Curves and Surfaces: Theory and Applications CIS510

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Chapter 1

Introduction

1.1 The Need for Affine Geometry

Suppose we have a particle moving in 3-space and that we want to describe the trajectory of this particle.

If one looks up a good textbook on dynamics, such as Greenwood [**?**], one finds out that the particle is modeled as a point, and that the position of this point x is determined with respect to a "frame" in $\mathbb{R}^3$ by a vector.

A frame is a pair

$$(O, (\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}))$$

consisting of an origin O (which is a point) together with a basis of three vectors $(\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3})$.

For example, the standard frame in $\mathbb{R}^3$ has origin $O = (0, 0, 0)$ and the basis of three vectors $\overrightarrow{e_1} = (1, 0, 0)$, $\overrightarrow{e_2} = (0, 1, 0)$, and $\overrightarrow{e_3} = (0, 0, 1)$.

The position of a point x is then defined by the "unique vector" from O to x.

But wait a minute, this definition seems to be defining frames and the position of a point without defining what a point is!

Well, let us identify points with elements of $\mathbb{R}^3$.

If so, given any two points $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$, there is a unique *free vector* denoted $\overrightarrow{ab}$ from a to b, the vector $\overrightarrow{ab} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$.

Note that

$$b = a + \overrightarrow{ab},$$

addition being understood as addition in $\mathbb{R}^3$.

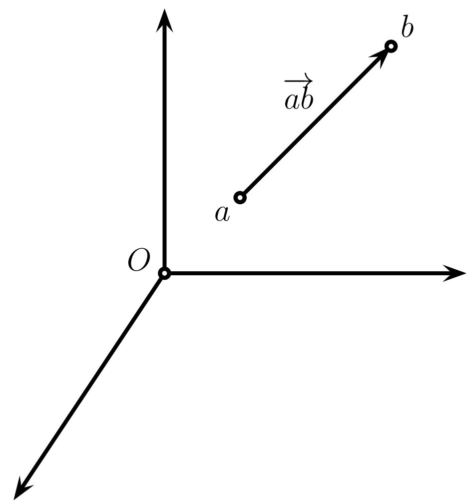


Figure 1.1: Points and free vectors

Then, in the standard frame, given a point $x = (x_1, x_2, x_3)$, the position of x is the vector $\overrightarrow{Ox} = (x_1, x_2, x_3)$, which coincides with the point itself.

What if we pick a frame with a different origin, say $\Omega = (\omega_1, \omega_2, \omega_3)$, but the same basis vectors $(\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3})$?

This time, the point $x = (x_1, x_2, x_3)$ is defined by two position vectors:

$\overrightarrow{Ox} = (x_1, x_2, x_3)$ in the frame $(O, (\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}))$, and

$\overrightarrow{\Omega x} = (x_1 - \omega_1, x_2 - \omega_2, x_3 - \omega_3)$ in the frame $(\Omega, (\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}))$.

This is because

$$\overrightarrow{Ox} = \overrightarrow{O\Omega} + \overrightarrow{\Omega x} \quad \text{and} \quad \overrightarrow{O\Omega} = (\omega_1, \omega_2, \omega_3).$$

We note that in the second frame $(\Omega, (\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}))$, points and position vectors are no longer identified.

This gives us evidence that **points are not vectors**. Inspired by physics, it is important to define points and properties of points that are frame invariant.

An undesirable side-effect of the present approach shows up if we attempt to define linear combinations of points.

If we consider the change of frame from the frame

$$(O, (\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}))$$

to the frame

$$(\Omega, (\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3})),$$

where

$$\overrightarrow{O\Omega} = (\omega_1, \omega_2, \omega_3),$$

given two points a and b of coordinates (a_1, a_2, a_3) and (b_1, b_2, b_3) with respect to the frame $(O, (\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}))$ and of coordinates (a'_1, a'_2, a'_3) and (b'_1, b'_2, b'_3) of with respect to the frame $(\Omega, (\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}))$, since

$$(a'_1, a'_2, a'_3) = (a_1 - \omega_1, a_2 - \omega_2, a_3 - \omega_3)$$

and

$$(b'_1, b'_2, b'_3) = (b_1 - \omega_1, b_2 - \omega_2, b_3 - \omega_3),$$

the coordinates of $\lambda a + \mu b$ with respect to the frame $(O, (\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}))$ are

$$(\lambda a_1 + \mu b_1, \lambda a_2 + \mu b_2, \lambda a_3 + \mu b_3),$$

but the coordinates

$$(\lambda a'_1 + \mu b'_1, \lambda a'_2 + \mu b'_2, \lambda a'_3 + \mu b'_3)$$

of $\lambda a + \mu b$ with respect to the frame $(\Omega, (\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}))$ are

$$\begin{gathered}(\lambda a_1 + \mu b_1 - (\lambda + \mu)\omega_1,\\ \lambda a_2 + \mu b_2 - (\lambda + \mu)\omega_2,\\ \lambda a_3 + \mu b_3 - (\lambda + \mu)\omega_3)\end{gathered}$$

which are different from

$$(\lambda a_1 + \mu b_1 - \omega_1, \lambda a_2 + \mu b_2 - \omega_2, \lambda a_3 + \mu b_3 - \omega_3),$$

unless $\lambda + \mu = 1$.

Thus, we discovered a major difference between vectors and points: the notion of linear combination of vectors is basis independent, but the notion of linear combination of points is frame dependent.

In order to salvage the notion of linear combination of points, some restriction is needed: the scalar coefficients must add up to 1.

A clean way to handle the problem of frame invariance and to deal with points in a more intrinsic manner is to make a clearer distinction between points and vectors.

We duplicate $\mathbb{R}^3$ into two copies, the first copy corresponding to points, where we forget the vector space structure, and the second copy corresponding to free vectors, where the vector space structure is important.

Furthermore, we make explicit the important fact that the vector space $\mathbb{R}^3$ acts on the set of points $\mathbb{R}^3$: Given any **point** $a = (a_1, a_2, a_3)$ and any **vector** $\overrightarrow{v} = (v_1, v_2, v_3)$, we obtain the **point**

$$a + \overrightarrow{v} = (a_1 + v_1, a_2 + v_2, a_3 + v_3),$$

which can be thought of as the result of translating a to b using the vector $\overrightarrow{v}$.

This action $+\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ satisfies some crucial properties. For example,

$$\begin{aligned} a + \overrightarrow{0} &= a, \\ (a + \overrightarrow{u}) + \overrightarrow{v} &= a + (\overrightarrow{u} + \overrightarrow{v}), \end{aligned}$$

and for any two points a, b, there is a unique free vector $\overrightarrow{ab}$ such that

$$b = a + \overrightarrow{ab}.$$

It turns out that the above properties, although trivial in the case of $\mathbb{R}^3$, are all that is needed to define the abstract notion of affine space (or affine structure).

This will be done rigorously in Chapter 5, but first, we will take an informal look at polynomial curves.

When we want to stress that we are dealing at the same time with points and vectors, we use the notation $\mathbb{A}^n$ for $\mathbb{R}^n$. We call $\mathbb{A}^n$ the *real affine space of dimension n*.

We need the concept of *affine combination, or barycenter*. Assume for simplicity that we are in $\mathbb{R}^3$ (really, $\mathbb{A}^3$), and that we use the standard frame $(O, (\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}))$.

Given any two points $a, b \in \mathbb{A}^3$ of coordinates (a_1, a_2, a_3) and (b_1, b_2, b_3), for any real number $\lambda \in \mathbb{R}$, we define the point

$$(1-\lambda)a + \lambda b$$

as the point of coordinates

$$((1-\lambda)a_1 + \lambda b_1,\ (1-\lambda)a_2 + \lambda b_2,\ (1-\lambda)a_3 + \lambda b_3).$$

This is the point

$$a + \lambda \overrightarrow{ab},$$

which is located "λ of the way from a" on the line determined by a and b (or a itself when $a = b$).

More generally, given n points $a_1, \ldots, a_n \in \mathbb{A}^3$, for any n reals $\lambda_i \in \mathbb{R}$ such that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1,$$

the *affine combination (or barycenter) of the points* $a_1, \ldots, a_n$ *w.r.t. the weights* $\lambda_1, \ldots, \lambda_n$ is the point

$$\lambda_1 a_1 + \cdots + \lambda_n a_n$$

also denoted by

$$\sum_{i=1}^{n} \lambda_i a_i$$

of coordinates

$$\left(\sum_{i=1}^{n} \lambda_i a_i^1, \sum_{i=1}^{n} \lambda_i a_i^2, \sum_{i=1}^{n} \lambda_i a_i^3 \right),$$

where a_i has the coordinates (a_i^1, a_i^2, a_i^3).

Barycenters can be characterized more geometrically as follows:

Assume we have n *weighted points*, (a_i, λ_i), where $\sum_{i=1}^{n} \lambda_i = 1$. For *any* choice of a point $b \in \mathbb{A}^3$, we can form the point

$$g = b + \lambda_1 \overrightarrow{ba_1} + \cdots + \lambda_n \overrightarrow{ba_n}.$$

It can be shown that g *does not depend* on the choice of the point b.

This uniquely defined point, g, *is* the barycenter of the weighted points (a_i, λ_i).

If we set $b = g$, we see that g is characterized by the fact that

$$\lambda_1 \overrightarrow{ga_1} + \cdots + \lambda_n \overrightarrow{ga_n} = 0.$$

Intuitively , g is "balances" the forces $\lambda_i \overrightarrow{ga_i}$. We can think of the λ_i's as (normalized) electric charges.

The case where $\lambda_i = 1/n$ for $i = 1, \ldots, n$, corresponds to the *centroid* (or *center of gravity*) of the points $a_1, \ldots, a_n$.

The barycenter of n weighted points can be computed by repeatedly computing barycenters of two points.

When $\lambda_i \geq 0$ for all i (and, of course, $\sum_{i=1}^{n} \lambda_i = 1$), the affine combination $\sum_{i=1}^{n} \lambda_i a_i$ is called a *convex combination* of the points $a_1, \ldots, a_n$.

Given any two points a, b, the set of all convex combinations $(1 - \lambda)a + \lambda b$ (recall that $0 \leq \lambda \leq 1$) is the *line segment* with endpoints a and b, denoted $[a, b]$.

A subset S of $\mathbb{A}^3$ is *convex* if it contains all affine combinations of (finitely many) points of S.

It can be shown that this is equivalent to the fact that S contains all affine combinations of any pair of points of S, that is, whenever $a, b \in S$, then the entire line segment $[a, b]$ is contained in S.

An *affine frame in* $\mathbb{A}^n$ is a pair

$$(a_0, (\overrightarrow{e_1}, \ldots, \overrightarrow{e_n})),$$

where a_0 is the origin of the frame and $(\overrightarrow{e_1}, \ldots, \overrightarrow{e_n})$ is a basis of $\mathbb{R}^n$. We also say that the $n+1$ points

$$(a_0, a_1, \ldots, a_n)$$

form an affine basis, where

$$a_i = a_0 + \overrightarrow{e_i},$$

for $i = 1, \ldots, n$.

We say that $m+1$ (ordered) points $a_0, a_1, \ldots, a_m$ are *affinely independent* in $\mathbb{A}^n$ iff the m vectors

$$\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m}$$

are linearly independent in $\mathbb{R}^n$.

Of course, this implies $m \leq n+1$.

An affine frame in $\mathbb{A}$ is any pair (r, s) of distinct numbers $r, s \in \mathbb{R}$.

