Chapter 1

Basics of the Differential Geometry of Curves

1.1 Introduction: Parameterized Curves

Properties of curves can be classified into *local properties* and *global properties*.

Local properties are the properties that hold in a small neighborhood of a point on a curve. Curvature is a local property.

Local properties can be studied more conveniently by assuming that the curve is parameterized locally.

Thus, it is important and useful to study parameterized curves.

In order to study the global properties of a curve, such as the number of points where the curvature is extremal, the number of times that a curves wraps around a point, or convexity properties, topological tools are needed.

A proper study of global properties of curves really requires the introduction of the notion of a manifold, a concept beyond the scope of this class.

Let $\mathcal{E}$ be some normed affine space of finite dimension, for the sake of simplicity, the Euclidean space $\mathbb{E}^2$ or $\mathbb{E}^3$.

Recall that the Euclidean space $\mathbb{E}^m$ is obtained from the affine space $\mathbb{A}^m$ by defining on the vector space $\mathbb{R}^m$ the standard inner product

$$(x_1, \ldots, x_m) \cdot (y_1, \ldots, y_m) = x_1 y_1 + \cdots + x_m y_m.$$

The corresponding Euclidean norm is

$$\|(x_1, \ldots, x_m)\| = \sqrt{x_1^2 + \cdots + x_m^2}.$$

Inspired by a kinematics view, we can define a curve as a continuous map $f\colon\,]a, b[\to \mathcal{E}$, from an open interval $I =]a, b[$ of $\mathbb{R}$ to the affine space $\mathcal{E}$.

From this point of view, we can think of the parameter $t \in]a, b[$ as time, and the function f gives the position $f(t)$ at time t of a moving particle.

The image $f(I) \subseteq \mathcal{E}$ of the interval I is the trajectory of the particle.

In fact, only asking that f be continuous turns out to be too liberal, as rather strange curves turn out to be definable, such as "square-filling curves", due to Peano, Hilbert, Sierpinski, and others (see the problems).

A very pretty square-filling curve due to Hilbert is defined by a sequence (h_n) of polygonal lines $h_n\colon [0,1] \to [0,1] \times [0,1]$ starting from the simple pattern h_0 (a "square cap" $\sqcap$) shown on the left in figure 8.1.

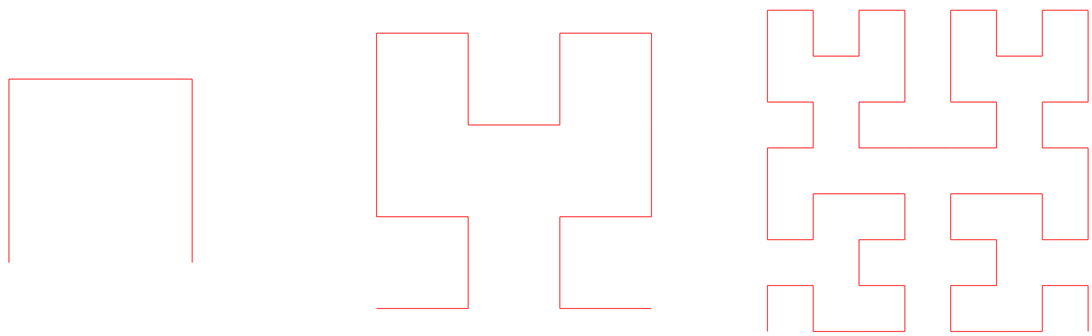

Figure 1.1: A sequence of Hilbert curves h_0, h_1, h_2

It can be shown that the sequence (h_n) converges (pointwise) to a continuous curve $h\colon [0,1] \to [0,1] \times [0,1]$ whose trace is the entire square $[0,1] \times [0,1]$.

The Hilbert curve h is nowhere differentiable. It also has infinite length! The curve h_6 is shown in figure 8.2:

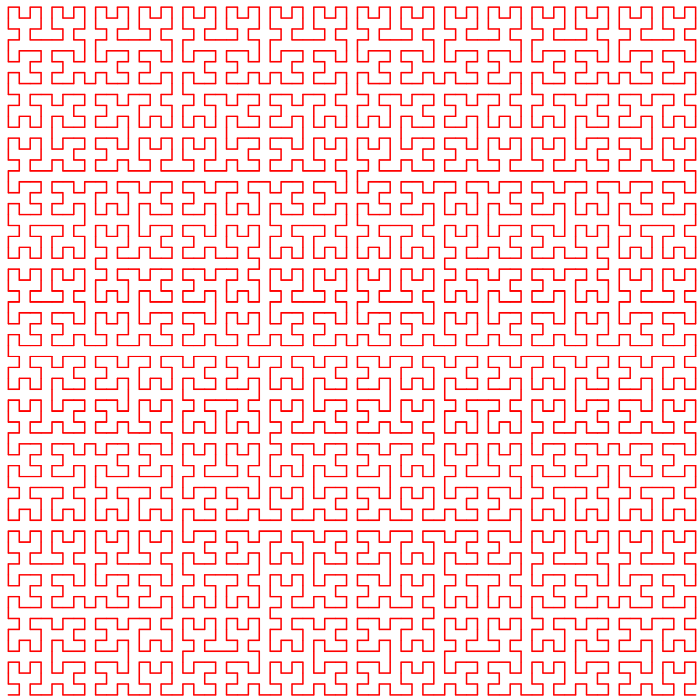

Figure 1.2: The Hilbert curve h_6

Actually, there are many fascinating curves that are only continuous, fractal curves being a major example (see Edgar [**?**]), but for our purposes, we need the existence of the tangent at every point of the curve (except perhaps for finitely many points).

This leads us to require that $f\colon\ ]a, b[\ \to \mathcal{E}$ be at least continuously differentiable, or as we said earlier, at least a C^1-function.

However, asking that $f\colon\ ]a,b[\ \to \mathcal{E}$ be a C^p-function for $p \geq 1$, still allows unwanted curves.

For example, the plane curve defined such that

$$f(t) = \begin{cases} (0,\ e^{1/t}) & \text{if } t < 0\ ; \\ (0,\ 0) & \text{if } t = 0\ ; \\ (e^{-1/t},\ 0) & \text{if } t > 0\ ; \end{cases}$$

is a C^∞-function, but $f'(0) = 0$, and thus the tangent at the origin is undefined.

What happens is that the curve has a sharp "corner" at the origin.

Similarly, the plane curve defined such that

$$f(t) = \begin{cases} (-e^{1/t},\ e^{1/t}\sin(e^{-1/t})) & \text{if } t < 0\ ; \\ (0,\ 0) & \text{if } t = 0\ ; \\ (e^{-1/t},\ e^{-1/t}\sin(e^{1/t})) & \text{if } t > 0\ ; \end{cases}$$

shown in Figure 8.3 is a C^∞-function, but $f'(0) = 0$. In this case, the curve oscillates more and more around the origin.

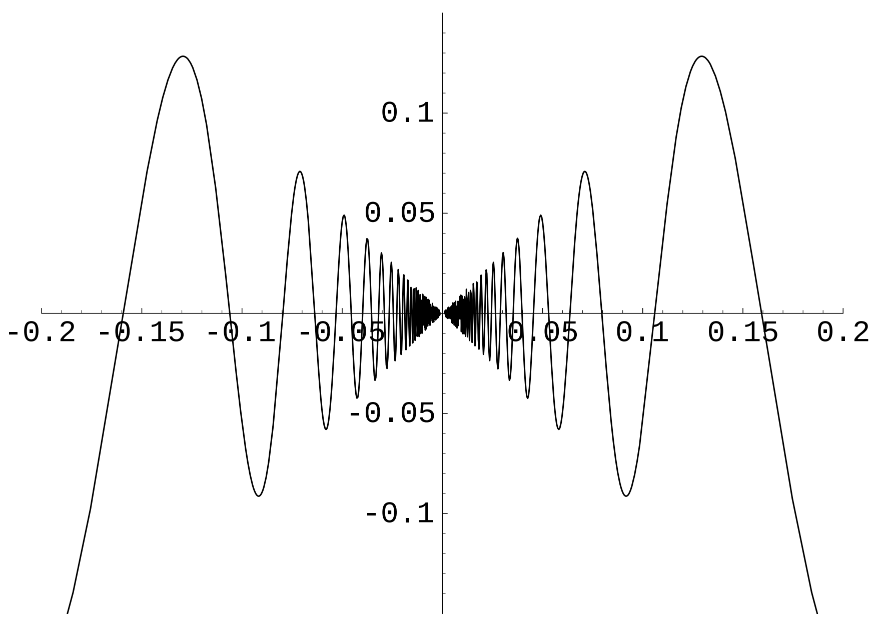

Figure 1.3: Stationary point at the origin

The problem with the above examples is that the origin is a singular point for which $f'(0) = 0$ (a stationary point).

Although it is possible to define the tangent when f is sufficiently differentiable and when for every $t \in]a, b[$, $f^{(p)}(t) \neq 0$ for some $p \geq 1$ (where $f^{(p)}$ denotes the p-th derivative of f), a systematic study is rather cumbersome.

Thus, we will restrict our attention to curves having only regular points, that is, for which $f'(t) \neq 0$ for every $t \in]a, b[$.

However, we will allow functions $f\colon\,]a, b[\, \to \mathcal{E}$ that are not necessarily injective, unless stated otherwise.

Definition 1.1.1 An *open curve (or open arc) of class* C^p is a map $f\colon\,]a, b[\, \to \mathcal{E}$ of class C^p, with $p \geq 1$, where $]a, b[$ is an open interval (allowing $a = -\infty$ or $b = +\infty$). The set of points $f(]a, b[)$ in $\mathcal{E}$ is called the *trace of the curve* f.

A point $f(t)$ is *regular at* $t \in\,]a, b[$ iff $f'(t)$ exists and $f'(t) \neq 0$, and *stationary* otherwise. A *regular open curve (or regular open arc) of class* C^p is an open curve of class C^p, with $p \geq 1$, such that every point is regular, i.e. $f'(t) \neq 0$ for every $t \in\,]a, b[$.

Note that definition 8.1.1 is stated for an open interval $]a, b[$, and thus, f may not be defined at a or b.

If we want to include the boundary points at a and b in the curve (when $a \neq -\infty$ and $b \neq +\infty$), we use the following definition.

Definition 1.1.2 A *curve (or arc) of class* C^p is a map $f\colon [a, b] \to \mathcal{E}$, with $p \geq 1$, such that the restriction of f to $]a, b[$ is of class C^p, and where $f^{(i)}(a) = \lim_{t \to a, t > a} f^{(i)}(t)$ and $f^{(i)}(b) = \lim_{t \to b, t < b} f^{(i)}(t)$ exist, where $0 \leq i \leq p$.

A *regular curve (or regular arc) of class* C^p is a curve of class C^p, with $p \geq 1$, such that every point is regular, i.e. $f'(t) \neq 0$ for every $t \in [a, b]$. The set of points $f([a, b])$ in $\mathcal{E}$ is called the *trace of the curve* f.

It should be noted that even if f is injective, the trace $f(I)$ of f may be self-intersecting.

