Algebraic Geometry Since 1980

by

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Chapter 1

Vanishing Theorems and Some Applications

1.1 Divisors, Curves: Nef, Big, Ample (and all that)

We begin by reviewing some basic notions, such as divisors, and by introducing some slight generalizations such as $\mathbb{Q}$-divisors. In this chapter, we assume that we are dealing with schemes of finite type over some algebraically closed field, k, of characteristic zero. By the Lefschetz Principle, we may assume that $k = \mathbb{C}$. Moreover, we also assume that our schemes are normal.

A *prime divisor* is an integral subscheme of codimension 1 (Recall: integral means reduced and irreducible). A *divisor* (or *Weil divisor*) is any $\mathbb{Z}$-linear combination of prime divisors.

A *Cartier divisor (or C-divisor)* is a divisor that it cut out locally by one equation.

A $\mathbb{Q}$-*Cartier divisor*, D, is a divisor so that

$$(\exists N \in \mathbb{Z})(N \neq 0 \quad \text{and} \quad ND \quad \text{is Cartier}).$$

A $\mathbb{Q}$-*divisor* is a $\mathbb{Q}$-linear combination of $\mathbb{Q}$-Cartier divisors. A $\mathbb{Q}$-divisor is *effective* iff D is of the form $D = \sum_i q_i D_i$ with $q_i > 0$ for all i (we assume $D_i \neq D_j$ whenever $i \neq j$). We write $D \geq E$ iff $D - E$ is effective.

We have the notion of *linear equivalence* for (ordinary) C-divisors. Suppose X is a *proper* scheme. If D and D' are C-divisors, then they are *numerically equivalent*, denoted $D \equiv D'$, iff for every integral curve, $C \subseteq X$, we have

$$D \cdot C = D' \cdot C.$$

(Recall that $\mathcal{O}_X(D)$ is the line bundle associated with D, so $\mathcal{O}_X(D) \restriction C$ is a line bundle on C. We take $D \cdot C$ to be the degree of the line bundle $\mathcal{O}_X(D) \restriction C$.)

If X is locally factorial (everywhere) then we know that

$$\mathrm{WDiv}(X) = \mathrm{CDiv}(X)$$

and the same holds for Q-divisors. We say that X is $\mathbb{Q}$-*factorial* iff every $\mathbb{Q}$-divisor is $\mathbb{Q}$-Cartier. Set

$$\mathrm{Num}(X) = \mathrm{CDiv}(X)/\equiv,$$

the *numerical class group of* X. Now, over $\mathbb{C}$, if X is a proper, normal, connected variety, we get the complex analytic space, X_h, (with $\mathcal{O}_{X_h} = \mathbb{C}$-analytic functions on X) and we have the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{X_h} \xrightarrow{e^{2\pi i -}} \mathcal{O}_{X_h}^* \longrightarrow 0.$$

If we apply cohomology, using GAGA, we get the long exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}_{X_h}) \longrightarrow H^1(X,\mathcal{O}_{X_h}^*) \xrightarrow{c} H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O}_{X_h}) \longrightarrow \cdots,$$

We know that $\mathrm{Pic}(X) = H^1(X,\mathcal{O}_{X_h}^*)$ and the map, $c\colon \mathrm{Pic}(X) \to H^2(X,\mathbb{Z})$, plays a special role. We get

$$0 \longrightarrow H^1(X,\mathcal{O}_{X_h})/H^1(X,\mathbb{Z}) \longrightarrow \mathrm{Pic}(X) \xrightarrow{c} H^2(X,\mathbb{Z}).$$

Let

$$\mathrm{Pic}^0(X) = H^1(X,\mathcal{O}_{X_h})/H^1(X,\mathbb{Z}),$$

a complex torus. Observe that the image of $\mathrm{Pic}(X)$ in $H^2(X,\mathbb{Z})$ is the same as the image of $\mathrm{Num}(X)$ in $H^2(X,\mathbb{Z})$; in fact $\mathrm{Num}(X) \subseteq H^2(X,\mathbb{Z})$. It follows that $\mathrm{Num}(X)$ is a finitely generated torsion-free abelian group (Neron-Severi).

Numerical equivalence also makes sense for $\mathbb{Q}$-divisors. (Check that $(mD\cdot C = m(D\cdot C)$.) Thus, we set

$$(D\cdot C) = \frac{1}{m}(mD\cdot C), \quad m > 0.$$

A C-divisor, D, is *very ample* iff the rational map, $\varphi_D\colon X \to \mathbb{P}(H^0(X,\mathcal{O}_X(D)))$ is a morphism and an immersion, with

$$\mathcal{O}_X(D) = \varphi_D^*(\mathcal{O}_{\mathbb{P}(1)}).$$

A C-divisor, D, is *ample* iff there is some integer, $m > 0$, so that mD is ample iff for all $m >> 0$, mD is very ample.

Recall Serre's characterizations of ampleness (from FAC). Here, we assume that X is a scheme of finite type that is proper.

(I) D is ample iff there is some $m >> 0$ such that mD is ample iff for all $m >> 0$, mD is ample.

(II) (Vanishing Criterion) D is ample iff for every coherent $\mathcal{O}_X$-module, $\mathcal{F}$,

$$(\exists n_0 = n_0(\mathcal{F}))(\forall p > 0)(H^p(X, \mathcal{F} \otimes \mathcal{O}_X(nD) = (0) \quad \text{when} \quad n \geq n_0).$$

(III) (Global Sections Criterion) D is ample iff for every coherent $\mathcal{O}_X$-module, $\mathcal{F}$,

$$(\exists n_0 = n_0(\mathcal{F}))(\forall n \geq n_0)(\mathcal{F} \otimes \mathcal{O}_X(nD) \quad \text{is generated by its global sections}).$$

Definition 1.1 A $\mathbb{Q}$-C-divisor is *nef* (*numerically effective*) iff for every integral curve, C, of X (a proper scheme), we have

$$D \cdot C \geq 0.$$

We say that D is *semi-ample* iff for all $m >> 0$, $\mathcal{O}_X(mD)$ is generated by its sections.

We say that D is *big* iff for all $K > 0$, there is some $m >> 0$ so that

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(mD)) > K m^{\dim X}.$$

Note that ample implies semi-ample.

The Hirzebruch-Riemann-Roch Theorem (for short, HRR) connects these concepts. In order to state the Hirzebruch-Riemann-Roch Theorem we need some preparation including the definition of Chern classes, of Chern characters and of the Todd polynomial.

Let $\mathcal{F}$ be either a holomorphic vector bundle on a smooth projective variety or a C^∞ vector bundle on a complex, compact, manifold, X. In both cases, Chern classes exist. Following Hirzebruch's axiomatic approach, the Chern classes, $c_i(\mathcal{F})$, turn out to exist and to be uniquely characterized by the following four axioms:

(1) $c_i(\mathcal{F}) \in H^{2i}(X, \mathbb{Z})$

(2) (Naturality) Say $\pi\colon Y \to X$ is a morphism of varieties (both "good", in the sense specified above) and write $c(\mathcal{F})(t) = 1 + c_1(\mathcal{F}) + c_2(\mathcal{F}) + \cdots$, the *Chern polynmial* for the v.b., $\mathcal{F}$, on X. Then,

$$c(\pi^*\mathcal{F})(t) = \pi^*(c(\mathcal{F})(t)).$$

(3) (Whitney sum) If $\mathcal{F}$ and $\mathcal{G}$ are both v.b.'s on X, then

$$c(\mathcal{F} \amalg \mathcal{G})(t) = c(\mathcal{F})(t) \amalg c(\mathcal{G})(t).$$

(4) (Normalization) If $X = \mathbb{P}^n$ and $\mathcal{F} = \mathcal{O}_X(1)$, the vector bundle corresponding to the hyperplane divisor, H, on $\mathbb{P}^n$, then

$$c(\mathcal{O}_X(1))(t) = 1 + Ht.$$

Say $\mathcal{L}$ is a line bundle on X, a C^∞ manifold. Then, there are lots of C^∞ sections and they give rise to a C^∞ map, $\varphi_{\mathcal{L}}\colon X \hookrightarrow \mathbb{P}^N$, with $\mathcal{L} = \varphi_{\mathcal{L}}^*(\mathcal{O}_{\mathbb{P}^N}(1))$. By Axiom (3),

$$\begin{aligned} c(\mathcal{L})(t) &= c(\varphi_{\mathcal{L}}^*(\mathcal{O}_{\mathbb{P}^N}(1)))(t) \\ &= \varphi_{\mathcal{L}}^*(c(\mathcal{O}_{\mathbb{P}^N}(1))(t)) \\ &= \varphi_{\mathcal{L}}^*(1+Ht) \\ &= 1+\varphi_{\mathcal{L}}^*(H)t. \end{aligned}$$

We deduce

$$\begin{aligned} c_1(\mathcal{L}) &= \varphi_{\mathcal{L}}^*(H) \\ c_i(\mathcal{L}) &= 0 \qquad \text{if} \quad i>1. \end{aligned}$$

Say $\mathcal{F}$ is a vector bundle on X. Then, there is a fibre space, $Y \xrightarrow{\pi} X$, so that $\pi^{-1}(x)$ is equal to the flag manifold on the vector space $\mathcal{F}_x$ (with $\dim \mathcal{F}_x = \mathrm{rk}\,\mathcal{F}$). It follows that Y is the flag manifold over X. Then, it is known that

(1) $\pi^*\mathcal{F} = L_1 \coprod \cdots \coprod L_q$, with $q = \mathrm{rk}\,\mathcal{F}$ and the L_j's are line bundles over Y.

(2) $\pi^*(H^\bullet(X,\mathbb{Z})) \longrightarrow H^\bullet(Y,\mathbb{Z})$ is a monomorphism (Borel).

But then, as $c(\pi^*\mathcal{F})(t) = \pi^*(c(\mathcal{F}(t))$ and by (1), $\pi^*\mathcal{F} = L_1 \coprod \cdots \coprod L_q$, using Axiom (3), we get

$$\pi^*(c(\mathcal{F}(t)) = \prod_{j=1}^{q} c(L_j)(t).$$

However, we know that $c(L_j) = 1+\gamma_j t$, with $\gamma_j = c_1(L_j) \in H^2(Y,\mathbb{Z})$, so

$$\prod_{j=1}^{q} c(L_j)(t) = \prod_{j=1}^{q}(1+\gamma_j t).$$

Now, as π^* is a monomorphism we can view π^* as an inclusion and we get

$$c(\mathcal{F})(t) = \prod_{j=1}^{q}(1+\gamma_j t).$$

(Here $q = \mathrm{rk}\,\mathcal{F}$.) The γ_j's are called the *Chern roots* of $\mathcal{F}$. But, we have

$$\prod_{j=1}^{q}(1+\gamma_j t) = \sum_{k=0}^{q} \sigma_k(\gamma_1,\ldots,\gamma_q)t^k,$$

where $\sigma_k(\gamma_1,\ldots,\gamma_q)$ is the k^{th} elementary symmetric function of the γ_j's. Consequently,

$$c_k(\mathcal{F}) = \sigma_k(\gamma_1,\ldots,\gamma_q).$$

In particular, $c_1(\mathcal{F}) = \gamma_1 + \cdots + \gamma_k$. Using Chern roots, we obtain the following useful computational rules:

(0) (Splitting Principle) Given a rank q vector bundle, V, make believe V splits as $V = \coprod_{j=1}^{q} L_j$ (for some line bundles, L_j), write $\gamma_j = c_1(L_j)$, the γ_j are the *Chern roots* of V. Then,

$$c(V)(t) = \prod_{j=1}^{q}(1 + \gamma_j t).$$

(1) $c(V^D)(t) = \prod_{j=1}^{q}(1-\gamma_j t)$ when $c(V)(t) = \prod_{j=1}^{q}(1+\gamma_j t)$. That is, $c_i(V^D) = (-1)^i c_i(V)$.

(2) If $0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$ is exact, then $c(V)(t) = c(V')(t)c(V'')(t)$.

(3) If $c(V)(t) = \prod_{j=1}^{q}(1 + \gamma_j t)$ and $c(W)(t) = \prod_{j=1}^{q'}(1 + \delta_j t)$, then

$$c(V \otimes W)(t) = \prod_{j,k=1}^{q,q'}(1 + (\gamma_j + \delta_k)t).$$

(4) If $c(V)(t) = \prod_{j=1}^{q}(1 + \gamma_j t)$, then

$$c\Big(\bigwedge^r V\Big)(t) = \prod_{1\leq i_1<\cdots<i_r\leq q}(1 + (\gamma_{i_1} + \cdots + \gamma_{i_r})t).$$

In particular, when $r = q$, there is just one factor in the polynomial, it has degree 1, it is $1 + (\gamma_1 + \cdots + \gamma_q)t$. By (2). we get

$$c_1\Big(\bigwedge^q V\Big)(t) = c_1(V) \quad \text{and} \quad c_l\Big(\bigwedge^q V\Big)(t) = 0 \quad \text{if} \quad l \geq 2.$$

(5) If $c(V)(t) = \prod_{j=1}^{q}(1 + \gamma_j t)$, then

$$c(\mathcal{S}^r V)(t) = \prod_{\substack{m_j\geq 0 \\ m_1+\cdots+m_q=r}}(1 + (m_1\gamma_1 + \cdots + m_q\gamma_q)t).$$

(6) If $\mathrm{rk}(V) \leq q$, then $\deg(c(V)(t)) \leq q$ (where $\deg(c(V)(t)$ is the degree of $c(V)(t)$ as a polynomial in t).

(7) Suppose we know $c(V)$, for some vector bundle, V, and L is a line bundle. Write $c = c_1(L)$. Then, the Chern classes of $V \otimes L$ are

$$c_l(V \otimes L) = \sigma_l(\gamma_1 + c, \gamma_2 + c, \cdots, \gamma_r + c),$$

where $r = \mathrm{rk}(V)$ and the γ_j are the Chern roots of V. This is because the Chern polynomial of $V \otimes L$ is

$$c(V \otimes L)(t) = \prod_{i=1}^{r}(1 + (\gamma_i + c)t).$$

Here is a method due to Griffith for computing Chern classes. Suppose $\mathcal{F}$ is a vector bundle generated by its global sections and say $\mathrm{rk}(\mathcal{F}) = r$. Pick, $\sigma_1, \ldots, \sigma_r$, some generic global sections of $\mathcal{F}$ and form $\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_{r-k+1}$ (a section of $\bigwedge^{r-k+1} \mathcal{F}$). Then, the cycle of zeros of $\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_{r-k+1}$ carries $c_k(\mathcal{F})$. From this, we draw two conclusions:

(A) $c_{\mathrm{rk}(\mathcal{F})}(\mathcal{F})$, the top Chern class of $\mathcal{F}$, is carried by the zeros of any generic section of $\mathcal{F}$.

(B) If $k = 1$, pick all r global sections and find the zeros of $\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_r$ (a section of $\bigwedge^r \mathcal{F} = \det(\mathcal{F})$). This cycle of zeros carries $c_1(\mathcal{F})$.

If $\mathcal{F}$ is a vector bundle and if $\gamma_1, \ldots, \gamma_q$ are its Chern roots define the *Chern character,* $\mathrm{ch}(\mathcal{F})(t)$, of $\mathcal{F}$ by

$$\begin{aligned}
\mathrm{ch}(\mathcal{F})(t) &= \sum_{j=1}^{q} e^{\gamma_j t} = \sum_{j=1}^{q} \sum_{i=0}^{\infty} \frac{\gamma_j^i t^i}{i!} \\
&= \sum_{i=0}^{\infty} \frac{1}{i!} \left(\sum_{j=1}^{q} \gamma_j^i \right) t^i \\
&= \sum_{i=0}^{\infty} \frac{1}{i!} s_i(\gamma_1, \ldots, \gamma_q) t^i
\end{aligned}$$

where $s_i(\gamma_1, \ldots, \gamma_q) = \sum_{j=1}^{q} \gamma_j^i$. If we let $\mathrm{ch}(\mathcal{F})(t) = \sum_{j \geq 0} \mathrm{ch}_j(\mathcal{F}) t^j$, we get

$$\mathrm{ch}_0(\mathcal{F}) = \mathrm{rk}(\mathcal{F}), \qquad \mathrm{ch}_j(\mathcal{F}) = \frac{1}{j!} s_j(\mathcal{F}), \quad j \geq 1.$$

Using Newton's formula

$$s_k - c_1 p_{k-1} + c_2 p_{k-2} + \cdots + (-1)^k k c_k = 0,$$

for $k \geq 1$ with $c_j = \sigma_j(\gamma_1, \ldots, \gamma_q)$, we can compute recursively the $\mathrm{ch}_j(\mathcal{F})$ in terms of the $c_i(\mathcal{F})$'s. We can also check that

$$\begin{aligned}
\mathrm{ch}(\mathcal{F} \coprod \mathcal{G})(t) &= \mathrm{ch}(\mathcal{F})(t) + \mathrm{ch}(\mathcal{G})(t) \\
\mathrm{ch}(\mathcal{F} \otimes \mathcal{G})(t) &= \mathrm{ch}(\mathcal{F})(t) \mathrm{ch}(\mathcal{G})(t).
\end{aligned}$$

Again, given a vector bundle, $\mathcal{F}$, of rank q, if $\gamma_1, \ldots, \gamma_q$ are the Chern roots of $\mathcal{F}$, we define the *Todd polynomial of* $\mathcal{F}$ as

$$\mathrm{Td}(\mathcal{F})(t) = \prod_{j=1}^{q} \frac{\gamma_j t}{1 - e^{-\gamma_j t}}.$$

We write $\mathrm{Td}(\mathcal{F})(t) = 1 + \mathrm{Td}_1(\mathcal{F}) t + \mathrm{Td}_2(\mathcal{F}) t^2 + \cdots$. If X is a manifold with $d = \dim X$, we have the tangent bundle, T_X, and we let

$$\mathrm{Td}(X) = \mathrm{Td}(T_X)$$

and $T(X)$, the *Todd genus of* X, is the degree d piece of $\mathrm{Td}(X)$. Hirzebruch proved that there is one and only one power series in the Chern classes so that

$$T(\mathbb{P}^n_{\mathbb{C}}) = 1, \quad \text{for all} \quad n \geq 0.$$

Theorem 1.1 *(Hirzebruch-Riemann-Roch (1954)) If X is a non-singular projective variety over $\mathbb{C}$ of dimension n (also true for a compact, complex manifold–Atiyah-Singer) and E is a rank r vector bundle on X, then*

$$\chi(X, \mathcal{O}_X(E)) = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{C}} H^i(X, \mathcal{O}_X(E)) = \deg_n(\mathrm{ch}(E)\mathrm{Td}(X)).$$

Let us work out some examples.

(1) $\dim X = 1$ and $\mathrm{rk}\, E = 1$, *i.e.*, X is a curve and E is a line bundle. Then, $c_1(E) \in H^2(X, \mathbb{Z}) = \mathbb{Z}$ and in this case, we know that $c_1(E) = \deg E$. Now, it is known that the top Chern class, $c_n(E)$ is given by

$$c_1(E) = \chi_{\mathrm{EP}}(X),$$

where $\chi_{\mathrm{EP}}(X)$ is the Euler-Poincaré characteristic of X, so in this case,

$$c_1(T_X) = 2 - 2g,$$

with $g =$ the genus of the curve C. Alternately, $\bigwedge^1 T_X = T_X = -K_X$, so

$$c_1(T_C) = -c_1(K_X) = -\deg K_X = -(2g - 2) = 2 - 2g.$$

We have

$$\mathrm{Td}(X) = 1 + \frac{1}{2} c_1(T_X) t \quad \text{and} \quad \mathrm{ch}(X) = 1 + (\deg E)t,$$

so

$$\deg_1(\mathrm{ch}(E)\mathrm{Td}(X)) = \deg E + \frac{1}{2} c_1(T_X) = \deg E + 1 - g.$$

Therefore, HRR says that

$$\chi(X, \mathcal{O}_X(E)) = \deg E + 1 - g,$$

which, of course, is the original Riemann-Roch Theorem.