An affine frame in $\mathbb{A}^2$ consists of any three points forming a nondegenerate triangle.

An affine frame in $\mathbb{A}^3$ consists of any four points forming a nondegenerate tetrahedron.

Every point $t \in \mathbb{A}$ is expressed as

$$t = (1 - t)0 + t1$$

in terms of the affine frame $(0, 1)$.

In term of an arbitrary frame (r, s),

$$t = \left(\frac{s - t}{s - r}\right) r + \left(\frac{t - r}{s - r}\right) s.$$

An *affine map* $f\colon \mathbb{A}^p \to \mathbb{A}^q$ is a function that preserves affine combinations, i.e.,

$$f\left(\sum_{i=1}^{n} \lambda_i a_i\right) = \sum_{i=1}^{n} \lambda_i f(a_i)$$

for any n points $a_i \in \mathbb{A}^p$ and any scalars λ_i such that $\lambda_1 + \cdots + \lambda_n = 1$.

A special case of an affine map is a *translation*. This is the case of an affine map, t, for which there is a vector, $\overrightarrow{u}$, so that

$$t(a) = a + \overrightarrow{u}$$

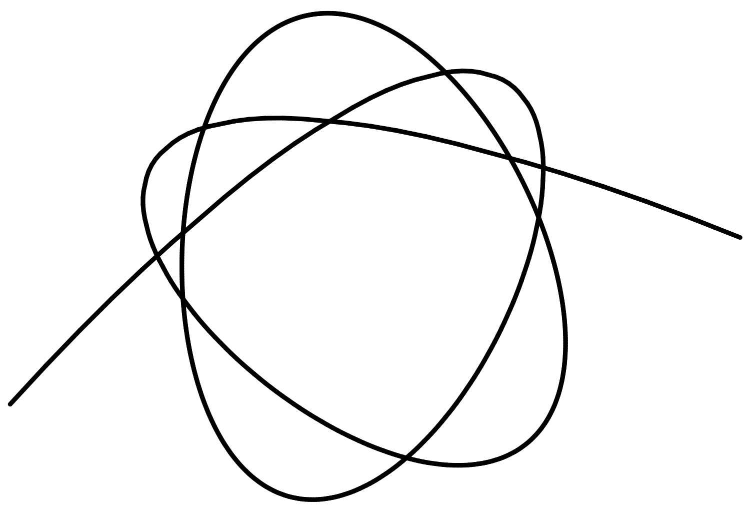
for all $a \in \mathbb{A}^3$.

Note that affine maps are more general than linear maps, because translations are *not* linear (unless $\overrightarrow{u} = 0$).

It is easily shown that every affine map can be written as the composition of a translation and of a linear map.

(Strictly speaking, instead of linear map, we should say an affine map that has a *fixed point*, i.e., a point a so that $f(a) = a$.)

Polynomial Curves and Spline Curves



Chapter 2

Introduction to the Algorithmic Geometry of Polynomial Curves

2.1 Parameterized Polynomial Curves

Recall that every point $\bar{t} \in \mathbb{A}$ is expressed as

$$\bar{t} = (1 - t)0 + t1$$

in terms of the affine frame $(0, 1)$.

A parameterized polynomial curve is defined as follows.

Definition 2.1.1 A (parameterized) *polynomial curve, F, of degree at most m* is a map $F\colon \mathbb{A} \to \mathcal{E}$ (where $\mathcal{E} = \mathbb{R}^2$ or $\mathbb{R}^3$) such that there exists real polynomials P_1, P_2 (resp. P_1, P_2, P_3), of degree at most m, so that for every $t \in \mathbb{A}$,

$$\begin{aligned} F_1(t) &= P_1(t) \\ F_2(t) &= P_2(t) \\ F_3(t) &= P_3(t) \end{aligned}$$

(dropping F_3 and P_3 when defining a curve in $\mathbb{A}^2$).

Given any affine frame (r, s) for $\mathbb{A}$ with $r < s$, a (parameterized) *polynomial curve segment $F[r, s]$ of degree (at most) m* is the restriction $F\colon [r, s] \to \mathcal{E}$ of a polynomial curve $F\colon \mathbb{A} \to \mathcal{E}$ of degree at most m.

The set of points $F(\mathbb{A})$ in $\mathcal{E}$ is called the *trace of the polynomial curve F*, and similarly, the set of points $F([r, s])$ in $\mathcal{E}$ is called the *trace of the polynomial curve segment $F[r, s]$*.

The definition can easily be extended to higher dimensional spaces (if $\mathcal{E} = \mathbb{A}^n$, use n polynomials $P_1, \ldots, P_n$ of degree at most m).

Intuitively, a polynomial curve is obtained by bending and twisting the affine line $\mathbb{A}$ using a polynomial map.

It should be noted that the maximum degree d of the polynomials $P_1, \ldots, P_n$ defining a polynomial curve F of degree m is not necessarily equal to m, and that it is only required that $d \leq m$.

For notational simplicity, we also denote the polynomials P_i by F_i.

We will now try to gain some insight into polynomial curves by determining the shape of the traces of plane polynomial curves (curves living in $\mathcal{E} = \mathbb{A}^2$) of degree $m \leq 3$. On the way, we will introduce a major technique of CAGD, *blossoming*.

We begin with $m = 1$. A polynomial curve F of degree ≤ 1 is of the form

$$\begin{aligned} x(t) &= F_1(t) = a_1 t + a_0, \\ y(t) &= F_2(t) = b_1 t + b_0. \end{aligned}$$

If both $a_1 = b_1 = 0$, the trace of F reduces to the single point (a_0, b_0). Otherwise, $a_1 \neq 0$ or $b_1 \neq 0$, and we can eliminate t between x and y, getting the implicit equation

$$a_1 y - b_1 x + a_0 b_1 - a_1 b_0 = 0,$$

which is the equation of a straight line.

Let us now consider $m = 2$, that is, quadratic curves. A polynomial curve F of degree ≤ 2 is of the form

$$
\begin{aligned}
x(t) &= F_1(t) = a_2 t^2 + a_1 t + a_0,\\
y(t) &= F_2(t) = b_2 t^2 + b_1 t + b_0.
\end{aligned}
$$

Since we already considered the case where $a_2 = b_2 = 0$, let us assume that $a_2 \neq 0$ or $b_2 \neq 0$.

We first show that by a change of coordinates (amounting to a rotation), we can always assume that either $a_2 = 0$ or $b_2 = 0$. If $a_2 \neq 0$ and $b_2 \neq 0$, after a rotation and a translations of the axes, and a change of parameter, we get a parametric representation of the form

$$
\begin{aligned}
X(u) &= au,\\
Y(u) &= bu^2,
\end{aligned}
$$

with $b > 0$. The corresponding implicit equation is

$$Y = \frac{b}{a^2} X^2.$$

This is a parabola, passing through the origin, and having the Y-axis as axis of symmetry. The diagram below shows the parabola defined by the following parametric equations

$$F_1(t) = 2t,$$
$$F_2(t) = t^2.$$

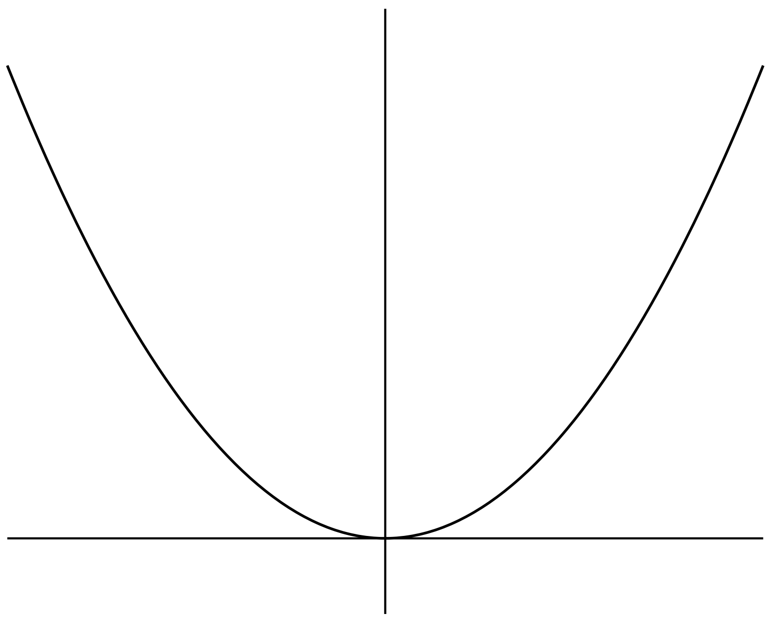


Figure 2.1: A parabola

Intuitively, the previous degenerate case (of a straight line) corresponds to $\frac{b}{a^2} = \infty$.

Conversely, since by an appropriate change of coordinates, every parabola is defined by the implicit equation $Y = aX^2$, every parabola can be defined as the parametric polynomial curve

$$\begin{aligned} X(u) &= u, \\ Y(u) &= au^2. \end{aligned}$$

We now show that there is another way of specifying quadratic polynomial curves which yields a very nice geometric algorithm for constructing points on these curves. The general philosophy is to *linearize* (or more exactly, multilinearize) polynomials. As a warm up, let us begin with straight lines.

In the case of an affine map $F\colon \mathbb{A} \to \mathbb{A}^3$, given any affine frame (r, s) for $\mathbb{A}$, where $r \neq s$, every point $F(t)$ on the line defined by F is obtained by a single interpolation step

$$F(t) = \left(\frac{s-t}{s-r}\right) F(r) + \left(\frac{t-r}{s-r}\right) F(s),$$

as illustrated in the following diagram, where $\frac{t-r}{s-r} = \frac{1}{3}$

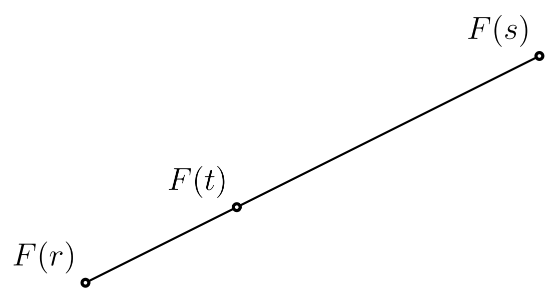


Figure 2.2: Linear Interpolation

We would like to generalize the idea of determining the point $F(t)$ on the the line defined by $F(r)$ and $F(s)$ by an interpolation step, to determining the point $F(t)$ on a polynomial curve F, by several interpolation steps from some (finite) set of given points related to the curve F.

For this, it is first necessary to turn the polynomials involved in the definition of F into multiaffine maps, that is, maps that are affine in each of their arguments. We now show how to turn a quadratic polynomial into a biaffine map.

As an example, consider the polynomial

$$F(X) = X^2 + 2X - 3.$$

Observe that the function of two variables

$$f_1(x_1, x_2) = x_1x_2 + 2x_1 - 3$$

gives us back the polynomial $F(X)$ on the diagonal, in the sense that $F(X) = f_1(X, X)$, for all $X \in \mathbb{R}$, but f_1 is also affine in each of x_1 and x_2. Note that

$$f_2(x_1, x_2) = x_1x_2 + 2x_2 - 3$$

is also biaffine, and $F(X) = f_2(X, X)$, for all $X \in \mathbb{R}$.

It would be nicer if we could find a unique biaffine function f such that $F(X) = f(X, X)$, for all $X \in \mathbb{R}$, and of course, such a function should satisfy some additional property.

