$, where $\varphi(p) = p$ for every $p \in U$, as a parametrization. Note that, in this case, there is a single parametrization, namely $\varphi$, for every point $p$ in $U$. 
Example 4.2.

For $U$ an open subset of $\mathbb{E}^n$ and

$$f : U \rightarrow \mathbb{E}^m$$

the graph of $f$, $\Gamma(f)$, is defined to be the subspace

$$\Gamma(f) = \{(x, f(x)) \in U \times \mathbb{E}^m\}.$$
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If $f$ is smooth, then $\Gamma(f)$ is a manifold of dimension $n$ in $\mathbb{E}^{n+m}$.

Indeed, if we let

$$\varphi : U \rightarrow \Gamma(f) \quad \text{and} \quad \psi : \Gamma(f) \rightarrow U$$

such that

$$\varphi(x) = (x, f(x)) \quad \text{and} \quad \psi((x, f(x)) = x,$$

then $\varphi$ and $\psi$ are smooth and inverse to each other, and hence they are homeomorphisms.
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The derivative, $(d\varphi)_x$, of $\varphi$ at $x$, which is equal to $(\text{id}_n, (df)_x)$, is clearly injective. So, $\Gamma(f)$ is a manifold in $E^{n+m}$. That’s why many of the familiar surfaces from calculus, for instance, an elliptic or a hyperbolic paraboloid, which are graphs of functions, are manifolds.
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Example 4.3.

For any two positive integers, $m$ and $n$, let $M_{m,n}(\mathbb{R})$ be the vector space of all $m \times n$ matrices.

Since $M_{m,n}(\mathbb{R})$ is isomorphic to $\mathbb{R}^{mn}$, we give it the topology of $\mathbb{R}^{mn}$.

The general linear group, $\text{GL}(n, \mathbb{R})$, is by definition the set of matrices

$$\text{GL}(n, \mathbb{R}) = \{ A \in M_n(\mathbb{R}) \mid \det(A) \neq 0 \} = \det^{-1}(\mathbb{R} - \{0\}).$$

$\text{GL}(n, \mathbb{R})$ is a manifold in $\mathbb{R}^{n^2}$. 
Indeed, since the function  
\[ \text{det} : M_n(\mathbb{R}) \to \mathbb{R} \]

is continuous and \( \mathbb{R} - \{0\} \) is open in \( \mathbb{R} \), and since \( \text{GL}(n, \mathbb{R}) \) is the inverse image of (the open set) \( \mathbb{R} - \{0\} \) under the function \( \text{det} \), \( \text{GL}(n, \mathbb{R}) \) is an open set of \( M_n(\mathbb{R}) \approx \mathbb{R}^{n^2} \).

From Example 4.1, we conclude that \( \text{GL}(n, \mathbb{R}) \) is a manifold in \( \mathbb{R}^{n^2} \).

The following two lemmas provide the link with the definition of an abstract manifold:
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**Lemma 4.1.** Given an \( m \)-dimensional manifold \( M \) in \( \mathbb{R}^N \), for every \( p \in M \) there are two open sets \( O, W \subseteq \mathbb{R}^N \), with \( 0_N \in O \) and \( p \in (M \cap W) \), and a smooth diffeomorphism

\[
\varphi: O \to W
\]

such that

\[
\varphi(0_N) = p \quad \text{and} \quad \varphi(O \cap (\mathbb{R}^m \times \{0_{N-m}\})) = M \cap W.
\]
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The next lemma is easily shown from Lemma 4.1. It is a key technical result used to show that interesting properties of maps between manifolds do not depend on parametrizations.

**Lemma 4.2.** Given an $m$-dimensional manifold $M$ in $\mathbb{E}^N$, for every $p \in M$ and any two parametrizations, $\varphi_1 : \Omega_1 \to U_1$ and $\varphi_2 : \Omega_2 \to U_2$ of $M$ at $p$, if $U_1 \cap U_2 \neq \emptyset$, the map

$$\varphi_2^{-1} \circ \varphi_1 : \varphi_1^{-1}(U_1 \cap U_2) \to \varphi_2^{-1}(U_1 \cap U_2)$$

is a smooth diffeomorphism.
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The maps $\varphi_2^{-1} \circ \varphi_1$ and $\varphi_1^{-1} \circ \varphi_2$ are called transition maps.
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Using Definition 4.1, it may be quite hard to prove that a space is a manifold. Therefore, it is handy to have alternate characterizations such as those given in the next Proposition.