(2) Again, $\dim X = 1$ but this time, $\mathrm{rk}\, E = r \geq 1$. Then, $c_1(E) = c_1(\bigwedge^r E) = c_1(\det E)$, so

$$\mathrm{ch}(E) = r + \deg(\det E)t$$

and we get

$$\chi(X, \mathcal{O}_X(E)) = \deg(\det E) + r(1 - g).$$

(3) $\dim X = 2$ and $\mathrm{rk}\, E = 1$, *i.e.*, X is a non-singular surface and E is a line bundle. Then,

$$\mathrm{ch} E) = 1 + c_1(E)t + \frac{1}{2}c_1(E)^2t^2$$

and

$$\mathrm{Td}(X) = 1 + \frac{1}{2}c_1(X)t + \frac{1}{12}(c_1^2(X) + \chi_{\mathrm{EP}}(X))t^2.$$

Also, $c_1(X) = c_1(T_X) = c_1(\bigwedge^2 T_X) = -K_X$. If we write $D = c_1(E)$ for the divisor corresponding to E, then

$$\deg_2(\mathrm{ch}(E)\mathrm{Td}(X)) = \frac{1}{2}D^2 - \frac{1}{2}K_X \cdot D + \frac{1}{12}(K_X^2 + \chi_{\mathrm{EP}}(X)).$$

It follows that

$$\chi(X, \mathcal{O}_X(E)) = \frac{1}{12}(K_X^2 + \chi_{\mathrm{EP}}(X)) + \frac{1}{2}D \cdot (D - K_X).$$

(4) $\dim X = 3$ and $\mathrm{rk}\, E = 1$, *i.e.*, X is a non-singular 3-fold and E is a line bundle. Then,

$$\mathrm{ch} E) = 1 + Dt + \frac{1}{2}D^2t^2 + \frac{1}{6}D^3t^3$$

and

$$\begin{aligned} \mathrm{Td}(X) &= 1 + \frac{1}{2}c_1(X)t + \frac{1}{12}(c_1^2(X) + c_2(X))t^2 + \frac{1}{12}c_1(X)c_2(X)t^3 \\ &= 1 - \frac{1}{2}K_X t + \frac{1}{12}(K_X^2(X) + c_2(X))t^2 - \frac{1}{12}K_X \cdot c_2(X)t^3. \end{aligned}$$

It follows that

$$\deg_3(\mathrm{ch}(E)\mathrm{Td}(X)) = \frac{1}{6}D^2 - \frac{1}{4}K_X \cdot D^2 + \frac{1}{12}D \cdot (K_X^2 + c_2(X)) - \frac{1}{24}K_X \cdot c_2(X).$$

Here is a useful conclusion of HRR for a line bundle, E, with corresponding divisor, D. If $\dim X = n$, as

$$\mathrm{ch} E) = 1 + Dt + \frac{1}{2}D^2t^2 + \cdots + \frac{1}{n!}D^nt^n$$

and

$$\mathrm{Td}(X) = 1 + \mathrm{Td}_1(X)t + \cdots + \mathrm{Td}_n(X)t^n,$$

we see that

$$\deg_n(\mathrm{ch}(E)\mathrm{Td}(X)) = \frac{1}{n!}D^n + O(D^{n-1}.$$

In particular, as $E^{\otimes m} = \mathcal{O}_X(mD)$, in this case, we get

$$\chi(X, \mathcal{O}_X(mD)) = \left(\frac{1}{n!}D^n\right)m^n + O(m^{n-1}).$$

We know that very ample $\Longrightarrow$ ample $\Longrightarrow$ semi-ample and semi-ample $\Longleftrightarrow$ $\mathcal{O}_X(mD)$ is generated by its global sections.

What does this mean? A global section, $\sigma \in H^0(X, \mathcal{O}_X(mD))$, corresponds to an effective divisor, $\widetilde{D}$, with $\widetilde{D} \sim D$ (*i.e.* $\widetilde{D}$ is linearly equivalent to D). Furthermore, $\sigma(x) = 0$ iff $x \in \widetilde{D}$. Therefore, $\mathcal{O}_X(mD)$ is generated by its global sections iff for every $x \in X$, there is some effective divisor, $\widetilde{D} \in |mD|$, with $x \notin \widetilde{D}$ iff no $x \in X$ is a basepoint of $|mD|$. (Here, $|mD|$ is the linear system associated with mD.)

Proposition 1.2 *On a proper (projective) variety, X, ample implies big and semi-ample implies nef.*

Proof. If D is ample, then for all $m >> 0$,

$$\chi(X, \mathcal{O}_X(mD)) = \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(mD)).$$

By HRR,

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(mD)) = \left(\frac{1}{n!}D^n\right) m^n + O(m^{n-1}) > Km^n$$

if $K = \frac{1}{n!}D^n > 0$. So, we need to prove $D^n > 0$. Although we only need the easy direction of the *Nakai-Moishezon criterion*, we state this criterion since it is a useful fact to know anyway:

Nakai-Moishezon Criterion: Say X is proper and D is a divisor on X. Then, D is ample iff $D^{\dim Y} \cdot Y > 0$, for every integral subscheme, Y, of X.

Now, apply the above criterion to $Y = D^{n-1}$. Then, $D^n = D \cdot Y = D \cdot D^{n-1} > 0$ as D is ample, which concludes this part of the proof. (We really don't need the Nakai-Moishezon Criterion. Say D is ample. Then, mD is very ample for $m >> 0$. Let Y be an integral subscheme with $\dim Y = r \leq n$. We have a closed immersion

$$\varphi_{mD} \colon X \hookrightarrow \mathbb{P}^N.$$

So, $D^r \mapsto H^r$ and $Y \mapsto$ a closed subvariety of $\mathbb{P}^N$ and $(mD)^r \cdot Y > 0$ becomes $\deg(\varphi_{mD}(Y)) > 0$, and we are done.)

Let us now prove that semi-ample implies nef. Assume D is semi-ample and let C be any curve in X. Look at $(mD) \cdot C = m(D \cdot C)$ with $m > .0$. Now, $m(D \cdot C)$ is the divisor of $\mathcal{O}_X(mD) \restriction C$ on C and as $\mathcal{O}_X(mD)$ is generated by its global sections, $\mathcal{O}_X(mD) \restriction C$ is generated by its global sections on C. It follows that $\deg(\mathcal{O}_X(mD) \restriction C) \geq 0$ which implies $mD \cdot C \geq 0$ and thus, $D \cdot C \geq 0$. As this holds for every curve, C, we conclude that D is nef. $\square$

Corollary 1.3 *Say Y and X are projective varieties and let $\pi \colon Y \to X$ be a proper morphism. If D is nef on X, then $\pi^* D$ is nef on Y (and similarly for ample).*

Proof. Recall the projection formula

$$(\pi^* D \cdot C) = (D \cdot \pi_* C),$$

(for any irreducible curve, C, on X) where

$$\pi_* C = \begin{cases} 0 & \text{if } \pi(C) = \text{point} \\ d\pi(C) & \text{if } \pi(C) \text{ is a curve and } d = (K(C)\colon K(\pi(C))). \end{cases}$$

Take any curve on Y and any divisor, D, on X, with D nef. Then, we have

$$(\pi^* D \cdot C) = (D \cdot \pi_* C) = \begin{cases} 0 & \\ dD \cdot \pi(C) & \geq 0 \end{cases}$$

and we are done. □

Sorites:

1. If X and Y are proper and $\pi\colon Y \to X$ is a finite morphism, then $\pi^*(\text{ample}) = \text{ample}$.
2. D is ample on X iff $D \restriction$ (every irreducible component of X) is ample.
3. Suppose D is ample and E is any Cartier divisor. Then, for all small enough $t \in \mathbb{Q}$, we have $D + tE$ is again ample (use Serre's characterization).
4. The sum of two amples is ample. By (3) and (4) , we see that the ample divisors form an open cone in $N^1(X)_{\mathbb{Q}}$.
5. nef + nef = nef (ample + nef = nef).
6. If D is very ample and E is any Cartier divisor, then $mD + E$ is very ample if $m >> 0$.
7. ample + nef = ample.
8. If D is very ample and E is generated by its sections, then $D + E$ is very ample (use the Segre morphism).

Here is a useful lemma:

Lemma 1.4 *Say X is proper and D is ample on X ($n = \dim X$). Then,*

$$D^r \cdot H^{n-r} > 0 \quad \textit{for} \quad 0 \leq r \leq n.$$

Proof. It follows from the easy direction of the Nakai-Moishezon criterion. □

The Cone of Curves. Say X is a proper scheme. If C and $\widetilde{C}$ are two curves on X, then C is numerically equivalent to $\widetilde{C}$ (written $C \equiv \widetilde{C}$) iff for every Cartier divisor, C, we have $D \cdot C = D \cdot \widetilde{C}$.

Write $N_1(X)_{\mathbb{Z}}$ for the free group of curves modulo $\equiv$ and set

$$\begin{aligned} N_1(X)_{\mathbb{Q}} &= N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \\ N_1(X)_{\mathbb{R}} &= N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}. \end{aligned}$$

We have the nondegenerate pairings

$$N_1(X)_{\mathbb{Z},\mathbb{Q},\mathbb{R}} \otimes N^1(X)_{\mathbb{Z},\mathbb{Q},\mathbb{R}} \longrightarrow \mathbb{Z}, \mathbb{Q}, \mathbb{R}.$$

If we use the norm topology on $N_1(X)_{\mathbb{Q},\mathbb{R}}$ and $N^1(X)_{\mathbb{Q},\mathbb{R}}$, then these spaces are ρ-dimensional vector spaces (with ρ = Picard number of X). Define $\mathrm{NE}(X) \subseteq N_1(X)_{\mathbb{R}}$ as the cone consisting of all equivalence classes of linear combinations

$$\sum_{j=1}^{m} a_j C_j, \quad a_j \in \mathbb{R},\ a_j > 0,$$

each C_j an irreducible curve.

Theorem 1.5 *If X is projective and D is a Cartier divisor on X (the theorem also holds for $\mathbb{Q}$-cartier, $\mathbb{Q}$-divisors), then*

(1) D is ample iff for every curve $C \in \overline{\mathrm{NE}(X)}$, if $C \neq 0$ then $D \cdot C > 0$.

(2) Suppose H is an ample divisor on X, then for any $k \geq 0$,

$$K_k = \{C \in N_1(X) \mid (H \cdot C) \leq k\}$$

is compact and contains only finitely many classes of irreducible curves, C.

Proof. (1) We know that D nef implies that $D \cdot C \geq 0$ on $\overline{\mathrm{NE}(X)}$. Now, suppose $C \neq 0$ and $D \cdot C = 0$. Since the above pairing is nondegenerate, there is some E such that $(E \cdot C) < 0$. Loook at $D + tE$, for t small ($t \in \mathbb{Q}$). Then, $(D + tE) \cdot C < 0$. Yet, $D + tE$ is ample for t small and so, $(D + tE) \cdot C \geq 0$, a contradiction. Therefore, $D \cdot C > 0$.

Conversely, write

$$K = \{C \in \overline{\mathrm{NE}(X)} \mid \|C\| = 1\}.$$

The set K is compact as $N_1(X)_{\mathbb{R}}$ is finite dimensional. The function, $f_D \colon K \to \mathbb{R}$ *via* $f_D(C) = D \cdot C$ is continuous and by hypothesis, $f_D > 0$ on K. Consequently, there is some $a \in \mathbb{Q}$ such that $0 < a < f_C(C)$ for all $C \in K$. Similarly, the function $f_H \colon K \to \mathbb{R}$ is continuous on K and, by the forward part already proved, $f_H > 0$ on K. Thus, there is some $b \in \mathbb{Q}$ such that $b > f_H(C) > 0$, for all $C \in K$. Look at $D - \frac{a}{b}H$. For $C \in K$,

$$\left(D - \frac{a}{b}H\right) \cdot C = D \cdot C - \frac{a}{b}(H \cdot C) \geq D \cdot C - a,$$

by choice of b. But, $D \cdot C > a$ (by choice of a), so

$$\Big(D - \frac{a}{b}H\Big) \cdot C \geq 0, \quad \text{for all} \quad C \in K.$$

Therefore,

$$\bigcup_{r>0} rK = \overline{\mathrm{NE}(X)}$$

and $D - \frac{a}{b}H$ is nef. But, $\frac{a}{b}H$ is $\mathbb{Q}$-ample, so

$$D = \Big(D - \frac{a}{b}H\Big) + \frac{a}{b}H$$

where the first term on the right hand side is nef and the second first term on the right hand side is ample. It follows that D is ample.

Let us now prove that ampe + nef = ample. We know that $D^r \cdot H^{n-r} > 0$, where H is the embedding divisor of X and $n = \dim X$ (by the useful lemma). Say H is given and D is nef, then $D \restriction Y$ is still nef for all integral schemes, Y, inside X. By the above

$$(D \restriction Y)^s \cdot (H \restriction Y)^{t-s} > 0,$$

with $t = \dim Y$, that is,

$$D^s \cdot H^{t-s} \cdot Y > 0, \qquad 0 \leq s \leq t.$$

Now,

$$(D + H)^t \cdot Y = \sum_{j=0}^{t} \binom{t}{j} D^j \cdot H^{j-j} \cdot Y > H^t \cdot Y > 0,$$

by Nakai-Moishezon. Therefore, $D + H$ is ample.

(2) Write

$$K_k = \{c \in N_1(X) \mid (H \cdot C) \leq k\}.$$

We need to show that K_k is compact and contains but finitely many classes of irreducible curves. Let ρ = Picard number of $X = \dim N^1(X)_{\mathbb{R}} < \infty$. Pick $D_1, \ldots, D_\rho$, a basis for $N^1(X)_{\mathbb{R}}$ and let $D^{(1)}, \ldots, D^{(\rho)}$ be the dual basis in $N_1(X)_{\mathbb{R}}$. For our K of part (1) and $C \in K$, we know that there is some $M_0 > 0$ so that,

$$(m_0 H \pm D) \cdot C > 0, \qquad \text{for all } C \in K.$$

It follows that

$$|D_j \cdot C| < m_0 |H \cdot C|, \qquad \text{for all } C \in K.$$

Thus, if $(H \cdot C) \leq k$, this bounds the coefficients of the expression of C in terms of $D^{(1)}, \ldots, D^{(\rho)}$. The closed bounded subset of $N_1(X)_{\mathbb{R}}$ resulting is then compact as $\rho < \infty$.

A curve, C, in K_k belongs to $N_1(\mathbb{Z})_{\mathbb{Z}} \cap K_k$ and as $N_1(X)_{\mathbb{Z}}$ is discrete, the previous set is finite. $\square$

Corollary 1.6 *If D is a real nef divisor, then D is arbitrarily approximable by a $\mathbb{Q}$-Cartier ample $\mathbb{Q}$-divisor. Hence, on a projective scheme, X, the real nef cone is the closure of the ample $\mathbb{Q}$-cone.*

Proof. If H is the very ample embedding divisor, pick $t \in \mathbb{Q}$, small and look at $D + tH$. This divisor and ample, so by Kleimann, $(D + tH) \cdot C > 0$, for any $C \in \overline{\mathrm{NE}(X)}$, $C \neq 0$. We can approximate D by a $\mathbb{Q}$-divisor, $\widetilde{D}$, so that

$$(\widetilde{D} + tH) \cdot C > 0 \qquad \text{in} \quad \overline{\mathrm{NE}(X)} - \{0\}.$$

By Kleimann, $\widetilde{D} + tH$ is ample. But D is close to $\widetilde{D} + tH$ as t is small. $\square$

Remark: (nef & big) + nef = nef & big.

Say D is nef and big and E is nef. Of course, $D + E$ is nef. Again, $\frac{1}{m}E$ is nef. So, as

$$m\left(D + \frac{1}{m}E\right) = mD + E,$$

if $n = \dim X$, we get

$$m^n\left(D + \frac{1}{m}E\right)^n = (mD + E)^n = \sum_{j=1}^{n} \binom{n}{j} m^j D^j E^{n-j} > m^n D^n.$$

But, $m^n D^n > K m^n$, as D is nef and big, which implies that $D + \frac{1}{m}E$ is nef and big. It follows that $D + \frac{1}{m}E + \frac{1}{m}E$ is nef and big and so on, and thus, $D + E$ is nef and big.

Theorem 1.7 *Say X is a proper and of finite type, $\mathcal{F}$ is a coherent X-module and D is a Cartier divisor. Then,*

(1) $h^i(X, \mathcal{F} \otimes \mathcal{O}_X(mD)) = O(m^{\dim X})$, for all i.

(2) If D is nef and $i > 0$, then
$h^i(X, \mathcal{F} \otimes \mathcal{O}_X(mD)) = O(m^{\dim X - 1})$.
(Here, $h^i(X, \mathcal{F}) = \dim H^i(X, \mathcal{F})$.)

(3) $h^0(X, \mathcal{F} \otimes \mathcal{O}_X(mD)) = \frac{D^n}{n!} m^n + O(m^{n-1})$, where $n = \dim X$.

Proof. By HRR, (2) $\Longrightarrow$ (3).

(1) We achieve a reduction. First, every coherent sheaf, $\mathcal{F}$, possesses a finite filtration

$$\mathcal{F} = \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \cdots \supseteq \mathcal{F}_r = (0)$$

in which the successive quotients $\mathcal{F}_j/\mathcal{F}_{j+1}$ have support on an integral subscheme of X and are torsion-free there. An obvious induction on r gets us to the case where X is integral and torsion-free. Matsukata proved that such a sheaf, $\mathcal{F}$, when restricted to a suitable dense open, U, of X is actually free, say $\mathcal{O}_U^r$. So,

$$\mathcal{F} \upharpoonright U = \mathcal{F} \otimes_{O_X} \mathcal{O}_U \underset{\theta}{\widetilde{\longrightarrow}} \mathcal{O}_U^r.$$

The choice of θ is equivalent to giving an embedding $\mathcal{F} \hookrightarrow K(X)^r$. Look at $\mathcal{G} = \mathcal{F} \cap \mathcal{O}_X^r$ (inside $K(X)^r$). We have the two exact sequences

$$0 \longrightarrow \mathcal{G} \stackrel{i}{\longrightarrow} \mathcal{F} \longrightarrow \mathcal{G}_1 \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{G} \stackrel{j}{\longrightarrow} \mathcal{O}_X^r \longrightarrow \mathcal{G}_2 \longrightarrow 0.$$

Since i is an isomorphism on U, we deduce that $\operatorname{supp} \mathcal{G}_l$ is a proper closed subset of X and so, $\dim \operatorname{supp} \mathcal{G}_l < \dim X$, for $l = 1, 2$. If we use induction on $n = \dim X$, then the dimensions of the cohomology vector spaces of the $\mathcal{G}_l$ grow at most like $O(m^{n-1})$. Therefore, the dimension of the cohomology of $\mathcal{F}$ grows like that of $\mathcal{G}$ which, in turn, grows like the dimension of $\mathcal{O}_X^r$ and as r is fixed, the latter grows like the dimension of the cohomology of $\mathcal{O}_X$. So, we are reduced to the case $X = \mathcal{O}_X$ with X integral.