It turns out that requiring f to be symmetric is just what's needed. We say that a function f of two arguments is *symmetric* iff

$$f(x_1, x_2) = f(x_2, x_1),$$

for all x_1, x_2. To make f_1 (and f_2) symmetric, simply form

$$f(x_1, x_2) = \frac{f_1(x_1, x_2) + f_1(x_2, x_1)}{2} = x_1x_2 + x_1 + x_2 - 3.$$

The symmetric biaffine function

$$f(x_1, x_2) = x_1x_2 + x_1 + x_2 - 3$$

is called the (affine) *blossom, or polar form, of F.*

For an arbitrary polynomial

$$F(X) = aX^2 + bX + c$$

of degree ≤ 2, we obtain a unique symmetric, biaffine map

$$f(x_1, x_2) = ax_1x_2 + b\,\frac{x_1 + x_2}{2} + c$$

such that $F(X) = f(X, X)$, for all $X \in \mathbb{R}$, called the polar form, or blossom, of F.

Note that the fact that f is symmetric allows us to view the arguments of f as a multiset (the order of the arguments x_1, x_2 is irrelevant).

Every $t \in \mathbb{A}$ can be expressed uniquely as a barycentric combination of r and s, say $t = (1-\lambda)r + \lambda s$, where $\lambda \in \mathbb{R}$.

Let us compute

$$f(t_1, t_2) = f((1-\lambda_1)r + \lambda_1 s,\ (1-\lambda_2)r + \lambda_2 s).$$

Since $\lambda_i = \frac{t_i - r}{s-r}$, for $i = 1, 2$, we get

$$\begin{aligned} f(t_1, t_2) = & \left(\frac{s-t_1}{s-r}\right)\left(\frac{s-t_2}{s-r}\right) f(r, r) \\ & + \left[\left(\frac{s-t_1}{s-r}\right)\left(\frac{t_2-r}{s-r}\right) + \left(\frac{t_1-r}{s-r}\right)\left(\frac{s-t_2}{s-r}\right)\right] f(r, s) \\ & + \left(\frac{t_1-r}{s-r}\right)\left(\frac{t_2-r}{s-r}\right) f(s, s). \end{aligned}$$

The coefficients of $f(r,r)$, $f(r,s)$ and $f(s,s)$ are obviously symmetric biaffine functions, and they add up to 1, as it is easily verified by expanding the product

$$\left(\frac{s-t_1}{s-r}+\frac{t_1-r}{s-r}\right)\left(\frac{s-t_2}{s-r}+\frac{t_2-r}{s-r}\right)=1.$$

Thus, we showed that every symmetric biaffine map $f\colon \mathbb{A}^2 \to \mathbb{A}^3$ is completely determined by the sequence of three points $f(r,r)$, $f(r,s)$ and $f(s,s)$ in $\mathbb{A}^3$, where $r \neq s$ are elements of $\mathbb{A}$.

Conversely, it is clear that given any sequence of three points $a, b, c \in \mathbb{A}^3$, the map

$$
\begin{aligned}
(t_1, t_2) \mapsto \; & \left(\frac{s-t_1}{s-r}\right)\left(\frac{s-t_2}{s-r}\right) a \\
& + \left[\left(\frac{s-t_1}{s-r}\right)\left(\frac{t_2-r}{s-r}\right) + \left(\frac{t_1-r}{s-r}\right)\left(\frac{s-t_2}{s-r}\right)\right] b \\
& + \left(\frac{t_1-r}{s-r}\right)\left(\frac{t_2-r}{s-r}\right) c
\end{aligned}
$$

is symmetric biaffine, and that $f(r, r) = a$, $f(r, s) = b$, $f(s, s) = c$.

The points $f(r,r)$, $f(r,s)$ and $f(s,s)$, are called *control points*, or *Bézier control points*, and as we shall see, they play a major role in the de Casteljau algorithm and its extensions.

If we let $r = 0$ and $s = 1$, then $t_1 = \lambda_1$ and $t_2 = \lambda_2$, and thus, the polynomial function corresponding to $f(t_1, t_2)$ beeing obtained by letting $t_1 = t_2 = t$, we get

$$F(t) = f(t,t) = \\ (1-t)^2\, f(0,0) + 2(1-t)t\, f(0,1) + t^2\, f(1,1).$$

The polynomials

$$(1-t)^2,\ 2(1-t)t,\ t^2$$

are known as the *Bernstein polynomials of degree* 2. Thus, $F(t)$ is also determined by the control points $f(0,0)$, $f(0,1)$, and $f(1,1)$, and the Bernstein polynomials.

Observe that the computation of

$$f(t_1, t_2) = f((1 - \lambda_1)r + \lambda_1 s,\ (1 - \lambda_2)r + \lambda_2 s),$$

that we performed above, can be turned into an algorithm, known as the *de Casteljau algorithm*.

Given any $t \in \mathbb{A}$, we will show how to construct geometrically the point $F(t) = f(t, t)$ on the polynomial curve F. Let $t = (1 - \lambda)r + \lambda s$. Then, $f(t, t)$ is computed as follows:

$$\begin{array}{ccc}
 & 1 & 2 \\
f(r,r) & & \\
 & f(r,t) & \\
f(r,s) & & f(t,t) \\
 & f(t,s) & \\
f(s,s) & &
\end{array}$$

The algorithm consists of two stages.

Geometrically, the algorithm consists of a diagram consisting of two polylines, the first one consisting of the two line segments

$$(f(r,r),\ f(r,s)) \quad \text{and} \quad (f(r,s),\ f(s,s)),$$

and the second one of the single line segment

$$(f(t,r),\ f(t,s)),$$

with the desired point $f(t,t)$ determined by λ. Each polyline given by the algorithm is called a *shell*, and the resulting diagram is called a *de Casteljau diagram*.

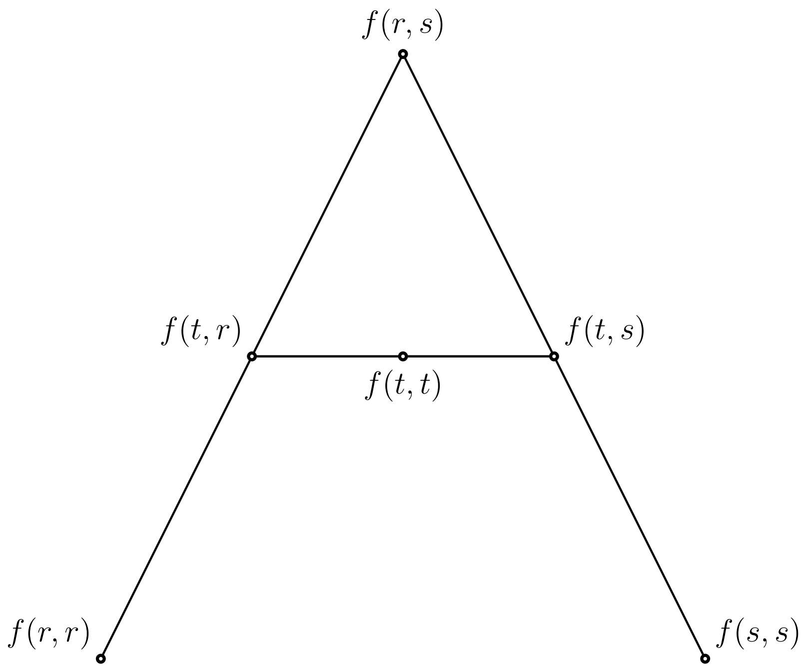


Figure 2.3: A de Casteljau diagram

The first polyline is also called a *control polygon* of the curve. Note that the shells are nested nicely. Actually, when t is outside $[r, s]$, we still obtain two polylines and a de Casteljau diagram, but the shells are not nicely nested. The following diagram illustrates the de Casteljau algorithm.

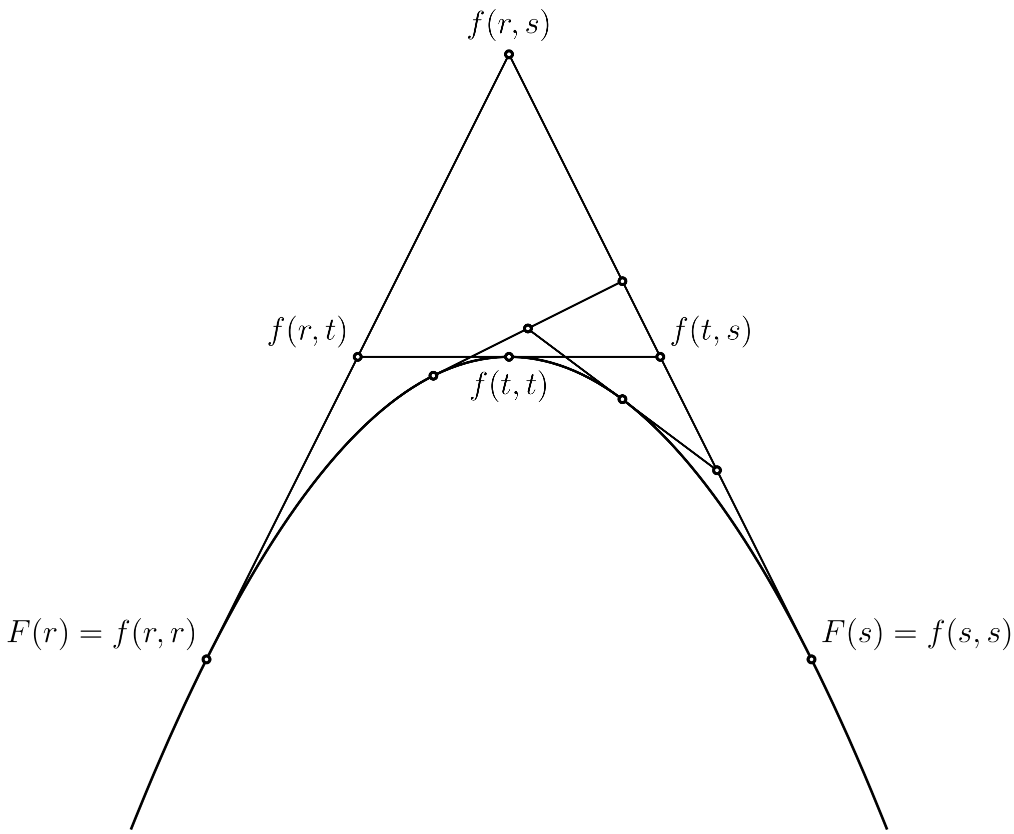


Figure 2.4: The de Casteljau algorithm

The above example shows the construction of the point $F(t)$ corresponding to $t = 1/2$, on the curve F, for $r = 0$, $s = 1$. It also shows the construction of another point on the curve, assuming different control points.

The parabola of the previous example is actually given by the parametric equations

$$F_1(t) = 2t,$$
$$F_2(t) = -t^2.$$

The polar forms are

$$f_1(t_1, t_2) = t_1 + t_2,$$
$$f_2(t_1, t_2) = -t_1 t_2.$$

The de Casteljau algorithm can also applied to compute any polar value $f(t_1, t_2)$:

$$
\begin{array}{lll}
 & 1 & 2 \\
f(r, r) & & \\
 & f(r, t_1) & \\
f(r, s) & & f(t_1, t_2) \\
 & f(t_1, s) & \\
f(s, s) & &
\end{array}
$$

The only difference is that we use different λ's during each of the two stages.

