IMPA - Instituto de Matemática Pura e Aplicada, Rio de Janeiro, RJ, Brazil

Introduction to Computational Manifolds and Applications

Part 1 - Foundations

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CURVES

Parametric curves

Properties of curves can be classified into **local properties** and **global properties**.

Local properties are the properties that hold in a small neighborhood of a point on the curve.

For instance, *curvature* is a local property.

Local properties can be more conveniently studied by assuming that the curve is parametrized locally.

A proper study of global properties of curves really requires the introduction of the notion of a manifold.

We will do that later

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Recall that the Euclidean space $\mathbb{E}^m$ is obtained from the vector space $\mathbb{R}^m$ by defining the standard inner product

$$(x_1, \ldots, x_m) \cdot (y_1, \ldots, y_m) = x_1 y_1 + \cdots + x_m y_m \,.$$

The corresponding Euclidean norm is

$$\|(x_1, \ldots, x_m)\| = \sqrt{x_1^2 + \cdots + x_m^2} \,.$$

Let $\mathcal{E} = \mathbb{E}^m$, for some $m \geq 2$. Typically, $m = 2$ or $m = 3$.

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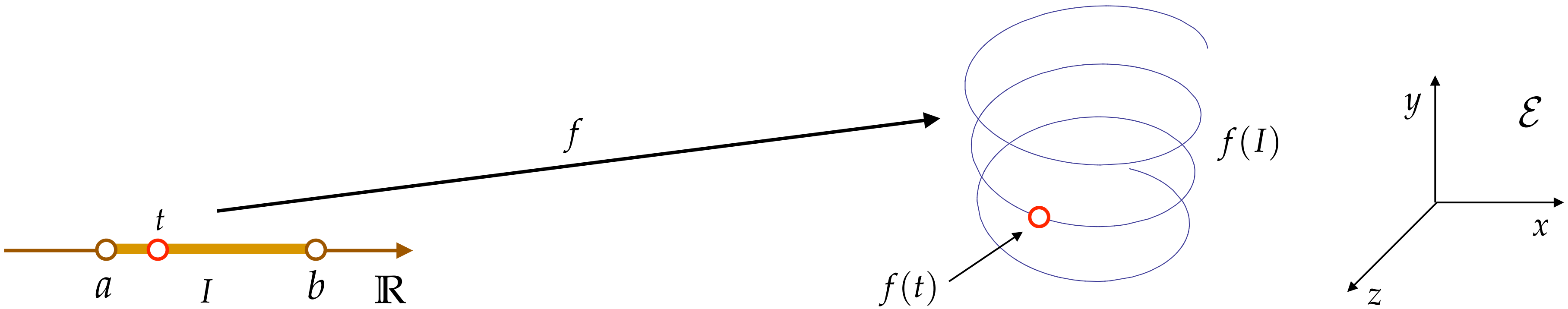
Parametric curves

From a kinematics point of view, a curve can be defined as a continuous map

$$f :]a,b[\to \mathcal{E},$$

from an open interval $I =]a,b[$ of $\mathbb{R}$ to the Euclidean space $\mathcal{E}$.



We can think of the parameter $t \in]a,b[$ as time, and the function f gives the position $f(t)$ of a moving particle, at time t. The image $f(I) \subseteq \mathcal{E}$ of the interval I is the trajectory of the particle.