**Proposition 4.3.** A subset, $M \subseteq \mathbb{E}^{m+k}$, is an $m$-dimensional manifold if and only if either

1. For every $p \in M$, there is some open subset, $W \subseteq \mathbb{E}^{m+k}$, with $p \in W$ and a (smooth) submersion, $f : W \rightarrow \mathbb{E}^k$, so that $W \cap M = f^{-1}(0)$, or

2. For every $p \in M$, there is some open subset, $W \subseteq \mathbb{E}^{m+k}$, with $p \in W$ and a (smooth) map, $f : W \rightarrow \mathbb{E}^k$, so that $(df)_p$ is surjective and $W \cap M = f^{-1}(0)$. 


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Condition (2), although apparently weaker than condition (1), is in fact equivalent to it.

This is because to say that $(df)_p$ is surjective means that the Jacobian matrix of $(df)_p$ has rank $k$, which means that some determinant is nonzero, and because the determinant function is continuous this must hold in some open subset $W_1 \subseteq W$ containing $p$.

Consequently, the restriction, $f_1$, of $f$ to $W_1$ is indeed a submersion and

$$f_1^{-1}(0) = W_1 \cap f^{-1}(0) = W_1 \cap (W \cap M) = W_1 \cap M.$$  

The proof is based on two technical lemmas that are proved using the inverse function theorem.
Lemma 4.4. Let $U \subseteq \mathbb{E}^m$ be an open subset of $\mathbb{E}^m$ and pick some $a \in U$. If

$$f : U \rightarrow \mathbb{E}^n$$

is a smooth immersion at $a$, i.e., if $df_a$ is injective (and hence, $m \leq n$), then there is an open set, $V \subseteq \mathbb{E}^n$, with $f(a) \in V$, an open subset, $U' \subseteq U$, with $a \in U'$ and $f(U') \subseteq V$, an open subset $O \subseteq \mathbb{E}^{n-m}$, and a diffeomorphism, $\theta : V \rightarrow (U' \times O)$, so that

$$\theta(f(x_1, \ldots, x_m)) = (x_1, \ldots, x_m, 0, \ldots, 0),$$

for all

$$(x_1, \ldots, x_m) \in U'.$$
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$O \subseteq \mathbb{E}^{n-m}$

$U \subseteq \mathbb{E}^m$

$U' \subseteq U$

$(U' \times O) \subseteq \mathbb{E}^n$

$\theta$

$f(a)$

$f(U')$

$V \subseteq \mathbb{E}^n$
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**Lemma 4.6.** Let $W \subseteq \mathbb{E}^m$ be an open subset of $\mathbb{E}^m$ and pick some $a \in W$. If

$$f : W \rightarrow \mathbb{E}^n$$

is a smooth submersion at $a$, i.e., if $df_a$ is surjective (and hence, $m \geq n$), then there is an open set, $V \subseteq W \subseteq \mathbb{E}^m$, with $a \in V$, and a diffeomorphism, $\psi$, with domain $O \subseteq \mathbb{E}^m$, so that

$$\psi(O) = V \quad \text{and} \quad f(\psi(x_1, \ldots, x_m)) = (x_1, \ldots, x_n),$$

for all

$$(x_1, \ldots, x_m) \in O.$$
Using Lemmas 4.5 and 4.6, we can prove the following theorem:
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**Theorem 4.7.** A nonempty subset, $M \subseteq \mathbb{E}^N$, is an $m$-manifold (with $1 \leq m \leq N$) iff any of the following conditions hold:

(1) For every $p \in M$, there are two open subsets $\Omega \subseteq \mathbb{E}^m$ and $U \subseteq M$, with $p \in U$, and a smooth function

$$\varphi : \Omega \to \mathbb{E}^N$$

such that $\varphi$ is a homeomorphism between $\Omega$ and $U = \varphi(\Omega)$, and $(d\varphi)_0$ is injective, where

$$p = \varphi(0).$$

(2) For every $p \in M$, there are two open sets $O, W \subseteq \mathbb{E}^N$, with $0_N \in O$ and $p \in (M \cap W)$, and a smooth diffeomorphism $\varphi : O \to W$, such that $\varphi(0_N) = p$ and

$$\varphi(O \cap (\mathbb{E}^m \times \{0_{N-m}\})) = M \cap W.$$
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(3) For every $p \in M$, there is some open subset, $W \subseteq \mathbb{E}^N$, with $p \in W$ and a smooth submersion, $f : W \to \mathbb{E}^{N-m}$, so that $W \cap M = f^{-1}(0)$.