A nice geometric interpretation of the polar value $f(t_1, t_2)$ can be obtained. For this, we need to look closely at the intersection of two tangents to a parabola. Let us consider the parabola given by

$$
\begin{aligned}
x(t) &= at \\
y(t) &= bt^2.
\end{aligned}
$$

The equation of the tangent to the parabola at $(x(t),\ y(t))$ is

$$x'(t)(y - y(t)) - y'(t)(x - x(t)) = 0,$$

that is,

$$ay - 2btx + abt^2 = 0.$$

To find the intersection of the two tangents to the parabola corresponding to $t = t_1$ and $t = t_2$, we solve the system of linear equations

$$\begin{aligned} ay - 2bt_1x + abt_1^2 &= 0 \\ ay - 2bt_2x + abt_2^2 &= 0, \end{aligned}$$

and we easily find that

$$\begin{aligned} x &= a\,\frac{t_1 + t_2}{2}, \\ y &= bt_1t_2. \end{aligned}$$

Thus, the polar form $f(t_1, t_2)$ of the polynomial function defining a parabola gives precisely the intersection point of the two tangents at $F(t_1)$ and $F(t_2)$ to the parabola.

Let us now consider $m = 3$, that is, cubic curves.

A polynomial curve F of degree ≤ 3 is of the form

$$x(t) = F_1(t) = a_3t^3 + a_2t^2 + a_1t + a_0,$$
$$y(t) = F_2(t) = b_3t^3 + b_2t^2 + b_1t + b_0.$$

Since we already considered the case where $a_3 = b_3 = 0$, let us assume that $a_3 \neq 0$ or $b_3 \neq 0$. If $a_3 \neq 0$ and $b_3 \neq 0$, let $\rho = \sqrt{a_3^2 + b_3^2}$, and consider the matrix R given below:

$$R = \begin{pmatrix} \frac{b_3}{\rho} & -\frac{a_3}{\rho} \\ \frac{a_3}{\rho} & \frac{b_3}{\rho} \end{pmatrix}$$

Under the change of coordinates

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix},$$

we get

$$\begin{aligned} x_1(t) &= \frac{a_2b_3 - a_3b_2}{\rho}\, t^2 + \frac{a_1b_3 - a_3b_1}{\rho}\, t + \frac{a_0b_3 - a_3b_0}{\rho}, \\ y_1(t) &= \rho t^3 + \frac{a_2a_3 + b_2b_3}{\rho}\, t^2 + \frac{a_1a_3 + b_1b_3}{\rho}\, t + \frac{a_0a_3 + b_0b_3}{\rho}. \end{aligned}$$

The effect of this rotation is that the curve now "stands straight up" (since $\rho > 0$).

Case 1. $a_2b_3 = a_3b_2$.

Then we have a degenerate case where $x_1(t)$ is equal to a linear function. If $a_1b_3 = a_3b_1$ also holds, then $x_1(t)$ is a constant and $y_1(t)$ can be arbitrary, since its leading term is ρt^3, and we get the straight line

$$X = \frac{a_0b_3 - a_3b_0}{\rho}.$$

If $a_1b_3 - a_3b_1 \neq 0$, let us assume that $a_1b_3 - a_3b_1 > 0$, the other case being similar. Then, we can eliminate t between $x_1(t)$ and $y_1(t)$, and we get an implicit equation of the form

$$y = a'x^3 + b'x^2 + c'x + d',$$

with $a' > 0$. Using some change of coordinates, we get the implicit equation

$$Y = aX^3 + bX,$$

with $a > 0$. This curve is symmetric with respect to the Y-axis.

Its shape will depend on the variations of sign of its derivative

$$Y' = 3aX^2 + b.$$

Also, since $Y'' = 6aX$, and $Y''(0) = 0$, the origin is an inflexion point.

If $b > 0$, then $Y'(X)$ is always strictly positive, and $Y(X)$ is strictly increasing with X. It has a flat S-shape, the slope b of the tangent at the origin beeing positive.

If $b = 0$, then $Y'(0) = 0$, and 0 is a double root of Y', which means that the origin is an inflexion point. The curve still has a flat S-shape, and the tangent at the origin is the X-axis.

If $b < 0$, then $Y'(X)$ has two roots,

$$X_1 = +\sqrt{\frac{-b}{3a}}, \qquad X_2 = -\sqrt{\frac{-b}{3a}}.$$

Then, $Y(X)$ is increasing when X varies from $-\infty$ to X_1, decreasing when X varies from X_1 to X_2, and increasing again when X varies from X_2 to $+\infty$. The curve has an S-shape, the slope b of the tangent at the origin beeing negative.

The following diagram shows the cubic of implicit equation

$$y = 3x^3 - 3x.$$

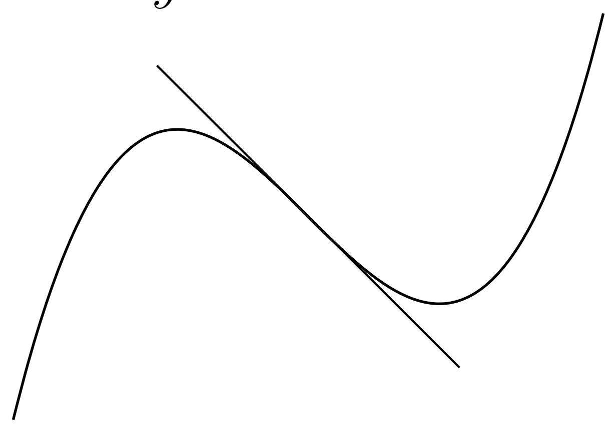


Figure 2.5: “S-shaped” Cubic

In all three cases, note that a line parallel to the Y-axis intersects the curve in a single point. This is the reason why we get a parametric representation.

Case 2. $a_2 b_3 - a_3 b_2 \neq 0$.

In this case, we say that we have a *nondegenerate cubic* (recall that $\rho > 0$).

Lemma 2.1.2 *Given any nondegenerate cubic polynomial curve F, i.e., any polynomial curve of the form*

$$\begin{aligned} x(t) &= F_1(t) = a_2 t^2, \\ y(t) &= F_2(t) = b_3 t^3 + b_2 t^2 + b_1 t, \end{aligned}$$

where $b_3 > 0$, after the translation of the origin given by

$$\begin{aligned} x &= X - \frac{b_1 a_2}{b_3}, \\ y &= Y - \frac{b_1 b_2}{b_3}, \end{aligned}$$

the trace of F satisfies the implicit equation

$$a_2 \left(\frac{a_2}{b_3} Y - \frac{b_2}{b_3} X \right)^2 + \frac{b_1 a_2}{b_3} X^2 = X^3.$$

Furthermore, if $b_1 \leq 0$, then the curve defined by the above implicit equation is equal to the trace of the polynomial curve F, and when $b_1 > 0$, the curve defined by the above implicit equation, excluding the origin $(X, Y) = (0, 0)$, is equal to the trace of the polynomial curve F. The origin $(X, Y) = (0, 0)$ is called a singular point of the curve defined by the implicit equation.

Thus, lemma 2.1.2 shows that every nondegenerate polynomial cubic is defined by some implicit equation of the form

$$c(aY - bX)^2 + cdX^2 = X^3,$$

with the exception that when $d > 0$, the singular point $(X, Y) = (0, 0)$ must be excluded from the trace of the polynomial curve.

The case where $d > 0$ is another illustration of the mismatch between the implicit and the explicit representation of curves. Again, this mismatch can be resolved if we treat these curves as complex curves.

The reason for choosing the origin at the singular point, is that if we intersect the trace of the polynomial curve with a line of slope m passing through the singular point, we discover that we get a nice parametric representation of the polynomial curve in terms of the parameter m.

Lemma 2.1.3 *For every nondegenerate cubic polynomial curve F, there is some parametric definition G of the form*

$$\begin{aligned} X(m) &= c(a\,m - b)^2 + cd, \\ Y(m) &= m(c(a\,m - b)^2 + cd), \end{aligned}$$

such that F and G have the same trace, which is also the set of points on the curve defined by the implicit equation

$$c(aY - bX)^2 + cdX^2 = X^3,$$

excluding the origin $(X, Y) = (0, 0)$, when $d > 0$.

Furthermore, unless it is a tangent at the origin to the trace of the polynomial curve F (which only happens when $d \leq 0$), every line of slope m passing through the origin $(X, Y) = (0, 0)$ intersects the trace of the polynomial curve F in a single point other than the singular point $(X, Y) = (0, 0)$.

The line $aY - bX = 0$ is an axis of symmetry for the curve, in the sense that for any two points (X, Y_1) and (X, Y_2) such that

$$Y_1 + Y_2 = \frac{2b}{a}X,$$

(X, Y_1) belongs to the trace of F iff (X, Y_2) belongs to the trace of F. The tangent at the point

$$(X, Y) = \left(cd, \frac{bcd}{a}\right)$$

of the trace of F (also on the axis of symmetry) is vertical.

We can now specify more precisely what is the shape of the trace of F, by studying the changes of sign of the derivative of $Y(m)$. We treat the case where $c > 0$, the case $c < 0$ being similar.

Case 1: $3d > b^2$.

In this case, we must have $d > 0$, which means that the singular point $(X, Y) = (0, 0)$ is not on the trace of the cubic.

Cubic of equation $3(Y-X)^2+6X^2=X^3$:

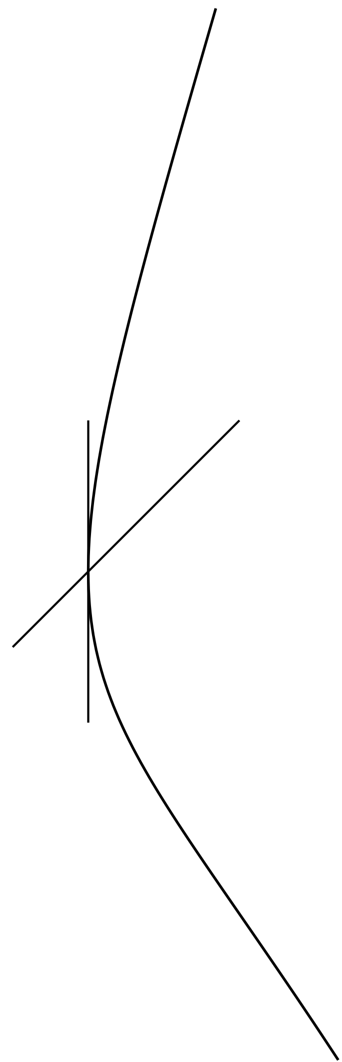


Figure 2.6: "Humpy" Cubic ($3d > b^2$)

Case 2: $b^2 \geq 3d > 0$.

In this case, since $d > 0$, the singular point $(X, Y) = (0, 0)$ is not on the trace of the cubic either.