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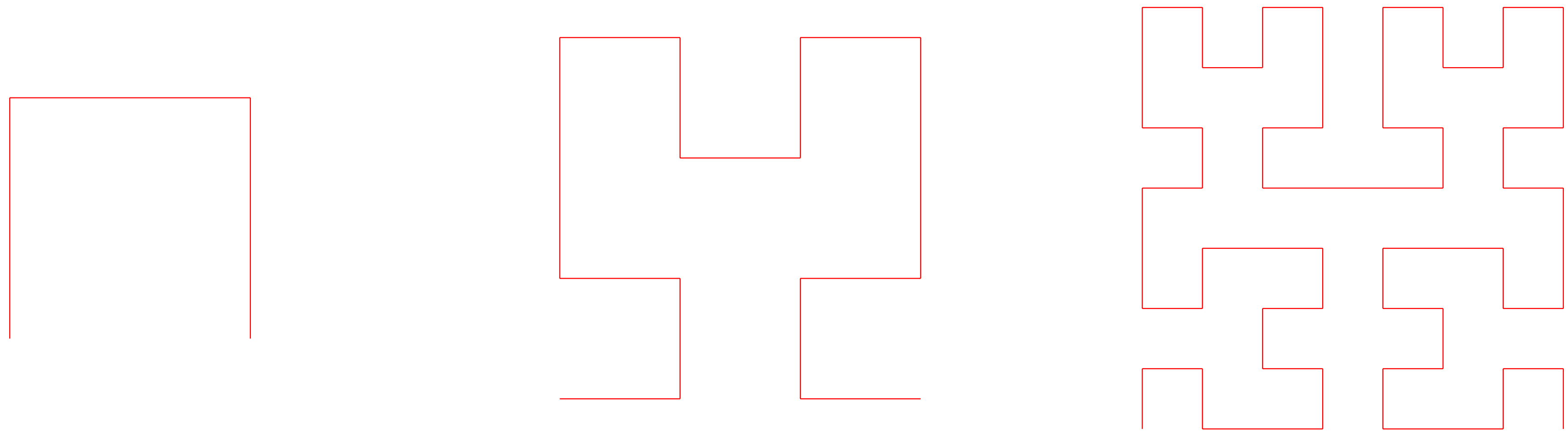
Parametric curves

In fact, only asking that f be continuous turns out to be too liberal, as rather strange curves turn out to be definable, such as "square-filling curves", due to Peano, Hilbert, Sierpinski, and others.

A very pretty square-filling curve due to Hilbert is defined by a sequence (h_n) of polygonal lines $h_n : [0,1] \to [0,1] \times [0,1]$ starting from the simple pattern h_0 (a "square cap" $\sqcap$) shown on the left below:



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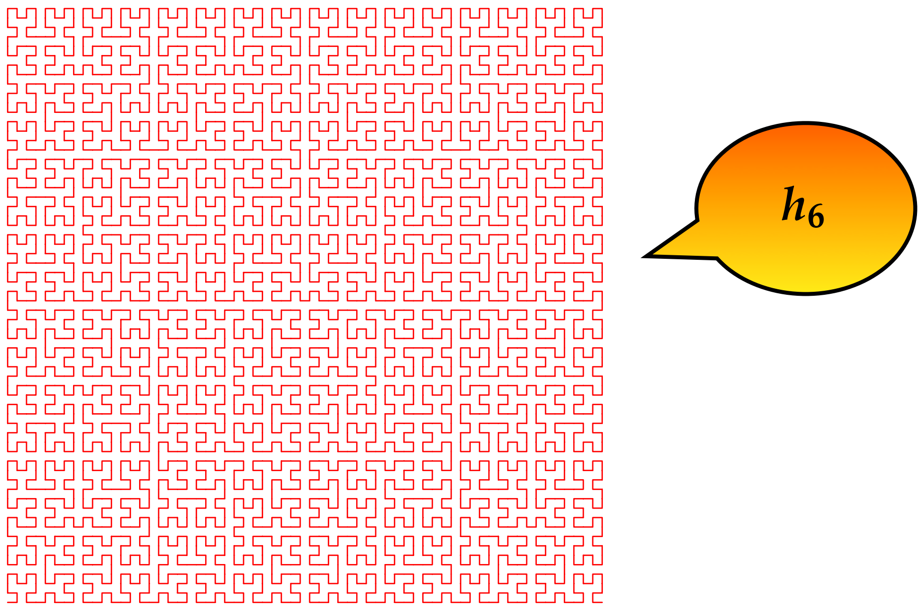
Parametric curves

It can be shown that the sequence (h_n) converges (pointwise) to a continuous curve

$$h : [0,1] \to [0,1] \times [0,1]$$

whose trace is the entire square $[0,1] \times [0,1]$. Curve h is nowhere differentiable and has infinite length!



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Parametric curves

Actually, there are many fascinating curves that are only continuous, *fractal curves* being a major example, but for our purposes, we need the existence of the tangent at every point (except perhaps for finitely many points).

This leads us to require that

$$f :]a, b[\to \mathcal{E}$$

be at least continuously differentiable. We also say that f is a C^1-function.

However, asking that $f :]a, b[\to \mathcal{E}$ be a C^p-function for $p \geq 1$, still allows unwanted curves.