Look at

$$\mathfrak{I}_1 = \mathcal{O}_X(-D) \cap \mathcal{O}_X \quad \text{and} \quad \mathfrak{I}_2 = \mathcal{O}_X(D) \cap \mathcal{O}_X,$$

two coherent ideals of $\mathcal{O}_X$. Let Y_i be the subscheme of X cut out by $\mathfrak{I}_i$. Note, $\mathfrak{I}_1(D) = \mathfrak{I}_2$. We may assume $Y_1, Y_2 \neq X$ (else, the argument is easier). Consider

$$0 \longrightarrow \mathfrak{I}_1(mD) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_{Y_1}(mD) \longrightarrow 0$$

and

$$0 \longrightarrow \mathfrak{I}_2((m-1)D) \longrightarrow \mathcal{O}_X((m-1)D) \longrightarrow \mathcal{O}_{Y_2}((m-1)D) \longrightarrow 0,$$

which are exact (and $\mathfrak{I}_1(mD) = \mathfrak{I}_2((m-1)D)$). We will use induction on $n = \dim X$. Apply cohomology to both sequences. We get exact sequences

$$\cdots \longrightarrow H^i(X, \mathfrak{I}_1(mD)) \longrightarrow H^i(X, \mathcal{O}_X(mD)) \longrightarrow H^i(Y_1, \mathcal{O}_{Y_1}(mD)) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow H^i(X, \mathfrak{I}_2((m-1)D)) \longrightarrow H^i(X, \mathcal{O}_X((m-1)D)) \longrightarrow H^i(Y_2, \mathcal{O}_{Y_2}((m-1)D)) \longrightarrow \cdots,$$

Consequently,

$$\begin{aligned} h^i(X, \mathcal{O}_X(mD)) &\leq h^i(X, \mathfrak{I}_1(mD)) + h^i(Y_1, \mathcal{O}_{Y_1}(mD)) \\ &\leq h^i(X, \mathfrak{I}_2((m-1)D)) + O(m^{n-1}) \end{aligned}$$

and

$$\begin{aligned} h^i(X, \mathfrak{I}_2(mD)) &\leq h^i(X, \mathcal{O}_X((m-1)D)) + h^{i-1}(Y_2, \mathcal{O}_{Y_2}((m-1)D)) \\ &\leq h^i(X, \mathcal{O}_X((m-1)D)) + O(m^{n-1}). \end{aligned}$$

Therefore,

$$h^i(X, \mathcal{O}_X(mD)) \leq h^i(X, \mathcal{O}_X((m-1)D)) + O(m^{n-1}),$$

that is

$$h^i(X, \mathcal{O}_X(mD)) - h^i(X, \mathcal{O}_X((m-1)D)) \leq O(m^{n-1}).$$

If we write all these inequalities for $j = 1, \ldots, i$ and add them up, we get

$$h^i(X, \mathcal{O}_X(mD)) = mO(m^{n-1}) = O(m^n),$$

establishing (1).

(2) Again, this case reduces to $X = \mathcal{O}_X$ with X integral but now, D is nef. We use induction on $\dim X$. If $i \geq 2$, we can repeat the entire argument (word for word, mutatis mutandis). Consequently

$$h^i(X, \mathcal{O}_X(mD)) = O(m^{n-1}), \quad i \geq 2.$$

Look at $\chi(X, \mathcal{O}_X(mD))$. Using the case $i \geq 2$, it is of the form

$$h^0(X, \mathcal{O}_X(mD)) - h^1(X, \mathcal{O}_X(mD)) + O(m^{n-1}).$$

By HRR, it is also of the form

$$\frac{D^n}{n!} m^n + O(m^{n-1}).$$

There are two cases:

(1) $h^0(X, \mathcal{O}_X(mD)) = (0)$ (all m). In this case,

$$-h^1(X, \mathcal{O}_X(mD)) = \frac{D^n}{n!} m^n + O(m^{n-1}).$$

If $m >> 0$, we have $D^n \geq 0$ as D is nef, so both sides must be zero. Therefore, $D^n = 0$ and $h^1(X, \mathcal{O}_X(mD)) = 0 = O(m^{n-1})$.

(2) There is some m_0 such that $h^0(X, \mathcal{O}_X(m_0 D)) \neq (0)$. In this case, there exists an effective divisor, E, with $E \equiv m_0 D$ and $\dim \operatorname{supp} E < \dim X$ and

$$0 \longrightarrow \mathcal{O}_X(-m_0 D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_E \longrightarrow 0$$

is exact. It follows that

$$0 \longrightarrow \mathcal{O}_X((m - m_0)D) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_E(mD) \longrightarrow 0$$

is exact. Consequently,

$$\begin{aligned} h^1(X, mD) &\leq h^1(E, \mathcal{O}_E(mD)) + h^1(X, (m - m_0)D) \\ &\leq O(m^{n-2}) + h^1(X, (m - m_0)D) \end{aligned}$$

(since $\dim E \leq \dim D$ and D is nef). We get

$$h^1(X, mD) - h^1(X, (m - m_0)D) = O(m^{n-2}).$$

Write all these inequalities for $m, m - m_0, m - 2m_0, \ldots$ and add them up. We get

$$h^1(mD) = O(m^{n-1}),$$

as claimed. $\square$

Corollary 1.8 *Let X be a projective variety and let D be a $\mathbb{Q}$-Cartier, $\mathbb{Q}$-divisor which is nef and big. Then, there exists an effective $\mathbb{Q}$-divisor, E_0, so that for all $t \in \mathbb{Q}$, with $0 < t < 1$, there is some ample divisor, $H(t)$, with*

$$D = H(t) + tE_0.$$

Proof. We may assume that D is an $\mathbb{Z}$-divisor. Let H be the embedding divisor in X, which is ample, then we have the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-H) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_H \longrightarrow 0.$$

By tensoring with $\mathcal{O}_X(mD)$, we get

$$0 \longrightarrow \mathcal{O}_X(mD - H) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_H(mD) \longrightarrow 0$$

is exact. By Theorem 1.7(1), it follows that

$$h^0(H, \mathcal{O}_X(mD)) = O(m^{n-1}),$$

with $n = \dim X$. As D is nef and big we have

(a) $\chi(X, \mathcal{O}_X(mD)) > Km^n$ (as D is big) and

(b) $h^0(X, \mathcal{O}_X(mD))$ grows like $\chi(X, \mathcal{O}_X(mD))$ (by Theorem 1.7(2), as D is nef).

Therefore, if $m >> 0$, then $h^0(H, \mathcal{O}_X(mD - H)) \neq (0)$. Let E be effective with $E \equiv mD - H$. Now,

$$D = (1-t)D + tD = \left[(1-t)D + \frac{t}{m}H\right] + t\left(\frac{1}{m}E\right).$$

If we set $E_0 = \frac{1}{m}E$, then we have an effective $\mathbb{Q}$-divisor and as $t > 0$, $\frac{1}{m}H$ is ample. Also $(1-t)D$ is nef because D is. Consequently,

$$(1-t)D + \frac{t}{m}H = H(t)$$

is ample and $D = H(t) + tE_0$, as required. $\square$

Say $\pi\colon X \to Y$ is a proper morphism. Notice that π contracts a curve, C, iff $\pi_*(C) = 0$ and $\pi_*(C)$ is a numerical criterion, by nondegeneracy of our pairing. Write $\mathrm{NE}(\pi)$ for the convex subcone of $\mathrm{NE}(X)$ generated by the curves contracted by π. Clearly,

$$\mathrm{NE}(\pi) = \mathrm{NE}(X) \cap \mathrm{Ker}\ \pi_*,$$

so $\mathrm{NE}(\pi)$ is a closed convex subcone of $\mathrm{NE}(X)$.

For which π does $\mathrm{NE}(\pi)$ provide information determining or quasi-determining π?

Claim: No chance unless the fibres of π are connected.

First, we claim that if $\pi_*\mathcal{O}_X = \mathcal{O}_Y$, then the fibres of π are connected (see Hartshorne's book). The converse is "almost true". Assume characteristic 0 and Y normal. If the fibres are connected, then $\pi_*\mathcal{O}_X = \mathcal{O}_Y$. Make the Stein factorization. For this, note that $\pi_*\mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-module and an $\mathcal{O}_Y$-algebra. So, we can make $\widetilde{Y} = \mathcal{S}pec\, \pi_*\mathcal{O}_X$ and there is a factorization

$$X \xrightarrow{\pi'} \widetilde{Y} \xrightarrow{g} Y$$

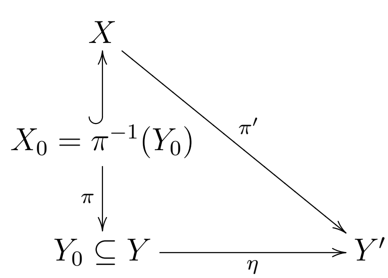
of π (the Stein factorization). Now, as $\pi'_*\mathcal{O}_X = \mathcal{O}_{\widetilde{Y}}$, the fibres of π' are connected (by the previous argument). But, g is a finite morphism.

Claim: g is an isomorphism.

We have $\deg g = 1$ at the general point, i.e., X and Y birational and g is bijective. But, for any open affine, $U \subseteq Y$, $H^0(g^{-1}(U), \mathcal{O}_{\widetilde{Y}})$ is a finite $H^0(U, \mathcal{O}_Y)$-module and $K(Y) = (K(\widetilde{Y}))$ algebra. By normality, $H^0(g^{-1}(U), \mathcal{O}_{\widetilde{Y}}) = H^0(U, \mathcal{O}_Y)$. Therefore, g is an isomorphism. As g contracts no curves, π contracts C iff π' contracts C.

Theorem 1.9 *Say X, Y, Y' are proper schemes and $\pi\colon X \to Y$ and $\pi'\colon X \to Y'$ are morphisms. Assume $\pi_*\mathcal{O}_X = \mathcal{O}_Y$.*

(a) Say there exists $y_0 \in Y$ such that π' contracts $\pi^{-1}(y_0)$. Then, there exists an open $Y_0 \ni y_0$ and a morphism, $\eta\colon Y_0 \to Y'$, so that the diagram



commutes: $\pi' \restriction X_0$ factors through π (by η).

(b) If every fibre of π is contracted by π', then π' factors through π.

Proof. Let $\alpha\colon X \to Y \prod Y'$ be the morphism (π, π') (with $(\alpha(x) = (\pi(x), \pi'(x)))$. Since α is proper, $\mathrm{Im}\,\alpha = Z$ is closed in $Y \prod Y'$. Because $\pi_* \mathcal{O}_X = \mathcal{O}_Y$, π is surjective. (If $U \subseteq Y$ is open, then $\mathcal{O}_X(\pi^{-1}(U)) = \mathcal{O}_Y(U) \neq (0)$ implies $\pi^{-1}(U) \neq \emptyset$.) Let $p = pr_1 \restriction Z$ and $q = pr_2 \restriction X$.

$$\begin{array}{ccc} X & \xrightarrow{\ \alpha\ } & Z \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle p} & \\ & Y & \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\ \alpha\ } & Z \\ & {\scriptstyle\pi'}\searrow \quad \swarrow{\scriptstyle q} & \\ & Y' & \end{array}$$

Now, $\pi^{-1}(y_0) \subseteq \pi'^{-1}(*)$, for some $* \in Y'$. Therefore, α contracts $\pi^{-1}(y_0)$. As $\pi^{-1}(y_0) = \alpha^{-1}(p^{-1}(y_0))$ and α contracts the left-hand side, we see that $p^{-1}(y_0)$ is a single point. Now, the locus of points in Y where p^{-1} blows things up is Zariski closed and $\neq Y$ as y_0 does not belong to this locus. So, there is some open Y_0, with $y_0 \in Y_0$ and $p\colon p^{-1}(Y_0) \to Y_0$ is a finite morphism. Write $Z_0 = p^{-1}(Y_0)$ and $X_0 = \pi^{-1}(Y_0)$. Observe that if we can prove that

$$\mathcal{O}_{Z_0} \subseteq \alpha_* \mathcal{O}_{X_0}$$

then we will have

$$\mathcal{O}_{Y_0} \subseteq p_* \mathcal{O}_{Z_0} \subseteq p_* \alpha_* \mathcal{O}_{X_0} = \pi_* \mathcal{O}_{X_0} = \mathcal{O}_{Y_0}$$

and so, $p_* \mathcal{O}_{Z_0} = \mathcal{O}_{Y_0}$. However, $\mathcal{O}_{Z_0} \subseteq \alpha_* \mathcal{O}_{X_0}$ holds because α is surjective and Z_0 is open in Z, the image of X. Consequently, p is a finite morphism on Z_0 and $p_* \mathcal{O}_{Z_0} = \mathcal{O}_{Y_0}$. So, the factorization is

$$\begin{array}{ccccc} Y_0 & \underset{p^{-1}}{\overset{\cong}{\longrightarrow}} & Z_0 & \xrightarrow[q\restriction Z_0]{} & Y' \\ & {\scriptstyle\pi}\nwarrow & & \nearrow{\scriptstyle\pi'} & \\ & & X_0 & & \end{array}$$

Observe that η is unique.

For (b), cover Y by these opens and get a morphism, p, finite over all of Y. Then, repeat the above by replacing Y_0 by Y. $\square$

Recall that a convex subcone, $\widetilde{\Gamma}$, of a cone, Γ, is *extremal* iff $\frac{\alpha+\beta}{2} \in \widetilde{\Gamma}$ implies that $\alpha, \beta \in \widetilde{\Gamma}$. This means that Γ lies in one of the two (closed) half spaces determined by any hyperplane containing $\widetilde{\Gamma}$.

Lemma 1.10 *(Hironaka's Lemma) Say X, Y, Y' are projective varieties and $\pi\colon X \to Y$ and $\pi'\colon X \to Y'$ are morphisms.*

(1) The subcone $\mathrm{NE}(\pi)$ *is always extremal in* $\mathrm{NE}(X)$.

(2) If $\pi_* \mathcal{O}_X = \mathcal{O}_Y$ and if $\mathrm{NE}(\pi) \subseteq \mathrm{NE}(\pi')$, *then there exists a unique morphism, $\eta\colon Y \to Y'$, so that π' factors through π via η.*

(3) If $\pi_* \mathcal{O}_X = \mathcal{O}_Y$, then the morphism π is uniquely determined by $\mathrm{NE}(\pi)$ *(up to isomorphism).*

Proof. (1) Let $\alpha = \sum_i a_i A_i$ and $\beta = \sum_j b_j B_j$ be two members of $\mathrm{NE}(\pi)$, with $a_i, b_j \geq 0$ and say that $\frac{\alpha+\beta}{2} \in \mathrm{NE}(\pi)$. Then, $\alpha + \beta = \sum_k d_k D_K$, with $d_k \geq 0$ and $\pi(D_k) = \mathrm{point}_k$. So,

$$\pi_*\Big(\sum_i a_i A_i + \sum_j b_j B_j\Big) = 0 \quad \text{in} \quad N_1(Y)_{\mathbb{R}},$$

that is,

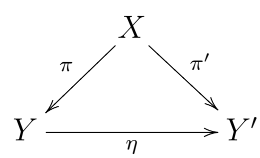
$$\sum_i a_i \pi_*(A_i) + \sum_j b_j \pi_*(B_j) = 0 \quad \text{in} \quad N_1(Y)_{\mathbb{R}}.$$

Assume that B_{j_0} is not contracted, that is, $\pi_* B_{j_0}$ is a curve in Y. As Y is projective, there is a some hyperplane, H, with $H \cdot \pi_* B_{j_0} > 0$ (here, we may assume $b_{j_0} > 0$). But, $A_i \cdot H \geq 0$ and $B_j \cdot H \geq 0$, for all i, j, a contradiction. Therefore, all the A_i and B_j are contracted, as required.

(2) As $\pi_* \mathcal{O}_X = \mathcal{O}_Y$, the morphism π is surjective and so, the fibres of π are connected.

Claim: Every fibre of π is contracted by π'.

Pick p and q in any fibre of π. As $\pi^{-1}(\text{point})$ is projective, p and q may be connected by a chain of curves. Each curve is in the same fibre, hence contracted by π and (by hypothesis) contracted by π'. We conclude that $\pi'(p) = \pi'(q)$. Therefore, $\pi(\text{fibre of } \pi) =$ a point and by the rigidity lemma, there is a unique $\eta\colon Y \to Y'$ such that the following diagram commutes:



(3) Given two morphisms π and π' with $\mathrm{NE}(\pi) = \mathrm{NE}(\pi')$, by applying (2) we get $\eta\colon Y \to Y'$ and $\xi\colon Y' \to Y$ with $\eta \circ \xi$ and $\xi \circ \eta$, two morphisms besides $\mathrm{id}_{Y'}$ and id_Y and so, $\eta \circ \xi = \mathrm{id}_{Y'}$ and $\xi \circ \eta = \mathrm{id}_Y$, as required. □

Mori's program has roughly two goals:

(1) Give a geometric condition under which an extremal subcone, E, gives a contracting morphism, π ($E = \mathrm{NE}(\pi)$).

(2) Show that after finitely many contractions, you have a "minimal model" and it is reasonably simple.

Examples.

(1) The case where $N_1(X)_{\mathbb{R}}$ is one-dimensional. If so, $X = \mathbb{P}^r$ and $N^1(X)_{\mathbb{Z}}$ is generated by the hyperplane, H. It follows that $N^1(X)_{\mathbb{R}} \cong \mathbb{R}$ and so, $N_1(X)_{\mathbb{R}} \cong \mathbb{R}$ and $\mathrm{NE}(X) = \mathbb{R}_{\geq 0} = \overline{\mathrm{NE}(X)}$. The two extremal subcones are (0) and $\mathbb{R}_{\geq 0}$. In the first case, π is the constant morphism, $\pi\colon \mathbb{P}^r \to \mathrm{pt}$ and in the second case the identity, $\pi = \mathrm{id}\colon \mathbb{P}^r \longrightarrow \mathbb{P}^r$.

(2) $X = \mathbb{P}^r \prod \mathbb{P}^r$. In this case, $N^1(X)_{\mathbb{R}} \cong \mathbb{R} \amalg \mathbb{R}$ and so, $N_1(X)_{\mathbb{R}} \cong \mathbb{R} \amalg \mathbb{R}$. There are four extremal subcones:

(a) (0), which corresponds to id.

(b) $\mathbb{R} \amalg \mathbb{R}$, in which case π contracts all points to a point.

(c) $\mathbb{R}$ (first component), in which case $\mathbb{P}^r \amalg \mathbb{P}^s \xrightarrow{pr_2} \mathbb{P}^s$.

(d) $\mathbb{R}$ (second component), in which case $\mathbb{P}^r \amalg \mathbb{P}^s \xrightarrow{pr_1} \mathbb{P}^r$.

(3) A ruled surface, $X = \mathbb{P}(E)$, where E is a rank 2 vector bundle over C, where C is a smooth projective curve. In other words, X is a $\mathbb{P}^1$ bundle over C (with group PGL(1)). By Tsen's Theorem, there exists a section, σ. The main point is this:

Proposition 1.11 *If $X = \mathbb{P}(E)$ is a ruled surface, where E is a rank 2 vector bundle over a smooth projective curve, C, then there is a one-to-one correspondence between sections, σ, of $\pi\colon X \to C$ and exact sequences*

$$0 \longrightarrow \ker \longrightarrow \mathcal{O}_C(E) \longrightarrow \mathcal{L} \longrightarrow 0$$

where $\mathcal{L}$ is a line bundle over C (= rank 1, locally free $\mathcal{O}_C$-module). In this correspondence, $\mathcal{L} = \sigma^*\mathcal{O}_X(1)$ and $\ker \cong \pi_*(\mathcal{O}_X(-C_0) \otimes \mathcal{O}_X(1))$, where $C_0 = \sigma(C)$. Also, $\mathcal{O}_X(-C_0) \otimes \mathcal{O}_X(1) = \pi^*(\ker)$.

Proof. The functorial definition of $\mathbb{P}(E)$ says that the section, $\sigma\colon C \to X$, corresponds to our surjection, $\mathcal{O}_C(E) \longrightarrow \mathcal{L} = \sigma^*\mathcal{O}_X(1)$, where $\mathcal{L}$ is a rank 1 locally free bundle (because $C = \mathbb{P}(\mathcal{L})$). Let $C_0 = \sigma(C)$, then

$$0 \longrightarrow \mathcal{O}_X(-C_0) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{C_0} \longrightarrow 0$$

is exact. Twist by $\mathcal{O}_X(1)$ to get

$$0 \longrightarrow \mathcal{O}_X(-C_0) \otimes \mathcal{O}_X(1) \longrightarrow \mathcal{O}_X(1) \longrightarrow \mathcal{O}_{C_0}(1) \longrightarrow 0$$

is exact. If we apply π_*, we get

$$0 \longrightarrow \pi_*(\mathcal{O}_X(-C_0) \otimes \mathcal{O}_X(1)) \longrightarrow \mathcal{O}_C(E) \longrightarrow \pi_*\mathcal{O}_{C_0}(1) \longrightarrow R^1\pi_*(\mathcal{O}_X(-C_0) \otimes \mathcal{O}_X(1)).$$

The following hold:

(a) On C_0, π and σ are inverse. Therefore, $\pi_* = \sigma^*$ on C_0 and so, $\mathcal{L} = \pi_*\mathcal{O}_{C_0}(1)$.