Cubic of equation $3(Y - 2X)^2 + 3X^2 = X^3$:

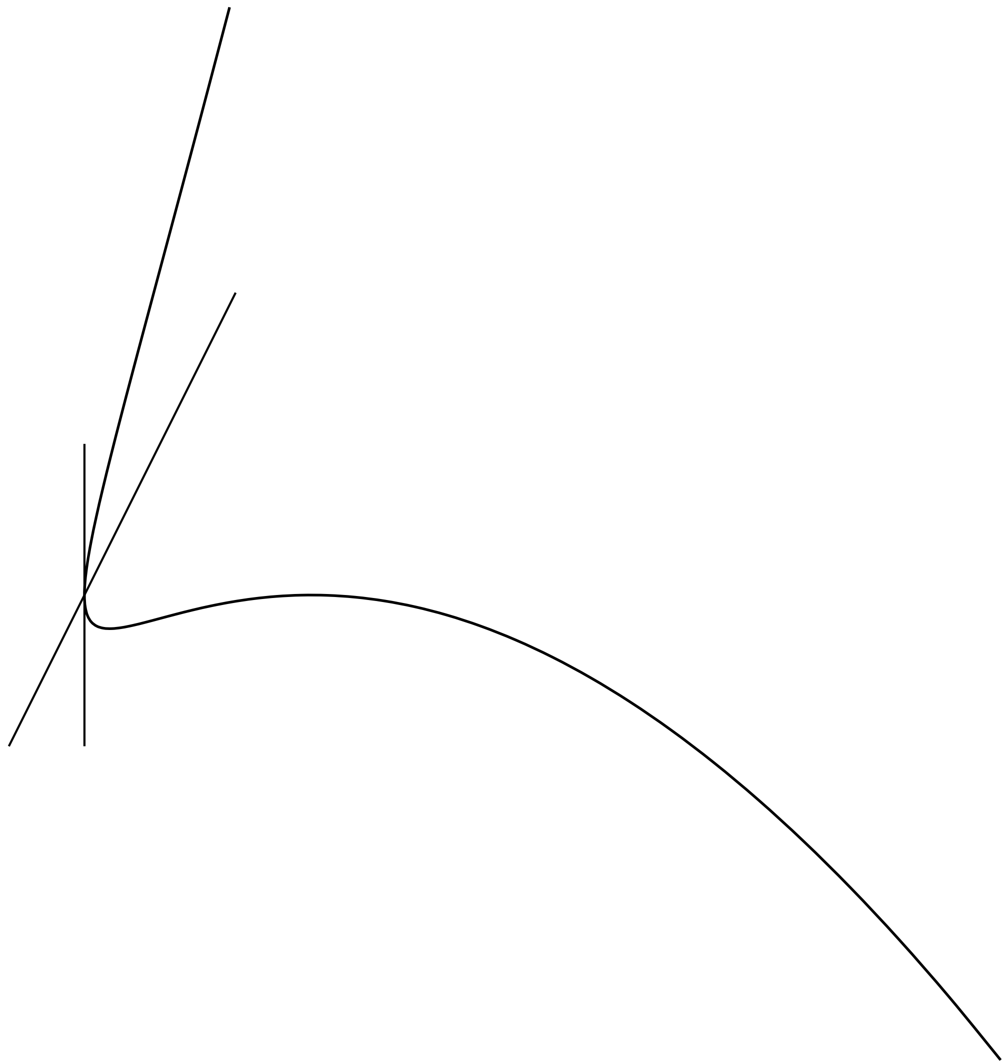


Figure 2.7: "Humpy" Cubic ($b^2 \geq 3d > 0$)

Case 3: $d = 0$ (a *cuspidal cubic*).

In this case, we have $b^2 - 3d > 0$.

Cubic of equation $3(Y - X)^2 = X^3$:

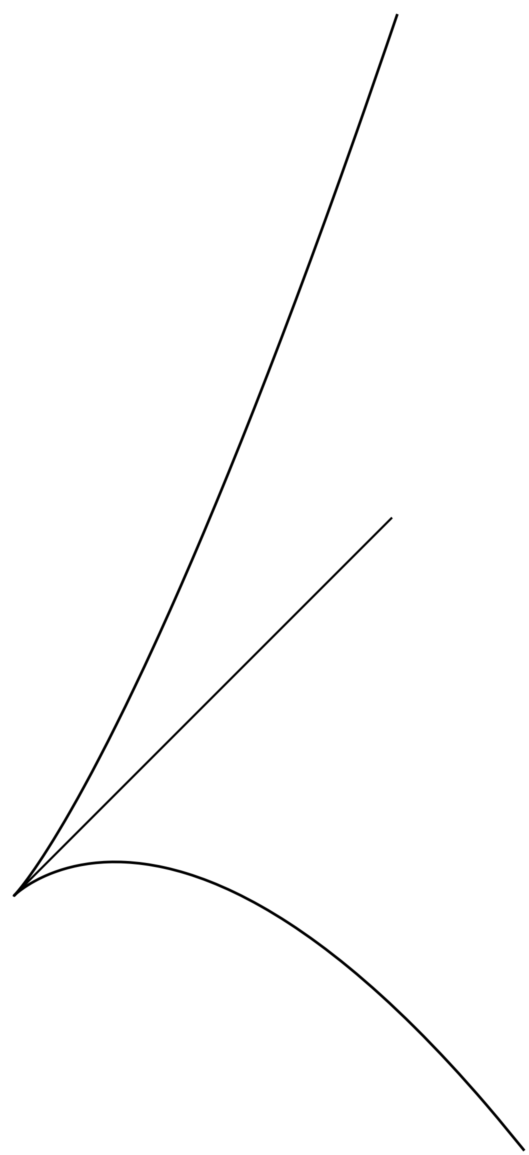


Figure 2.8: Cuspidal Cubic ($d = 0$)

Case 4: $d < 0$ (a *nodal cubic*).

In this case, $b^2 - 3d > 0$, and $Y'(m)$ has two roots m_1 and m_2. Furthermore, since $d < 0$, the singular point $(X, Y) = (0, 0)$ belongs to the trace of the cubic, Since $d < 0$, the polynomial $X(m) = c(a\,m - b)^2 + cd$, has two distinct roots, and thus, the cubic is self-intersecting at the singular point $(X, Y) = (0, 0)$.

Cubic of equation $\frac{3}{4}(Y - X)^2 - 3X^2 = X^3$:

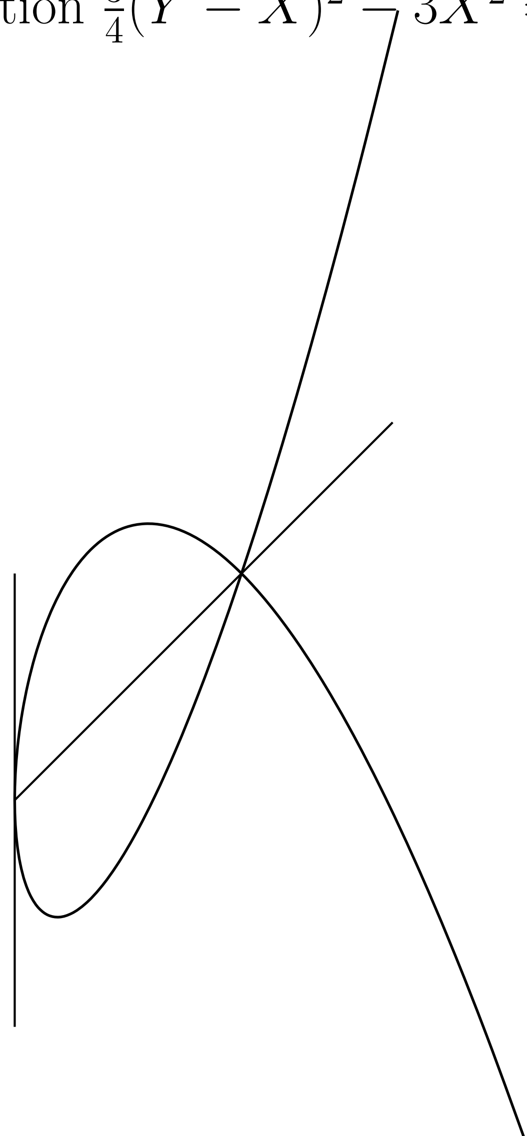


Figure 2.9: Nodal Cubic ($d < 0$)

One will observe the progression of the shape of the curve, from "humpy" to "loopy", through "cuspy".

Remark: The implicit equation

$$c(aY - bX)^2 + cdX^2 = X^3$$

of a nondegenerate polynomial cubic (with the exception of the singular point) is of the form

$$\varphi_2(X, Y) = X^3,$$

where $\varphi_2(X, Y)$ is a homogeneous polynomial in X and Y of total degree 2 (in the case of a degenerate cubic of equation $y = aX^3 + bX^2 + cX + d$, the singular point is at infinity. To make this statement precise, projective geometry is needed).

Using some algebraic geometry, it can be shown that the (nondegenerate) cubics that can be represented by parametric rational curves of degree 3 (i.e., fractions of polynomials of degree ≤ 3) are exactly those cubics whose implicit equation is of the form

$$\varphi_2(X, Y) = \varphi_3(X, Y),$$

where $\varphi_2(X, Y)$ and $\varphi_3(X, Y)$ are homogeneous polynomial in X and Y of total degree respectively 2 and 3.

These cubics have a singular point at the origin. Thus, the polynomial case is obtained in the special case where $\varphi_3(X,Y) = X^3$.

Furthermore, there are some cubics that cannot be represented even as rational curves. For example, the cubics defined by the implicit equation

$$Y^2 = X(X-1)(X-\lambda),$$

where $\lambda \neq 0, 1$, cannot be parameterized rationally. Such cubics are *elliptic curves*.

Returning to polynomial cubics, inspired by our treatment of quadratic polynomials, we would like to extend blossoming to polynomials of degree 3. First, we need to define the polar form (or blossom) of a polynomial of degree 3. Given any polynomial of degree ≤ 3,

$$F(X) = aX^3 + bX^2 + cX + d,$$

the polar form of F is a symmetric triaffine function $f\colon \mathbb{A}^3 \to \mathbb{A}$, that is, a function which takes the same value for all permutations of x_1, x_2, x_3, i.e., such that

$$\begin{aligned} f(x_1, x_2, x_3) &= f(x_2, x_1, x_3) = f(x_1, x_3, x_2) = \\ & f(x_2, x_3, x_1) = f(x_3, x_1, x_2) = f(x_3, x_2, x_1), \end{aligned}$$

which is affine in each argument, and such that

$$F(X) = f(X, X, X),$$

for all $X \in \mathbb{R}$. We easily verify that f must be given by

$$\begin{aligned} & f(x_1, x_2, x_3) = \\ & \quad a x_1 x_2 x_3 + b\,\frac{x_1 x_2 + x_1 x_3 + x_2 x_3}{3} + c\,\frac{x_1 + x_2 + x_3}{3} + d. \end{aligned}$$

Then, given a polynomial cubic curve $F\colon \mathbb{A} \to \mathbb{A}^3$, determined by three polynomials F_1, F_2, F_3 of degree ≤ 3, we can determine their polar forms f_1, f_2, f_3, and we obtain a symmetric triaffine map $f\colon \mathbb{A}^3 \to \mathbb{A}^3$, such that $F(X) = f(X, X, X)$, for all $X \in \mathbb{A}$. Again, let us pick an affine basis (r, s) in $\mathbb{A}$, with $r \neq s$, and let us compute

$$
\begin{aligned}
&f(t_1, t_2, t_3) = \\
&\quad f((1-\lambda_1)r + \lambda_1 s,\ (1-\lambda_2)r + \lambda_2 s,\ (1-\lambda_3)r + \lambda_3 s).
\end{aligned}
$$