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Parametric curves

For example, the plane curve given by

$$f(t) = \begin{cases} (0, e^{1/t}) & \text{if } t < 0; \\ (0, 0) & \text{if } t = 0; \\ (e^{-1/t}, 0) & \text{if } t > 0; \end{cases}$$

is a C^∞-function, but $f'(0) = 0$, and thus the tangent at the origin is undefined.

What happens is that the curve has a sharp "corner" at the origin.

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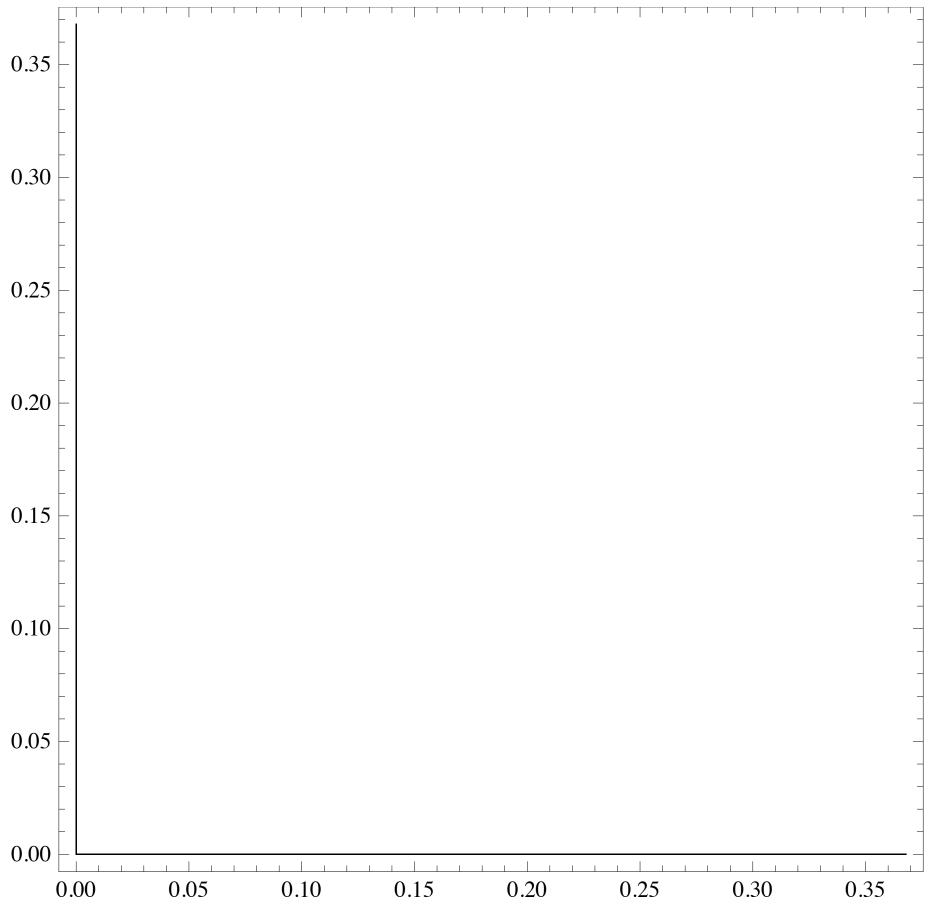
Parametric curves



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Parametric curves

Similarly, the plane curve defined such that

$$f(t) = \begin{cases} (-e^{1/t}, e^{1/t}\sin(e^{-1/t})) & \text{if } t < 0; \\ (0, 0) & \text{if } t = 0; \\ (e^{-1/t}, e^{-1/t}\sin(e^{1/t})) & \text{if } t > 0; \end{cases}$$

is a C^∞-function, but $f'(0) = 0$. In this case, the curve oscillates more and more around the origin.