Consider the curve $f\colon \mathbb{R} \to \mathbb{E}^2$ defined such that,

$$f_1(t) = \frac{t(1+t^2)}{1+t^4},$$
$$f_2(t) = \frac{t(1-t^2)}{1+t^4}.$$

The trace of this curve is called the "lemniscate of Bernoulli", and it has a self-intersection at the origin.

The map f is continuous, and in fact bijective, but its inverse f^{-1} is not continuous.

Self-intersection is due to the fact that

$$\lim_{t\to-\infty} f(t) = \lim_{t\to+\infty} f(t) = f(0).$$

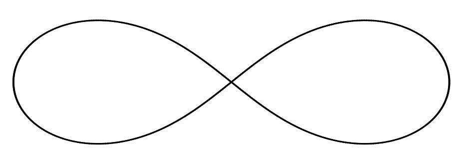

Figure 1.4: Lemniscate of Bernoulli

If we consider a curve $f\colon [a, b] \to \mathcal{E}$ and we assume that f is injective on the entire *closed* interval $[a, b]$, then the trace $f([a, b])$ of f has no self-intersection.

Such curves are usually called *Jordan arcs*, or *simple arcs*.

Because $[a, b]$ is compact, f is in fact a homeomorphism between $[a, b]$ and $f([a, b])$.

Many fractal curves are only continuous Jordan arcs that are not differentiable.

We can also define closed curves. A simple way to do so is to say that a closed curve is a curve $f\colon [a,b] \to \mathcal{E}$ such that $f(a) = f(b)$.

However, this does not ensure that the derivatives at a and b agree, something quite undesirable. A better solution is to define a closed curve as an open curve $f\colon \mathbb{R} \to \mathcal{E}$, where f is periodic.

Definition 1.1.3 A *closed curve (or closed arc) of class* C^p is a map $f\colon \mathbb{R} \to \mathcal{E}$, such that f is of class C^p, with $p \geq 1$, and such that f is *periodic*, which means that there is some $T > 0$, such that $f(x+T) = f(x)$ for all $x \in \mathbb{R}$.

A *regular closed curve (or regular closed arc) of class* C^p is a closed curve of class C^p, with $p \geq 1$, such that every point is regular, i.e. $f'(t) \neq 0$ for every $t \in \mathbb{R}$. The set of points $f([0,T])$ (or $f(\mathbb{R})$) in $\mathcal{E}$ is called the *trace of the curve* f.

A closed curve is a *Jordan curve (or a simple closed curve)* iff f is injective on the interval $[0, T[$. A Jordan curve has no self-intersection.

The ellipse defined by the map $t \mapsto (a \cos t,\ b \sin t)$ is an example of a closed curve of type C^∞, which is a Jordan curve. In this example, the period is $T = 2\pi$.

It is possible that the trace of a curve be defined by many parameterizations, as illustrated by the unit circle, which is the trace of the parameterized curves $f_k\colon]0, 2\pi[\to \mathcal{E}$ (or $f_k\colon [0, 2\pi] \to \mathcal{E}$), where $f_k(t) = (\cos kt,\ \sin kt)$, with $k \geq 1$.

A clean way to handle this phenomenon is to define a notion of *geometric arc curve*. Such a treatment is given in Berger and Gostiaux [**?**].

For our purposes, it will be sufficient to define a notion of change of parameter which does not change the "geometric shape" of the trace.

Recall that a *diffeomorphism* $g\colon]a, b[\to]c, d[$ *of class* C^p from an open interval $]a, b[$ to another open interval $]c, d[$, is a bijection such that both $g\colon]a, b[\to]c, d[$ and its inverse $g^{-1}\colon]c, d[\to]a, b[$ are C^p-functions.

This implies that $f'(c) \neq 0$ for every $c \in]a, b[$.

Definition 1.1.4 Two regular curves $f\colon]a, b[\to \mathcal{E}$ and $g\colon]c, d[\to \mathcal{E}$ of class C^p, with $p \geq 1$, are C^p*-equivalent* iff there is a diffeomorphism $\theta\colon]a, b[\to]c, d[$ of class C^p such that $f = g \circ \theta$.

It is immediately verified that definition 8.1.4 yields an equivalence relation on open curves.

Definition 8.1.4 is adpated to curves, by extending the notion of C^p-diffeomorphism to closed intervals in the obvious way.

Remark: Using definition 8.1.4, we could define a *geometric curve (or arc) of class C^p* as an equivalence class of (parameterized) curves. This what is done in Berger and Gostiaux [**?**].

From now on, in most cases, we will drop the word "regular" when referring to regular curves, and simply say curves.

Also, when we refer to a point $f(t)$ on a curve, we mean that $t \in]a, b[$ for an open curve $f\colon]a, b[\to \mathcal{E}$, and $t \in [a, b]$ for a curve $f\colon [a, b] \to \mathcal{E}$.

In the case of a closed curve $f\colon \mathbb{R} \to \mathcal{E}$, we can assume that $t \in [0, T]$, where T is the period of f, and thus, closed curves will simply be treated as curves in the sequel.

1.2 Tangent Lines and Osculating Planes

We begin with the definition of a tangent line.

Definition 1.2.1 For any open curve $f\colon\ ]a,b[\ \to \mathcal{E}$ of class C^p (or curve $f\colon [a,b] \to \mathcal{E}$ of class C^p), with $p \geq 1$, given any point $M_0 = f(t)$ on the curve, if f is locally injective at M_0 and for any point $M_1 = f(t+h)$ near M_0, if the line $T_{t,h}$ determined by the points M_0 and M_1 has a limit T_t when $h \neq 0$ approaches 0, we say that T_t is the *tangent line to f in $M_0 = f(t)$ at t.*

More precisely, if there is an open interval $]t-\eta,\, t+\eta[\ \subseteq\ ]a,b[$ (with $\eta > 0$) such that $M_1 = f(t+h) \neq f(t) = M_0$ for all $h \neq 0$ with $h \in]-\eta,\, \eta[$ and the line $T_{t,h}$ determined by the points M_0 and M_1 has a limit T_t when $h \neq 0$ approaches 0 (with $h \in]-\eta,\, \eta[$), then T_t is the tangent line to f in M_0 at t.

For simplicity, we will often say tangent, instead of tangent line.

The definition is simpler when f is a simple curve (there is no danger that $M_1 = M_0$ when $h \neq 0$).

In this chapter, it is notationally more convenient to denote the vector **ab** as $b - a$. The following lemma shows why regular points are important.

Lemma 1.2.2 *For any open curve $f\colon]a, b[\to \mathcal{E}$ of class C^p (or curve $f\colon [a, b] \to \mathcal{E}$ of class C^p), with $p \geq 1$, given any point $M_0 = f(t)$ on the curve, if M_0 is a regular point at t, then the tangent line to f in M_0 at t exists and is determined by the derivative $f'(t)$ of f at t.*

Remarks: If $f'(t) = 0$, the above argument breaks down.

However, if f is a C^p-function and $f^{(p)}(t) \neq 0$ for some $p \geq 2$, where p is the smallest integer with that property, we can show that the line $T_{t,h}$ has the limit determined by M_0 and the derivative $f^{(p)}(t)$. Thus, the tangent line may still exist at a stationary point.

For example, the curve f defined by the map $t \mapsto (t^2, t^3)$ is a C^∞-function, but $f'(0) = 0$. Nevertheless, the tangent at the origin is defined for $t = 0$ (it is the x-axis).

However, some strange things can happen at a stationary point.

Note that the tangent at a point can exist, even when the derivative f' is not continuous at this point. For example, the C^0-curve f defined such that

$$f(t) = \begin{cases} (t,\, t^2 \sin(1/t)) & \text{if } t \neq 0; \\ (0,0) & \text{if } t = 0; \end{cases}$$

and shown in Figure 8.5 has a tangent at $t = 0$.

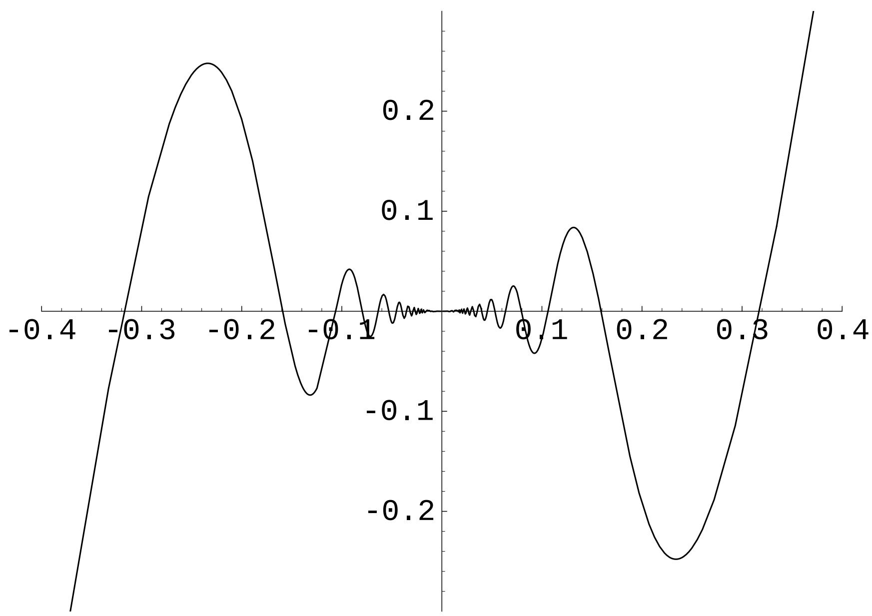

Figure 1.5: Curve with tangent at O and yet f' discontinuous at O

Indeed, $f(0) = (0, 0)$, and $\lim_{t\to 0} t\sin(1/t) = 0$, and the derivative at $t = 0$ is the vector $(1, 0)$. For $t \neq 0$,

$$f'(t) = (1, 2t\sin(1/t) - \cos(1/t)),$$

which has no limit when t tends to 0. Thus, f' is discontinuous at 0.

What happens is that f oscillates more and more near the origin, but the amplitude of the oscillations decreases.

The notion of tangent is intrinsic to the geometric curve defined by f. We now consider osculating planes.

Definition 1.2.3 For any open curve $f\colon]a,b[\to \mathcal{E}$ of class C^p (or curve $f\colon [a,b] \to \mathcal{E}$ of class C^p), with $p \geq 2$, given any point $M_0 = f(t)$ on the curve, if the tangent T_t at M_0 exists, the point $M_1 = f(t+h)$ is not on T_t for $h \neq 0$ small enough, and the plane $P_{t,h}$ determined by the tangent T_t and the point M_1 has a limit P_t when $h \neq 0$ approaches 0, we say that P_t is the *osculating plane to* f *in* $M_0 = f(t)$ *at* t.

More precisely, if the tangent T_t at M_0 exists, there is an open interval $]t-\eta, t+\eta[\subseteq]a,b[$ (with $\eta > 0$) such that the point $M_1 = f(t+h)$ is not on T_t for every $h \neq 0$ with $h \in]-\eta, +\eta[$, and the plane $P_{t,h}$ determined by the tangent T_t and the point M_1 has a limit P_t when $h \neq 0$ approaches 0 (with $h \in]-\eta, +\eta[$), we say that P_t is the osculating plane to f in $M_0 = f(t)$ at t.