(4) For every $p \in M$, there is some open subset, $W \subseteq \mathbb{E}^N$, with $p \in W$, and $N - m$ smooth functions, $f_i : W \to \mathbb{E}$, so that the linear forms $(df_1)_p, \ldots, (df_{N-m})_p$ are linearly independent and

$$W \cap M = f_1^{-1}(0) \cap \cdots \cap f_{N-m}^{-1}(0).$$

Condition (4) says that locally (that is, in a small open set of $M$ containing $p \in M$), $M$ is "cut out" by $N - m$ smooth functions, $f_i : W \to \mathbb{E}$, in the sense that the portion of the manifold $M \cap W$ is the intersection of the $N - m$ hypersurfaces, $f_i^{-1}(0)$, (the zero-level sets of the $f_i$) and that this intersection is "clean", which means that the linear forms $(df_1)_p, \ldots, (df_{N-m})_p$ are linearly independent.
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Example 4.4.

The sphere

$$S^n = \{ x \in \mathbb{E}^{n+1} \mid \|x\|_2^2 - 1 = 0 \}$$

is an $n$-dimensional manifold in $\mathbb{E}^{n+1}$. Indeed, the map $f : \mathbb{E}^{n+1} - \{0\} \rightarrow \mathbb{E}$ given by

$$f(x) = \|x\|_2^2 - 1$$

is a submersion, since

$$(df)_x(y) = \begin{pmatrix} 2x_1 & 2x_2 & \cdots & 2x_{n+1} \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n+1} \end{pmatrix}, \text{ for all } x, y \in (\mathbb{E}^{n+1} - \{0\}).$$
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Using condition (3) of Theorem 4.7, namely,

(3) For every $p \in M$, there is some open subset, $W \subseteq \mathbb{E}^N$, with $p \in W$ and a smooth submersion, $f : W \to \mathbb{E}^{N-m}$, so that $W \cap M = f^{-1}(0)$,

with

$$M = S^n \quad \text{and} \quad W = \mathbb{E}^{n+1} - \{0\},$$

we get $W \cap M = S^n = f^{-1}(0)$. So, by Theorem 4.7., $S^n$ is indeed an $n$-manifold in $\mathbb{E}^{n+1}$. 
Example 4.5.

Recall that the orthogonal group, $O(n)$, is the group of all real $n \times n$ matrices, $R$, such that

$$RR^\top = R^\top R = I$$

and

$$\det(R) = \pm 1,$$

and that the rotation group, $SO(n)$ is the subgroup of $O(n)$ consisting of all matrices in $O(n)$ such that

$$\det(R) = +1.$$

The group $SO(n)$ is an $\left(\frac{n \cdot (n - 1)}{2}\right)$-dimensional manifold in $\mathbb{R}^{n^2}$. 

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To see why, recall that

$$GL^+(n) = \{ A \in GL(n) \mid \det(A) > 0 \}$$

is an open set of $\mathbb{R}^{n^2}$. Now, note that $A^T A - I$ is a symmetric matrix, for all $A \in GL^+(n)$.

So, let

$$f : GL^+(n) \to S(n)$$

be given by

$$f(A) = A^T A - I,$$

where $S(n) \approx \mathbb{R}^{\frac{n(n+1)}{2}}$ is the vector space consisting of all $n \times n$ real symmetric matrices.
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It is easy to show (using directional derivatives) that

$$(df)_A(H) = A^T H + H^T A \in S(n).$$

But then, $(df)_A$ is surjective for all $A \in SO(n)$, because if $S$ is any symmetric matrix, we see that

$$df(A) \left( \frac{AS}{2} \right) = A^T \left( \frac{AS}{2} \right) + \left( \frac{AS}{2} \right)^T A = \frac{1}{2} (A^T AS + S^T A^T A) = \frac{1}{2} (S + S) = S.$$

As $SO(n) = f^{-1}(0)$, we can use condition (3) of Theorem 4.7 again, with $W = GL^+(n)$ and $M = SO(n)$, to conclude that $SO(n)$ is indeed a $\frac{n(n-1)}{2}$-manifold in $\mathbb{R}^{n^2}$. 