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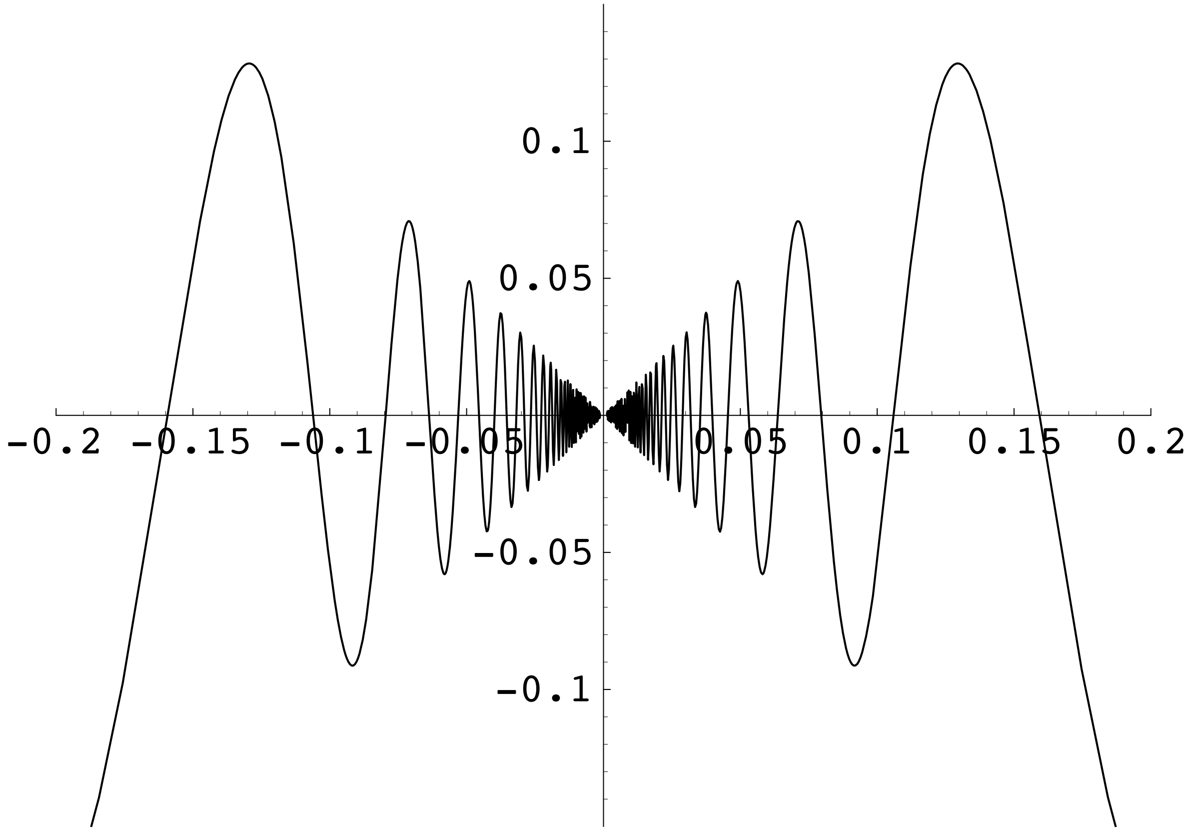
Parametric curves



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Parametric curves

The problem with the above examples is that the origin is a singular point for which $f'(0) = 0$ (a *stationary point*).

Although it is possible to define the tangent when f is sufficiently differentiable and when for every $t \in]a, b[$, $f^{(p)}(t) \neq 0$ for some $p \geq 1$ (where $f^{(p)}$ denotes the p-th derivative of f), a systematic study is rather cumbersome.

Thus, we will restrict our attention to curves having only regular points, that is, for which $f'(t) \neq 0$ for every $t \in]a, b[$.

However, we allow functions $f :]a, b[\to \mathcal{E}$ that are not necessarily injective, unless stated otherwise.

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Parametric curves

Definition 1.1. An *open curve (or open arc) of class C^p* is a map

$$f :]a, b[\to \mathcal{E}$$

of class C^p, with $p \geq 1$, where $]a, b[$ is an open interval (allowing $a = -\infty$ or $b = +\infty$). The set of points

$$f(]a, b[)$$

in $\mathcal{E}$ is called the *trace of the curve f*. A point $f(t)$ is *regular at $t \in]a, b[$* iff $f'(t)$ exists and $f'(t) \neq 0$, and *stationary* otherwise. A *regular open curve (or regular open arc) of class C^p* is an open curve of class C^p, with $p \geq 1$, such that every point is regular, i.e., $f'(t) \neq 0$ for every $t \in]a, b[$.