(b) $R^1\pi_*(\mathcal{O}_X(-C_0) \otimes \mathcal{O}_X(1)) = (0)$.

On each fibre, $\pi^{-1}(c) = F = \mathbb{P}^1$, $\mathcal{O}_X(-C_0) \otimes \mathcal{O}_X(1)$ is just $\mathcal{O}_{\mathbb{P}^1}(-C_0 \cdot F + \Delta)$, where Δ is the divisor induced on F by $\mathcal{O}_X(1)$. As $\deg \Delta > 0$ and $C_0 \cdot F = 1$, we deduce that the degree of $\mathcal{O}_X(-C_0) \otimes \mathcal{O}_X(1)$ on F is non-negative and independent of F. As

$$H^1(F, \mathcal{O}_X(-C_0) \otimes \mathcal{O}_X(1)) = (0),$$

for every c, we have

$$H^1(F, (\mathcal{O}_X(-C_0) \otimes \mathcal{O}_X(1))_c) = (0).$$

But the above is just

$$\overline{R^1\pi_*(\mathcal{O}_X(-C_0) \otimes \mathcal{O}_X(1))_c} \otimes \kappa(c)$$

(by the formal functions Theorem) and, by Nakayama and denseness, we get $R^1\pi_*(\mathcal{O}_X(-C_0) \otimes \mathcal{O}_X(1)) = (0)$. Therefore,

$$\ker = \pi_*(\mathcal{O}_X(-C_0) \otimes \mathcal{O}_X(1)).$$

Let us abbreviate $\mathcal{O}_X(-C_0) \otimes \mathcal{O}_X(1)$ as $\mathfrak{m}$. We know that $\mathfrak{m} \cdot F$ $(= \deg(\mathfrak{m} \restriction F))$ = constant ≥ 0 and so, $H^0(\pi^{-1}(c), \mathfrak{m} \cdot \pi^{-1}(c))$ has dimension $= \deg + 1$ (by RR on $\pi^{-1}(c)$). Grauert's Theorem implies that $\pi_*\mathfrak{m}$ is locally free of rank $\dim H^0 = \deg + 1$. But the rank is equal to 1 and thus, $\deg = 0$ and $\mathfrak{m} = \pi^*(\text{divisor}) = \pi^*(\pi_*\mathfrak{m})$. $\square$

If E is a bundle on C and if we twist by $\mathcal{O}_C(D)$, we have

$$\begin{aligned} c_1(E \otimes \mathcal{O}_C(D)) &= c_1\left(\bigwedge^2 (E \otimes \mathcal{O}_C(D))\right) \\ &= c_1(E) + 2c_1(D) \\ &= c_1(E) + 2\deg D. \end{aligned}$$

Consequently, we can adjust E by tensoring with a line bundle so that

(a) $H^0(C, \mathcal{O}_C(E)) \neq (0)$, yet

(b) $H^0(C, \mathcal{O}_C(E) \otimes M) \neq (0)$ if $\deg M < 0$.

We have $X = \mathbb{P}(E) = \mathbb{P}(E \otimes M)$ and therefore, we may assume (a) and (b). Such an E is said to be "normalized".

Say E is a normalized bundle, then there is a nonzero section, $s \in H^0(C, \mathcal{O}_C(E))$, and this s gives an exact sequence

$$0 \longrightarrow \mathcal{O}_C \stackrel{s}{\longrightarrow} \mathcal{O}_C(E) \longrightarrow \mathcal{L} \longrightarrow 0.$$

Claim: $\mathcal{L}$ is a line bundle on C.

We need only check $\mathcal{L}$ is torsion-free as C is a smooth curve. Let $T = \text{torsion}(\mathcal{L})$, and pull back T to $\mathcal{O}_C(E)$; let $\mathcal{F}$ be the corresponding subsheaf of $\mathcal{O}_C(E)$. Now, as $\mathcal{O}_C(E)$ is torsion-free, $\mathcal{F}$ must be torsion-free and so, $\mathcal{F}$ is a bundle. But, if $\mathcal{F}$ is a line bundle, it contains $\mathcal{O}_C$ and $\mathcal{F} \neq \mathcal{O}_C$, else $T = (0)$. Therefore, $\deg \mathcal{F} > 0$. As a consequence, $E \otimes \mathcal{F}^{-1}$ has a section and yet, $\deg \mathcal{F}^{-1} < 0$, contradicting (b) and proving the Claim.

Now, $\mathcal{O}_C = \ker = \pi_*(\mathcal{O}_X(-C_0) \otimes \mathcal{O}_X(1))$ implies that $\mathcal{O}_X = \mathcal{O}_X(-C_0) \otimes \mathcal{O}_X(1)$ and for this s, $\mathcal{O}_X(C_0) = \mathcal{O}_X(1)$. We have the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C_0) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{C_0} \longrightarrow 0$$

and if we tensor it with $\mathcal{O}_{C_0}$, we get

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(1) \longrightarrow \mathcal{O}_{C_0}(C_0^2) \longrightarrow 0.$$

If we push it down by π_*, we get

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C(E) \longrightarrow \mathcal{O}_{C_0}(C_0^2) \longrightarrow 0.$$

Also recall that $c_1(E) = \deg \bigwedge^2 E = C_0^2$. Define

$$-e = \deg \bigwedge^2 E = C_0^2.$$

This is an invariant of X. Now, on X, $\mathrm{Num}(X)$ is free of rank 2 and the class of $\mathcal{O}_X(1)(= C_0)$ and the class of F are a basis, so $K_X = \alpha F + \beta C_0$.

The adjunction formula says that

$$\begin{aligned} \deg K_F &= F \cdot (K_X + F) \\ -2 &= F \cdot K_X + F^2 \\ -2 &= F \cdot K_X = \beta. \end{aligned}$$

Thus, $\beta = -2$. Furthermore,

$$\begin{aligned} \deg K_{C_0} &= C_0 \cdot (C_0 + K_X) \\ 2g-2 &= C_0^2 + C_0 \cdot (-2C_0 + \alpha F) \\ 2g-2 &= -C_0^2 + \alpha \\ 2g-2 &= e + \alpha, \end{aligned}$$

so $\alpha = 2g - 2 - e$. Consequently,

$$K_X = -2C_0 + (2g-2-e)F.$$

We check that

$$K_X^2 = 4C_0^2 - 4(2g-2-e) = 8(1-g).$$

Also

$$\begin{aligned} c_2(X) &= \chi_{\mathrm{top}}(X) = \chi_{\mathrm{top}}(F)\chi_{\mathrm{top}}(C) \\ &= 2(2-2g) \\ &= 4(1-g) \end{aligned}$$

and

$$\frac{1}{12}(K_X^2 + c_2) = \mathrm{Td}(X) = 1 - g.$$

Now, look at the Leray spectral sequence

$$H^p(C, R^q\pi_*\mathcal{O}_X) \Longrightarrow H^\bullet(X, \mathcal{O}_X).$$

We have

$$(\widehat{R^q\pi_*\mathcal{O}_X})_c \otimes \kappa(c) = H^q(\pi^{-1}(c), \mathcal{O}_X \restriction \pi^{-1}(c)) = \begin{cases} \mathbb{C} & \text{if } q = 0 \\ (0) & \text{if } q > 0. \end{cases}$$

Therefore,

$$R^q\pi_*\mathcal{O}_X = \begin{cases} \mathcal{O}_C & \text{if } q = 0 \\ (0) & \text{if } q > 0. \end{cases}$$

Consequently,

$$H^p(C, \mathcal{O}_C) \cong H^p(X, \mathcal{O}_X) \quad \text{for all} \quad p \geq 0,$$

from the Leray SS. So,

$$\begin{aligned} H^0(C, \mathcal{O}_C) &= \mathbb{C} \\ H^1(C, \mathcal{O}_C) &= \mathbb{C}^g \qquad g = \text{genus}\, C \\ H^p(C, \mathcal{O}_C) &= (0), \qquad p \geq 2 \end{aligned}$$

and

$$\begin{aligned} \dim H^0(C, \mathcal{O}_C) &= 1 \\ \dim H^1(C, \mathcal{O}_C) &= q = g \\ \dim H^2(C, \mathcal{O}_C) &= p_g = 0. \end{aligned}$$

So, HRR checks. We know that $H^0(C, \mathcal{O}_X) \neq (0)$, yet $H^0(C, \mathcal{O}_C \otimes \mathcal{O}_C(M)) = (0)$ if $\deg M < 0$.

Take M with $\deg M = -1$. The sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C(E) \longrightarrow \mathcal{O}_C(C_0^2) \longrightarrow 0$$

is exact and if we twist with $\mathcal{O}_C(M)$, we get

$$0 \longrightarrow \mathcal{O}_C(M) \longrightarrow \mathcal{O}_C(E) \otimes \mathcal{O}_C(M) \longrightarrow \mathcal{O}_C(C_0^2) \otimes \mathcal{O}_C(M) \longrightarrow 0.$$

If we apply cohomology, we get

$$0 \longrightarrow H^0(C, \mathcal{O}_C(C_0^2) \otimes \mathcal{O}_C(M)) \longrightarrow H^1(C, \mathcal{O}_C(M)).$$

By Riemann-Roch on C

$$-h^1(\mathcal{O}_C(M)) = -1 + 1 - g = -g,$$

that is, $g = h^1(\mathcal{O}_C(M))$, which implies $h^0(\mathcal{O}_C(C_0^2) \otimes \mathcal{O}_C(M)) \geq g$. By Riemann-Roch on C,

$$h^0(\mathcal{O}_C(C_0^2) \otimes \mathcal{O}_C(M)) \geq C_0^2 - 1 + 1 - g = C_0^2 - g.$$

Therefore, $g \geq c_0^2 - g$, that is, $2g \geq C_0^2 = -e$, namely

$$e \geq -2g.$$

(Actually, Nagata, 1960, showed $e \geq -g$.)

Say X is just a surface and look on the divisor side. We have $\mathrm{Amp}(X) \subseteq \mathrm{NE}(X)$ and so,

(1) $\mathrm{nef}(X) = \overline{\mathrm{Amp}(X)} \subseteq \overline{\mathrm{NE}(X)}$.

Say Γ is an irreducible curve on X and $\Gamma^2 = 0$. Pick an effective "curve", $\widetilde{C}$ (really, a 0-cycle) on X. Either Γ is an irreducible component of $\widetilde{C}$ or not. If not, $\Gamma \cdot \widetilde{C} \geq 0$. Let

$$\overline{\mathrm{NE}(X)}_{\Gamma \geq 0} = \{\widetilde{C} \in \overline{\mathrm{NE}(X)} \mid \Gamma \cdot \widetilde{C} \geq 0\}.$$

Then, we have

(2a) $\overline{\mathrm{NE}(X)}$ = the cone spanned by Γ and $\overline{\mathrm{NE}(X)}_{\Gamma \geq 0}$ and

(2b) Γ is the boundary of $\overline{\mathrm{NE}(X)}$.

(2c) If $\Gamma^2 < 0$, then Γ is extremal.

Back to ruled surfaces. The group $\mathrm{Num}(X)$ is generated by $\mathcal{O}_X(1)$ and F and we know that $F^2 = 0$ and F is nef. It follows that F is on the boundary of $\overline{\mathrm{NE}(X)}$.

Use the class, ξ, of $\mathcal{O}_X(1)$ and the class, f, of F as a basis (f as abscissae and ξ as ordinate). Then we have a bijection, $\mathrm{Num}(X)_{\mathbb{R}} \longrightarrow \mathbb{R}^2$. Vectors with $y = 0$ and $x \geq 0$ are one boundary of $\overline{\mathrm{NE}(X)}$. To find the other boundary of $\overline{\mathrm{NE}(X)}$ (and $\mathrm{Nef}(X)$) we need information about E. This is a question of "stability" for vector bundles on a curve, C.

Definition 1.2 Let E be a vector bundle of rank r on our curve, C. We say that E is *unstable* on C iff E possesses a subbundle, F, so that

$$\mu(F) = \frac{\deg F}{\mathrm{rk}\, F} > \mu(E) = \frac{\deg E}{\mathrm{rk}\, E}.$$

The vb E is *semi-stable* if it is not unstable, that is, for all F as above,

$$\mu(F) \leq \mu(E)$$

and E is *stable* iff for all F as above

$$\mu(F) < \mu(E).$$

If

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

is an exact sequence of bundles on C, then we have

$$\mu(F) \leq \mu(E) \quad \text{iff} \quad \mu(G) \geq \mu(E)$$

and

$$\mu(F) < \mu(E) \quad \text{iff} \quad \mu(G) > \mu(E).$$

Let X be a ruled surface and take $X = \mathbb{P}(E)$, so that $\deg E \equiv 0\,(2)$. Then, normalize E, for our purposes, so that $\deg E = 0$.

Case (A). E is unstable (*e.g.*, $E = \mathcal{O}_C(2) \amalg \mathcal{O}_C(-2)$). Here,

$$\mu(E) = \frac{\deg E}{2} = 0.$$

Unstability means that there is some line subbundle, F, with $\mu(F) = \deg F > \mu(E) = 0$. Note that $\mu(E/F = G) < 0$. We have the exact sequence

$$0 \longrightarrow \mathcal{O}_X(F) \longrightarrow \mathcal{O}_X(E) \longrightarrow \mathcal{L} = \mathcal{O}_X(G) \longrightarrow 0$$

and on X, we have our C_0, corresponding to the above exact sequence, with $C_0^2 = \deg \mathcal{L} = \deg G < 0$. Here, C_0 plays the role of Γ and so, C_0 is an extremal ray in $\overline{\mathrm{NE}}(X)$. This ray must be our other boundary.

As E is unstable, there is a quotient, L, of E with $\deg L < 0$ and we have an exact sequence

$$0 \longrightarrow \ker \longrightarrow E \longrightarrow L \longrightarrow 0,$$

so L corresponds to a section, D, of $\pi\colon \mathbb{P}(E) \to C$, and $D = \alpha f + \beta\xi$. But, $D \cdot f = 1$, so $\beta = 1$ and $D = \alpha f + \xi$. It follows that $\alpha = D \cdot \xi = \deg L < 0$ and so, $\alpha < 0$.

Recall that

(1) $\mathrm{Nef}(X) \subseteq \overline{\mathrm{NE}(X)}$ and

(2) $\Gamma^2 \leq 0$ (Γ an irreducible curve) imply that

   (a) Γ and $\{C' \mid \Gamma \cdot C' \geq 0\}$ generate $\overline{\mathrm{NE}(X)}$.
   (b) Γ is on the boundary of $\overline{\mathrm{NE}(X)}$.

(3) $\Gamma^2 < 0$ implies Γ is extremal.

Since $D^2 = 2\alpha < 0$, we deduce that $\alpha f + \xi$ is extremal and on the boundary of $\overline{\mathrm{NE}(X)}$. Of course, F is an effective curve and the x-axis is another boundary of $\overline{\mathrm{NE}(X)}$.

What about $\mathrm{Nef}(X)$?

Then, $\Delta = \gamma f + \delta\xi$ is on $\partial\,\mathrm{Nef}(X)$ iff Δ is perpendicular to the boundary of $\overline{\mathrm{NE}(X)}$. Thus,

$\Delta \cdot f = 0$, which yields $\delta = 0$ (on the first boundary)

$\Delta \cdot (\alpha f + \xi) = 0$, which yields $\gamma + \delta\alpha = 0$ (on the second boundary), i.e., $\gamma = -\delta\alpha$.

Consequently,

$$\Delta = \delta(-\alpha f + \xi),$$

is on the boundary of $\mathrm{Nef}(X)$.

Case (B) E is semi-stable.

Since we are in characteristic 0, one finds all the bundles $S^m E$ are semi-stable ($m \geq 1$). Say A is some line bundle on C, with $\deg A = a$ and suppose that

$$H^0(C, S^m(E) \otimes_{\mathcal{O}_C} A) \neq (0)$$

for some m. A nonzero section corresponds to a map

$$0 \longrightarrow \mathcal{O}_C \longrightarrow S^m E \otimes A$$

and we get the exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow S^m E \otimes A \longrightarrow M \longrightarrow 0.$$

If we twist by A^D, we get

$$0 \longrightarrow A^D \longrightarrow S^m E \longrightarrow M \otimes_{\mathcal{O}_X} A^D \longrightarrow 0$$

is exact and semi-stability implies $\deg A^D \leq 0$. Thus $\deg A \geq 0$, that is, $a \geq 0$. Pick some irreducible curve, Γ, on X, then as a divisor, $\mathcal{O}_X(\Gamma) \sim \mathcal{O}_X(m) \otimes$ fibres, for some $m \geq 1$ and some fibres $= \pi^* A$. It follows that Γ is the zero divisor of a section, s, in $\mathcal{O}_X(m) \otimes \pi^* A$. But,

$$\pi^*(\mathcal{O}_X(m) \otimes \pi^* A) = S^m E \otimes A$$

and

$$\Gamma(C, S^m E \otimes A) = \Gamma(C, \pi_*(\mathcal{O}_X(m) \otimes \pi^* A)) = \Gamma(X, \mathcal{O}_X(m) \otimes \pi^* A).$$

Whenever $s \in \Gamma(X, \mathcal{O}_X(m) \otimes \pi^* A)$, we also have $s \in \Gamma(C, S^m E \otimes A)$, so $a \geq 0$, where $a = \deg A$. As $\Gamma = m\xi + af$, we deduce that Γ belongs to the first quadrant of the (f, ξ)-plane and $f = 0$ is still a boundary. Therefore, $\overline{\mathrm{NE}(X)}$ is equal to the first quadrant including its boundaries.

As $\mathrm{Nef}(X)$ = closure of $\mathrm{Amp}(X)$, we see that $\mathrm{Nef}(X)$ is also the first quadrant with its boundaries.

Question: Is the ξ-axis in $\mathrm{NE}(X)$? That is, does there exist Γ so that $\Gamma = m\xi$ for some m?

Here, we must have $a = 0$. This implies E and all the S^mE are semi-stable but not stable.

Narasimhan and Seshadri gave a characterization of *stable* bundles using representations of $\pi_1(C)$ and Hartshorne (AVB) used this to show if $g(C) \geq 2$, then there is some vector bundle, E, of rank 2 on C, semi-stable, so that

$$H^0(C, S^mE \otimes A) = (0)$$

for all $m \geq 1$, provided $\deg A \leq 0$. (Almost all E on the boundary of the moduli space of vb's work.) But, by the above, the ξ-axis is not given by any Γ and therfore in this case, $\mathrm{NE}(X) \neq \overline{\mathrm{NE}(X)}$.

Mumford's Example: Let X, E, V be as before ($\mathrm{NE}(X) \neq \overline{\mathrm{NE}(X)}$). Take D to be a divisor representing ξ. Then, $D \cdot Z > 0$ (with $Z \in \mathrm{NE}(X)$) and yet, $D \cdot D = 0$. We claim that D is not ample, as otherwise, by Kleiman, $D \cdot D > 0$, as $D \in \overline{\mathrm{NE}(X)}$. Therefore, in Nakai-Moshezon, we need to take D^n's, wrong otherwise.

1.2 The Kodaira & Akizuki-Nakano Vanishing Theorems–Part I. Coverings

First, we consider the easiest case: cyclic covers.

Proposition 1.12 *If X is affine and $s \in \mathbb{C}[X]$, with $s \not\equiv 0$, for any $m \geq 1$, there is a finite and flat morphism, $\pi\colon Y \to X$, and there is some $s' \in \mathbb{C}[Y]$, so that $(s')^m = \pi^* s$. Moreover, Y is ramified exactly along $(s)_0$.*

Proof. Make $X \prod \mathbb{A}^1$ and let t be the coordinate on $\mathbb{A}^1$. Look at Y = the locus of $t^m - \pi^* s = 0$ on $X \prod \mathbb{A}^1$ and take $\pi = pr_1 \restriction Y$. Then, set $s' = t \restriction Y$ to get $(s')^m = \pi^* s$; flatness is clear.