Again, the definition is simpler when f is a simple curve. The following lemma gives a simple condition for the existence of the osculating plane at a point.

Lemma 1.2.4 *For any open curve $f\colon]a,b[\to \mathcal{E}$ of class C^p (or curve $f\colon [a,b] \to \mathcal{E}$ of class C^p), with $p \geq 2$, given any point $M_0 = f(t)$ on the curve, if $f'(t)$ and $f''(t)$ are linearly independent (which implies that M_0 is a regular point at t), then the osculating plane to f in M_0 at t exists and is determined by the first and second derivatives $f'(t)$ and $f''(t)$ of f at t.*

When $f'(t)$ and $f''(t)$ exist and are linearly independent, it is sometimes said that f is *biregular at t*, and that $f(t)$ *is a biregular point at t*.

From the kinematics point of view, the osculating plane at time t is determined by the position of the moving particle $f(t)$, the velocity vector $f'(t)$, and the acceleration vector $f''(t)$.

Remarks: If the curve f is a plane curve, then the osculating plane at every regular point is the plane containing the curve.

Even when $f'(t)$ and $f''(t)$ are linearly dependent, the osculating plane may still exists, for instance, if there are two derivatives $f^{(p)}(t) \neq 0$ and $f^{(q)}(t) \neq 0$ that are linearly independent, with $p < q$, the smallest integers with that property.

The notion of osculating plane is intrinsic to the geometric curve defined by f.

It should also be noted that the notions of tangent and osculating plane are affine notions, that is, preserved under affine bijections.

We now consider the notion of arc length. For this, we assume that the affine space $\mathcal{E}$ is a normed affine space of finite dimension with norm $\| \ \|$.

1.3 Arc Length

For simplicity, we can assume that $\mathcal{E} = \mathbb{E}^n$.

Given an interval $[a, b]$ (where $a \neq -\infty$ and $b \neq +\infty$), a *subdivision* of $[a, b]$ is any finite increasing sequence $t_0, \ldots, t_n$, such that $t_0 = a$, $t_n = b$, and $t_i < t_{i+1}$, for all i, $0 \leq i \leq n - 1$, where $n \geq 1$.

Given any curve $f\colon [a, b] \to \mathcal{E}$ of class C^p, with $p \geq 0$, for any subdivision $\sigma = t_0, \ldots, t_n$ of $[a, b]$, we obtain a polygonal line $f(t_0), f(t_1), \ldots, f(t_n)$ with endpoints $f(a)$ and $f(b)$, and we define the length of this polygonal line as

$$l(\sigma) = \sum_{i=0}^{n-1} \|f(t_{i+1}) - f(t_i)\| \,.$$

Definition 1.3.1 For any curve $f\colon [a, b] \to \mathcal{E}$ of class C^p, with $p \geq 0$, if the set $\mathcal{L}(f)$ of the lengths $l(\sigma)$ of the polygonal lines induced by all subdivisions $\sigma = t_0, \ldots, t_n$ of $[a, b]$ is bounded, we say that f is *rectifiable*, and we call the least upper bound $l(f)$ of the set $\mathcal{L}(f)$ the *length of* f.

It is obvious that $\|f(b) - f(a)\| \leq l(f)$. If $g = f \circ \theta$ is a curve C^p-equivalent to f, where θ is a C^p-diffeomorphism, since $\theta'(t) \neq 0$, θ is a strictly increasing or decreasing function, and thus, the set of sums of the form $l(\sigma)$ is the same for both f and g.

Thus, the notion of length is intrinsic to the geometric curve defined by f. This is false if θ is not strictly increasing or decreasing.

Lemma 1.3.2 *For any curve* $f\colon [a, b] \to \mathcal{E}$ *of class* C^p*, with* $p \geq 1$*,* f *is rectifiable.*

Remark: In fact, lemma 8.3.2 can be shown under the hypothesis that f is of class C^0, and that $f'(t)$ exists and $\|f'(t)\| \leq M$ for some $M \geq 0$, for all $t \in [a, b]$.

Definition 1.3.3 For any open curve $f\colon\,]a, b[\, \to \mathcal{E}$ of class C^p (or curve $f\colon [a, b] \to \mathcal{E}$ of class C^p), with $p \geq 1$, for any closed interval $[t_0, t] \subseteq\,]a, b[$ (or $[t_0, t] \subseteq [a, b]$, in the case of a curve), letting $f_{[t_0,t]}$ be the restriction of f to $[t_0, t]$, the length $l(f_{[t_0,t]})$ (which exists, by lemma 8.3.2) is called the *arc length of* $f_{[t_0,t]}$.

For any fixed $t_0 \in \,]a,b[$ (or any fixed $t_0 \in [a,b]$, in the case of a curve), we define the function $s\colon \,]a,b[\, \to \mathbb{R}$ (or $s\colon [a,b] \to \mathbb{R}$, in the case of a curve), called *algebraic arc length w.r.t.* t_0, as follows:

$$s(t) = \begin{cases} l(f_{[t_0,t]}) & \text{if } [t_0,t] \subseteq \,]a,b[\ ; \\ -l(f_{[t_0,t]}) & \text{if } [t,t_0] \subseteq \,]a,b[\ ; \end{cases}$$

(and similarly in the case of a curve, except that $[t_0,t] \subseteq [a,b]$ or $[t,t_0] \subseteq [a,b]$).

For the sake of brevity, we will often call s the arc length, rather than algebraic arc length w.r.t. t_0.

Lemma 1.3.4 *For any open curve $f\colon \,]a,b[\, \to \mathcal{E}$ of class C^p (or curve $f\colon [a,b] \to \mathcal{E}$ of class C^p), with $p \geq 1$, for any fixed $t_0 \in \,]a,b[$ (or $t_0 \in [a,b]$, in the case of a curve), the algebraic arc length $s(t)$ w.r.t. t_0 is of class C^p, and furthermore, $s'(t) = \|f'(t)\|$.*

Thus, the arc length is given by the integral

$$s(t) = \int_{t_0}^{t} \|f'(u)\| \, du.$$

In particular, when $\mathcal{E} = \mathbb{E}^n$ and the norm is the Euclidean norm, we have

$$s(t) = \int_{t_0}^{t} \sqrt{f_1'(u)^2 + \cdots + f_n'(u)^2} \, du.$$

where $f = (f_1, \ldots, f_n)$.

The number $\|f'(t)\|$ is often called the *speed* of $f(t)$ at time t. For every regular point at t, the unit vector

$$t = \frac{f'(t)}{\|f'(t)\|}$$

is called the *unit tangent (vector) at t.*

Now, if $f\colon]a,b[\to \mathcal{E}$ (or $f\colon [a,b]\to \mathcal{E}$) is a regular curve of class C^p, with $p\geq 1$, since $s'(t)=\|f'(t)\|$, and $f'(t)\neq 0$ for all $t\in]a,b[$ (or $t\in [a,b]$), we have $s'(t)>0$ for all $t\in]a,b[$ (or $t\in [a,b]$).

The mean-value theorem implies that s is injective, and by theorem **??** (2), $s\colon]a,b[\to]s(a),s(b)[$ (or $s\colon [a,b]\to [s(a),s(b)]$) is a diffeomorphism of class C^p.

In particular, the curve $f\circ\varphi\colon]s(a),s(b)[\to \mathcal{E}$ (or $f\circ\varphi\colon [s(a),s(b)]\to \mathcal{E}$), with $\varphi = s^{-1}$, is C^p-equivalent to the original curve f, but it is parameterized by the arc length $s\in]s(a),s(b)[$ (or $s\in [s(a),s(b)]$).

As a consequence it can be shown that

$$\|g'(s)\| = 1,$$

i.e., that when a regular curve is parameterized by arc length, its velocity vector has unit length.

Remark: If a curve f (or a closed curve) is of class C^p, for $p \geq 1$, and it is a Jordan arc, then the algebraic arc length $s\colon [a,b] \to \mathbb{R}$ w.r.t. t_0 is strictly increasing, and thus injective. Thus, s^{-1} exists, and the curve can still be parameterized by arc length as $g = f \circ s^{-1}$. However, $g'(s)$ only exists when $s(t)$ corresponds to a regular point at t.

Thus, it still seems necessary to restrict our attention to regular curves, in order to avoid complications.

We now consider the notion of curvature. In order to do so, we assume that the affine space $\mathcal{E}$ has a Euclidean structure (an inner product), and that the norm on $\mathcal{E}$ is the norm induced by this inner product.

For simplicity, we assume that $\mathcal{E} = \mathbb{E}^n$.

1.4 Curvature and Osculating Circles (Plane Curves)

In a Euclidean space, orthogonality makes sense, and we can define normal lines and normal planes. We begin with plane curves, i.e. the case where $\mathcal{E} = \mathbb{E}^2$.

Definition 1.4.1 Given any regular plane curve $f\colon]a, b[\to \mathcal{E}$ (or $f\colon [a, b] \to \mathcal{E}$) of class C^p, with $p \geq 1$, the *normal line N_t to f at t* is the line through $f(t)$ and orthogonal to the tangent line T_t to f at t. Any nonnull vector defining the direction of the normal line N_t is called a *normal vector to f at t*.

From now on, we also assume that we are dealing with curves f that are biregular for all t. This means that $f'(t)$ and $f''(t)$ always exist and are linearly independent.

A fairly intuitive way to introduce the notion of curvature is to study the variation of the normal line N_t to a curve f at t, in a small neighborhood of t.

The intuition is that the normal N_t to f at t rotates around a certain point, and that the "speed" of rotation of the normal measures how much the curve bends around t.

In other words, the rate at which the normal turns corresponds to the curvature of the curve at t.

Another way to look at it, is to focus on the point around which the normal turns, the center of curvature C at t, and to consider the radius $\mathcal{R}$ of the circle centered at C and tangent to the curve at $f(t)$ (i.e., tangent to the tangent line to f at t).

Intuitively, the smaller $\mathcal{R}$ is, the faster the curve bends, and thus, the curvature can be defined as $1/\mathcal{R}$.

Let us assume that some origin O is chosen in the affine plane, and to simplify the notations, for any curve f, let us denote $f(t) - O$ as $\overrightarrow{M(t)}$, or $\overrightarrow{M}$, for any point P, denote $P - O$ as $\overrightarrow{P}$, and denote $f'(t)$ as $\overrightarrow{M'(t)}$, or $\overrightarrow{M'}$.

