Algebra

by

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To Peter A. Cassileth and to the memories of Stanley M.K. Chung and Ralph C. Marcove, physicians and friends all. Being mortal and denied the gift of life, he gives and they gave the next best thing: Time

To Anne, Mia, Philippe and Sylvie

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Preface

A book on "Abstract" or "Modern" Algebra is a commonplace thing in today's mathematical milieu. Even a book for *well-prepared, serious* beginning graduate students who intend to become research mathematicians is not so strange any longer. But, the genesis of this book, which *is* intended for serious, *well-prepared* graduate students, is somewhat strange.

To begin with, it is a reworking of notes for a year long graduate course I gave several years ago–not in itself a strange thing. But, I possess no such notes nor did I ever make any and I never lecture with a written *aide memoir* of any sort. Rather, my method is to work out fully during lecture (at the board) each proof and example. Students will thus see what are the "inner workings" of the subject. Of course, this is pedagogically to their advantage and, furthermore, it slows me down.

Then where did the notes (to be reworked) come from? They were provided by my friend and colleague Jean H. Gallier (of the Computer Science Department at Penn). Determined to augment his mathematical knowledge, he began several years ago to audit some of my graduate courses. "Audit" for him means faithfully attending lectures, doing all the problem assignments, participating in each bi-weekly problem session (where he takes his turn presenting problems), writing excellent notes from my oral presentation *and rendering these notes in LaTeX form*.[1] That this book will appear is, in large measure, his doing. While I have been responsible for its writing, he has on occasion introduced results and/or alternate proofs that have rendered some material more perspicacious from a student's point of view–these have improved the text. He is in every sense a joint author, save that errors are solely my responsibility. There is no way I can thank him adequately here in plain words and I won't try except to say, *Je te remercie vivement, mon ami Jean, pour tout ton travail*.

Others should be thanked as well–in particular the members of the class that attended the course from which the book is formed.[2] By their interest and attention to detail, they kept me on my toes. One particular member of that class deserves special mention: Mathew Cross.[3] Mathew started the index and set the original 115 problems in LaTeX. He lightened our burden by a considerable amount.

The content of the book follows rather closely the oral lectures–with just a few exceptions. These are: In Chapter 3, the section on Integral Dependence is now augmented by proofs of all results, the original lectures had statements only of some of these (due to exigencies of time) and Gallier insisted on a full treatment. In Chapter 4, the sections on Norms and Traces as well as Kummer Theory and Transcendental Extensions are likewise augmented by full proofs. In Chapter 5, there is now more to the section on (co)homological functors and there are full proofs in the last section on the Koszul Complex. Otherwise, the material is just (a smoothed out version of) what was presented. One will have to move fast to present it to students in one year, at least I did.

But the heart of the book is the Problem section. Here, I've attempted to simulate at the beginning graduate level some of the features of real mathematical work. There is a jumbling of the problems *vis a*

[1] One must realize he maintains a full research and teaching schedule, directs Ph.D. students, attends to administrative duties and has a family life in addition to this "auditing"!

[2] The members of the class were: A. Bak, D. Boyarchenko, S. Brooks, M. Campbell, S. Corry, M. Cross, C. Daenzer, C. Devena, J. Gallier, S. Guerra, C. Hoelscher, T. Jaeger, J. Long, S. Mason, T. Zhu.

[3] Mathew spells his name with but one "t"; there is no misprint.

vis subject matter just as in real research one never knows what kind of mathematics will be needed in the solution of a problem. There is no hint of the level of difficulty of a problem (save for the few problems where suggestions are offered), and anyway the notion of difficulty is ill-defined. And, the problems refer to each other, just as in real work one is constantly reminded of past efforts (successful or not). In effect, as suggested in the preface for students, one should begin with the problems and use the text as a means to fill in knowledge as required to do them (as well as to do other problems assigned by an instructor in this course or another course).

This brings me to the text material itself. There is no attempt to be encyclopedic. After all, the material is a faithful copy of what was actually covered in a year and any competent instructor can add material that has been omitted. I regret not covering the Wederburn-Artin Theory of DCC rings, the Brauer Group, and some basic material on group representations. What is covered, however, is to my mind central to the education of any prospective mathematician who aspires to contribute to what is now the mainstream of mathematical endeavor. Also, while there are over 150 problems filling some 55 pages of text (some of the problems are rather long being multi-part), other problems of an instructor's choosing can certainly be assigned. As to the attribution of the origins of these problems, I have assigned names when they are known to me. If no name is assigned, the problem comes from some source in my past (perhaps one of my own teachers or their teachers) and in no way do I claim it as my own. Good problems from all sources are the treasure hoard of practicing mathematicians in their role as passers on of our common heritage.