Since $\lambda_i = \frac{t_i - r}{s - r}$, for $i = 1, 2, 3$, we get

$$
\begin{aligned}
f(t_1, t_2, t_3) &= \left(\frac{s-t_1}{s-r}\right)\left(\frac{s-t_2}{s-r}\right)\left(\frac{s-t_3}{s-r}\right) f(r, r, r) \\
&+ \left[\left(\frac{s-t_1}{s-r}\right)\left(\frac{s-t_2}{s-r}\right)\left(\frac{t_3-r}{s-r}\right) + \left(\frac{s-t_1}{s-r}\right)\left(\frac{t_2-r}{s-r}\right)\left(\frac{s-t_3}{s-r}\right)\right. \\
&\qquad \left. + \left(\frac{t_1-r}{s-r}\right)\left(\frac{s-t_2}{s-r}\right)\left(\frac{s-t_3}{s-r}\right)\right] f(r, r, s) \\
&+ \left[\left(\frac{s-t_1}{s-r}\right)\left(\frac{t_2-r}{s-r}\right)\left(\frac{t_3-r}{s-r}\right) + \left(\frac{t_1-r}{s-r}\right)\left(\frac{s-t_2}{s-r}\right)\left(\frac{t_3-r}{s-r}\right)\right. \\
&\qquad \left. + \left(\frac{t_1-r}{s-r}\right)\left(\frac{t_2-r}{s-r}\right)\left(\frac{s-t_3}{s-r}\right)\right] f(r, s, s) \\
&+ \left(\frac{t_1-r}{s-r}\right)\left(\frac{t_2-r}{s-r}\right)\left(\frac{t_3-r}{s-r}\right) f(s, s, s).
\end{aligned}
$$

The coefficients of $f(r,r,r)$, $f(r,r,s)$, $f(r,s,s)$, and $f(s,s,s)$, are obviously symmetric triaffine functions, and they add up to 1, as it is easily verified by expanding the product

$$\left(\frac{s-t_1}{s-r}+\frac{t_1-r}{s-r}\right)\left(\frac{s-t_2}{s-r}+\frac{t_2-r}{s-r}\right)\left(\frac{s-t_3}{s-r}+\frac{t_3-r}{s-r}\right)=1.$$

Thus, we showed that every symmetric triaffine map $f\colon \mathbb{A}^3 \to \mathbb{A}^3$ is completely determined by the sequence of four points $f(r,r,r)$, $f(r,r,s)$, $f(r,s,s)$, and $f(s,s,s)$ in $\mathbb{A}^3$, where $r \neq s$ are elements of $\mathbb{A}$.

Conversely, it is clear that given any sequence of four points $a,b,c,d \in \mathbb{A}^3$, the map

$$
\begin{aligned}
(t_1, t_2, t_3) \mapsto & \left(\frac{s-t_1}{s-r}\right)\left(\frac{s-t_2}{s-r}\right)\left(\frac{s-t_3}{s-r}\right) a \\
+ \Big[& \left(\frac{s-t_1}{s-r}\right)\left(\frac{s-t_2}{s-r}\right)\left(\frac{t_3-r}{s-r}\right) + \left(\frac{s-t_1}{s-r}\right)\left(\frac{t_2-r}{s-r}\right)\left(\frac{s-t_3}{s-r}\right) \\
& + \left(\frac{t_1-r}{s-r}\right)\left(\frac{s-t_2}{s-r}\right)\left(\frac{s-t_3}{s-r}\right)\Big] b \\
+ \Big[& \left(\frac{s-t_1}{s-r}\right)\left(\frac{t_2-r}{s-r}\right)\left(\frac{t_3-r}{s-r}\right) + \left(\frac{t_1-r}{s-r}\right)\left(\frac{s-t_2}{s-r}\right)\left(\frac{t_3-r}{s-r}\right) \\
& + \left(\frac{t_1-r}{s-r}\right)\left(\frac{t_2-r}{s-r}\right)\left(\frac{s-t_3}{s-r}\right)\Big] c \\
+ & \left(\frac{t_1-r}{s-r}\right)\left(\frac{t_2-r}{s-r}\right)\left(\frac{t_3-r}{s-r}\right) d
\end{aligned}
$$

is symmetric triaffine, and that
$f(r, r, r) = a$, $f(r, r, s) = b$, $f(r, s, s) = c$,
and $f(s, s, s) = d$.

The points $f(r, r, r)$, $f(r, r, s)$, $f(r, s, s)$, and $f(s, s, s)$, are called *control points*, or *Bézier control points*. They play a major role in the de Casteljau algorithm and its extensions.

Note that the polynomial curve defined by f passes through the two points $f(r, r, r)$ and $f(s, s, s)$, but not through the other control points. If we let $r = 0$ and $s = 1$, so that $\lambda_1 = t_1$, $\lambda_2 = t_2$, and $\lambda_3 = t_3$, the polynomial function associated with $f(t_1, t_2, t_3)$ is obtained by letting $t_1 = t_2 = t_3 = t$, and we get

$$\begin{aligned} F(t) = f(t, t, t) = (1-t)^3\, f(0, 0, 0) + 3(1-t)^2 t\, f(0, 0, 1) \\ + 3(1-t)t^2\, f(0, 1, 1) + t^3\, f(1, 1, 1). \end{aligned}$$

The polynomials

$$(1-t)^3,\ 3(1-t)^2 t,\ 3(1-t)t^2,\ t^3,$$

are the *Bernstein polynomials of degree* 3. They form a basis of the vector space of polynomials of degree ≤ 3.

Thus, the point $F(t)$ on the curve can be expressed in terms of the control points $f(r, r, r)$, $f(r, r, s)$, $f(r, s, s)$, and $f(s, s, s)$, and the Bernstein polynomials. However, it is more useful to extend the de Casteljau algorithm.

Lemma 2.1.4 *Given any sequence of four points a, b, c, d in $\mathcal{E}$, there is a unique polynomial curve $F\colon \mathbb{A} \to \mathcal{E}$ of degree 3, whose polar form $f\colon \mathbb{A}^3 \to \mathcal{E}$ satisfies the conditions $f(r, r, r) = a$, $f(r, r, s) = b$, $f(r, s, s) = c$, and $f(s, s, s) = d$ (where $r, s \in \mathbb{A}$, $r \neq s$). Furthermore, the polar form f of F is given by the formula*

$$
\begin{aligned}
f(t_1, t_2, t_3) &= \left(\frac{s - t_1}{s - r}\right)\left(\frac{s - t_2}{s - r}\right)\left(\frac{s - t_3}{s - r}\right) a \\
&+ \left[\left(\frac{s - t_1}{s - r}\right)\left(\frac{s - t_2}{s - r}\right)\left(\frac{t_3 - r}{s - r}\right) + \left(\frac{s - t_1}{s - r}\right)\left(\frac{t_2 - r}{s - r}\right)\left(\frac{s - t_3}{s - r}\right)\right. \\
&\qquad \left. + \left(\frac{t_1 - r}{s - r}\right)\left(\frac{s - t_2}{s - r}\right)\left(\frac{s - t_3}{s - r}\right)\right] b \\
&+ \left[\left(\frac{s - t_1}{s - r}\right)\left(\frac{t_2 - r}{s - r}\right)\left(\frac{t_3 - r}{s - r}\right) + \left(\frac{t_1 - r}{s - r}\right)\left(\frac{s - t_2}{s - r}\right)\left(\frac{t_3 - r}{s - r}\right)\right. \\
&\qquad \left. + \left(\frac{t_1 - r}{s - r}\right)\left(\frac{t_2 - r}{s - r}\right)\left(\frac{s - t_3}{s - r}\right)\right] c \\
&+ \left(\frac{t_1 - r}{s - r}\right)\left(\frac{t_2 - r}{s - r}\right)\left(\frac{t_3 - r}{s - r}\right) d.
\end{aligned}
$$

It is easy to generalize the de Casteljau algorithm to polynomial cubic curves. Let us assume that the cubic curve F is specified by the control points $f(r, r, r) = b_0$, $f(r, r, s) = b_1$, $f(r, s, s) = b_2$, and $f(s, s, s) = b_3$ (where $r, s \in \mathbb{A}$, $r < s$). Given any $t \in [r, s]$, the computation of $F(t)$ can be arranged in a triangular array, as shown below, consisting of three stages:

$$
\begin{array}{cccc}
 & 1 & 2 & 3 \\
f(r,r,r) & & & \\
 & f(r,r,t) & & \\
f(r,r,s) & & f(t,t,r) & \\
 & f(r,t,s) & & f(t,t,t) \\
f(r,s,s) & & f(t,t,s) & \\
 & f(t,s,s) & & \\
f(s,s,s) & & &
\end{array}
$$

The above computation is usually performed for $t \in [r, s]$, but it works just as well for any $t \in \mathbb{A}$, even outside $[r, s]$. When t is outside $[r, s]$, we usually say that $F(t) = f(t, t, t)$ is computed by *extrapolation*.

In order to describe the above computation more conveniently as an algorithm, let us denote the control points $b_0 = f(r,r,r)$, $b_1 = f(r,r,s)$, $b_2 = f(r,s,s)$ and $b_3 = f(s,s,s)$, as $b_{0,0}$, $b_{1,0}$, $b_{2,0}$, and $b_{3,0}$, and the intermediate points $f(r,r,t)$, $f(r,t,s)$, $f(t,s,s)$ as $b_{0,1}$, $b_{1,1}$, $b_{2,1}$, the intermediate points $f(t,t,r), f(t,t,s)$ as $b_{0,2}, b_{1,2}$, and the point $f(t,t,t)$ as $b_{0,3}$. Note that in $b_{i,j}$, the index j denotes the stage of the computation, and $F(t) = b_{0,3}$.

Then the triangle representing the computation is as follows:

$$
\begin{array}{cccc}
 & 1 & 2 & 3 \\
b_0 = b_{0,0} & & & \\
 & b_{0,1} & & \\
b_1 = b_{1,0} & & b_{0,2} & \\
 & b_{1,1} & & b_{0,3} \\
b_2 = b_{2,0} & & b_{1,2} & \\
 & b_{2,1} & & \\
b_3 = b_{3,0} & & &
\end{array}
$$

Then, we have the following inductive formula for computing $b_{i,j}$:

$$b_{i,j} = \left(\frac{s-t}{s-r}\right) b_{i,j-1} + \left(\frac{t-r}{s-r}\right) b_{i+1,j-1},$$

where $1 \leq j \leq 3$, and $0 \leq i \leq 3 - j$.

We have $F(t) = b_{0,3}$.

As will shall see, the above formula generalizes to any degree m. When $r \leq t \leq s$, each interpolation step computes a convex combination, and $b_{i,j}$ lies between $b_{i,j-1}$ and $b_{i+1,j-1}$. In this case, geometrically, the algorithm constructs the three polylines

$$\begin{gathered}(b_0,\, b_1),\, (b_1,\, b_2),\, (b_2,\, b_3)\\ (b_{0,1},\, b_{1,1}),\, (b_{1,1},\, b_{2,1})\\ (b_{0,2},\, b_{1,2})\end{gathered}$$

called *shells*, and with the point $b_{0,3}$, they form a diagram called a *de Casteljau diagram*.