The normal line N_t to f at t is the set of points P such that

$$\overrightarrow{M'} \cdot \overrightarrow{MP} = 0,$$

or equivalently

$$\overrightarrow{M'} \cdot \overrightarrow{P} = \overrightarrow{M'} \cdot \overrightarrow{M}.$$

Similarly, for any small $\delta \neq 0$ such that $f(t+\delta)$ is defined, the normal line $N_{t+\delta}$ to f at $t+\delta$ is the set of points Q such that

$$\overrightarrow{M'(t+\delta)} \cdot \overrightarrow{Q} = \overrightarrow{M'(t+\delta)} \cdot \overrightarrow{M(t+\delta)}.$$

Thus, the intersection point P of N_t and $N_{t+\delta}$ if it exists, is given by the equations

$$\begin{aligned} \overrightarrow{M'} \cdot \overrightarrow{P} &= \overrightarrow{M'} \cdot \overrightarrow{M} \\ \overrightarrow{M'(t+\delta)} \cdot \overrightarrow{P} &= \overrightarrow{M'(t+\delta)} \cdot \overrightarrow{M(t+\delta)}. \end{aligned}$$

We can show that if it exists, P is the intersection of the two lines of equations

$$\begin{aligned}\overrightarrow{M'} \cdot \overrightarrow{MP} &= 0 \\ \overrightarrow{M''} \cdot \overrightarrow{MP} &= \|\overrightarrow{M'}\|^2.\end{aligned}$$

Thus, if $\overrightarrow{M'}$ and $\overrightarrow{M''}$ are linearly independent which is equivalent to saying that $f'(t)$ and $f''(t)$ are linearly independent, i.e., f is biregular at t, the above two equations have a unique solution P.

Also, the above analysis shows that the intersection of the two normals N_t and $N_{t+\delta}$, for $\delta \neq 0$ small enough, has a limit C (really, $C(t)$).

This limit is called the *center of curvature of f at t*.

It is possible to compute the distance $\mathcal{R} = \|\overrightarrow{MC}\|$, the *radius of curvature at* t, and the coordinates of C, given any affine frame for the plane.

It is worth noting that the equation

$$\overrightarrow{M''} \cdot \overrightarrow{P} = \overrightarrow{M''} \cdot \overrightarrow{M} + \overrightarrow{M'} \cdot \overrightarrow{M'}$$

is obtained by taking the derivative of the equation

$$\overrightarrow{M'} \cdot \overrightarrow{P} = \overrightarrow{M'} \cdot \overrightarrow{M}$$

with respect to t.

This observation can be used to compute the coordinates of the center of curvature, but first, we show that the radius of curvature has a very simple expression when the curve is parameterized by arc length.

Then, letting

$$\overrightarrow{n} = \frac{\overrightarrow{M''}}{\|\overrightarrow{M''}\|}$$

be the unit vector associated with the acceleration vector $\overrightarrow{M''}$, we get

$$\mathcal{R} = \frac{1}{\|\overrightarrow{M''}\|} = \frac{1}{\|f''(s)\|}.$$

Thus, the radius of curvature is the inverse of the norm of the acceleration vector $f''(s)$.

We define the *curvature* κ as the inverse of the radius of curvature $\mathcal{R}$, that is, as

$$\kappa = \|f''(s)\|.$$

In summary, when the curve f is parameterized by arc length, we found that the curvature κ and the radius of curvature $\mathcal{R}$ are defined by the equations

$$\boxed{\kappa = \|f''(s)\|, \quad \mathcal{R} = \frac{1}{\kappa}.}$$

We now come back to the general case.

Assuming that $\overrightarrow{M'}$ and $\overrightarrow{M''}$ are linearly independent, we can write $\overrightarrow{MC} = \alpha\overrightarrow{M'} + \beta\overrightarrow{M''}$, for some unique α, β.

Since C is determined by the equations

$$\begin{aligned} \overrightarrow{M'} \cdot \overrightarrow{MC} &= 0 \\ \overrightarrow{M''} \cdot \overrightarrow{MC} &= \|\overrightarrow{M'}\|^2, \end{aligned}$$

we get the system

$$\begin{aligned} (\overrightarrow{M'} \cdot \overrightarrow{M'})\,\alpha + (\overrightarrow{M'} \cdot \overrightarrow{M''})\,\beta &= 0 \\ (\overrightarrow{M'} \cdot \overrightarrow{M''})\,\alpha + (\overrightarrow{M''} \cdot \overrightarrow{M''})\,\beta &= \|\overrightarrow{M'}\|^2, \end{aligned}$$

and we also note that

$$\mathcal{R}^2 = \overrightarrow{MC} \cdot \overrightarrow{MC} = \overrightarrow{MC} \cdot (\alpha\overrightarrow{M'} + \beta\overrightarrow{M''}) = \beta\|\overrightarrow{M'}\|^2.$$

The reader can verify that we find

$$\beta = \frac{\|\overrightarrow{M'}\|^4}{\|\overrightarrow{M'}\|^2\|\overrightarrow{M''}\|^2 - (\overrightarrow{M'} \cdot \overrightarrow{M''})^2},$$

and thus,

$$\mathcal{R}^2 = \frac{\|\overrightarrow{M'}\|^6}{\|\overrightarrow{M'}\|^2\|\overrightarrow{M''}\|^2 - (\overrightarrow{M'} \cdot \overrightarrow{M''})^2}.$$

However, if we remember about the cross-product of vectors and the Lagrange identity, we have

$$\|\overrightarrow{M'}\|^2\|\overrightarrow{M''}\|^2 - (\overrightarrow{M'} \cdot \overrightarrow{M''})^2 = \|\overrightarrow{M'} \times \overrightarrow{M''}\|^2,$$

and thus,

$$\mathcal{R} = \frac{\|\overrightarrow{M'}\|^3}{\|\overrightarrow{M'} \times \overrightarrow{M''}\|} = \frac{\|f'(t)\|^3}{\|f'(t) \times f''(t)\|},$$

and the curvature is given by

$$\kappa = \frac{\|\overrightarrow{M'} \times \overrightarrow{M''}\|}{\|\overrightarrow{M'}\|^3} = \frac{\|f'(t) \times f''(t)\|}{\|f'(t)\|^3}.$$

In summary, when the curve f is not necessarily parameterized by arc length, we found that the curvature κ and the radius of curvature $\mathcal{R}$ are defined by the equations

$$\boxed{\kappa = \frac{\|f'(t) \times f''(t)\|}{\|f'(t)\|^3}, \quad \mathcal{R} = \frac{1}{\kappa}.}$$

Note that from an analytical point of view, the curvature has the advantage of being defined at every regular point, since $\kappa = 0$ when either $f''(t) = \overrightarrow{0}$ or $f''(t)$ is collinear to $f'(t)$, whereas at such points, the radius of curvature goes to $+\infty$.

We leave as an exercise to show that the curvature is an *invariant* intrinsic to the geometric curve defined by f. In view of the above considerations, we give the following definition of the curvature, which is more intrinsic.

Definition 1.4.2 For any regular plane curve $f\colon]a,b[\to \mathcal{E}$ (or $f\colon [a,b] \to \mathcal{E}$) of class C^p parameterized by arc length, with $p \geq 2$, the *curvature* κ *at* s is defined as the nonnegative real number $\kappa = \|f''(s)\|$. For every s such that $f''(s) \neq \overrightarrow{0}$, letting $\overrightarrow{n} = \frac{f''(s)}{\|f''(s)\|}$ be the unit vector associated with $f''(s)$, we have $f''(s) = \kappa\overrightarrow{n}$, the point C defined such that $C - f(s) = \overrightarrow{n}/\kappa$ is the *center of curvature at* s, and $\mathcal{R} = 1/\kappa$ is the *radius of curvature at* s.

The circle of center C and of radius $\mathcal{R}$ is called the *osculating circle to* f *at* s. When $f''(s) = \overrightarrow{0}$, by convention, $\mathcal{R} = \infty$, and the center of curvature is undefined.

Actually it is possible to define the notion of osculating circle more geometrically as a limit, in the spirit of the definition of a tangent.

Definition 1.4.3 Given any plane curve $f\colon]a, b[\to \mathcal{E}$ (or $f\colon [a, b] \to \mathcal{E}$) of class C^p, with $p \geq 1$, given any point $M_0 = f(t)$ on the curve, if f is locally injective at M_0, the tangent T_t to f at t exists, and the circle $\Sigma_{t,h}$ tangent to T_t and passing through M_1 has a limit Σ_t when $h \neq 0$ approaches 0 we say that Σ_t is the *osculating circle to f in $M_0 = f(t)$ at t.*

More precisely if there is an open interval $]t-\eta,\, t+\eta[\subseteq]a, b[$ (with $\eta > 0$) such that, $M_1 = f(t+h) \neq f(t) = M_0$ for all $h \neq 0$ with $h \in]-\eta,\, \eta[$, the tangent T_t to f at t exists, and the circle $\Sigma_{t,h}$ tangent to T_t and passing through M_1 has a limit Σ_t when $h \neq 0$ approaches 0 (with $h \in]-\eta,\, \eta[$), we say that Σ_t is the osculating circle to f in $M_0 = f(t)$ at t.

It is not hard to show that if the center of curvature C (and thus, the radius of curvature $\mathcal{R}$) exists at t, then the osculating circle at t is indeed the circle of center C and radius $\mathcal{R}$ (also called *circle of curvature at t*).

Remarks: It is possible that the osculating circle exists at a point t, but that the center of curvature at t is undefined.

Consider the curve defined such that

$$f(t) = \begin{cases} (t, t^2 + t^3 \sin(1/t)) & \text{if } t \neq 0; \\ (0, 0) & \text{if } t = 0. \end{cases}$$

and shown in Figure 8.6.

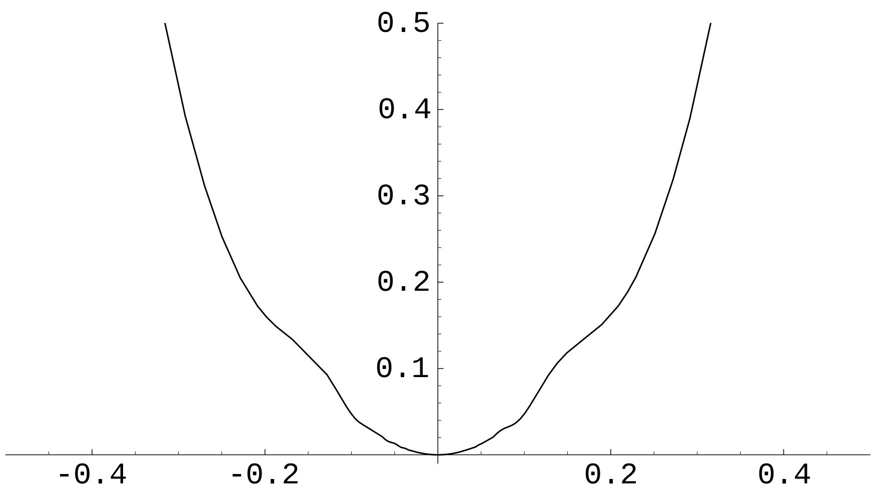

Figure 1.6: Osculating circle at O exists and yet $f''(0)$ is undefined

We leave as an exercise to show that the osculating circle for $t = 0$ is the circle of center $(0, 1/2)$, but $f''(0)$ is undefined, so that the center of curvature is undefined at $t = 0$.

In general, the osculating circle intersects the curve in another point besides the point of contact, which means that near the point of contact, one of the two branches of the curve is outside the osculating circle, and the other branch is inside.

This property fails for points on an axis of symmetry for the curve, such as the points on the axes of an ellipse.

Osculating circles give a very good approximation of the curve around each (biregular) point.

We will see in later examples that plotting enough osculating circles gives the illusion that the curve is plotted, when in fact it is not!

Recalling that we denoted the (unit) tangent vector $f'(s)$ at s as $\overrightarrow{t}$, and the unit normal vector $\frac{f''(s)}{\|f''(s)\|}$ as $\overrightarrow{n}$, we can also show the following lemma, confirming the geometric characterization of the center of curvature.