Proposition 1.13 *(Global case) Let X be an irreducible variety, L be a line bundle on X and $m \geq 1$ be any integer and let $s \in \Gamma(X, L^{\otimes m})$, with $s \not\equiv 0$. Then, there is an irreducible Y and a morphism, $\pi\colon Y \to X$, finite and flat, a section, $\sigma \in \Gamma(Y, \pi^* L)$, so that $\sigma^m = \pi^* s$ and if X is smooth then Y can be taken to be smooth. Moreover, if $D = (s)_0$, then π is an isomorphism, $(\sigma)_0 \xrightarrow{\sim} D$, and if D is smooth we can find σ with $(\sigma)_0$ smooth.*

Proof. (1) (a la Grothendieck) The result holds in the affine case. Since s is a section of an m^{th} power, these affine pieces glue. The rest of the statements are local computations.

(2) Another argument: Since L is a line bundle on X we can make

$$V(L) = \mathit{Spec}_{\mathcal{O}_X}(\mathrm{Sym}\, L^D),$$

the total space of $\mathbb{L}$ and let $p\colon V(L) \to X$. There is a tautological section of p^*L over $\mathbb{L}$. We need a section, σ, so that $\sigma(\xi) \in (p^*L)_\xi$, for all $\xi \in \mathbb{L}$. But, $(p^*L)_\xi = L_{p(\xi)}$ and $\xi \in \mathbb{L}$ so ξ is a pair

$$\xi = (p(\xi), \text{vector in } L_{p(\xi)})$$

and we can set $\sigma(\xi) =$ second component of ξ. Let T be the tautological section. Consequently, $T(\xi) = \xi$ itself. We need a map $\mathbb{L} \longrightarrow p^*L$. But, $p^*L = \mathbb{L} \otimes L$. Now, as everything is affine, we need a map

$$\mathrm{Sym}(L^D) \longrightarrow \mathrm{Sym}(L^D) \otimes_{\mathcal{O}_X} L,$$

that is, a map

$$\mathcal{O}_X \amalg L^D \amalg L^{D^2} \amalg \cdots \longrightarrow L \amalg \mathcal{O}_X \amalg L^D \amalg L^{D^2} \amalg \cdots .$$

The lefthand side is a summand of the righthand side so the desired map exists. (Our T is locally the t of the previous proposition.) In $\mathbb{L}$, look at the locus of $T^m - \pi^*s = 0$. This is Y and in Y we have

$$T^m = \pi^*s.$$

The rest of the statements are purely local. □

We will also need roots of bundles.

Theorem 1.14 *(Bloch-Gieseker Covers) Say X is a quasi-projective irreducible algebraic variety, $m \geq 1$ is an integer, and L is a line bundle on X. Then, there exists a finite flat morphism, $\pi\colon Y \to X$, with Y irreducible and a line bundle, N, on Y so that*

$$N^{\otimes m} \cong \pi^*L \qquad (\textit{on } Y).$$

If X is smooth, we can take Y smooth. If X is reduced, we can take Y reduced. If D is a simple normal-crossing divisor (SNC) on X, we can arrange π^*D is again SNC. If $\dim X \geq 2$ and the D_i's are the irreducible components of D (an SNC divisor), then we can arrange that the π^*D_i are the irreducible components of π^*D.

Proof. We do a reduction. Suppose the result is known for $L = f^*\mathcal{O}_{\mathbb{P}^r}(1)$ where $f\colon X \to \mathbb{P}^r$ is a quasi-finite morphism. Then, given any L, there are R and S of the form $f^*\mathcal{O}_{\mathbb{P}^r}(1)$, $g^*\mathcal{O}_{\mathbb{P}^r}(1)$, so that $L = R \otimes S^D$. There is Y_1 so that $R = m^{\text{th}}$ power of Y_1 (*via* μ^*) ,

$$\mu^*L = \mu^*R \otimes (\mu^*S)^D.$$

Now, take an m^{th} root of μ^*S and get

$$\pi\colon Y_2 \xrightarrow{\nu} Y_1 \xrightarrow{\mu} X$$

and $\pi^*L = m^{\text{th}}$ power $\otimes\, m^{\text{th}}$ power. This shows existence. In the case that $L = f^*\mathcal{O}_{\mathbb{P}^r}(1)$ consider the map

$$\nu\colon \mathbb{P}^r \longrightarrow \mathbb{P}^r$$

given by

$$\nu(T_0, \ldots, T_r) = (T_0^m, \ldots, T_r^m)$$

and the Cartesian diagram

$$\begin{array}{ccc} Y = X \prod\limits_{\mathbb{P}^r} \mathbb{P}^r & \xrightarrow{pr_2} & \mathbb{P}^r \\ {\scriptstyle \pi = pr_1}\downarrow & & \downarrow{\scriptstyle \nu} \\ X & \xrightarrow[f]{} & \mathbb{P}^r \end{array}$$

The variety Y is finite, flat over X by pulling back ν and

$$\begin{aligned} \pi^*L &= \pi^*(f^*\mathcal{O}_{\mathbb{P}^r}(1)) \\ &= pr_2^*(\nu^*(\mathcal{O}_{\mathbb{P}^r}(1))) \\ &= pr_2^*(\mathcal{O}_{\mathbb{P}^r}(m)) \\ &= pr_2^*(\mathcal{O}_{\mathbb{P}^r}(1)^{\otimes m}) \\ &= (pr_2^*(\mathcal{O}_{\mathbb{P}^r}(1)))^{\otimes m}, \end{aligned}$$

so we set $N = pr_2^*(\mathcal{O}_{\mathbb{P}^r}(1))$. Now, twist ν by any $\sigma \in \mathrm{GL}(r+1)$ and form Y_σ as the fibred product $X \prod\limits_{\mathbb{P}^r} \mathbb{P}^r$, with ν replaced by $\nu_{[\sigma]} = \sigma \circ \nu$:

$$\begin{array}{ccc} Y_\sigma & \xrightarrow{pr_2} & \mathbb{P}^r \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \nu_{[\sigma]}} \\ X & \xrightarrow[f]{} & \mathbb{P}^r \end{array}$$

We will show that Y_σ is irreducible last.

Since we are in characteristic 0, each $Y_\sigma \longrightarrow X$ is generically reduced (X is intergral). To show Y_σ is everywhere reduced is local. So, we may assume $X = \operatorname{Spec} A$, where A is a domain and $Y = \operatorname{Spec} B$, with B flat (Argument due to Mike Roth). By generic reducedness, there is some $\alpha \in A$ such that B_α is reduced. Pick $\beta \in B$, with β nilpotent. Under $A \longrightarrow A_\alpha$, the element β must go to 0. So, there is some t such that $\alpha^t\beta = 0$. Now, $\alpha^t\colon A \to A$ is injective, so tensor with B. As B is flat over A we deduce that α^t is injective on B and so, $\beta = 0$.

Recall Kleiman's Theorem (Hartshorne, Chapter III): Say X is a homogeneous variety for the algebraic group G and say $Y \longrightarrow X$ and $Z \longrightarrow X$ are morphisms. Then, there is some open $U \subseteq G$ so that, for all $\sigma \in U$, $Y_\sigma \prod\limits_X Z$ is nonsingular for the expected dimension, that is, $\dim Y + \dim Z - \dim X$.

Kleiman's Theorem implies Y_σ is nonsingular for any $\sigma \in U$, where U is an open in $\mathrm{GL}(r+1)$. The same kind of argument (DX) get the nonsingularity of the pullback of a divisor in the covering and normal crossing, too.

Now, for the irreducibility of Y_σ. Recall Bertini's Theorem (Hartshorne, Chapter II): Let $f\colon X \to \mathbb{P}^r$ be a morphism, assume that d is chosen with $d < \dim \overline{f(X)}$, where X is irreducible. Then, for a Zariski open set of $(r-d)$-planes, L, the variety $f^{-1}(L)$ is irreducible. $\square$

From this and the Stein factorization we get *Zariski's connectedness Theorem*:

Say X is proper and irreducible and $f\colon X \to \mathbb{P}^r$ is a morphism. Assume $d < \dim f(X)$ and let L be any $(r-d)$-plane of $\mathbb{P}^r$. Then, $f^{-1}(L)$ is connected. If X is not proper, then assume f is a proper morphism over some open U, of $\mathbb{P}^r$. Then, connectness still holds provided L is parametrized by U.

One also has the Fulton-Hansen connectedness Theorem:

Let X be proper and let $f\colon X \to \mathbb{P}^r \prod \mathbb{P}^r$ be a morphism. If $\dim f(X) > r$, then $f^{-1}(\Delta)$ is connected (where Δ is the diagonal in $\mathbb{P}^r \prod \mathbb{P}^r$).

Theorem 1.15 *(Irreducibility of Generic Graphs) Say $f\colon X \to \mathbb{P}^r \prod \mathbb{P}^r$ is given, with $\mathrm{dm}\, \overline{f(X)} > r$, then there is some open, $U \subseteq \mathrm{GL}(r+1)$, so that for all $\sigma \in U$, $f^{-1}(\Gamma_\sigma)$ is irreducible.*

Proof. Take $\sigma = (a_{ij}) \in \mathrm{GL}(r+1)$ let $L_\sigma \subseteq \mathbb{P}^r \prod \mathbb{P}^r$ be given by the equations

$$y_i = \sum_{j=0}^{r} a_{ij} x_j, \quad 0 \le i \le r.$$

Then (easy), $L_\sigma \xrightarrow{\sim} \Gamma_\sigma$. Look at the plane (L_{id}) given by $y_i = x_i$ and observe that $d < r$ implies $2r - d > r$. In Bertini, such L's are admissible. By an elementary argument, we can prove that all L's near L_{id} are of the form L_σ for $\sigma \in U$ here U is some open in $\mathrm{GL}(r+1)$. By Bertini, $f^{-1}(L_\sigma)$ is irreducible and thus, $f^{-1}(\Gamma_\sigma)$ is also irreducible.

Here is our situation:

$$\begin{array}{ccc} Y_\sigma & \xrightarrow{pr_2} & \mathbb{P}^r \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \nu_{[\sigma]}} \\ X & \xrightarrow[\varphi]{} & \mathbb{P}^r \end{array}$$

Make believe all these are sets. Then,

$$Y_\sigma = \{(\xi, \eta) \mid \varphi(\xi) = \eta(\nu(\eta))\}$$

and

$$\begin{aligned}(\varphi,\nu)(\Gamma_{\sigma^{-1}}) &= \{(\xi,\eta)\mid(\varphi,\nu)(\xi,\eta)\in\Gamma_{\sigma^{-1}}\}\\ &= \{(\xi,\eta)\mid(\varphi(\xi),\nu(\eta))\in\Gamma_{\sigma^{-1}}\}\\ &= \{(\xi,\eta)\mid\sigma^{-1}(\varphi(\xi))=\nu(\eta)\}\\ &= Y_\sigma.\end{aligned}$$

Consequently, on some open subset of $\mathrm{GL}(r+1)$, we have $(\varphi,\nu)^{-1}(\Gamma_{\sigma^{-1}})=Y_\sigma$, proving that Y_σ is irreducible. $\square$

1.3 The Kodaira & Akizuki-Nakano Vanishing Theorems–Part II

Recall the *Lefschetz Hyperplane Theorem* (Griffith & Harris):

Say X is a complex, projective, nonsingular variety and D is an effective, ample divisor which is nonsingular. Then, the restriction map $r_i\colon H^i(X,\mathbb{Z})\to H^i(D,\mathbb{Z})$ is an isomorphism if $i\leq n-2$ and an injection if $i=n-1$ (where $n=\dim X$).

Injectivity lemma.

Say X and Y are projective varieties, with X normal, $f\colon Y\to X$ is a finite, flat morphism, and E is a vector bundle on X (we are in characteristic 0). Then, the canonical map

$$H^j(X,\mathcal{O}_X(E))\longrightarrow H^j(Y,f^*\mathcal{O}_X(E))$$

is injective for all j.

Proof. We can normalize Y and not change anything. By Leray, we have isomorphisms

$$H^j(X,f_*f^*(\mathcal{O}_X(E)))\xrightarrow{\sim}H^j(Y,f^*(\mathcal{O}_X(E))).$$

Note that

$$f^*\mathcal{O}_X(E)=f^*_{\mathrm{space}}\mathcal{O}_X(E)\otimes_{f^*_{\mathrm{space}}\mathcal{O}_X}\mathcal{O}_Y.$$

The projection formula yields

$$f_*f^*(\mathcal{O}_X(E))=\mathcal{O}_X(E)\otimes_{\mathcal{O}_X}f_*\mathcal{O}_Y.$$

Because of characteristic 0, we have a trace map

$$\mathrm{Tr}_{Y/X}\colon f_*\mathcal{O}_Y\longrightarrow\mathcal{O}_X$$

and we have an injection $\mathcal{O}_X\hookrightarrow f_*\mathcal{O}_Y$. This gives a splitting

$$f_*\mathcal{O}_Y=\mathcal{O}_X\amalg\mathcal{E}.$$

If we tensor with $\mathcal{O}_X(E)$, we get

$$f_*f^*(\mathcal{O}_X(E)) = \mathcal{O}_X(E) \amalg \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{E}.$$

When we apply cohomology, we get

$$H^j(X, f_*f^*(\mathcal{O}_X(E))) = H^j(X, \mathcal{O}_X(E)) \amalg H^j(X, \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{E}),$$

so we get an injection

$$H^j(X, \mathcal{O}_X(E) \hookrightarrow H^j(X, f_*f^*(\mathcal{O}_X(E))) \cong H^j(Y, f^*\mathcal{O}_X(E)),$$

as desired. □

Theorem 1.16 *(Kodaira Vanishing Theorem) Suppose X is a complex, nonsingular, projective, algebraic variety of dimension $n = \dim X$. For any ample line bundle, L, on X, we have*

$$H^k(X, \mathcal{O}_X(L) \otimes_{\mathcal{O}_X} \omega_X) = (0) \quad \textit{if} \quad k > 0.$$

By Serre Duality, the latter space is dual to $H^{n-k}(X, \mathcal{O}_X(L^D))$. Therefore, the conclusion of Theorem 1.16 is equivalent to

$$H^k(X, \mathcal{O}_X(L^D)) = (0) \quad \text{if} \quad k < n.$$

Proof. Begin with Hodge theory:

$$H^j(X, \mathbb{C}) \cong \coprod_{p+q=j} H^q(X, \Omega_X^p) = \coprod_{p+q=j} H^{p,q}(X).$$

We also have (Lefschetz)

$$H^j(D, \mathbb{C}) \cong \coprod_{p+q=j} H^q(D, \Omega_X^p) = \coprod_{p+q=j} H^{p,q}(D).$$

By tensoring up by $\mathbb{C}$ over $\mathbb{Z}$ in Lefschetz, we get maps

$$r_i \colon H^i(X, \mathbb{C}) \to H^i(D, \mathbb{C}),$$

with r_i an isomorphism if $i \leq n-2$ and an injection if $i = n-1$. By Hodge and Lefschetz, we have maps

$$r_{p,q} \colon H^{p,q}(X) \to H^{p,q}(D),$$

with $r_{p,q}$ an isomorphism if $p + q \leq n-2$ and an injection if $p + q = n-1$.

Look at $L^{\otimes m}$ for $m >> 0$. There exists a section, $\sigma \in \Gamma(X, \mathcal{O}_X(L^{\otimes m}))$ so that $D = (\sigma)_0$ is an effective nonsingular (very) ample divisor on X. Make $Y \longrightarrow X$, the m-fold cyclic covering of X, branched along D. Then, $\pi^*(D)$ is a nonsingular, ample divisor on nonsingular Y. By the injectivity lemma, if Kodaira holds for Y, then it will hold for X. Therefore, we may assume our original L is represented by a smooth effective divisor, D.

Apply "Holomorphic Lefschetz" for $p = 0$, $q = j$. Then,

$$r_{0,j} \colon H^{0,j}(X) \to H^{0,j}(D),$$

with $r_{p,q}$ an isomorphism if $j \leq n-2$ and an injection if $j = n-1$. Here, $H^{0,j}(X) = H^j(X, \mathcal{O}_X)$ and $H^{0,j}(D) = H^j(D, \mathcal{O}_D)$. But, the sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D) \longrightarrow 0$$

is exact, ie.,

$$0 \longrightarrow \mathcal{O}_X(L^D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D) \longrightarrow 0$$

is exact. If we apply cohomology we get

$$H^j(X, \mathcal{O}_X) \longrightarrow H^j(D, \mathcal{O}_D) \longrightarrow H^{j+1}(X, \mathcal{O}_X(-D)) \longrightarrow H^{j+1}(X, \mathcal{O}_X) \longrightarrow H^{j+1}(D, \mathcal{O}_D).$$

By taking $j \leq n-2$ and using $r_{0,j}$ we get our theorem. $\square$

Remark: The Lefschetz Hyperplane Theorem can be understood from the point of view of algebraic topology in the following way: Let Y be our smooth divisor in the smooth (complex) X and let $U = X - Y$, our affine open. It is known that by triangulation there is a fundamental system of neighborhoods of Y in X, all which deformation retract to Y; call them Y_i. From this, we see that

$$H^k(X, Y; \mathbb{Z}) = \varinjlim_i H^k(X, Y_i; \mathbb{Z}).$$

By excision, we get

$$H^k(X, Y_i; \mathbb{Z}) \cong H^k(U, U \cap Y_i; \mathbb{Z}).$$

Now, U is a smooth open oriented manifold of real dimension $2n$ (where $n = \dim_{\mathbb{C}} X$) and we have a relative version of Poincaré Duality, namely

$$K^k(U, U - K; \mathbb{Z}) \cong H_{2n-k}(K, \mathbb{Z}),$$

where $K \subseteq U$ is compact and K is a deformation retract of an open of U. For example, $K_i = U - U \cap Y_i$ is such a K, Consequently,

$$H^k(U, U \cap Y_i; \mathbb{Z}) = H^k(U, U - K_i; \mathbb{Z}) \cong H_{2n-k}(K_i, \mathbb{Z}),$$

and so,

$$\varinjlim_i H_{2n-k}(K_i, \mathbb{Z}), = H^k(X, Y; \mathbb{Z}).$$

As every $(2n-k)$-chain lies in K_i for some i, we get

$$H_{2n-k}(U, \mathbb{Z}) \cong H^k(X, Y; \mathbb{Z}).$$

Now, we have the exact sequence of relative cohomology

$$\cdots \longrightarrow H^k(X,Y;\mathbb{Z}) \longrightarrow H^k(X,\mathbb{Z}) \longrightarrow H^k(Y,\mathbb{Z}) \longrightarrow H^{k+1}(X,Y;\mathbb{Z}) \longrightarrow \cdots$$

Using our previous isomorphisms, we get

$$\cdots \longrightarrow H_{2n-k}(U,\mathbb{Z}) \longrightarrow H^k(X,\mathbb{Z}) \longrightarrow H^k(Y,\mathbb{Z}) \longrightarrow H_{2n-k-1}(U,\mathbb{Z}) \longrightarrow \cdots$$

Therefore, the Lefschetz Hyperplane Theorem holds iff $H_{2n-k}(U,\mathbb{Z}) = (0)$ when $k \leq n-1$, that is, iff $H_l(U,\mathbb{Z}) = (0)$, for $l \geq n+1$.

In fact, Andreotti, Frankel (1959) and Milnor (1963) showed using Morse Theory:

Theorem 1.17 *(Andreotti, Frankel, Milnor) Every affine, smooth, complex, n-dimensional algebraic variety (even analytic) has the homotopy type of a CW-complex of real dimension at most n.*

In order to prove a sharper vanishing theorem, we need some preliminaries on differentials with logarithmic poles.

Let X be a smooth, complex variety and let D be a smooth Cartier divisor on X. Write $\Omega^1_X(\log D)$ for the sheaf of 1-forms on X having at most poles of order 1 along D (and no ther poles). Write $\Omega^p_X(\log D) = \bigwedge^p \Omega^1_X(\log D)$. That is, if $z_1, \ldots, z_{n-1}, z_n$ are local coordinates near D, where D is defined locally by $z_n = 0$, then $\Omega^1_X(\log D)$ is spanned by

$$dz_1, \ldots, dz_{n-1}, \frac{dz_n}{z_n}$$

locally. Similarly for $\Omega^p_X(\log D)$.