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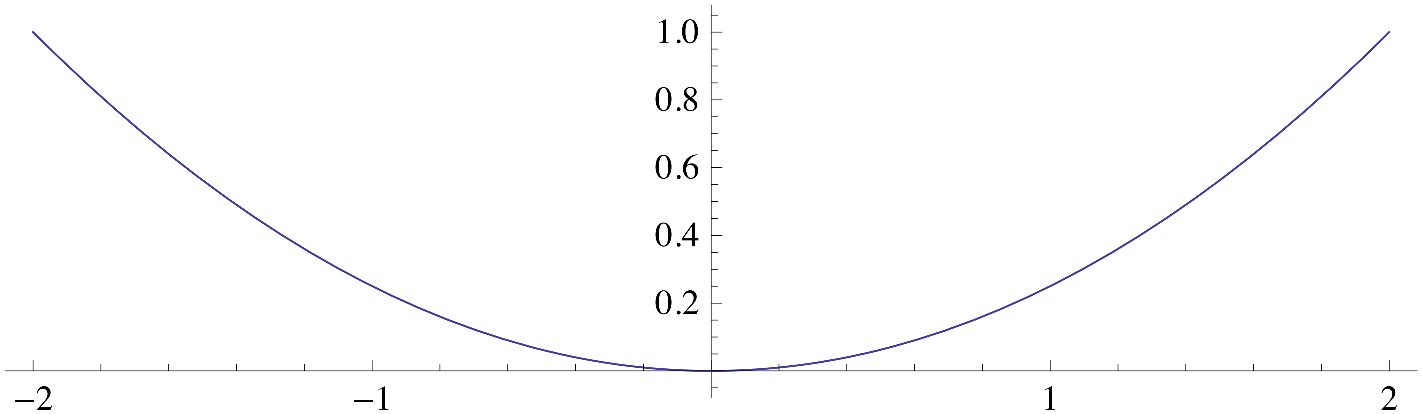
For example, a parabola is defined by the map

$$f(t) = (2t, t^2)\,.$$

The trace of this curve corresponding to the interval $(-1, 1)$ is shown below:



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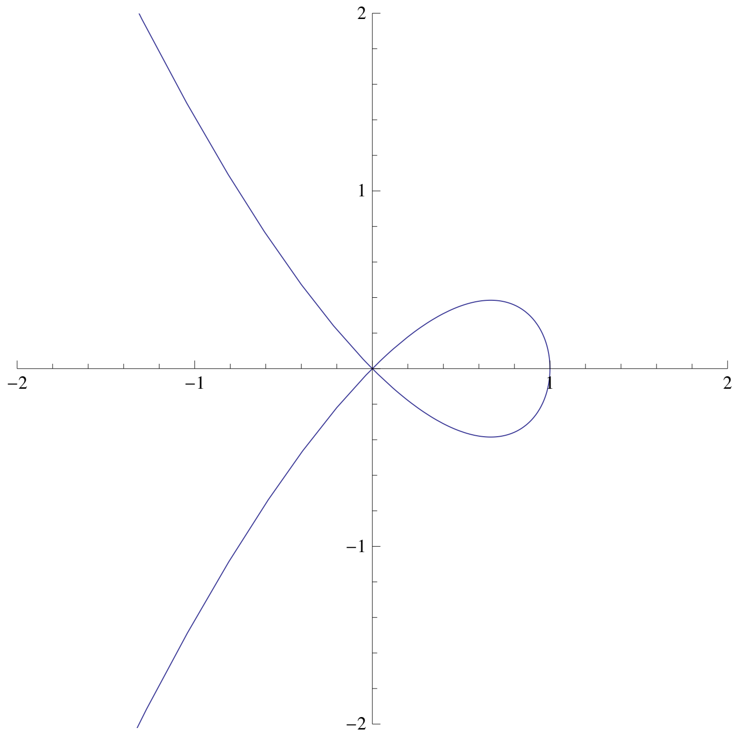
The curve defined by

$$f(t) = \left(1 - t^2, t(1 - t^2)\right)$$

is known as a nodal cubic.