Lemma 1.4.4 *For any regular plane curve* $f\colon]a,b[\to \mathcal{E}$ *(or* $f\colon [a,b] \to \mathcal{E}$*) of class* C^p *parameterized by arc length, with* $p \geq 2$*, assuming that* $f''(s) \neq \overrightarrow{0}$*, the center of curvature is on a curve* c *of class* C^0 *defined such that* $c(s) - f(s) = \mathcal{R}\overrightarrow{n}$*, where* $\mathcal{R} = 1/\|f''(s)\|$ *and* $\overrightarrow{n} = f''(s)/\|f''(s)\|$*, and whenever* $\mathcal{R}'(s) \neq 0$*,* c *is regular at* s *and* $c'(s) = \mathcal{R}'\overrightarrow{n}$*, which means that the normal to* f *at* s *is the tangent to* c *at* s*.*

In other words, for every s where κ'/κ^2 is defined and not equal to zero, the point $c(s)$ is regular.

This is not the case for points for which the curvature is minimal or maximal. The example of an ellipse is typical (see below).

The curve c defined in lemma 8.4.4 is called the *evolute* of the curve f.

Summarizing the discussion before definition 8.4.2, we also have the following lemma:

Lemma 1.4.5 *For any regular plane curve* $f\colon]a,b[\to \mathcal{E}$ *(or* $f\colon [a,b]\to \mathcal{E}$*) of class* C^p*, with* $p\geq 2$*, the curvature at* t *is given by the expression*

$$\kappa = \frac{\|f'(t)\times f''(t)\|}{\|f'(t)\|^3}.$$

Furthermore, whenever $f'(t)\times f''(t)\neq \overrightarrow{0}$*, the center of curvature* C *defined such that* $C - f(t) = \overrightarrow{n}/\kappa$*, is the limit of the intersection of any normal* $N_{t+\delta}$ *and the normal* N_t *at* t*, when* $\delta\neq 0$ *small enough approaches* 0*.*

Lemma 8.4.5 gives us a way of calculating the curvature at any point, for any (regular) paramerization of a curve.

Let us now determine the coordinates of the center of curvature (when defined). Let $(O, \overrightarrow{i}, \overrightarrow{j})$ be an othonormal frame for the plane, and let the curve be defined by the map $f(t) = O + u(t)\overrightarrow{i} + v(t)\overrightarrow{j}$.

The equation of the normal to f at t is

$$(x - u)u' + (y - v)v' = 0$$

or

$$u'x + v'y = uu' + vv'.$$

As we noted earlier, the center of curvature is obtained by intersecting this normal with the line whose equation is obtained by taking the derivative of the equation of the normal w.r.t. t.

Thus, the center of curvature is the solution of the system

$$\begin{aligned} u'x + v'y &= uu' + vv', \\ u''x + v''y &= uu'' + vv'' + u'^2 + v'^2. \end{aligned}$$

We leave as an exercise to verify that the solution is given by:

$$x = u - \frac{v'(u'^2 + v'^2)}{u'v'' - v'u''},$$
$$y = v + \frac{u'(u'^2 + v'^2)}{u'v'' - v'u''},$$

provided that $u'v'' - v'u'' \neq 0$.

One will also check that the radius of curvature is given by

$$\mathcal{R} = \frac{(u'^2 + v'^2)^{3/2}}{|u'v'' - v'u''|}.$$

We now give a few examples. If f is a straight line, then $f''(t) = \overrightarrow{0}$, and thus the curvature is null for every point of a line.

A circle of radius a can be defined by

$$x = a \cos t,$$
$$y = a \sin t.$$

We get

$$\mathcal{R} = \frac{(u'^2 + v'^2)^{3/2}}{|u'v'' - v'u''|} = a,$$

and $\kappa = 1/a$.

Thus, as expected, the circle has constant curvature $1/a$, where a is its radius, and the center of curvature is reduced to a single point, the center of the circle.

An ellipse is defined by

$$x = a\,\cos\theta,$$
$$y = b\,\sin\theta.$$

We leave as an exercise to show that the radius of curvature is

$$\mathcal{R} = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab},$$

and, that the center of curvature is on the curve defined by

$$x = \frac{c^2}{a}\cos^3\theta, \quad y = -\frac{c^2}{b}\sin^3\theta.$$

This curve has four cusps, corresponding to the two maxima and minima of the curvature.

It is fun to plot the locus of the center of curvature and enough osculating circles to the ellipse.

Figure 8.7 shows 64 osculating circles of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

(with $a \geq b$), for $a = 4$, $b = 2$, and the locus of the center of curvature. Although the ellipse is not explicitly plotted, it seems to be present!

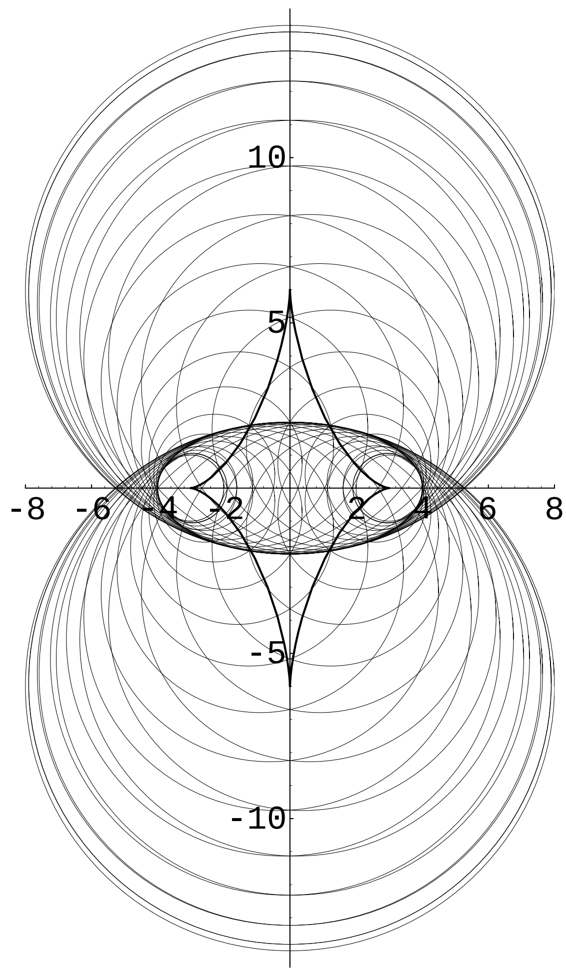

Figure 1.7: Osculating circles of an ellipse

The logarithmic spiral given in polar coordinates by

$$r = a\, e^{m\theta},$$

or by

$$\begin{aligned} x &= a\, e^{m\theta} \cos\theta, \\ y &= a\, e^{m\theta} \sin\theta, \end{aligned}$$

(with $a > 0$) is particularly interesting.

We leave as an exercise to show that the radius of curvature is

$$\mathcal{R} = a\sqrt{1+m^2}\, e^{m\theta},$$

and that the center of curvature is on the spiral (in fact, equal to the original spiral) defined by

$$\begin{aligned} x &= -ma\, e^{m\theta} \sin\theta, \\ y &= ma\, e^{m\theta} \cos\theta. \end{aligned}$$

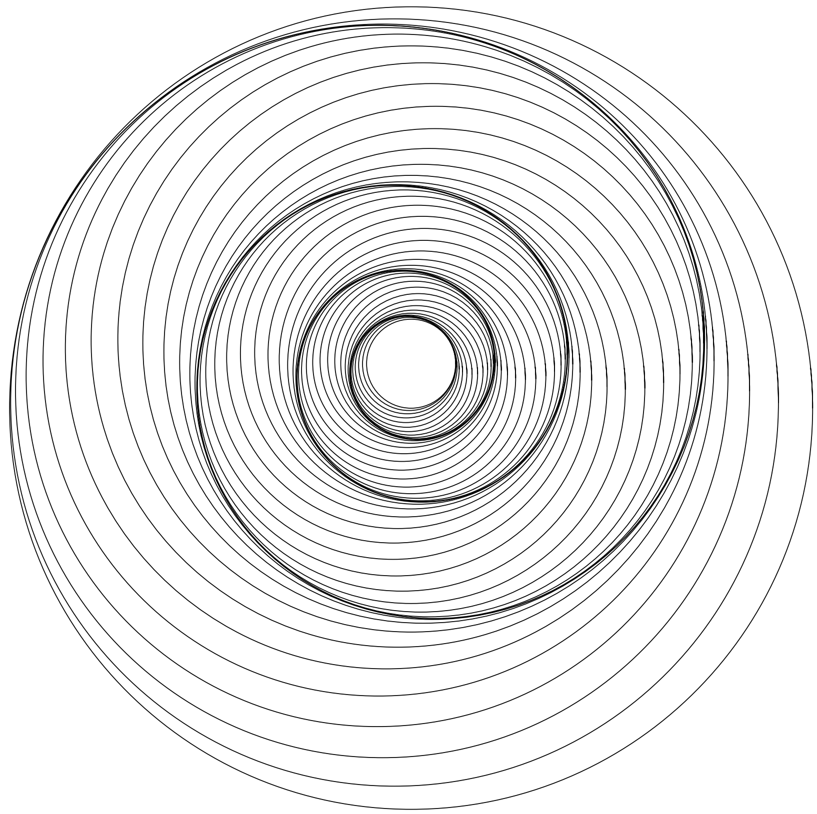

Figure 1.8: Osculating circles of a logarithmic spiral

Figure 8.8 shows 50 osculating circles of the logarithmic spiral given in polar coordinates by $r = a\, e^{m\theta}$, for $a = 0.6$ and $m = 0.1$.

The cardioïd given in polar coordinates by

$$r = a(1 + \cos\theta)$$

or by

$$x = a(1 + \cos\theta)\cos\theta,$$
$$y = a(1 + \cos\theta)\sin\theta,$$

is also an neat example.