Proposition 1.18 *If X is a smooth, complex, variety and D is a smooth C-dvisor on X then the following satements hold:*

(a) There is an exact sequence

$$0 \longrightarrow \Omega^p_X \longrightarrow \Omega^p_X(\log D) \stackrel{\restriction_D}{\longrightarrow} \Omega^{p-1}_D \longrightarrow 0.$$

(b) There is an exact sequence

$$0 \longrightarrow \Omega^p_X(\log D) \otimes \mathcal{O}_X(-D) \longrightarrow \Omega^p_X \stackrel{\text{res}}{\longrightarrow} \Omega^p_D \longrightarrow 0.$$

(c) If $\pi\colon Y \to X$ is the degree m cyclic cover branched along D and D' is the smooth Cartier divisor of Y isomorphic to D by π so that $\pi^*D = mD'$, then

$$\pi^*(\Omega_X^p(\log D)) = \Omega_Y(\log D').$$

Proof sketch. (a) The definition of the residue map is this: Map

$$dz_1 \wedge \cdots \wedge dz_{i_p} \qquad (i_p < n)$$

to 0 and map

$$f\left(dz_1 \wedge \cdots \wedge dz_{i_{p-1}} \wedge \frac{dz_n}{z_n}\right)$$

to

$$dz_1 \wedge \cdots \wedge dz_{i_{p-1}} \wedge \operatorname{res}\left(f\frac{dz_n}{z_n}\right).$$

Then, we can check that (a) holds by local computations as the maps are globally defined. Let's do it for $p = 1$. The kernel of res must be generated by $dz_1, \ldots .dz_{n-1}$ and $z_n \frac{dz_n}{z_n} (= dz_n)$ and therefore, Ω_X^p is the kernel (for $= 1$). A similar argument can be made for any p.

(b) Take generators for Ω_X^p (locally and for $p = 1$) , namely, $dz_1, \ldots, dz_n$. The kernel of $\restriction_D$ is spanned by $z_n dz_1, \ldots, z_n dz_{n-1}$ and dz_n, that is $z_n dz_1, \ldots, z_n dz_{n-1}$ and $z_n \frac{dz_n}{z_n}$ and these locally span $\Omega_X^p(\log D) \otimes \mathcal{O}_X(-D)$ (for $p = 1$).

(c) Consider $p = 1$. The local coordinates in Y near D' are

$$z_1, \ldots, z_{n-1}, (z_n)^{\frac{1}{m}}.$$

The local coordinates for $\Omega_Y^1(\log D')$ are

$$dz_1, \ldots, dz_{n-1}, \frac{d(z_n)^{\frac{1}{m}}}{(z_n)^{\frac{1}{m}}}.$$

But, by calculus

$$\frac{d(z_n)^{\frac{1}{m}}}{(z_n)^{\frac{1}{m}}} = \frac{1}{m}\frac{dz_n}{z_n}.$$

This gives (c) for $p = 1$. □

Theorem 1.19 *(Akizuki-Nakano Vanishing Theorem) Let X be a smooth, complex, projective variety of dimension n, and let L be an ample line bundle on X. Write A for the divisor representing $L\,(= \mathcal{O}_X(A))$. Then,*

$$H^q(X, \Omega_X^p \otimes L) = (0) \qquad \textit{if} \quad p + q > n.$$

(Note: Kodaira corresponds to the case $p = 1$.)

By Serre duality, the above statement is equivalent to

$$H^s(X, \Omega_X^r \otimes L^D) = (0) \qquad \text{if} \quad r + s < n.$$

Proof. We prove the Serre dual formulation. Since L is ample, for $m >> 0$, there exists $D \in |mA|$, with D smooth, effective, irreducible. Now, suppose we could prove

$$H^s(X, \Omega_X^r(\log D)) \otimes \mathcal{O}_X(-A)) = (0) \qquad \text{if} \quad r + s < n. \tag{†}$$

Then, we can use induction on $n = \dim X$ to finish the proof.

If $n = 0, 1$, the theorem holds (trivial for $n = 0$, by Kodaira for a curve). For the induction step, assume the theorem holds for $\Omega_D^{r-1} \otimes \mathcal{O}_X(-A)$ provided $s + r - 1 < n - 1$, *i.e.*, $s + r < n$. Then, by tensoring (a) with A and taking cohomology we get

$$H^{s-1}(D, \Omega_D^{r-1} \otimes \mathcal{O}_X(-A)) \longrightarrow H^s(X, \Omega_X^r \otimes \mathcal{O}_X(-A)) \longrightarrow H^s(X, \Omega_X^r(\log D) \otimes \mathcal{O}_X(-A)) \longrightarrow H^s(D, \Omega_D^{r-1} \otimes \mathcal{O}_X(-A))$$

The ends vanish by induction, (†) kills the $\log D$ group and our theorem follows in this case.

It remains to prove (†). Construct the cyclic cover $\pi \colon Y \to X$ of degree m, branched along D and write D' for the associated divisor in Y. By the Injectivity Lemma, we must prove

$$H^s(Y, \pi^*(\Omega_X^r(\log D) \otimes \mathcal{O}_Y(-A))) = (0) \qquad \text{if} \quad r + s < n.$$

By Proposition 1.18 (c),

$$\pi^*(\Omega_X^r(\log D) \otimes \mathcal{O}_Y(-A)) = \Omega_Y^r(\log D') \otimes \mathcal{O}_Y(-D').$$

Now apply Proposition 1.18 (b) to our groups:

$$0 \longrightarrow \Omega_Y^r(\log D') \otimes \mathcal{O}_Y(-D') \longrightarrow \Omega_Y^r \longrightarrow \Omega_{D'}^r \longrightarrow 0$$

is exact and by taking cohomology we get

$$\cdots \longrightarrow H^{s-1}(\Omega_Y^r) \xrightarrow{r_{r,s-1}} H^{s-1}(\Omega_{D'}^r) \longrightarrow H^s(\Omega_Y^r(\log D') \otimes \mathcal{O}_X(-D')) \longrightarrow H^s(\Omega_Y^r) \xrightarrow{r_{r,s}} H^s(\Omega_{D'}^r)$$

where $r + s < n$. Holomorphic Lefschetz says

1. $r_{r,s-1}$ is an isomorphism for $r+s-1 < n-1$ and

2. $r_{r,s}$ is an injection for $r+s < n-1$,

and therefore, (†) is proved. □

Bogomolov proved the following vanishing theorem:

Theorem 1.20 *(F. Bogomolov, 1978) Suppose X is a smooth, complex, projective variety, D is a SNC divisor and L is any line bundle on X. Then*

$$H^0(X, \Omega_X^p(\log D) \otimes L^D) = (0) \qquad \textit{if} \quad p < \kappa(L).$$

[Here, $\kappa(L)$ is the *Iitaka dimension of* L. That is, let

$$\underline{N}(L) = \{m \mid m \geq 0 \;\; H^0(X, L^{\otimes m}) \neq (0)\}.$$

Now, if $m \in \underline{N}(L)$ and $m > 0$, then we get a rational map $\varphi_m \colon X \dashrightarrow \mathbb{P}(H^0(X, L^{\otimes M}))$. Write $\overline{\varphi_m(X)}$ for the Zariski closure if the image of φ_m. Set

$$\kappa(L) = \max\{\dim \overline{\varphi_m(X)} \mid m > 0, \; m \in \underline{N}(L)\}$$

and if $\underline{N}(L) = \emptyset$, set $\kappa(L) = \infty$].

Example. If $L = \Omega_X$, then

$$\dim H^0(X, \omega_X^{\otimes m}) = P_m,$$

the m^{th} pluri-genus. Note that $P_1 = p_g$, the geometric genus. Then, $\kappa(\omega_X) =$ the Kodaira dimension of X (denoted $\mathrm{Kod}(X)$). We say that X is a variety of *general type* iff $\kappa(\omega_X) = \mathrm{Kod}(X) = \dim X$.

1.4 Rational Curves and the “Classification of Varieties”

Say $\pi \colon X \to Y$ is a rational map, then there exists a largest open set, $U \subseteq X$, where π is a morphism. Suppose Y is normal and proper. In fact, unless otherwise stated all X and Y are normal and irreducible. Let $\Gamma = \Gamma_{\pi \restriction U}$ be the graph of π restricted to U ($\Gamma \subseteq U \prod Y$) and let $\widetilde{X}$ be the closure of Γ in $X \prod Y$. Then, we have a birational morphism, $p \colon \widetilde{X} \to X$. Since Y is proper, p is proper. As Y is normal, Zariski's Connectednes Theorem implies the fibres of p are connected. Remember that $\dim p^{-1}(x)$ is always upper semi-continuous on X. Pick x where $p^{-1}(x)$ is a point, then there is a Zariski-closed set, V, with $x \in V$ and $\dim p^{-1}(\xi) = 0$ if $\xi \in V$. Over V, the morphism p is finite (it is proper and a quasi-finite). By a previous argument (normality + one-to-one + birational) p is an isomorphism over V. But then, by definition of U, we get $V \subseteq U$. Hence, we find $\xi \in U$ iff $p^{-1}(\xi)$ does not have positive dimension. Hence, we've proved

Theorem 1.21 *(Zariski's Main Theorem) If $\pi\colon X \to Y$ is a rational map with Y proper and normal, then π fails to be a morphism exactly where $p\colon \widetilde{X} \to X$ has a fibre of positive dimension. Moreover,* $\mathrm{codim}(X - U) \geq 2$ *(where U is the largest open set where π is a morphism).*

The second statement holds because $\pi^{-1}(y)$ having positive dimension and the place where this occurs having codimension 1 means means these fill out X, which would imply that π is nowhere defined, a contradiction.

Say $\pi\colon X \to Y$ is a birational morphim and write $E(\pi)$ for the locus

$$E(\pi) = \{x \mid \pi \text{ is not a local isom. at } x\}.$$

The set $E(\pi)$ is called the *exceptional locus* of π. If $\pi^{-1}(y)$ has at least two points, then the Connectednes Theorem implies that $\pi^{-1}(y)$ has a curve in it. Therefore, $E(\pi) = \pi^{-1}(\pi(E))$, where $E = E(\pi)$. In particular, as before, $\mathrm{codim}\, \pi(E(\pi)) \geq 2$. (We use normality and properness of Y.) Let's weaken the hypotheses.

Say Y is normal and locally $\mathbb{Q}$-factorial. This means each Weil divisor, D, on Y has a multiple in D which is a Cartier divisor and $\pi\colon X \to Y$ is a birational morphism.

Claim.

(1) $\mathrm{codim}\, \pi(E(\pi)) \geq 2$.

(2) Every component of $E(\pi)$ has codimension 1.

Pick x in some component of $E(\pi)$ and write $y = \pi(x)$. We know $\pi^*\colon K(Y) \to K(X)$ is an isomorphism—identify $K(X)$ and $K(Y)$. Then, our map gives a map $\mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y} \neq \mathcal{O}_{X,x}$ as $x \in E(\pi)$. Hence, there is some $t \in \mathfrak{m}_{X,x}$ and $t \notin \mathfrak{m}_{Y,y}$. Our t is a meromorphic function on Y. We can choose effective Weil divisors, D_1, D_2, so that $(t) = D_1 - D_2$ (*i.e.* $D_1 = (t)_0, D_2 = (t)_\infty$). There exists $m >> 0$ such that mD_1 and mD_2 are Cartier divisors. Therefore, mD_1 is given by $u = 0$ and mD_2 is given by $v = 0$ and thus,

$$t^m = \frac{u}{v}.$$

Claim. The elements u and v belong to $\mathfrak{m}_{Y,y}$.

If $v \notin \mathfrak{m}_{Y,y}$, then v is a unit and so, $t^m \in \mathcal{O}_{Y,y}$. As Y is normal, $t \in \mathcal{O}_{Y,y}$, a contradiction.

Now, $u = t^m v \in \mathfrak{m}_{X,x} \cap \mathcal{O}_{Y,y} = \mathfrak{m}_{Y,y}$. But, the locus, Z, (on Y) given by $u = 0$ and $v = 0$ has codimension 2 and both vanish on y, which implies $y \in Z$. Therefore, (1) is proved.

Now, look on X. We have $u = t^m v$, so $v = 0$ implies $u = 0$ on X and $\pi^{-1}(Z)$ is given by $v = 0$. But, $x \in \pi^{-1}(Z)$ implies that through x we have a component of codimension 1 and (2) follows.

Ramification Divisors.

Assume X, Y are smooth and $\pi\colon X \to Y$ is a morphism. We get a tangent map, $T_\pi\colon T_X \to \pi^* T_Y$, and if $\dim X = \dim Y = n$, we also have a map

$$\bigwedge^n T_\pi\colon \bigwedge^n T_X \longrightarrow \bigwedge \pi^* T_y.$$

Then, by dualizing, we get a map

$$\bigwedge^n T_\pi^D : \pi^* \bigwedge^n T_Y^D \longrightarrow \bigwedge^n T_X^D,$$

that is,

$$\bigwedge^n T_\pi^D : \pi^* \omega_Y \longrightarrow \omega_X.$$

Consequently, we get a map

$$\mathcal{O}_X \longrightarrow \omega_X \otimes \pi^* \omega_Y^D,$$

and so, we get a section, $\sigma \in \Gamma(X, \omega_X \otimes \pi^* \omega_Y^D)$, *i.e.*, a section $\sigma \in \Gamma(X, \mathcal{O}_X(K_X - \pi^* K_Y))$. Observe that $\sigma \equiv 0$ iff $X \longrightarrow Y$ is nowhere *étale*. So, in characteristic $p \neq 0$ we assume $K(X)$ is separable over $K(Y)$. Since $X \longrightarrow Y$ is generically *étale*, the zeros of σ give a divisor, $\mathrm{Ram}(\pi)$ called the *ramification divisor of* π on X. Then,

$$K_X = \pi^* K_Y + \mathrm{Ram}(\pi).$$

Birational Morphisms.

Suppose X and Y are projective, smooth and $\pi\colon X \to Y$ is a birational *morphism*. Then, there is a theorem of Grothendieck (Hartshorne, Chapter II) which says:

Theorem 1.22 *(Grothendieck) In the situation as above, there is some coherent $\mathcal{O}_Y$-ideal, $\mathfrak{I}$, such that X is the blow-up,* $\mathrm{Bl}_Y(\mathfrak{I})$, *of* $\mathfrak{I}$.

To define $\mathrm{Bl}_Y(\mathfrak{I})$ we proceed as follows: First, we make the graded sheaf of rings, $\mathrm{Pow}(\mathfrak{I})$, given by

$$\mathrm{Pow}(\mathfrak{I}) = \coprod_{j=0}^{\infty} \mathfrak{I}^j = \mathcal{O}_Y \coprod \mathfrak{I} \coprod \mathfrak{I}^2 \coprod \cdots$$

and then we make $\mathrm{Proj}(\mathrm{Pow}(\mathfrak{I}))$. By definition, $\mathrm{Bl}_Y(\mathfrak{I}) = \mathrm{Proj}(\mathrm{Pow}(\mathfrak{I}))$.

Moreover, $\pi^{-1}(\mathfrak{I})\mathcal{O}_X$ is an ideal of $\mathcal{O}_X$ which is a line bundle, that is, $\mathcal{O}_X(1)$ under a suitable embedding. That is, $\mathfrak{I}$ pulled back to X is (locally) principal. Now, we want to understand the relation between $E(\pi)$ and the support of $\mathcal{O}_X(1)$.

Let E be an effective divisor for $\mathcal{O}_X(1)$. Take an ample, H, on Y, then if $m >> 0$, $m\pi^*H - E$ is ample on X. So, through each point of $E(\pi)$, there is a curve, C, in $E(\pi)$ that π contracts. But, $0 < (m\pi^*H - E) \cdot C$, that is

$$m\pi^*H \cdot C - E \cdot C = mH \cdot \pi(C) - E \cdot C = -E \cdot C.$$

Consequently, C is contained in the support of E and as C is arbitrary, we conclude that $E(\pi) \subseteq \operatorname{supp} E$. In fact (Hartshorne, Chapter II, Exercise), we can choose $\mathfrak{I}$ so that $E(\pi) = \operatorname{supp} E$.

Notion of "Classification" of Varieties.

(1) Choose a notion of equivalence for varieties.

(2) Determine in each class a "simplest" variety.

(3) Show (or give a procedure) that (2) holds.

By experience, (1) must be coarser than isomorphism. It turns out that success seems to indicate that $X \approx Y$ should mean "birational".

The example of curves is "easy". Here birational equivalence of smooth curves *is* isomorphism.

For surfaces, birational equivalence is not isomorphism in general.

Theorem 1.23 *(Castelnuovo) For a smooth surface, X, and for a rational curve, C, on X there exists a birational morphism, $\pi\colon X \to Y$, contracting C iff $C^2 = -1$ (where Y is another smooth surface).*

Castelnuovo and Enriques "proved" that the process of contraction eventually stops. The result is

(1) A smooth surface, Y, unique example in the birational class and this happens iff X is not covered by rational curves and K_X is nef.

or

(2) A smooth Y, not a unique example in its birational class and this happens when X is covered by rational curves and K_X is not nef.

Example of (2): $\mathbb{P}^2$ and $\mathbb{P}^1 \prod \mathbb{P}^1$.

For higher dimensions, we can have K_X nef, yet $X \approx Y_1$, $X \approx Y_2$, both Y_1 and Y_2 are "minimal" birational yet not isomorphic.

Proposition 1.24 *Say $\pi\colon X \to Y$ is a birational morphism and π is proper, Y is smooth and π is not an isomorphism. Then, through every generic point of E (the exceptional divisor of π) there is a rational curve that π contracts. That is, each component of E is birationally ruled.*

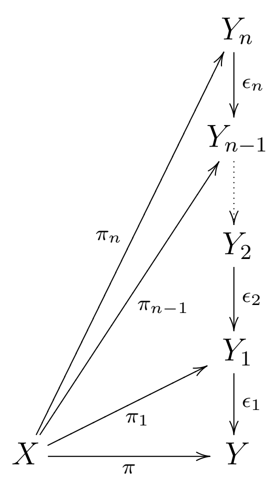
Proof. Preliminary reduction: First, we normalize X and we may assume that X is smooth in codimension 1. So, for any generic point, $x \in E$, by (1) above, x is a smooth point. Shrink X and Y to get

(a) X smooth

(b) E smooth, irreducible

(c) $\overline{\pi(E)}$ smooth.

Let $Y_1 = \mathrm{Bl}_{\overline{\pi(E)}}$ be the blow-up of Y along $\overline{\pi(E)}$ and let $\epsilon_1\colon Y_1 \to Y$ be the corresponding birational morphism. By the universality for blow-ups, π factors through a map, $\pi_1\colon X \to Y_1$. Also, if E_1 is the exceptional divisor for ϵ_1, then $\overline{\pi_1(E)} \subseteq E_1$. If $\mathrm{codim}(\overline{\pi_1(E)}) \geq 2$ (in Y_1), we can repeat this process. We get the following diagram in which $\mathrm{codim}(\overline{\pi_i(E)}) \geq 2$ (in Y_i) for all i, with $1 \leq i \leq n-1$:



We know that

$$K_{Y_1} = \epsilon_1^* K_Y + \gamma_1 E_1$$

where $\gamma_1 = \mathrm{codim}_Y(\overline{\pi(E)}) - 1$ and generally,

$$K_{Y_{i+1}} = \epsilon_{i+1}^* K_{Y_i} + \gamma_{i+1} E_{i+1},$$

with $1 \leq i \leq n-1$ and $Y_0 = Y$. As $\pi_n E \subseteq E_n$, we deduce that $\pi_n^* E_n - E$ is effective and this implies that

$$K_{Y_n} = \epsilon_n^* \cdots \epsilon_1^* K_Y + \gamma_1 E_1 + \cdots + \gamma_n E_n.$$

As π is birational, $\pi^*\mathcal{O}_Y(K_Y)$ is a subsheaf of $\mathcal{O}_X(K_X)$. This implies $\pi^*\mathcal{O}_Y(K_Y) + (\gamma_1 + \cdots + \gamma_n)E$ is a subsheaf of $\mathcal{O}_X(K_X)$. The later is coherent on X, so the ascending chain

$$\pi^*\mathcal{O}_Y(K_Y) \subseteq \pi^*\mathcal{O}_Y(K_Y + \gamma_1 E) \subseteq \cdots$$

stops, say at n. This implies $\mathrm{codim}(\pi_n(E))$ in Y_n is 1. Now, as $\pi_n(E)$ has codimension 1 in E_n, we deduce that E is birationally isomorphic to E_n. But, E_n is ruled, being the exceptional locus of a blow-up. □

Corollary 1.25 *Say π is a rational map from X to Y and*

(1) X is smooth

(2) X has no rational curve

(3) Y is proper.