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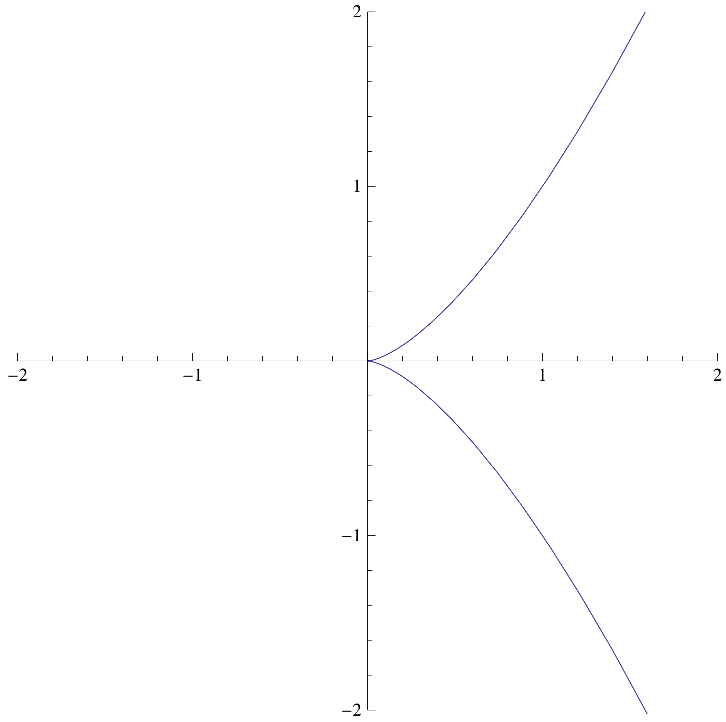
Parametric curves

The curve defined by

$$f(t) = (t^2, t^3)$$

is known as a cuspidal cubic.



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Parametric curves

Definition 1.2. A *curve (or arc) of class C^p* is a map

$$f : [a, b] \to \mathcal{E} ,$$

with $p \geq 1$, such that the restriction of f to $]a, b[$ is of class C^p, and where

$$f^{(i)}(a) = \lim_{t \to a, t > a} f^{(i)}(t) \qquad \text{and} \qquad f^{(i)}(b) = \lim_{t \to b, t < b} f^{(i)}(t)$$

exist, where $0 \leq i \leq p$. A *regular curve (or regular arc) of class C^p* is a curve of class C^p, with $p \geq 1$, such that every point is regular, i.e., $f'(t) \neq 0$ for every $t \in [a, b]$. The set of points

$$f([a, b])$$

in $\mathcal{E}$ is called the *trace of the curve f*.

It should be noted that even if f is injective, the trace $f(I)$ of f may be self-intersecting.

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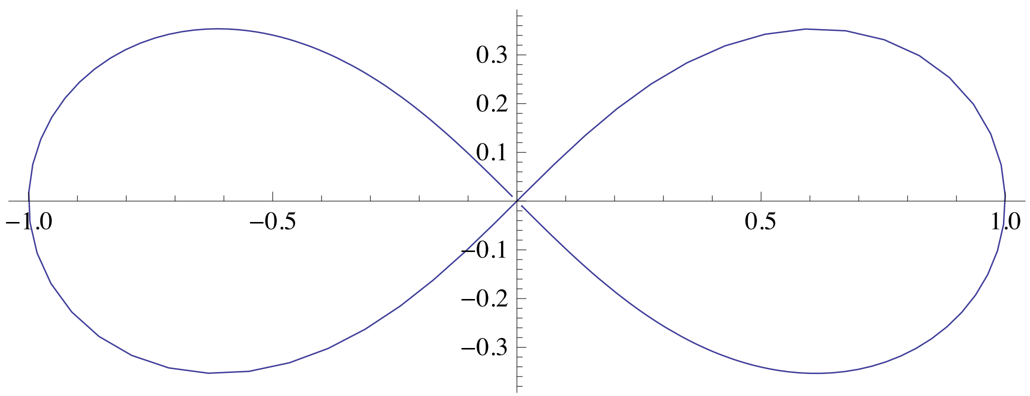
Parametric curves

Consider the curve $f : \mathbb{R} \to \mathbb{E}^2$ given by,

$$f(t) = \left(\frac{t(1+t^2)}{1+t^4}, \frac{t(1-t^2)}{1+t^4} \right) .$$

The trace of this curve is called the lemniscate of Bernoulli, and it has a self-intersection at the origin.