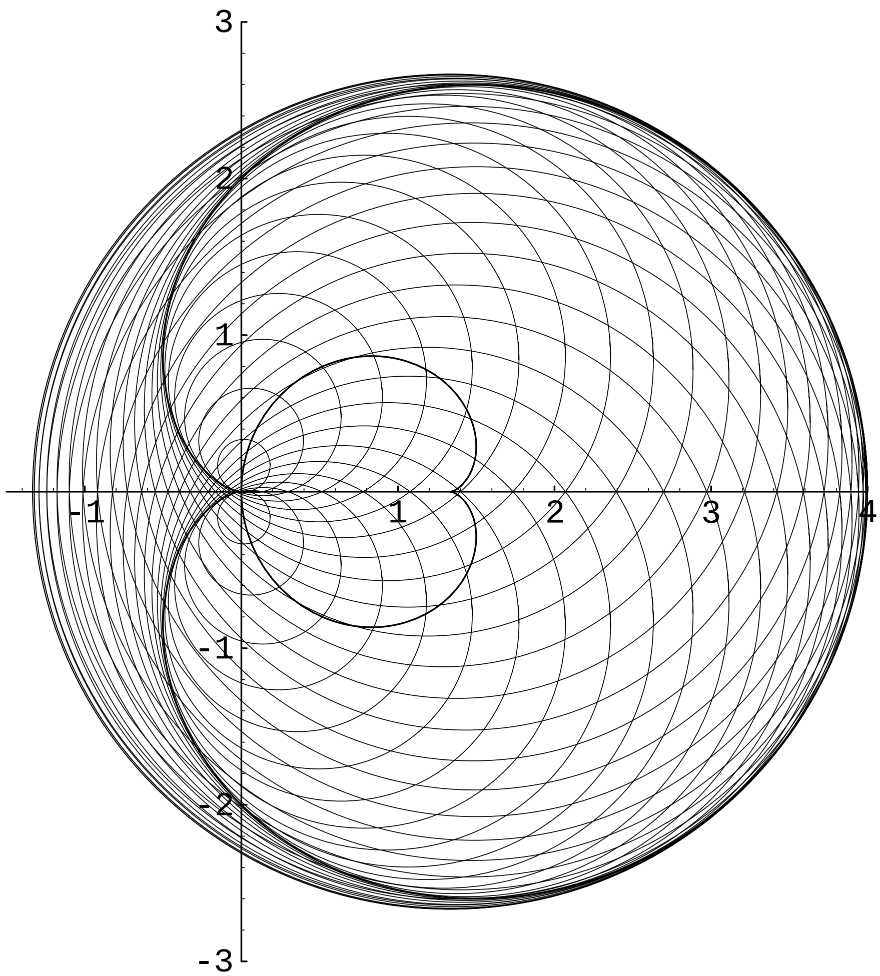

Figure 1.9: Osculating circles of a cardioïd

Figure 8.9 shows 50 osculating circles of the cardioïd given in polar coordinates by $r = a(1 + \cos\theta)$, for $a = 2$, and the locus of the center of curvature.

We leave as an exercise to show that the radius of curvature is

$$\mathcal{R} = \left|\frac{2a}{3}\cos(\theta/2)\right|,$$

and that the center of curvature is on the cardioïd defined by

$$\begin{aligned} x &= \frac{2a}{3} + \frac{a}{3}(1 - \cos\theta)\cos\theta, \\ y &= \frac{a}{3}(1 - \cos\theta)\sin\theta. \end{aligned}$$

We conclude our discussion of the curvature of plane curves with a brief look at the algebraic curvature.

Since a plane can be oriented, it is possible to give a sign to the curvature.

Let us assume that the plane is oriented by an othonormal frame $(O, \overrightarrow{i}, \overrightarrow{j})$, assumed to have a positive orientation, and that the curve f is parameterized by arc length.

Then, given any unit tangent vector $\overrightarrow{t}$ at s to a curve f, there exists a unit normal vector $\overrightarrow{\nu}$ such that $(O, \overrightarrow{t}, \overrightarrow{\nu})$ has positive orientation.

In fact, if θ is the angle (mod 2π) between $\overrightarrow{i}$ and $\overrightarrow{t}$, so that

$$\overrightarrow{t} = \cos\theta\, \overrightarrow{i} + \sin\theta\, \overrightarrow{j},$$

we have

$$\begin{aligned}\overrightarrow{\nu} = \cos(\theta + \pi/2)\, \overrightarrow{i} + \sin(\theta + \pi/2)\, \overrightarrow{j} \\ = -\sin\theta\, \overrightarrow{i} + \cos\theta\, \overrightarrow{j}.\end{aligned}$$

Note that this normal vector $\overrightarrow{\nu}$ is not necessarily equal to the unit normal vector $\overrightarrow{n} = f''(s)/\|f''(s)\|$: it can be of the opposite direction.

Furthermore, $\overrightarrow{\nu}$ exists for every regular point, even when $f''(s) = \overrightarrow{0}$, which is not true of $\overrightarrow{n}$.

We define the *algebraic curvature k at s* as the real number (negative, null or positive) such that

$$f''(s) = k\overrightarrow{\nu}.$$

We also define the *algebraic radius of curvature R* as $R = 1/k$.

Clearly, $\kappa = |k|$, and $\mathcal{R} = |R|$.

Thus, we also have

$$\overrightarrow{t'} = k\overrightarrow{\nu},$$

and it is immediately verified that the center of curvature is still given by

$$C - f(s) = R\overrightarrow{\nu}$$

and that

$$\overrightarrow{\nu'} = -k\overrightarrow{t}.$$

The algebraic curvature plays an important role in some global theorems of differential geometry.

It is also possible to prove that if $c\colon]a, b[\to \mathbb{R}$ is a continuous function, and $s_0 \in]a, b[$, then there is a unique curve $f\colon]a, b[\to \mathcal{E}$ such that $f(s_0)$ is any given point, $f'(s_0)$ is any given vector, and such that $c(s)$ is the algebraic curvature of f.

Roughly speaking, the algebraic curvature k determines the curve completely, up to rigid motion.

One should be careful that this results fails if we consider the curvature κ instead of the algebraic curvature k.

Indeed, it is possible that $k(s) = c(s) = 0$, and thus that $\kappa(s) = 0$.

Such points may be inflexion points, and counter-examples to the above result with κ instead of k are easily found.

However, if we require $c(s) > 0$ for all s, the above result holds for the curvature κ.

We now consider curves in affine Euclidean $3D$ spaces (i.e. $\mathcal{E} = \mathbb{E}^3$).

1.5 Normal Planes and Curvature ($3D$ Curves)

The first thing to do is to define the notion of a normal plane.

Definition 1.5.1 Given any regular $3D$ curve $f\colon\,]a, b[\to \mathcal{E}$ (or $f\colon [a, b] \to \mathcal{E}$) of class C^p, with $p \geq 2$, the *normal plane* N_t *to* f *at* t is the plane through $f(t)$ and orthogonal to the tangent line T_t to f at t. The intersection of the normal plane and of the osculating plane (if it exists) is called the *principal normal line to* f *at* t.

In order to get an intuitive idea of the notion of curvature, we need to look at the variation of the normal plane around t, since there are infinitely many normal lines to a given line in 3-space.

This time, we will see that the normal plane rotates around a line perpendicular to the osculating plane (called the *polar axis at* t).

The intersection of this line with the osculating plane is the center of curvature.

But now, not only does the normal plane rotates around an axis, so does the osculating plane, and the plane containing the tangent line and normal to the osculating plane, called the rectifying plane.

Thus, a second quantity called the torsion will make its appearance.

The treatment that we gave for the plane extends immediately to space (in $3D$).

Indeed, The normal plane N_t to f at t is the set of points P such that

$$\overrightarrow{M'} \cdot \overrightarrow{MP} = 0,$$

or equivalently

$$\overrightarrow{M'} \cdot \overrightarrow{P} = \overrightarrow{M'} \cdot \overrightarrow{M}.$$

Thus, the intersection points P of N_t and $N_{t+\delta}$ if they exist, are given by the equations

$$\begin{aligned} \overrightarrow{M'} \cdot \overrightarrow{P} &= \overrightarrow{M'} \cdot \overrightarrow{M} \\ \overrightarrow{M'(t+\delta)} \cdot \overrightarrow{P} &= \overrightarrow{M'(t+\delta)} \cdot \overrightarrow{M(t+\delta)}. \end{aligned}$$

As in the planar case, for δ very small, the intersection of the two planes N_t and $N_{t+\delta}$ is given by the equations

$$\begin{aligned} \overrightarrow{M'} \cdot \overrightarrow{MP} &= 0 \\ \overrightarrow{M''} \cdot \overrightarrow{MP} &= \|\overrightarrow{M'}\|^2. \end{aligned}$$

Thus, if $\overrightarrow{M'}$ and $\overrightarrow{M''}$ are linearly independent which is equivalent to saying that $f'(t)$ and $f''(t)$ are linearly independent, i.e., f is biregular at t, the above two equations define a unique line Δ orthogonal to the osculating plane.

This line is called the *polar axis at t*.

Since the line Δ is perpendicular to the osculating plane, it intersects the osculating plane in a single point C (really, $C(t)$), the *center of curvature of f at t*.

The distance $\mathcal{R} = \|\overrightarrow{MC}\|$, is the *radius of curvature at t*, and it inverse $\kappa = 1/\mathcal{R}$ is the *curvature at t*.

Note that C is on the normal line to the curve f at t contained in the osculating plane, i.e. on the principal normal at t.

1.6 The Frenet Frame ($3D$ Curves)

When $f'(t)$ and $f''(t)$ are linearly independent, we can find a unit vector in the plane spanned by $f'(t)$ and $f''(t)$ and orthogonal to the unit tangent vector

$$\overrightarrow{t} = f'(t)/\|f'(t)\|$$

at t, and equal to the unit vector

$$f''(t)/\|f''(t)\|$$

when $f'(t)$ and $f''(t)$ are orthogonal, namely the unit vector

$$\overrightarrow{n} = \frac{-(f'(t) \cdot f''(t))\, f'(t) + \|f'(t)\|^2\, f''(t)}{\| - (f'(t) \cdot f''(t))\, f'(t) + \|f'(t)\|^2\, f''(t)\|}.$$

The unit vector $\overrightarrow{n}$ is called the *principal normal vector to f at t*.

We define the *binormal vector $\overrightarrow{b}$ at t* as $\overrightarrow{b} = \overrightarrow{t} \times \overrightarrow{n}$.

Thus, the triple $(\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b})$ is a basis of orthogonal unit vectors.

It is usually called the *Frenet (or Frenet-Serret) frame at* t (this concept was introduced independently by Frenet in 1847, and Serret in 1850).

Definition 1.6.1 Given any biregular $3D$ curve $f\colon\,]a,b[\, \to \mathcal{E}$ (or $f\colon [a,b] \to \mathcal{E}$) of class C^p, with $p \geq 2$, the *Frenet frame (of Frenet trihedron) at* t is the triple $(\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b})$ consisting of the three orthogonal unit vectors such that $\overrightarrow{t} = f'(t)/\|f'(t)\|$ is the unit tangent vector at t,

$$\overrightarrow{n} = \frac{-(f'(t)\cdot f''(t))\, f'(t) + \|f'(t)\|^2\, f''(t)}{\|-(f'(t)\cdot f''(t))\, f'(t) + \|f'(t)\|^2\, f''(t)\|}$$

is a unit vector orthogonal to $\overrightarrow{t}$ called the *principal normal vector to* f *at* t, and $\overrightarrow{b} = \overrightarrow{t} \times \overrightarrow{n}$, is the *binormal vector at* t. The plane containing $\overrightarrow{t}$ and $\overrightarrow{b}$ is called the *rectifying plane at* t.

As we will see shortly, the principal normal $\overrightarrow{n}$ has a simpler expression when the curve is parameterized by arc length.

The calculations of $\mathcal{R}$ are still valid, since the cross-product $\overrightarrow{M'} \times \overrightarrow{M''}$ makes sense in 3-space, and thus, we have

$$\mathcal{R} = \frac{\|\overrightarrow{M'}\|^3}{\|\overrightarrow{M'} \times \overrightarrow{M''}\|} = \frac{\|f'(t)\|^3}{\|f'(t) \times f''(t)\|},$$

and the curvature is given by

$$\kappa = \frac{\|\overrightarrow{M'} \times \overrightarrow{M''}\|}{\|\overrightarrow{M'}\|^3} = \frac{\|f'(t) \times f''(t)\|}{\|f'(t)\|^3}.$$

For example, consider the curve given by

$$f(t) = (t, t^2, t^3)$$

known as the *twisted cubic*.