Then, π is defined everywhere.

Proof. Let U be the largest open subset of X where π is defined and write $\Gamma \subseteq X \prod Y$ be the graph of $\pi \restriction U$. As before, let $\widetilde{X}$ be the closure of Γ and write $p = pr_1 \restriction \widetilde{X}$. Then as Y is proper, so is $p\colon \widetilde{X} \to X$ and as $E = \mathrm{Exc}(p) \neq \emptyset$ the previous proposition applies so, through every generaic point of E there is a rational curve, C, and p contracts C. Thus, $pr_2(C)$ is either a point or rational curve in Y, but the second possibility yields a contradiction. It follows that pr_2 contracts C but then, C is a single point and $E = \emptyset$, which is absurd. Therefore, $U = X$ and we are done. □

Theorem 1.26 *Say X and Y are projective irreducible varieties, both smooth and $\pi\colon X \to Y$ is a birational morphism. Suppose π is not an isomorphism. Then, there is a rational curve $D \subseteq X$, so that*

(1) π contracts D.

(2) $K_X \cdot D < 0$.

Proof. (1) Write $E = \mathrm{Exc}(\pi)$, we know E is pure codimension 1 and $\pi(E)$ has codimension at least 2 in Y. Pick $y \in \pi(E)$. As Y is projective, there is an embdedding, $Y \hookrightarrow \mathbb{P}^N$, for some (large) N and Bertini's Theorem implies that any general hyperplane cuts Y in a smooth codimension 1 section. We can even pick the hyperplanes through y (DX). If we do this $\dim Y - 2$ times we get a smooth surface, $S \subseteq Y$, so that

(1) $y \in S$;

(2) $S \cap \pi(E)$ is a finite set of points.

Do this one more time in two different ways:

(a) a hyperplane through y, we get a smoth curve, C_0.

(b) a hyperplane omitting all of $\pi(E) \cap S$, obtaining a smooth curve, C.

By construction, $C \sim C_0$ implies

$$K_Y \cdot C = K_Y \cdot C_0.$$

If we let $C' = \pi^* C$ we see that C' is isomorphic to C and let C_0' be the proper transform of C_0, that is $C_0 = \overline{\pi^{-1}(C_0 - \{y\})}$. Recall that

$$K_X = \pi^* K_Y + \mathrm{Ram}(\pi)$$

and the support of $\mathrm{Ram}(\pi)$ is contained is equal to E. We get

$$K_X \cdot C' = \pi^* K_Y \cdot C' + \mathrm{Ram}(\pi) \cdot C = K_Y \cdot C$$

and so,

$$K_X \cdot C' = K_Y \cdot C.$$

Now,

$$K_X \cdot C_0' = \pi^* K_Y \cdot C_0' + \mathrm{Ram}(\pi) \cdot C_0 > \pi^* K_Y \cdot C_0' = K_Y \cdot C_0,$$

so

$$K_X \cdot C_0' > K_Y \cdot C_0.$$

It follows from all this that

$$K_X \cdot C_0' > K_X \cdot C'. \tag{$\dagger$}$$

Now, look at π^{-1} but restricted to S. It may happen that π^{-1} is not defined on points of $\pi(E)$. But, by surface theory (Hartshorne, Chapter V), we can blow up finitely many points of S to get a new surface, $\widetilde{S}$, and a birational morphism, $\epsilon\colon \widetilde{S} \to S$. We get a morphism, $g\colon \widetilde{S} \to X$ and let $C'' = \epsilon^* C \cong C$ and $\epsilon^* C_0 = C_0'' + \sum_i m_i E_i$, with $m_i \geq 0$, where the E_i are the components of the exceptional divisor of ϵ and C_0'' is the proper transform of C_0 under ϵ. We have $g_* C'' = C'$ and $g_* C_0'' = C_0'$. Then,

$$\pi^* C_0 = g_* C_0'' + \sum_i m_i g_*(E_0) = C_0' + \sum_i m_i g_*(E_i)$$

and we know that

$$K_X \cdot C' = K_X \cdot \pi^* C = K_x \cdot \pi^* C_0$$

because $C \sim C_0$ implies $\pi^* C \sim \pi^* C_0$ and

$$K_X \cdot \pi^* C_0 = K_X \cdot C_0' + \sum_i m_i K_X \cdot g_*(E_i).$$

By ($\dagger$), we have $\sum_i m_i K_X \cdot g_*(E_i) < 0$ and consequently:

(1) $m_i > 0$ for some i;

(2) $g_*(E_i)$ is a curve for this i, call it D.

As E is rational, D is rational.

(2) by following the last diagram (to be filled in) we see that $\pi(D) = g_*(E_i)$ is a point and so, $K_X \cdot D < 0$. $\square$

Corollary 1.27 *If $\pi\colon X \to Y$ is a birational morphism of smooth projective varieties and K_X is nef, then π is an isomorphism.*

We now go back to the "classification" of varieties. For simplicity assume all varieties are smooth.

(1) Let $\mathcal{C} =$ be the birational class (smooth varieties) and assume there is some $X_0 \in \mathcal{C}$ such that X_0 possesses no rational curves. Let $Z \in \mathcal{C}$ be any other variety and assume there is a rational map, $\pi\colon Z \dashrightarrow X_0$. Corollary 1.27 implies π is a *morphism*. Write $X \preceq Y$ iff there is a birational morphism $Y \longrightarrow X$. The above implies that (the equivalence class of) X_0 is minimal. If X_0 and $\widetilde{X}_0$ are minimal, with no rational curve in either of them, then Theorem 1.26 implies there is birational morphism, $\pi\colon X_0 \to \widetilde{X}_0$, and as there are no rational curves in $\widetilde{X}_0$, the map π must be an isomorphism. Therefore, X_0 is unique up to isomorphism and is a smallest element.

(2) Let $\mathcal{C} =$ be the birational class (smooth varieties) and assume there is some $X_0 \in \mathcal{C}$ with $K_X \cdot C \geq 0$ for all rational curves, C, in X_0. (This really does mean that K_{X_0} is nef.) Can there be some $Z \in \mathcal{C}$ and a birational morphism, $X_0 \longrightarrow Z$?

The theorem implies $X_0 \cong Z$ and so, X_0 is minimal.

Now, the idea is, for a smooth X, give a procedure (contraction of curves) to make K_X nef. These will be among the extremal rays of the cone $\overline{\mathrm{NE}}(X)$.

1.5 The Kawamata-Vichweg Vanishing Theorem-Part I—The Integral Vanishing Theorem

First, we have to discuss the *resolution of singularities à la Hironaka*.

Theorem 1.28 *(Hironaka, 1961) Let X be an irreducible, complex, algebraic variety and D be an effective divisor on X. Then the following assertions hold:*

(1) There exists a birational projective morphism, $\rho\colon \widetilde{X} \to X$, so that $\widetilde{X}$ is nonsingular and $\rho^*D + \mathrm{Exc}(\rho)$ is a divisor on $\widetilde{X}$ with support SNC.

(2) One can make ρ by a composition of blowings-up of nonsingular centers supported in $\mathrm{Sing}\, X$ or $\mathrm{Sing}\, Y$. Hence, ρ is an isomorphism over $X - (\mathrm{Sing}\, X \cup \mathrm{Sing}\, Y)$.

Remarks:

(1) This is usually called the "*embedded resolution*" or "*log resolution*" of the pair (X, D).

(2) Assertion (1) called the *Weak Hironaka Theorem* is usually sufficient for most applications. Simple short ($\sim$ 6 printed pages) were given by Bogomolov–Pantev and Abramovic-deJong. However, if we use the full strengh of (2) we can prove more.

Proposition 1.29 *Say (X, D) is a pair as in Hironaka's Theorem and assume X is smooth and projective. Then, if $\rho\colon \widetilde{X} \to X$ is "the" log resolution of (X, D), then*

(a) $\rho_*\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) = \mathcal{O}_X(K_X)$.

(b) $(R^p\rho_*)(\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}})) = (0),\ p > 0$.

(c) Take H ample on X, then there is some $p >> 0$ and some integers, $b_1, \ldots, b_t \geq 0$, so that $\rho_*(pH) - \sum_{j=1}^t b_j E_j$ is ample on $\widetilde{X}$ where the E_j are the exceptional divisors of the blow-ups.

Proof. It is clear that (a), (b), (c) will hold for a composition of blow-ups if they hold for one blow-up. But for a single blow-up, this follows from Hartshorne, Chapter II. □

Theorem 1.30 *((Integral) Kawamata–Vichweg Vanishing Theorem) Say X is a smooth, projective, irreducible, complex variety. If D is a big and nef divisor on X, then*

$$H^p(X, \mathcal{O}_X(K_X + D)) = (0), \quad p > 0;$$

that is, by Serre Duality

$$H^p(X, \mathcal{O}_X(-D)) = (0), \quad p < \dim X.$$

(Note that Kodaira's Theorem is just Kawamata–Vichweg Vanishing for D ample).

Does Akizuki-Nakano generalize to the case where D is big and nef?

Answer: **No**.

Here is a **Counter-Example**: Let $X = \mathrm{Bl}_P(\mathbb{P}^3)$, the blow-up of (complex) projective space $\mathbb{P}^3$ at a point, P, and let D be the pull-back of a general hyperplane on $\mathbb{P}^3$. Then, D is nef and big. Look at $H^2(X, \mathbb{C})$. By Poincaré Duality,

$$\dim H^2(X, \mathbb{C}) = \dim H^1(X, \mathbb{C}).$$

The right-hand side has dimension 2. Using Hodge theory, we have

$$H^2(X, \mathbb{C}) = H^{2,0} \amalg H^{1,1} \amalg H^{0,2}$$

and $H^{2,0} = H^0(X, \Omega_X^2)$, whose dimension is P_2. But, we know the birational invariance of P_2, so $\dim H^{2,0} = 0$ (as this holds for $\mathbb{P}^3$). It follows that $\dim H^{0,2} = 0$, so $\dim H^{1,1} = 2$ (with $H^{1,1} = H^1(X, \Omega_X)$). Now, $H^1(X, \Omega_D^1)$ has dimension 1 as $D = \mathbb{P}^2$. Recall the exact sequence

$$0 \longrightarrow \Omega_X^1(\log D) \otimes \mathcal{O}_X(-D) \longrightarrow \Omega_X^1 \longrightarrow \Omega_D^1 \longrightarrow 0$$

and apply cohomology. We get

$$H^0(D, \Omega_D^1) \longrightarrow H^1(X, \Omega_X^1(\log D) \otimes \mathcal{O}_X(-D)) \longrightarrow H^1(X, \Omega_X^1) \longrightarrow H^1(D, \Omega_D^1).$$

But, $H^0(D, \Omega_D^1) = (0)$ as $D = \mathbb{P}^2$. Therefore, $\dim H^1(X, \Omega_X^1(\log D) \otimes \mathcal{O}_X(-D)) \neq 0$. Now, we have the residue exact sequence

$$0 \longrightarrow \Omega_X^1 \longrightarrow \Omega_X^1(\log D) \longrightarrow \Omega_D^0 = \mathcal{O}_D \longrightarrow 0.$$

If we twist by $\mathcal{O}_X(-D)$, we get the exact sequence

$$0 \longrightarrow \Omega_X^1 \otimes \mathcal{O}_X(-D) \longrightarrow \Omega_X^1(\log D) \otimes \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_D(-D^2) \longrightarrow 0.$$

Take cohomology and get

$$H^0(D, \mathcal{O}_D(-D^2)) \longrightarrow H^1(X, \Omega_X^1 \otimes \mathcal{O}_X(-D)) \longrightarrow H^1(X, \Omega_X^1(\log D) \otimes \mathcal{O}_X(-D)) \longrightarrow H^1(D, \mathcal{O}_D(-D^2)).$$

But, $H^0(D, \mathcal{O}_D(-D^2)) = (0)$ and $H^1(D, \mathcal{O}_D(-D^2)) = (0)$. Consequently, $H^1(X, \Omega_X^1 \otimes \mathcal{O}_X(-D)) \neq (0)$, contradicting Akizuki-Nakano.

What is the problem? While $H^{0,q}(X)$ and $H^{q,0}(X)$ are birational invariants for smooth X, the $H^{p,q}$ for $p, q \geq 1$ are **not**.

In order to prove the Kawamata–Vichweg Vanishing Theorem we need a slight generalization of Kodaira's Theorem.

Lemma 1.31 *(Norimatsu) Let X be a smooth, projective, irreducible, complex variety and let A be an ample divisor and E an SNC divisor. Then,*

$$H^p(X, \mathcal{O}_X(K_X + A + E)) = (0) \quad \textit{if} \quad p > 0,$$

that is (Serre Duality)

$$H^p(X, \mathcal{O}_X(-A - E)) = (0) \quad \textit{if} \quad p < \dim X.$$

Proof. Write $E = E_1 + E_2 + \cdots + E_t$ and use induction on t. If $t = 0$, then $E = \emptyset$ and Norimatsu's Lemma is just Kodaira's Theorem. Assume the induction hypothesis holds if $t \leq k$ and look at $E = \sum_{i=1}^{k} E_i + E_{k+1}$. We have the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-A - \sum_{i=1}^{k+1} E_i) \longrightarrow \mathcal{O}_X(-A - \sum_{i=1}^{k} E_i) \longrightarrow \mathcal{O}_{E_{k+1}}(-A - \sum_{i=1}^{k} E_i) \longrightarrow 0.$$

By induction, the theorem holds for the two right-hand side sheaves if $p < \dim X$ and for E_{k+1} if $p < \dim X - 1$. The cohomology sequence finishes the proof. □

Proof of Theorem 1.30. As D is big, for some $m >> 0$, mD has the form $mD = H + N$, where H is ample and N is effective.

Step 1. Reduction to the case: N is a divisor whose support is SNC. We apply log-resolutions (of Hironaka) to the pair (X, N). Then $\rho^*N + \mathrm{Exc}(\rho)$ has support SNC. Then,

$$\rho^* mD = \rho^* H + \rho^* N,$$

but $\rho^* H$ may no longer be ample. Write $\rho^* N = \sum_{j=1}^t a_j E_j$, where $a_j \geq 0$ and the exceptional divisors are among the E_j's. We know there is $p >> 0$ so that

$$\rho^*(pH) - \sum_{j=1}^t b_j E_j$$

is ample for some $b_j \geq 0$, using (2) of Hironaka. Then,

$$\begin{aligned} \rho^*(pmD) &= \rho^*(pH) + \rho^*(pN) \\ &= \underbrace{\rho^*(pH) - \sum_{j=1}^t +b_j E_j}_{\text{ample}} + \underbrace{\sum_{j=1}^t (pa_j + b_j) E_j}_{\text{effective}}. \end{aligned}$$

On $\widetilde{X}$, we see that $pm(\rho^* D)$ is the sum of an ample plus an effective divisor and the support of N is an SNC divisor. We know that

$$\rho_*(\mathcal{O}_{\widetilde{X}})(K_{\widetilde{X}}) = \mathcal{O}_X(K_X)$$

and

$$R^p \rho_*(\mathcal{O}_{\widetilde{X}})(K_{\widetilde{X}}) = (0) \quad \text{if} \quad p > 0.$$

Suppose we know the theorem when our D has

$$mD = H + N,$$

where H is ample and N is nef and the support of N is SNC (for some $m >> 0$). Then, $\rho^* D$ is such a divisor on $\widetilde{X}$ and our theorem holds for $\widetilde{X}$ and $\rho^* D$, that is,

$$H^r(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-\rho^*(D)) = (0) \quad \text{if} \quad r < n = \dim \widetilde{X},$$

that is,

$$H^r(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}} + \rho^*(D))) = (0) \quad \text{if} \quad r > 0,$$

by Hironaka (2). Apply the Leray spectral sequence, as $R^q \rho_*(\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}} + \rho^* D)) = (0)$ if $q > 0$, by Hironaka and the projection formula we get

$$\rho_* \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}} + \rho^*(D)) = \mathcal{O}_X(K_X + D)$$

and we get

$$H^r(X, \mathcal{O}_X(K_X + D)) = (0), \quad r > 0,$$

as required.

Step 2. The case where D has the property that $mD = H + N$, with H ample, N effective and $\operatorname{supp} N$ is SNC, for some $m >> 0$.

In this case we will apply the following covering lemma:

Lemma 1.32 *(Kawamata's Covering Lemma) Say X is a smooth, quasi-projective variety and $m_1, \ldots, m_t$ are chosen positive integers. Given any SNC divisor, $E = \sum_{i=1}^t E_i$, there exists a flat, finite cover, $h\colon Y \to X$, so that $h^* E_i = m_i E_i'$ and $E' = \sum_{i=1}^t E_i'$ is an SNC divisor.*

Assume this for now. Then, take $N = \sum_{i=1}^t e_i E_i$, ($e_i > 0$ and the divisor $\sum_i E_i$ is SNC). Let $\epsilon = e_1 e_2 \cdots e_t > 0$ and write $\epsilon_i = \epsilon/e_i$, *i.e.*, $e_i \epsilon_i = \epsilon$. Take $m_i = m\epsilon_i$, for $i = 1, \ldots, t$. Go up to the Kawamata covering, Y. Write $D' = h^* D$ and $H' = h^* H$. The divisor H' is ample on Y and

$$\begin{aligned} mD' &= h^*(mD) = H' + h^* N \\ &= H' + \sum_{i=1}^t e_i (h^* E_i) \\ &= H' + \sum_{i=1}^t e_i m_i E_i' \\ &= H' + \sum_{i=1}^t m e_i \epsilon_i E_i' \\ &= H' + m\epsilon \sum_{i=1}^t E_i' \\ &= H' + m\epsilon E'. \end{aligned}$$

Consider $m\epsilon(D' - E')$, we have

$$m\epsilon(D' - E') = m\epsilon D' + H' - mD' = m(\epsilon - 1)D' + H' = \text{nef} + \text{ample} = \text{ample},$$

which implies that $D' - E' = A'$ is ample. But then, $D' = A' + E'$ is the sum of an ample plus an SNC divisor. By Norimatsu, we get the vanishing result:

$$H^r(Y, \mathcal{O}_Y(-A' - E')) = (0), \quad r < \dim X,$$

that is

$$H^r(Y, \mathcal{O}_Y(-D')) = (0), \quad r < \dim X.$$

But, $Y \longrightarrow X$ is a cover, so we use the injectivity lemma and this gives

$$H^r(X, \mathcal{O}_X(-D)) = (0), \quad r < \dim X,$$

the required vanishing. □

Proof of Kawamata's Covering Lemma. We can use induction on the number of components of our SNC divisor, $D = D_1 + \cdots + D_t$.

By Bloch-Gieseker, we get a cover $\widetilde{Y}$ (of X), $f\colon \widetilde{Y} \to X$ and $f^*(\mathcal{O}_X(D_1)) = \widetilde{L}^{\otimes m_1}$, where $\widetilde{L}^{\otimes m_1} = \mathcal{O}_{\widetilde{Y}}(B)$, but B is not necessarily effective. Then, as $f^*(\mathcal{O}_X(D_1))$ is an m_1^{th} power, we can make the cyclic cover, $h\colon Y \to \widetilde{Y}$, branched along $f^*D_1 = \widetilde{D}_1$ and

$$h^*\widetilde{D}_1 = m_1 D_1'$$

on Y. Now

(a) f^*D is still SNC.