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Parametric curves

The map f is continuous, and in fact bijective, but its inverse f^{-1} is **not** continuous.

Self-intersection is due to the fact that

$$\lim_{t \to -\infty} f(t) = \lim_{t \to +\infty} f(t) = f(0)\,.$$

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Parametric curves

If we consider a curve

$$f : [a, b] \to \mathcal{E}$$

and we assume that f is injective on the entire **closed** interval $[a, b]$, then the trace

$$f([a, b])$$

of f has no self-intersection. Such curves are usually called *Jordan arcs* or *simple arcs*.

Because $[a, b]$ is compact, f is in fact a homeomorphism between $[a, b]$ and $f([a, b])$.

Many fractal curves are only continuous Jordan arcs that are not differentiable.

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It is possible that the trace of a curve be defined by many parameterizations, as illustrated by the unit circle, which is the trace of the parameterized curves $f_k :]0, 2\pi[\to \mathcal{E}$ (or $f_k : [0, 2\pi] \to \mathcal{E}$), where

$$f_k(t) = (\cos kt, \sin kt)\,,$$

with $k \geq 1$.

A clean way to handle this phenomenon is to define a notion of *geometric arc curve*. For our purposes, it suffices to define a notion of *change of parameter* which does not change the "geometric shape" of the trace.

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Parametric curves

Recall that a *diffeomorphism $g :]a, b[\to]c, d[$ of class C^p* from an open interval $]a, b[$ to another open interval $]c, d[$ is a bijection, such that both $g :]a, b[\to]c, d[$ and its inverse $g^{-1} :]c, d[\to]a, b[$ are C^p-functions.

This implies that $g'(t) \neq 0$ for every $t \in]a, b[$.

Definition 1.3. Two regular curves $f :]a, b[\to \mathcal{E}$ and $h :]c, d[\to \mathcal{E}$ of class C^p, with $p \geq 1$, are *C^p-equivalent* iff there is a diffeomorphism $g \to]a, b[\to]c, d[$ of class C^p such that f is equal to $h \circ g$.

It is immediately verified that Definition 1.3 yields an equivalence relation on open curves.

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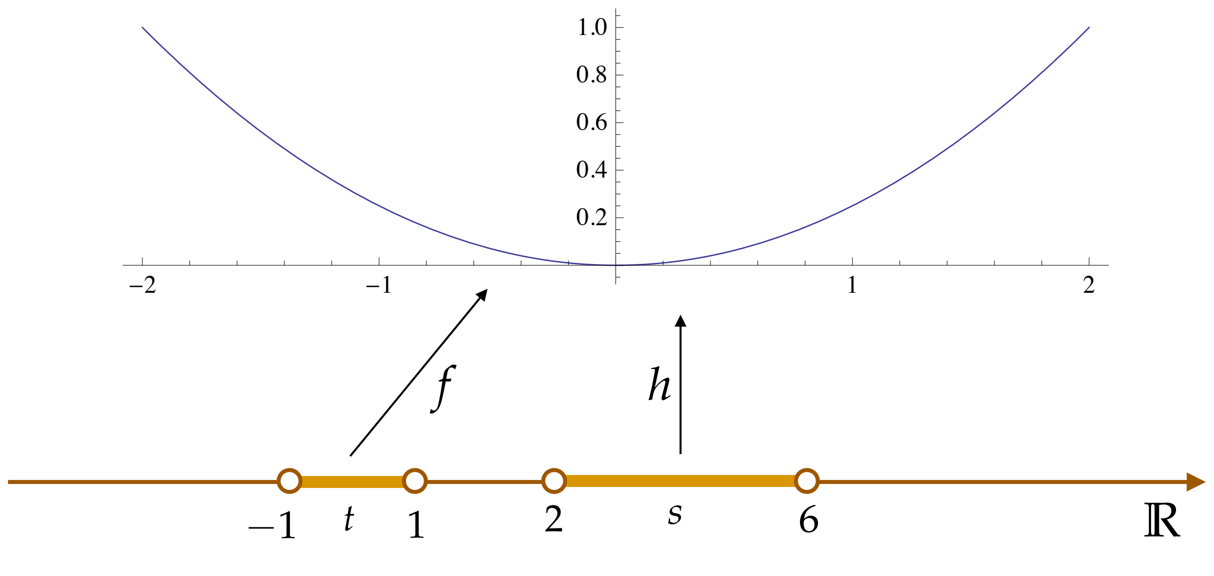
Parametric curves