I refer to the special symbols (DX) and the "curves ahead" road sign (appearing at odd places in the text) in the student preface; no repeat of the explanations I offer there is necessary. If you as instructor are lucky enough to have a class as interested and tough to satisfy as I did, you are lucky indeed and need no further assurance that mathematics will be in good hands in the future. I intend this book to be of service to such individuals as they begin their long climb to mathematical independence and maturity.

Tolda Santa Cotogna
Summer, 2006

For the Student

It may be surprising but the most important part of the book you now hold before you is the very last section–the one labeled "Problems". To learn mathematics one must *do* mathematics. Indeed, the best way to read this book is to turn immediately to the problem section and begin to do the problems. Of course, you will soon reach some unknown terminology or not have enough knowledge to meet the technical demands of a problem and this is where you turn to the text to fill in gaps, see ideas explained and techniques demonstrated. Then you plunge once more back into the problems and repeat the whole process.

The book is designed for serious, well-prepared students who plan on becoming research mathematicians. It presumes you have had previous acquaintance with algebra; in particular you have met the concepts of group, ring, field, vector space, homomorphism, isomorphism, and the elementary theorems about these things. No book on mathematics can be simply read, rather you must recreate the text yourself line by line checking at each stage all details including those omitted. This is slow work and, as you know, mathematics has very high density on the page.

In the text, you will find two special symbols: (DX) and a sign such as one sees on the road warning of dangerous curves ahead. The symbol (DX) stands for "diagnostic exercise", it means some elementary details have been omitted and that supplying them should be easy. However, if supplying them is not easy, then you should go back a page or two as something fundamental has skipped you by. In this way, the sign (DX) is like a medical test: failing it is sure to tell you if something is wrong (no false positives), however, if you pass it (supply the details), something still might be wrong. Just read on and anything wrong will surface later. As for the dangerous curves sign, it precedes counter-examples to naively made conjectures, it warns when things could go wrong if hypotheses are omitted, and generally forces you to slow down in the reading and recreating.

If you use this book in a course or even for self study, I recommend that you tackle the problems in a small group (two to four persons, total). This is because no person has a monopoly on ideas, a good idea or half-idea can germ in any head, and the working out of a problem by a committed group is akin to the actual way much research mathematics is accomplished. In your group, you want constant give and take, and there must be time to think alone so that a real contribution to the group's effort can be made.

The problems are all jumbled up by area and there is no signal given as to a problem's difficulty (exceptions are the few cases where hints or suggestions are given). In real mathematical life, no signs are given that a question being attacked involves a certain small area of mathematical knowledge or is hard or easy; any such sign is gleaned by virtue of experience and that is what you are obtaining by *doing* mathematics in these problems. Moreover, hard and easy are in the eyes of the beholder; they are not universal characteristics of a problem. About all one can say is that if a large number of people find a problem difficult, we may classify it so. However, we shouldn't be surprised when an individual solves it and claims that, "it was not that hard". In any case, guard against confusing mathematical talent either with overall intelligence or with mathematical speed. Some quick people are in fact talented, many are just quick. Don't be discouraged if you find yourself slower than another, the things that really count in doing mathematics (assuming talent) are persistence and courage.

I can think of no better lines to close with than these which come from B. Pasternak's poem entitled "*Night*"[4]

"And maybe in an attic
And under ancient slates
A man sits wakeful working
He thinks and broods and waits."

"He looks upon the planet,
As if the heavenly spheres
Were part of his entrusted
Nocturnal private cares."

"Fight off your sleep: be wakeful,
Work on, keep up your pace,
Keep vigil like the pilot,
Like all the stars in space."

"Work on, work on, creator–
To sleep would be a crime–
Eternity's own hostage,
And prisoner of Time."

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Summer, 2006

[4]From the collection entitled "*When It Clears Up*", 1956. Translated by Lydia Pasternak Slater (the poet's sister).