(b) Using (a) we see that H^*f^*D is also SNC. We continue by induction to obtain the result for $D_1 + \cdots + D_t$. □

Corollary 1.33 *(Generalized K-V Vanishing) Let X be a smooth, projective variety; H an ample divisor on X; D a Cartier divisor that is nef and assume there is some $k \geq 0$ such that $D^{n-k} \cdot H^k > 0$, where $n = \dim X$. Then,*

$$H^i(X, \mathcal{O}_X(K_X + D)) = (0) \quad i > k.$$

Proof. By induction on k. When $k = 0$, this is just Kawamata–Vichweg. Assume the induction hypothesis holds for varieties and integers $< k$. We may assume H is very ample and the divisor is smooth. The sequence

$$0 \longrightarrow \mathcal{O}_X(-H) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_H \longrightarrow 0$$

is exact. If we tensor with $\mathcal{O}_X(K_X + D + H)$, we get

$$0 \longrightarrow \mathcal{O}_X(K_X + D) \longrightarrow \mathcal{O}_X(K_X + D + H) \longrightarrow \mathcal{O}_H(K_X \cdot H + H \cdot H + D \restriction H) \longrightarrow 0.$$

By adjunction, the last term is $\mathcal{O}_H(K_H + D \restriction H\,(= D \cdot H))$. The hypothesis implies that the right-hand term is the induction term for the variety H ($\dim H = n - 1$) and the integer $k - 1$. The cohomology sequence and induction imply that

$$H^l(X, \mathcal{O}_H(K_H + D \restriction H)) = (0)$$

for $l > k - 1$. Then,

$$H^i(X, \mathcal{O}_X(K_X + (D + H))) = (0), \qquad i > 0,$$

since $D + H$ is ample, and the induction step is established. □

Definition 1.3 A morphism (between schemes), $f\colon Y \to X$ is an *alteration* iff it is generically finite and surjective.

Remark: De Jong's Theorem says: Every finite type scheme/k admits an alteration which is nonsingular.

Theorem 1.34 *(Grauert-Riemenschneider Vanishing Theorem) If $f\colon Y \to X$ is an alteration of (irreducible) varieties and if Y is smooth, then $R^p f_* \mathcal{O}(K_Y)$ vanishes if $p > 0$.*

For this, we need a lemma:

Lemma 1.35 *Say V and W are projective varieties, $f\colon V \to W$ is a morphism and A is ample on W. Given any coherent sheaf, $\mathcal{F}$, on V, so that*

$$H^j(V, \mathcal{F} \otimes \mathcal{O}_X(f^*(mA))) = (0)$$

for $j > 0$ and all $m >> 0$, we have

$$R^p f_* \mathcal{F} = (0), \quad \text{if} \quad p > 0.$$

Proof. Look at $R^j f_* \mathcal{F}$ (only finitely many j necessary). All these sheaves are coherent on W (by Serre). Then, as A is ample, we can arrange

$$H^t(W, (R^j f_* \mathcal{F}) \otimes \mathcal{O}_W(mA)) = (0)$$

for $t > 0$, $j \geq 0$ and $m >> 0$ and $(R^j f_* \mathcal{F}) \otimes \mathcal{O}_W(mA)$ is generated by its sections for all $j \geq 0$ and all $m >> 0$. If we apply the projection formula, we get

$$R^q f_*(\mathcal{F} \otimes \mathcal{O}_V(f^* mA)) = (R^q f_* \mathcal{F}) \otimes \mathcal{O}_W(mA),$$

for all $q \geq 0$. Therefore,

$$E_2^{q,q} = H^p(W, R^q f_*(\mathcal{F} \otimes \mathcal{O}_V(f^* mA))) = (0)$$

if $p > 0$ and $q >> 0$ ($m >> 0$). Consequently, the Leray SS degenerates and this implies

$$H^0(W, R^q f_*(\mathcal{F} \otimes \mathcal{O}_V(f^* mA))) \xrightarrow{\sim} H^q(V, \mathcal{F} \otimes \mathcal{O}_V) f^* mA)).$$

Thus, if $q > 0$, then the right-hand side is (0) (by hypothesis). This implies that the global sections of $R^q f_*(\mathcal{F} \otimes \mathcal{O}_V f^*(mA))$ vanish and so (by the projection formula), the global sections of $(R^q f_* \mathcal{F}) \otimes \mathcal{O}_W(mA)$ vanish for $q > 0$. As $(R^q f_* \mathcal{F}) \otimes \mathcal{O}_W(mA)$ is generated by global sections, we deduce that

$$(R^q f_* \mathcal{F}) \otimes \mathcal{O}_W(mA) = (0).$$

Therefore, $R^q f_* \mathcal{F} = (0)$, for $q > 0$. $\square$

Proof of Theorem 1.34. The theorem is local on X, therefore we may assume that X is affine. The idea is to "compactify" the situation $Y \longrightarrow X$. We can close up X to get $\overline{X} \subseteq \mathbb{P}^N$. Check that there is some $\overline{Y}$ (projective) and a morphism, $\overline{f} \colon \overline{Y} \to \overline{X}$, with $Y \hookrightarrow \overline{Y}$ (Y dense in $\overline{Y}$) so that the diagram

$$\begin{array}{ccc} Y & \hookrightarrow & \overline{Y} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ X & \hookrightarrow & \overline{X} \end{array}$$

is cartesian (easy). This means that

$$Y = \overline{Y} \prod_{\overline{X}} X.$$

By Hironaka, we can resolve $\overline{Y}$ and we get $\widetilde{Y}$. The morphism $\widetilde{Y} \longrightarrow \overline{X}$ is equal to $Y \longrightarrow X$ when restricted to Y. Moreover, by denseness

$$R^p \overline{f}(K_{\widetilde{Y}}) \restriction X = R^p f_*(K_Y).$$

Consequently, we may assume from the outset that X and Y are projective as well as smooth (and we still have an alteration). Now take A ample on X, for $m >> 0$, we have

(a) $f^*(mA) = \text{nef}$;

(b) $f^*(mA) = \text{big}$, as F is generically finite.

By Kawamata-Vichweg,

$$H^p(Y, \mathcal{O}_Y(K_Y) \otimes \mathcal{O}_Y(f^*(mA))) = (0)$$

if $p > 0$ and $m >> 0$. Then, the lemma implies

$$(R^p f_*)(\mathcal{O}_Y(K_Y)) = (0), \quad p > 0.$$

This concludes the proof. $\square$

Now, take X and a resolution, $\mu \colon X' \to X$. We can make $\mu_* \mathcal{O}_{X'}(K_{X'})$.

Claim: This coherent sheaf is *independent* of the resolution.

Take another resolution, $\nu \colon X'' \to X$ and look at the Cartesian diagram

$$\begin{array}{ccccc} & & X' \prod_X X'' & & \\ & {\scriptstyle pr_1}\swarrow & & \searrow{\scriptstyle pr_2} & \\ X' & & & & X'' \\ & {\scriptstyle \mu}\searrow & & \swarrow{\scriptstyle \nu} & \\ & & X & & \end{array}$$

So, $X''' = X' \prod_X X''$ is a again a resolution of X, say $\theta\colon X''' \to X$. Then,

$$\theta_*(K_{X'''}) = \mu_*(pr_1(K_{X'''})) = \nu_*((pr_2)_*(K_{X'''})).$$

By Hartshorne (Chapter II), as X', X'', X''' are all smooth and birationally equivalent, we get

$$\begin{aligned} pr_1(K_{X'''}) &= K_{X'} \\ pr_2(K_{X'''}) &= K_{X''}. \end{aligned}$$

Independence follows. □

In view of the independence result just established, set $\mathcal{K}_X = \mu_*(\mathcal{O}_{X'}(K_{X'}))$, for any resolution, $\mu\colon X' \to X$. The sheaf $\mathcal{K}_X$ is coherent on X and it is called the *Grauert-Riemenschneider canonical sheaf* of X.

Remark: The Kawamata-Vichweg Vanishing Theorem works for $\mathcal{K}_X$.

Proposition 1.36 *If X is an irreducible variety and D is nef and big on X, then*

$$H^p(X, \mathcal{K}_X \otimes \mathcal{O}_X(D)) = (0), \qquad p > 0.$$

Proof. Take a resolution, $\mu\colon X' \to X$, then $\mathcal{K}_X = \mu_*(K_{X'})$. The divisor $\mu^*(mD)$ is nef and big on X' and X' is smooth. Then, by Kawamata-Vichweg,

$$H^p(X', \mathcal{O}_{X'}(K_{X'}) \otimes \mu^*(mD)) = (0).$$

Observe that

$$R^q\mu_*(\mathcal{O}_{X'}(K_{X'}) \otimes \mu^*(mD)) = R^q\mu_*\mathcal{O}_{X'}(K_{X'}) \otimes \mathcal{O}_X(mD).$$

Grauert-Riemenschneider (Theorem 1.34) implies the above is zero for $q > 0$ and the Leray SS implies

$$H^p(X, \mathcal{K}_X \otimes \mathcal{O}_X(mD)) \cong H^p(X', \mathcal{O}_{X'}(K_{X'} + \mu^* mD)).$$

Take $m = 1$ and apply the Kawamata-Vichweg Vanishing Theorem to the right-hand side to finish the proof.

Rational Singularities.

Definition 1.4 A variety, X, has *rational singularities* iff

(1) X is normal and

(2) There exists a resolution, $\mu\colon X' \to X$, so that $R^p\mu_*\mathcal{O}_{X'} = (0)$, for all $p > 0$.

Any resolution works if one does:

$$\begin{array}{ccccc} & & X''' & & \\ & \overset{f}{\swarrow} & & \overset{g}{\searrow} & \\ X' & & & & X'' \\ & \underset{\mu}{\searrow} & & \underset{\nu}{\swarrow} & \\ & & X & & \end{array}$$

As X''', X', X'' are smooth, $R^q f_* \mathcal{O}_{X'''} = (0)$ and $R^q g_* \mathcal{O}_{X'''} = (0)$, for all $q > 0$. Also, $\mu \circ f = \nu \circ g$ implies (using the composed spectral sequence)

$$\begin{array}{ccc} R^\bullet(\mu \circ f)_* \mathcal{O}_{X'''} & = & R^\bullet(\nu \circ g)_* \mathcal{O}_{X'''} \\ \Uparrow & & \Uparrow \\ R^p \mu_*(R^q f_* \mathcal{O}_{X'''}) & & R^p \nu_*(R^q g_* \mathcal{O}_{X'''}) \end{array}$$

The rest is clear. (Rational singularities are also called *DuVal singularities*, after Duval who studied them for surfaces—1934.)

Proposition 1.37 *Suppose X has rational singularities and D is nef and big on X. Then,*

$$H^p(X, \mathcal{O}_X(-D)) = (0), \qquad p < \dim X.$$

Proof. Make a resolution of singularities, $\mu\colon X' \to X$, then $\mu^* D$ is big and nef. Apply the Kawamata-Vichweg Vanishing Theorem to $\mu^* D$: we get

$$H^p(X', \mu^*(-D)) = (0), \qquad p < \dim X'.$$

By the projection formula

$$R^p \mu_*(\mathcal{O}_{X'}(\mu^*(-D))) = R^p \mu_* \mathcal{O}_{X'} \otimes \mathcal{O}_X(-D),$$

and the right-hand side vanishes by rational singularities. The Leray SS tells us that

$$H^p(X, \mathcal{O}_X(-d)) \xrightarrow{\sim} H^p(X', \mathcal{O}_{X'}(-\mu^* D))$$

and the proposition follows. $\square$

Theorem 1.38 *(Fujita's Vanishing Theorem) Say X is a projective scheme of finite type, H is an ample line bundle on X and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module. There exists an $m_0 = m_0(\mathcal{F}, H)$, so that for all nef, D,*

$$H^p(X, \mathcal{F}(mH + D)) = (0), \qquad p > 0,\ m \geq m_0.$$

Remark: If $D = (0)$, this is Serre's ampleness criterion. The content of this theorem is that the result holds for *all* nef divisors with *the same* m_0.

Proof. If X is a curve, the theorem holds by Riemann-Roch. What about non-reduced, reducible, *etc.*?

Note that: H ample on X iff $H \restriction X_{\mathrm{red}}$ is ample on X_{red} and H is ample on X iff $H \restriction$ irred. components of X each are ample.

Therefore, we may assume that X is reduced and irreducible. We use induction on $\dim X$. Then it will be true of the support, $\mathrm{Supp}(\mathcal{F})$, since $\dim(\mathrm{Supp}(\mathcal{F})) < \dim X$.

Claim. Say there is an integer, a, so that the result is true for $\mathcal{F} = \mathcal{O}_X(aH)$, then the result holds for all $\mathcal{F}$ on X.

For, given $\mathcal{O}_X(b_i H)$, $i = 1, \ldots, t$, we can twist sufficiently high (depending on the b_i's) and get above m for aH, then the result holds for

$$\coprod_{i=1}^{t} \mathcal{O}_X(b_i H)^{l_i}.$$

Now, Serre proved (FAC): Given $\mathcal{F}$ on X, we have

$$\cdots \longrightarrow \mathcal{O}_X(b_2 H)^{p_2} \longrightarrow \mathcal{O}_X(b_1 H)^{p_1} \longrightarrow 0$$

is exact. If only finitely many b_i's appear, then using exact sequences and the result for the b_i's, we get m_0 for $\mathcal{F}$. If infinitely many terms appear, the cohomology for $\mathcal{F}$ uses in higher dimensions the cohomology for the K_i's where

$$K_i = \mathrm{Ker}\,(\mathcal{O}_X(b_i H) \longrightarrow K_{i-1})$$

and in high dimensions any cohomology on X is zero. We are reduced to the case: $\mathcal{F} = \mathcal{O}(aH)$, for $a >> 0$.

Now, take a resolution of singularities

$$\mu\colon X' \to X,$$

and look at

$$\mathcal{O}_{X'}(K_{X'}) \quad \text{and} \quad \mathcal{K}_X = \mu_* \mathcal{O}_{X'}(K_{X'})$$

where $\mathcal{K}_X$ is the Grauert-Riemenschneider canonical sheaf on X. As $a >> 0$, $\mu^*(\mathcal{O}_X(aH)) - K_{X'}$ is generated by its sections. Take, σ_1, a nontrivial section, we get

$$0 \longrightarrow \mathcal{O}_{X'} \stackrel{\text{``}\sigma''}{\longrightarrow} \mathcal{O}_{X'}(\mu^* \mathcal{O}_X(aH)) \otimes \mathcal{O}_{X'}(K_{X'})^D.$$

Therefore, we get

$$0 \longrightarrow \mathcal{O}_{X'}(K_{X'}) \longrightarrow \mathcal{O}_{X'}(\mu^* \mathcal{O}_X(aH)).$$

As μ_* is left exact, using the projection formula we get

$$0 \longrightarrow \mathcal{K}_X \stackrel{u}{\longrightarrow} \mathcal{O}_X(aH) \longrightarrow \operatorname{cok} u \longrightarrow 0.$$

Now, $\operatorname{cok} u$ has lower dimensional support. Were the theorem true when $\mathcal{F} = \mathcal{K}_X$, then we would be done using the cohomology sequence. Thus, we must show

$$H^p(X, \mathcal{K}_X \otimes \mathcal{O}_X(mH) \otimes \mathcal{O}_X(D)) = (0) \qquad (\dagger)$$

if $p > 0$, $m \geq m_0$ and all D (nef). Now, the sheaf inside this cohomology is

$$R^p\mu_*\mathcal{O}_{X'}(K_{X'}) \otimes \mu_*(\mu^*(mH + D)).$$

By the Grauert-Riemanschneider Theorem and Leray, we deduce that the cohomology group in (†) is

$$H^p(X', \mathcal{O}_{X'}(K_{X'}) + \mu^*(mH + D))$$

and $\mu^*(mH + D))$ is big and nef. So, by Kawamata-Vichweg Vanishing, this group vanishes (independently of D) and the proof is complete. □

Here is an interesting consequence of Fujita's Theorem:

Theorem 1.39 *Say X is projective, with $\dim X = n$. If $\mathcal{F}$ is a coherent sheaf on X, then $\dim H^p(X, \mathcal{F}(mD)) = O(m^{n-p})$ whenever D is nef.*

Proof. Pick H very ample on X and H should avoid all irreducible subvarieties corresponding to the associated primes of the given $\mathcal{F}$. Pick D, nef. Look at $0, D, 2D, \ldots, rD$, all nef. Then, Fujita's Theorem implies that

$$H^p(X, \mathcal{F}(H + rD)) = (0), \qquad p > 0.$$

Use induction on $\dim X$. For curves, the result holds by Riemann-Roch. Then, we have the exact sequence

$$0 \longrightarrow \mathcal{F}(rD) \longrightarrow \mathcal{F}(H + rD) \longrightarrow \mathcal{F}(H + rD) \restriction H \longrightarrow 0.$$

Apply cohomology and induction for $p \geq 1$; we get

$$\dim H^p(X, \mathcal{F}(rD)) \leq \dim H^{p-1}(H, \mathcal{F}(H + rD) \restriction H)$$

and on the right-hand side, this yields

$$O(r^{(n-1)-(p-1)}) = O(r^{n-p}),$$

as claimed. □

Question. Look at a curve and an ample divisor, D, on it. Thus, $\deg D > 0$. We know mD is very ample in general for $m >> 0$ but on a curve there is a uniform bound, $m \geq 2g + 1$.

Given X, with $\dim X > 1$ and D ample, is there some $m = m(X)$ such that mD is very ample?

The answer is **no**, even if X is a smooth projective surface. Here is an example due to Kollar.

Start with an elliptic curve, E, and make the surface, $S = E \prod E$. Let F_1, F_2 be the obvious fibres. Given n, write

$$A_n = F_1 + (n^2 - n + 1)F_2 - (n-1)\Delta,$$

a family of divisors on S. Observe that

$$F_1^2 = F_2^2 = \Delta^2(2-2g) = 0; F_1 \cdot F_2 = 1; F_i \cdot \Delta = 1.$$

Consequently,

$$\begin{aligned} A_n^2 &= 2[n(n^2-n+1) - n(n-1) - (n-1)(n^2-n+1)] \\ &= 2(n^2 - n + 1 - n^2 + n) = 2. \end{aligned}$$

Also, $A_n \cdot F_1 = n^2 - 2n + 2 > 0$ if $n \geq 1$, $A_n \cdot F_2 = 1 > 0$ and $A_n \cdot \Delta = n^2 + 1 > 0$. By Nakai-Moishezon, A_n is ample for $n \geq 1$.

Let $D = F_1 + F_2$ and look at $2D$. As $2D$ is ample there is a smooth $B \subseteq |2D|$. Now, take the cyclic cover of S of degree 2 branched along B, call it X. Let $\pi \colon X \to S$ and write $D_n = \pi^* A_n$.

Recall that for the cyclic cover of degree r,

$$\pi_* \mathcal{O}_X = \mathcal{O}_S \coprod \mathcal{O}_S(-B) \coprod \cdots \coprod \mathcal{O}_S(-(r-1)B).$$

For us,

$$\pi_* \mathcal{O}_X = \mathcal{O}_S \coprod \mathcal{O}_S(-B).$$

Then,

$$\pi_*(\mathcal{O}_X(nD_n)) = \mathcal{O}_S(nA_n) \coprod \mathcal{O}_S(nA_n - B).$$

There is a canonical injection

$$H^0(S, \mathcal{O}_S(nA_n)) \longrightarrow H^0(X, \mathcal{O}_X(nD_n)).$$

Were this injection an isomorphism, then nD_n could not be very ample (dimensions are too small). Therefore, the number corresponding to D_n to make is very ample is at least n. It remains to prove that

$$H^0(S, \mathcal{O}_S(nA_n - B)) = (0).$$

We have

$$\begin{aligned} (nA - B)^2 &= (nA - 2(F_1 + F_2))^2 \\ &= 2n^2 + 8 - 4n(A_n \cdot F_1 + A_n \cdot F_2) \\ &= 2n^2 + 8 - 4n(n^2 - 2n + 2 + 1) \\ &= -O(n^3) < 0 \qquad \text{if } n \geq 3. \end{aligned}$$

Therefore, out cohomology group vanishes.