For instance, consider the regular curves $f : [-1, 1] \to \mathcal{E}$ and $h : [2, 6] \to \mathcal{E}$ given by

$$f(t) = \left(2t, t^2\right) \qquad \text{and} \qquad h(s) = \left(s - 4, \frac{s^2 - 8s + 16}{4}\right).$$

Note that $f = h \circ g$, where $g : [-1, 1] \to [2, 6]$ is the diffeomorphism given by $g(r) = 2 \cdot r + 4$.



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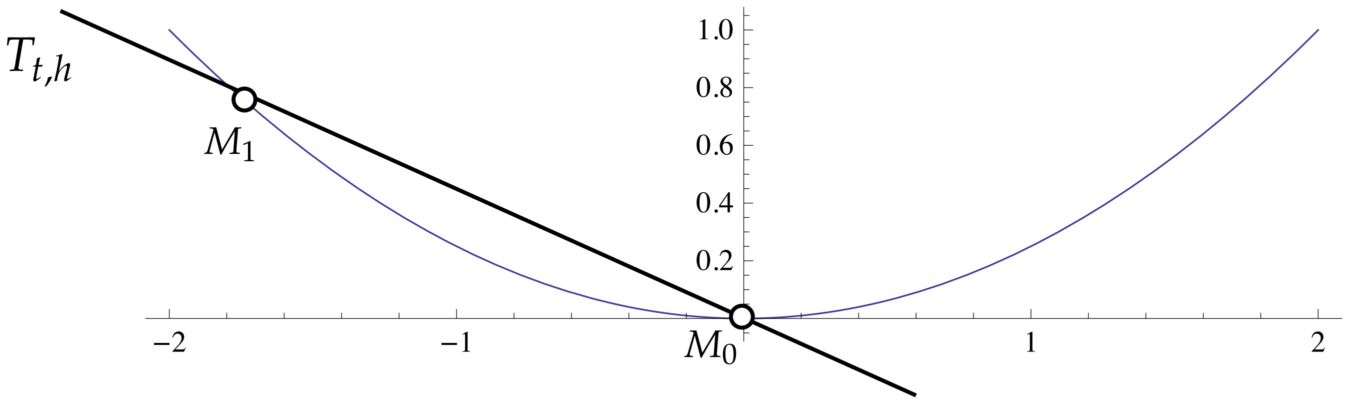
Parametric curves

Definition 1.4. For any open curve $f :]a, b[\to \mathcal{E}$ of class C^p (or curve $f : [a, b] \to \mathcal{E}$ of class C^p), with $p \geq 1$, given any point $M_0 = f(t)$ on the curve, if f is locally injective at M_0 and for any point $M_1 = f(t + h)$ near M_0, if the line $T_{t,h}$ determined by the points M_0 and M_1 has a limit T_t when $h \neq 0$ approaches 0, we say that T_t is the *tangent line to f in $M_0 = f(t)$ at t*.



Parametric curves

For simplicity, we will often say tangent, instead of tangent line.

The definition is simpler when f is a simple curve (there is no danger that $M_1 = M_0$ when $h \neq 0$).

The following lemma shows why regular points are important.

Lemma 1.1. For any open curve $f :]a, b[\to \mathcal{E}$ of class C^p (or curve $f : [a, b] \to \mathcal{E}$ of class C^p), with $p \geq 1$, given any point $M_0 = f(t)$ on the curve, if M_0 is a regular point at t, then the tangent line to f in M_0 at t exists and is determined by the derivative $f'(t)$ of f at t.

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If $f'(t) = 0$, the above argument breaks down.

However, if f is a C^p-function and $f^{(p)}(t) \neq 0$ for some $p \geq 2$, where p is the smallest integer with that property, we can show that the line $T_{t,h}$ has the limit determined by M_0 and the derivative $f^{(p)}(t)$. Thus, the tangent line may still exist at a stationary point.

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Parametric curves

For example, the curve f defined by the map $t \mapsto (t^2, t^3)$ is a C^∞-function, but $f'(0) = 0$. Nevertheless, the tangent at the origin is defined for $t = 0$ (it is the x-axis).