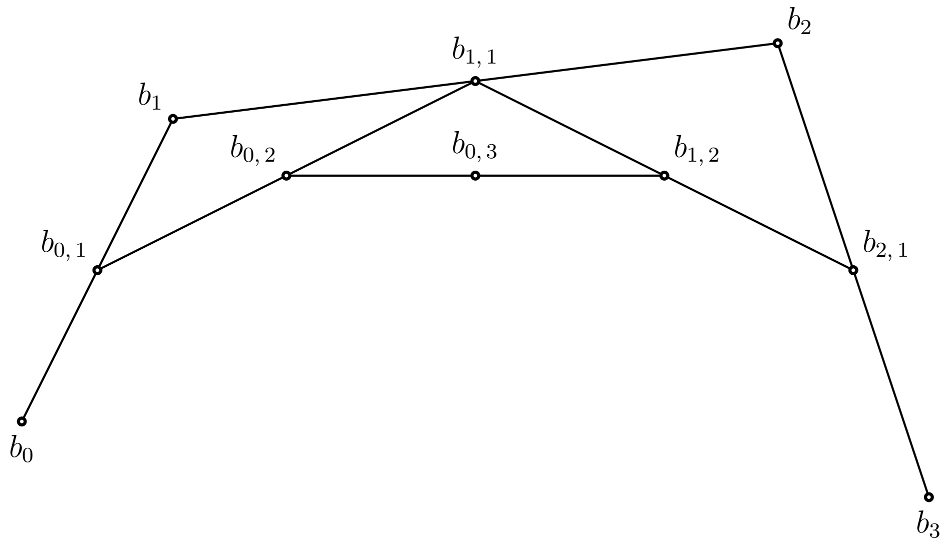


Figure 2.10: A de Casteljau diagram

Note that the shells are nested nicely. The polyline

$$(b_0,\, b_1),\ (b_1,\, b_2),\ (b_2,\, b_3)$$

is also called a *control polygon* of the curve. When λ is outside $[r, s]$, we still obtain three shells and a de Casteljau diagram, but the shells are not nicely nested. The following diagram illustrates the de Casteljau algorithm for computing the point $F(t)$ on a cubic, where $r = 0$, and $s = 6$:

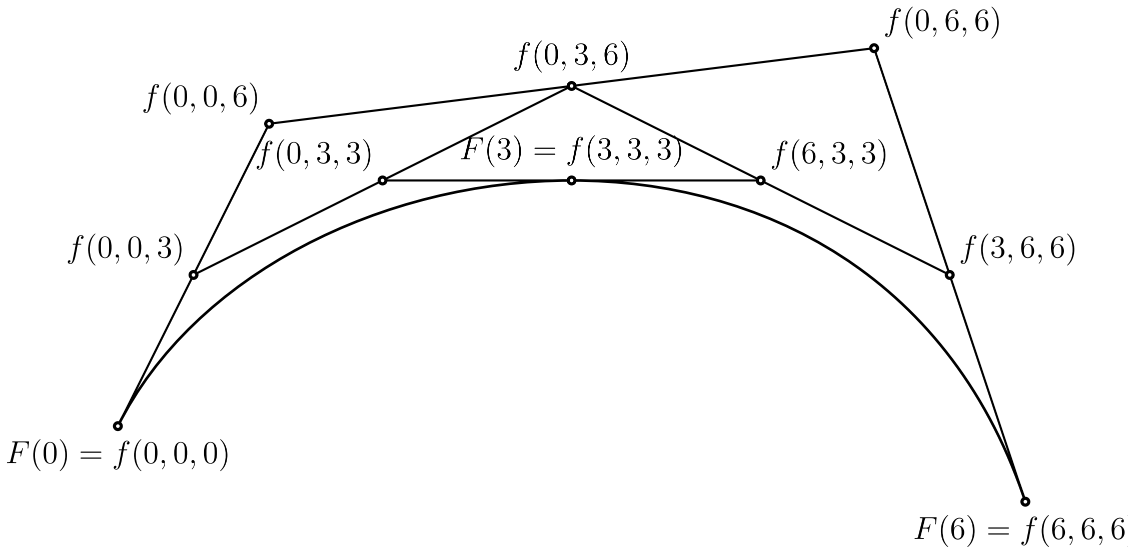


Figure 2.11: The de Casteljau algorithm for $t = 3$

The above example shows the construction of the point $F(3)$ corresponding to $t = 3$, on the curve F.

As in the quadratic case, the de Casteljau algorithm can also be used to compute any polar value $f(t_1, t_2, t_3)$ (which is not generally on the curve). All we have to do is to use a different ratio of interpolation λ_j during phase j, given by

$$\lambda_j = \frac{t_j - r}{s - r}.$$

The computation can also be represented as a triangle:

$$\begin{array}{cccc}
 & 1 & 2 & 3 \\
f(r,r,r) & & & \\
 & f(r,r,t_1) & & \\
f(r,r,s) & & f(t_1,t_2,r) & \\
 & f(r,t_1,s) & & f(t_1,t_2,t_3) \\
f(r,s,s) & & f(t_1,t_2,s) & \\
 & f(t_1,s,s) & & \\
f(s,s,s) & & &
\end{array}$$

This time, it is convenient to denote the intermediate points $f(r, r, t_1), f(r, t_1, s), f(t_1, s, s)$ as $b_{0,1}, b_{1,1}, b_{2,1}$, the intermediate points $f(t_1, t_2, r), f(t_1, t_2, s)$ as $b_{0,2}, b_{1,2}$, and the point $f(t_1, t_2, t_3)$ as $b_{0,3}$. Note that in $b_{i,j}$, the index j denotes the stage of the computation, and
$f(t_1, t_2, t_3) = b_{0,3}$.

Then the triangle representing the computation is as follows:

$$
\begin{array}{cccc}
 & 1 & 2 & 3 \\
b_0 = b_{0,0} & & & \\
 & b_{0,1} & & \\
b_1 = b_{1,0} & & b_{0,2} & \\
 & b_{1,1} & & b_{0,3} \\
b_2 = b_{2,0} & & b_{1,2} & \\
 & b_{2,1} & & \\
b_3 = b_{3,0} & & &
\end{array}
$$

We also have the following inductive formula for computing $b_{i,j}$:

$$
b_{i,j} = \left(\frac{s - t_j}{s - r} \right) b_{i,j-1} + \left(\frac{t_j - r}{s - r} \right) b_{i+1,j-1},
$$

where $1 \leq j \leq 3$, and $0 \leq i \leq 3 - j$. We have $f(t_1, t_2, t_3) = b_{0,3}$.

Thus, there is very little difference between this more general version of de Casteljau algorithm computing polar values, and the version computing the point $F(t)$ on the curve: just use a new ratio of interpolation at each stage.

Example 1. Consider the plane cubic defined as follows:

$$F_1(t) = 3t,$$
$$F_2(t) = 3t^3 - 3t.$$

The polar forms of $F_1(t)$ and $F_2(t)$ are:

$$f_1(t_1, t_2, t_3) = t_1 + t_2 + t_3,$$
$$f_2(t_1, t_2, t_3) = 3t_1t_2t_3 - (t_1 + t_2 + t_3).$$

With respect to the affine frame $r = -1, s = 1$, the coordinates of the control points are:

$$b_0 = (-3, 0)$$
$$b_1 = (-1, 4)$$
$$b_2 = (1, -4)$$
$$b_3 = (3, 0).$$

The curve has the following shape.

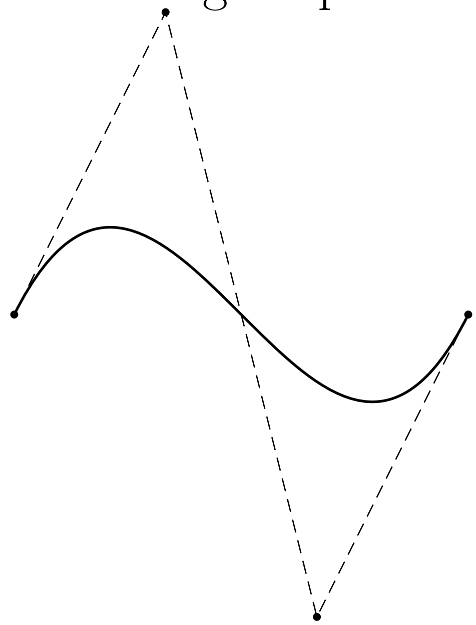


Figure 2.12: Bezier Cubic 1

The above cubic is an example of degenerate “S-shaped” cubic.

Example 2. Consider the plane cubic defined as follows:

$$F_1(t) = 3(t-1)^2 + 6,$$
$$F_2(t) = 3t(t-1)^2 + 6t.$$

Since

$$F_1(t) = 3t^2 - 6t + 9,$$
$$F_2(t) = 3t^3 - 6t^2 + 9t,$$

we get the polar forms

$$f_1(t_1, t_2, t_3) = (t_1t_2 + t_1t_3 + t_2t_3) - 2(t_1 + t_2 + t_3) + 9$$
$$f_2(t_1, t_2, t_3) = 3t_1t_2t_3 - 2(t_1t_2 + t_1t_3 + t_2t_3) + 3(t_1 + t_2 + t_3).$$

With respect to the affine frame $r = 0, s = 1$, the coordinates of the control points are:

$$b_0 = (9, 0)$$
$$b_1 = (7, 3)$$
$$b_2 = (6, 4)$$
$$b_3 = (6, 6).$$

The curve has the following shape.

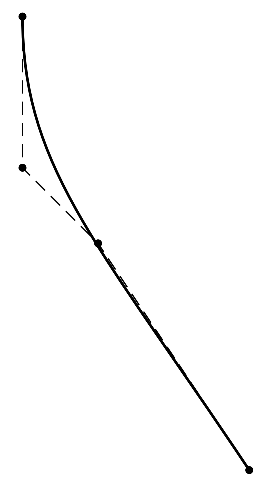


Figure 2.13: Bezier Cubic 2

We leave as an exercise to verify that this cubic corresponds to case 1, where $3d > b^2$. The axis of symmetry is $y = x$.

Example 3. Consider the plane cubic defined as follows:

$$\begin{aligned} F_1(t) &= 3(t-2)^2 + 3, \\ F_2(t) &= 3t(t-2)^2 + 3t. \end{aligned}$$

Since

$$\begin{aligned} F_1(t) &= 3t^2 - 12t + 15, \\ F_2(t) &= 3t^3 - 12t^2 + 15t, \end{aligned}$$

we get the polar forms

$$\begin{aligned} f_1(t_1, t_2, t_3) &= (t_1t_2 + t_1t_3 + t_2t_3) - 4(t_1 + t_2 + t_3) + 15 \\ f_2(t_1, t_2, t_3) &= 3t_1t_2t_3 - 4(t_1t_2 + t_1t_3 + t_2t_3) + 5(t_1 + t_2 + t_3). \end{aligned}$$

With respect to the affine frame $r = 0, s = 2$, the coordinates of the control points are:

$$\begin{aligned} b_0 &= (15, 0) \\ b_1 &= (7, 10) \\ b_2 &= (3, 4) \\ b_3 &= (3, 6). \end{aligned}$$

The curve has the following shape.