We have $f'(t) = (1, 2t, 3t^2)$ and $f''(t) = (0, 2, 6t)$.

At $t = 0$ (the origin), the Frenet frame $(\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b})$ consists of the three unit vectors $\overrightarrow{i} = (1, 0, 0)$, $\overrightarrow{j} = (0, 1, 0)$, and $\overrightarrow{k} = (0, 0, 1)$.

Thus, the osculating plane is the xy-plane, the normal plane is the yz-plane, and the rectifying plane is the xz-plane.

It is easily checked that

$$f' \times f'' = (6t^2, -6t, 2),$$

and the curvature at t is given by

$$\kappa(t) = \frac{2(9t^4 + 9t^2 + 1)^{1/2}}{(9t^4 + 4t^2 + 1)^{3/2}}.$$

In particular, $\kappa(0) = 2$, and the polar line is the vertical line in the yz-plane passing through the point $C = (0, 1/2, 0)$, the center of curvature.

When the curve is parameterized by arc length, $\overrightarrow{t} = f'(s)$, and we obtain the same results as in the planar case.

We get

$$\mathcal{R} = \frac{1}{\|\overrightarrow{M''}\|} = \frac{1}{\|f''(s)\|}.$$

Thus, the radius of curvature is the inverse of the norm of the acceleration vector $f''(s)$, and the curvature κ is

$$\kappa = \|f''(s)\|.$$

Again, as in the planar case, the curvature is an invariant intrinsic to the geometric curve defined by f.

We now consider how the rectifying plane varies. This will uncover the torsion.

We leave as an easy exercise to show that the osculating plane rotates around the tangent line, for points $t + \delta$ close enough to t.

1.7 Torsion (3D Curves)

Recall that the rectifying plane is the plane orthogonal to the principal normal at t passing through $f(t)$. Thus, its equation is

$$\overrightarrow{n} \cdot \overrightarrow{MP} = 0,$$

where $\overrightarrow{n}$ is the principal normal vector.

However, things get a bit messy when we take the derivative of $\overrightarrow{n}$, because of the denominator, and it is easier to use the vector

$$\overrightarrow{N} = -(f'(t) \cdot f''(t))\, f'(t) + \|f'(t)\|^2\, f''(t),$$

which is collinear to $\overrightarrow{n}$, but not necessarily a unit vector.

Still, we have $\overrightarrow{N} \cdot \overrightarrow{M'} = 0$, which is the important fact.

Since the equation of the rectifying plane is $\overrightarrow{N} \cdot \overrightarrow{MP} = 0$ or

$$\overrightarrow{N} \cdot \overrightarrow{P} = \overrightarrow{N} \cdot \overrightarrow{M},$$

by a familiar reasoning, we can easily prove that the intersection of these two planes is given by the equations

$$\begin{aligned} \overrightarrow{N} \cdot \overrightarrow{MP} &= 0, \\ \overrightarrow{N'} \cdot \overrightarrow{MP} &= \overrightarrow{N} \cdot \overrightarrow{M'} = 0, \end{aligned}$$

since $\overrightarrow{N} \cdot \overrightarrow{M'} = 0$.

Thus, if $\overrightarrow{N}$ and $\overrightarrow{N'}$ are linearly independent, the intersection of these two planes is a line in the rectifying plane, passing through the point $M = f(t)$.

We now have to take a closer look at $\overrightarrow{N'}$. It is easily seen that

$$\overrightarrow{N'} = -(\|\overrightarrow{M''}\|^2 + \overrightarrow{M'} \cdot \overrightarrow{M'''})\overrightarrow{M'} + (\overrightarrow{M'} \cdot \overrightarrow{M''})\overrightarrow{M''} + \|\overrightarrow{M'}\|^2 \overrightarrow{M'''}.$$

Thus, $\overrightarrow{N}$ and $\overrightarrow{N'}$ are linearly independent iff $\overrightarrow{M'}$, $\overrightarrow{M''}$ and $\overrightarrow{M'''}$ are linearly independent.

Now, since the line in question is in the rectifying plane, every point P on this line can be expressed as

$$\overrightarrow{MP} = \alpha \overrightarrow{b} + \beta \overrightarrow{t},$$

where α and β are related by the equation

$$(\overrightarrow{N'} \cdot \overrightarrow{b})\alpha + (\overrightarrow{N'} \cdot \overrightarrow{t})\beta = 0,$$

obtained from $\overrightarrow{N'} \cdot \overrightarrow{MP} = 0$.

However,

$$\overrightarrow{t} = \overrightarrow{M'}/\|\overrightarrow{M'}\|,$$

and it is immediate that

$$\overrightarrow{b} = \frac{\overrightarrow{M'} \times \overrightarrow{M''}}{\|\overrightarrow{M'} \times \overrightarrow{M''}\|}.$$

Recalling that the radius of curvature is given by

$$\mathcal{R} = \|\overrightarrow{M'}\|^3 / \|\overrightarrow{M'} \times \overrightarrow{M''}\|,$$

it is tempting to see what is the value of α when $\beta = \mathcal{R}$.

After some calculation, we get

$$\alpha = \frac{\|\overrightarrow{M'} \times \overrightarrow{M''}\|^2}{(\overrightarrow{M'}, \overrightarrow{M''}, \overrightarrow{M'''})}.$$

So finally, we have shown that the axis of rotation of the rectifying planes for $t + \delta$ close to t is determined by the vector

$$\overrightarrow{MP} = \alpha \overrightarrow{b} + \mathcal{R} \overrightarrow{t},$$

or equivalently, that

$$(\kappa \overrightarrow{t} + \tau \overrightarrow{b}) \cdot \overrightarrow{MP} = 0,$$

where κ is the curvature, and $\tau = -1/\alpha$ is called the *torsion at* t, and is given by

$$\tau = -\frac{(\overrightarrow{M'}, \overrightarrow{M''}, \overrightarrow{M'''})}{\|\overrightarrow{M'} \times \overrightarrow{M''}\|^2}.$$

Its inverse $\mathcal{T} = 1/\tau$ is called the *radius of torsion at* t.

In summary, we obtained the following formulae for the curvature and the torsion of a $3D$-curve:

$$\boxed{\kappa = \frac{\|f'(t) \times f''(t)\|}{\|f'(t)\|^3}, \quad \tau = -\frac{(f'(t), f''(t), f'''(t))}{\|f'(t) \times f''(t)\|^2}.}$$

The torsion is an *invariant* intrinsic to the geometric curve defined by f.

Going back to the example of the twisted cubic

$$f(t) = (t, t^2, t^3),$$

since $f'(t) = (1, 2t, 3t^2)$, $f''(t) = (0, 2, 6t)$, and $f'''(t) = (0, 0, 6)$, we get

$$(f', f'', f''') = 12,$$

and since

$$f' \times f'' = (6t^2, -6t, 2),$$

the torsion at t is given by

$$\tau(t) = -\frac{3}{9t^4 + 9t^2 + 1}.$$

In particular, $\tau(0) = -3$, and the rectifying plane rotates around the line through the origin and of direction

$$-\tau \overrightarrow{t} + \kappa \overrightarrow{b} = (3, 2, 0).$$

1.8 The Frenet Equations ($3D$ Curves)

Assuming that curves are parameterized by arc length, we are now going to see how κ and τ reappear naturally when we determine how the Frenet frame $(\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b})$ varies with s, and more specifically, in expressing $(\overrightarrow{t'}, \overrightarrow{n'}, \overrightarrow{b'})$ over the basis $(\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b})$.

We claim that

$$\begin{aligned}\overrightarrow{t'} &= \kappa \overrightarrow{n}, \\ \overrightarrow{n'} &= -\kappa \overrightarrow{t} - \tau \overrightarrow{b}, \\ \overrightarrow{b'} &= \tau \overrightarrow{n},\end{aligned}$$

where κ is the curvature, and τ turns out to be the torsion.

In matrix form, we can write the equations know as the *Frenet (or Frenet-Serret) equations* as

$$\boxed{(\overrightarrow{t'}, \overrightarrow{n'}, \overrightarrow{b'}) = (\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}) \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}}$$

We can now verify that τ agrees with the geometric interpretation given before.

The axis of rotation of the rectifying plane is the line given by the intersection of the two planes of equations

$$\overrightarrow{n} \cdot \overrightarrow{MP} = 0,$$
$$\overrightarrow{n'} \cdot \overrightarrow{MP} = 0,$$

and since

$$\overrightarrow{n'} = -\kappa \overrightarrow{t} - \tau \overrightarrow{b},$$

the second equation is equivalent to

$$(\kappa \overrightarrow{t} + \tau \overrightarrow{b}) \cdot \overrightarrow{MP} = 0.$$

This is exactly the equation that we found earlier with $\tau = -1/\alpha$, where

$$\alpha = \frac{\|\overrightarrow{M'} \times \overrightarrow{M''}\|^2}{(\overrightarrow{M'}, \overrightarrow{M''}, \overrightarrow{M'''})}.$$

Remark: Some authors, including Darboux ([**?**], Livre I, Chapter 1) and Élie Cartan ([**?**], Chapter VII, Section 2), define the torsion as $-\tau$, in which case

$$\tau = \frac{(\overrightarrow{M'}, \overrightarrow{M''}, \overrightarrow{M'''})}{\|\overrightarrow{M'} \times \overrightarrow{M''}\|^2},$$

and the Frenet equations take the form

$$(\overrightarrow{t'}, \overrightarrow{n'}, \overrightarrow{b'}) = (\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}) \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}$$

A possible advantage of this choice is the elimination of the negative sign in the expression for τ above, and the fact that it may be slightly easier to remember the Frenet matrix, since signs on descending diagonals remain the same.

Another possible advantage is that the Frenet matrix has a similar shape in higher dimension (≥ 4).

Books on CAGD seem to prefer this choice.

On the other hand, do Carmo [**?**] and Berger and Gostiaux [**?**] use the opposite convention (as we do).

It should also be noted that if we let

$$\omega = \tau \overrightarrow{t} + \kappa \overrightarrow{b},$$

often called the *Darboux vector*, then (abbreviating three equations in one using a slight abuse of notation)

$$(\overrightarrow{t'}, \overrightarrow{n'}, \overrightarrow{b'}) = \omega \times (\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}),$$

which shows that the vectors

$$\overrightarrow{t'}, \overrightarrow{n'}, \overrightarrow{b'}$$

are the velocities of the tips of the unit frame, and that the unit frame rotates around an instantaneous axis of rotation passing through the origin of the frame, whose direction is the vector $\omega = \tau \overrightarrow{t} + \kappa \overrightarrow{b}$.

We now summarize the above considerations in the following definition and lemma.

Definition 1.8.1 Given any biregular $3D$ curve $f\colon\,]a,b[\to \mathcal{E}$ (or $f\colon [a,b] \to \mathcal{E}$) of class C^p parameterized by arc length, with $p \geq 3$, given the Frenet frame $(\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b})$ at s, the *curvature* κ *at* s is the nonnegative real such that $\overrightarrow{t'} = \kappa \overrightarrow{n}$, the *torsion* τ *at* s is the real such that $\overrightarrow{b'} = \tau \overrightarrow{n}$, the *radius of curvature at* s is the nonnegative real $\mathcal{R} = 1/\kappa$, the *radius of torsion at* s is the real $\mathcal{T} = 1/\tau$, the *center of curvature at* s is the point C on the principal normal such that $C - f(s) = \mathcal{R}\overrightarrow{n}$, and the *polar axis at* s is the line orthogonal to the osculating plane passing through the center of curvature.

Again, we stress that the curvature κ and the torsion τ are intrinsic *invariants* of the geometric curve defined by f.

Lemma 1.8.2 *Given any biregular* 3*D curve* $f\colon\,]a,b[\, \to \mathcal{E}$ *(or* $f\colon [a,b] \to \mathcal{E}$*) of class* C^p *parameterized by arc length, with* $p \geq 3$*, given the Frenet frame* $(\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b})$ *at* s*, we have the Frenet (or Frenet-Serret) equations*

$$(\overrightarrow{t'}, \overrightarrow{n'}, \overrightarrow{b'}) = (\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}) \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$$

Given any parameterization for f*, the curvature* κ *and the torsion* τ *are given by the expressions*

$$\kappa = \frac{\|f'(t) \times f''(t)\|}{\|f'(t)\|^3}$$

and

$$\tau = -\frac{(f'(t), f''(t), f'''(t))}{\|f'(t) \times f''(t)\|^2}.$$

Furthermore, for δ *small enough, the normal plane at* $t+\delta$ *rotates around the polar axis, a line othogonal to the osculating plane and passing through the center of curvature, and the rectifying plane at* $t+\delta$ *rotates around the line defined by the point of contact at* t *and the vector* $-\tau\overrightarrow{t}+\kappa\overrightarrow{b}$ *(the Darboux vector).*

The torsion measures how the osculating plane rotates around the tangent.

If f is a biregular curve and if $\tau = 0$ for all t, then f is a plane curve.

As an example of the computation of the torsion, consider the circular helix defined by

$$f(t) = (a\cos t,\ a\sin t,\ kt).$$

It is easy to show that the curvature is given by

$$\kappa = \frac{a}{a^2 + k^2}$$

and that the torsion is given by

$$\tau = -\frac{k}{a^2 + k^2}.$$

Thus, both the curvature and the torsion are constant!

The intrinsic nature of the curvature and the torsion is illustrated by the following result.

If $c\colon\,]a, b[\, \to \mathbb{R}_+$ is a continuous positive C^1 function, $d\colon\,]a, b[\, \to \mathbb{R}$ is a continuous function, and $s_0 \in\,]a, b[$, then there is a unique biregular $3D$ curve $f\colon\,]a, b[\, \to \mathcal{E}$ such that $f(s_0)$ is any given point, $f'(s_0)$ is any given vector, $f''(s_0)$ is any given vector, and such that $c(s)$ is the curvature of f, and $d(s)$ is the torsion of f.

Roughly speaking, the curvature and the torsion determine a biregular curve completely, up to rigid motion.

The hypothesis that $c(s) > 0$ for all s is crucial, and the above result is false if this condition is not satisfied everywhere.

1.9 Osculating Spheres ($3D$ Curves)

We conclude our discussion of curves in 3-space by discussing briefly the notion of osculating sphere.

Definition 1.9.1 For any $3D$ curve $f\colon\,]a,b[\, \to \mathcal{E}$ (or $f\colon [a,b] \to \mathcal{E}$) of class C^p, with $p \geq 3$, given any point $M_0 = f(t)$ on the curve, if the polar axis at t exists, f is locally injective at M_0, and the sphere $\Sigma_{t,h}$ centered on the polar axis and passing through the points M_0 and $M_1 = f(t+h)$ has a limit Σ_t when $h \neq 0$ approaches 0, we say that Σ_t is the *osculating sphere to f in $M_0 = f(t)$ at t.*

More precisely, if the polar axis at t exists and if there is an open interval $]t-\eta,\, t+\eta[\, \subseteq\,]a,b[$ (with $\eta > 0$) such that the point $M_1 = f(t+h)$ is distinct from M_0 for every $h \neq 0$ with $h \in\,]-\eta,\, +\eta[$ and the sphere $\Sigma_{t,h}$ centered on the polar axis and passing through the points M_0 and M_1 has a limit Σ_t when $h \neq 0$ approaches 0 (with $h \in\,]-\eta,\, +\eta[$), we say that Σ_t is the osculating sphere to f in $M_0 = f(t)$ at t.

Again, the definition is simpler when f is a simple curve. The following lemma gives a simple condition for the existence of the osculating sphere at a point.

Lemma 1.9.2 *For any 3D curve $f\colon]a,b[\to \mathcal{E}$ (or $f\colon [a,b] \to \mathcal{E}$) of class C^p parameterized by arc length, with $p \geq 3$, given any point $M_0 = f(s)$ on the curve, if M_0 is a biregular point at s and if $\mathcal{R}'$ is defined, then the osculating sphere to f in M_0 at s exists and has its center Ω on the polar axis Δ, such that $\Omega - C = -\mathcal{T}\mathcal{R}'\overrightarrow{b}$, where $\mathcal{T}$ is the radius of torsion, $\mathcal{R}$ is the radius of curvature, C is the center of curvature, and $\overrightarrow{b}$ is the binormal, at s.*

When s varies, the polar axis generates a surface, and the center Ω of the osculating sphere generates a curve on this surface.

In general, this surface consists of the tangents to this curve (called *line of striction* of the ruled surface).

Figure 8.10 illustrates the Frenet frame, the polar axis, the center of curvature, and the osculating sphere. It also shows the osculating plane, the normal plane, and the rectifying plane.

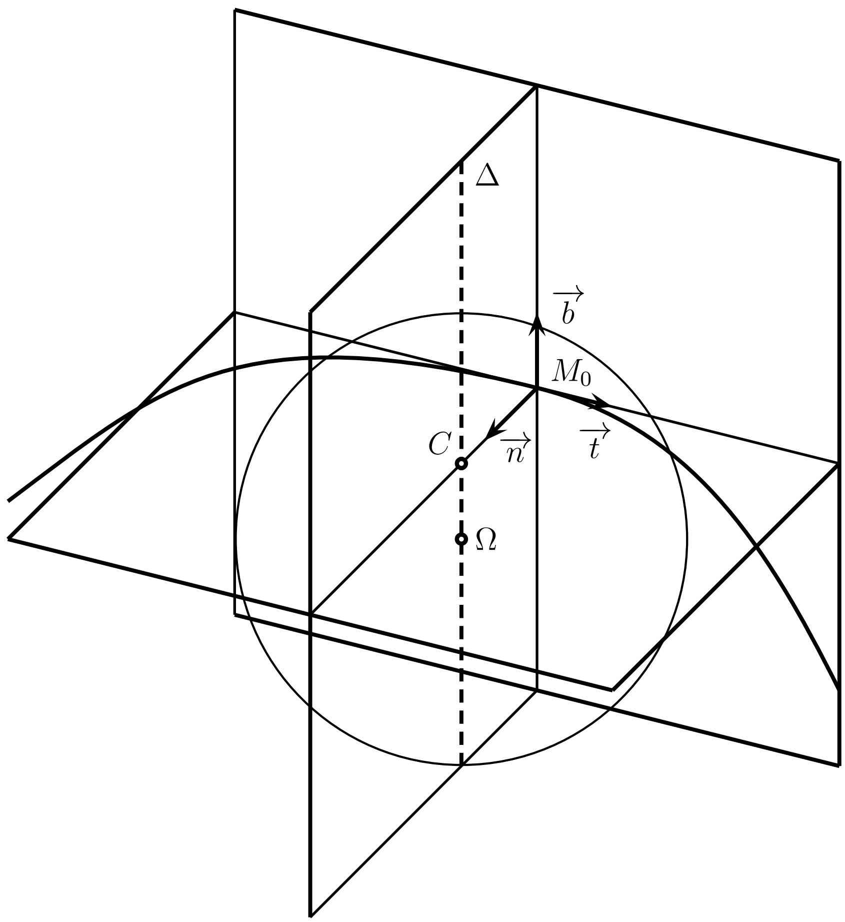

Figure 1.10: The Frenet Frame, polar axis, center of curvature, and osculating sphere

Finally, we discuss very briefly the case of curves in Euclidean spaces of dimension $n \geq 4$.

1.10 The Frenet Frame for nD Curves ($n \geq 4$)

Given a curve $f\colon\,]a,b[\, \to \mathcal{E}$ (or $f\colon [a,b] \to \mathcal{E}$) of class C^p parameterized by arc length, with $p \geq n$, where $\mathcal{E} = \mathbb{E}^n$ is a Euclidean space of dimension $n \geq 4$, for any s such that

$$f'(s), f''(s), \ldots, f^{(n-1)}(s)$$

are linearly independent, we can construct an orthonormal basis

$$(\overrightarrow{e_1}, \ldots, \overrightarrow{e_n})$$

where $(\overrightarrow{e_1}, \ldots, \overrightarrow{e_{n-1}})$ is constructed from

$$f'(s), f''(s), \ldots, f^{(n-1)}(s)$$

using the Gram-Schmidt orthonormalization procedure (see lemma **??**), and where

$$\overrightarrow{e_n} = \overrightarrow{e_1} \times \cdots \times \overrightarrow{e_{n-1}}.$$

The basis $(\overrightarrow{e_1}, \ldots, \overrightarrow{e_n})$ is a *generalized Frenet-Serret frame*.

Then, it is not difficult to show by induction that there are $n-1$ reals $\kappa_1, \ldots, \kappa_{n-1}$, the *generalized curvatures*, such that

$$
\begin{aligned}
&(\overrightarrow{e_1'}, \ldots, \overrightarrow{e_n'}) \\
&\quad = (\overrightarrow{e_1}, \ldots, \overrightarrow{e_n}) \begin{pmatrix} 0 & \kappa_1 & & & \\ -\kappa_1 & 0 & \kappa_2 & & \\ & -\kappa_2 & 0 & \ddots & \\ & & \ddots & \ddots & \kappa_{n-1} \\ & & & -\kappa_{n-1} & 0 \end{pmatrix}
\end{aligned}
$$

In fact, our previous reasoning is immediately extended to show that the limit of the intersection of the normal hyperplane at $t+\delta$ with the normal hyperplane at t (for δ small) with the osculating plane is a point C such that $C - f(t) = (1/\kappa_1)\overrightarrow{e_1}$.

Thus, we obtain a geometric interpretation for the curvature κ_1, and it is also possible to obtain an interpretation for the other κ_i.

The theorem showing that a curve is completely determined by the generalized curvatures $\kappa_1, \ldots, \kappa_{n-1}$ as functions of the arc length, with some suitable constraints, can also be generalized.

It should also be mentioned that it is possible to define a notion of affine normal and a notion of affine curvature without appealing to the concept of an inner product.

For some interesting applications, see Calabi, Olver, and Tannenbaum [**?**].