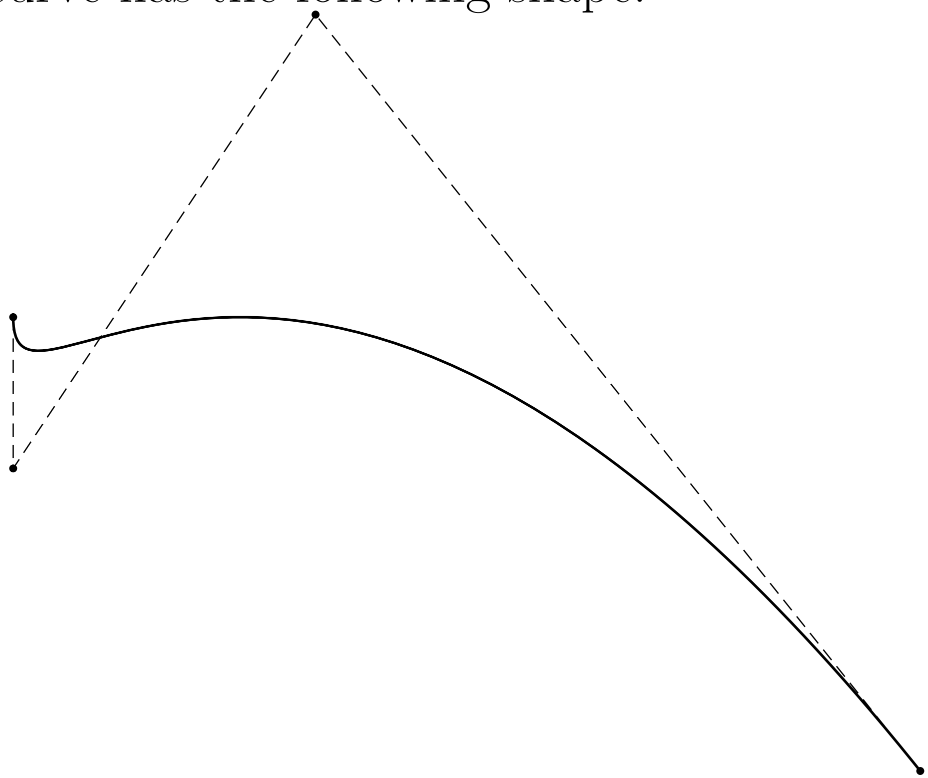


Figure 2.14: Bezier Cubic 3

We leave as an exercise to verify that this cubic corresponds to case 2, where $b^2 \geq 3d > 0$. The axis of symmetry is $y = 2x$.

It is interesting to see which control points are obtained with respect to the affine frame $r = 0, s = 1$:

$$\begin{aligned} b'_0 &= (15, 0) \\ b'_1 &= (11, 5) \\ b'_2 &= (8, 6) \\ b'_3 &= (6, 6). \end{aligned}$$

The second "hump" of the curve is outside the convex hull of this new control polygon. This shows that it is far from obvious, just by looking at some of the control points, to predict what the shape of the entire curve will be!

Example 4. Consider the plane cubic defined as follows:

$$F_1(t) = 3(t-1)^2,$$
$$F_2(t) = 3t(t-1)^2.$$

Since

$$F_1(t) = 3t^2 - 6t + 3,$$
$$F_2(t) = 3t^3 - 6t^2 + 3t,$$

we get the polar forms

$$f_1(t_1, t_2, t_3) = (t_1t_2 + t_1t_3 + t_2t_3) - 2(t_1 + t_2 + t_3) + 3$$
$$f_2(t_1, t_2, t_3) = 3t_1t_2t_3 - 2(t_1t_2 + t_1t_3 + t_2t_3) + (t_1 + t_2 + t_3).$$

With respect to the affine frame $r = 0, s = 2$, the coordinates of the control points are:

$$b_0 = (3, 0)$$
$$b_1 = (-1, 2)$$
$$b_2 = (-1, -4)$$
$$b_3 = (3, 6).$$

The curve has the following shape.

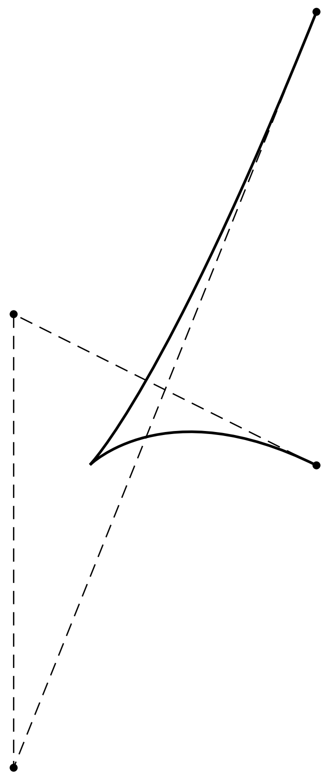


Figure 2.15: Bezier Cubic 4

We leave as an exercise to verify that this cubic corresponds to case 3, where $d = 0$, a cubic with a cusp at the origin. The axis of symmetry is $y = x$.

It is interesting to see which control points are obtained with respect to the affine frame $r = 0, s = 1$:

$$\begin{aligned} b'_0 &= (3, 0) \\ b'_1 &= (1, 1) \\ b'_2 &= (0, 0) \\ b'_3 &= (0, 0). \end{aligned}$$

Thus, $b'_2 = b'_3$. This indicates that there is a cusp at the origin.

Example 5. Consider the plane cubic defined as follows:

$$
\begin{aligned}
F_1(t) &= \frac{3}{4}(t-1)^2 - 3, \\
F_2(t) &= \frac{3}{4}t(t-1)^2 - 3t.
\end{aligned}
$$

Since

$$
\begin{aligned}
F_1(t) &= \frac{3}{4}t^2 - \frac{3}{2}t - \frac{9}{4}, \\
F_2(t) &= \frac{3}{4}t^3 - \frac{3}{2}t^2 - \frac{9}{4}t,
\end{aligned}
$$

we get the polar forms

$$
\begin{aligned}
f_1(t_1, t_2, t_3) &= \frac{1}{4}(t_1t_2 + t_1t_3 + t_2t_3) - \frac{1}{2}(t_1 + t_2 + t_3) - \frac{9}{4} \\
f_2(t_1, t_2, t_3) &= \frac{3}{4}t_1t_2t_3 - \frac{1}{2}(t_1t_2 + t_1t_3 + t_2t_3) - \frac{3}{4}(t_1 + t_2 + t_3).
\end{aligned}
$$

With respect to the affine frame $r = -1, s = 3$, the coordinates of the control points are:

$$
\begin{aligned}
b_0 &= (0, 0)\\
b_1 &= (-4, 4)\\
b_2 &= (-4, -12)\\
b_3 &= (0, 0).
\end{aligned}
$$

The curve has the following shape.

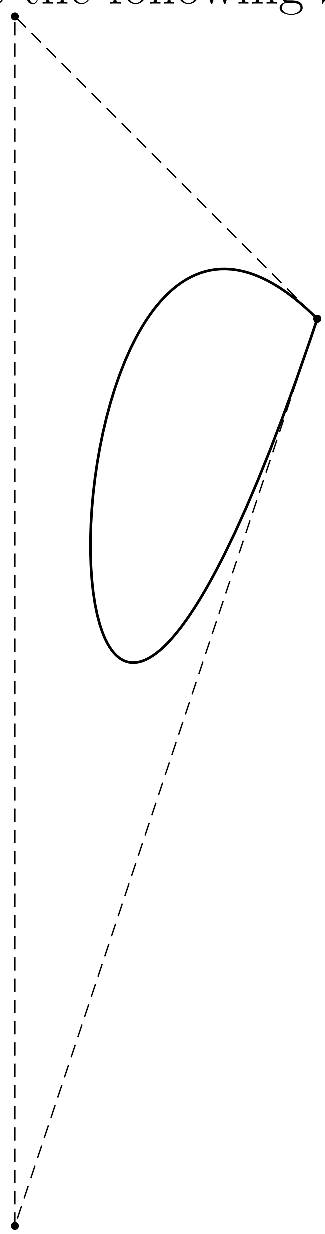


Figure 2.16: Bezier Cubic 5

Note that $b_0 = b_3$.

We leave as an exercise to verify that this cubic corresponds to case 4, where $d < 0$, a cubic with a node at the origin. The axis of symmetry is $y = x$. The two tangents at the origin are $y = -x$, and $y = 3x$ (this explains the choice of $r = -1$, and $s = 3$). Here is a more global view of the same cubic:

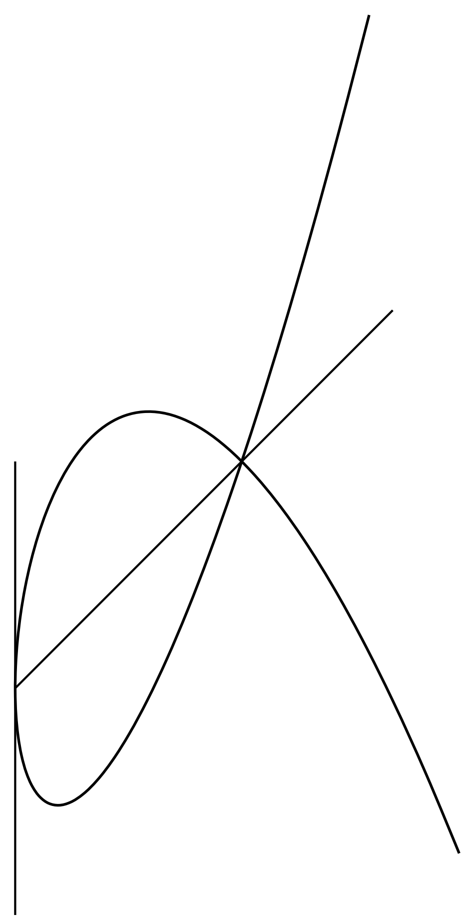


Figure 2.17: Nodal Cubic ($d < 0$)

The above examples suggest that it may be interesting, and even fun, to investigate which properties of the shape of the control polygon (b_0, b_1, b_2, b_3) determine the nature of the plane cubic that it defines. Try it!

Challenge: Given a planar control polygon (b_0, b_1, b_2, b_3), is it possible to find the singular point geometrically? Is it possible to find the axis of symmetry geometrically?

Chapter 3

Polynomial Curves and Control Points

3.1 Polar Forms and Control Points

The purpose of this short chapter is to show how polynomial curves of arbitrary degree are handled in terms of control points.

As we observed in the case of polynomial curves of degree ≤ 3, the key to the treatment of polynomial curves in terms of control points is that polynomials can be multilinearized.[1]

To be more precise, say that a map $f\colon \underbrace{\mathbb{A}^d \times \cdots \times \mathbb{A}^d}_{m} \to \mathbb{A}^n$ is *multiaffine* if it is affine in each of its arguments.

[1]The term "multilinearized" is technicaly incorrect, we should say "multiaffinized"!

A map $f\colon \underbrace{\mathbb{A}^d \times \cdots \times \mathbb{A}^d}_{m} \to \mathbb{A}^n$ is *symmetric* if it does not depend on the order of its arguments, i.e.,

$$f(a_{\pi(1)}, \ldots, a_{\pi(m)}) = f(a_1, \ldots, a_m)$$

for all $a_1, \ldots, a_m$, and all permutations π.

Then, for every polynomial $F(t)$ of degree m, there is a unique symmetric and multiaffine map
$f\colon \underbrace{\mathbb{A} \times \cdots \times \mathbb{A}}_{m} \to \mathbb{A}$ such that

$$F(t) = f(\underbrace{t, \ldots, t}_{m}), \quad \text{for all } t \in \mathbb{A}.$$

This is an old "folk theorem", probably already known to Newton. The proof is easy.

By linearity, it is enough to consider a monomial of the form x^k, where $k \leq m$.

The unique symmetric multiaffine map corresponding to x^k is

$$\frac{\sigma_k(t_1, \ldots, t_m)}{\binom{m}{k}},$$

where $\sigma_k(t_1, \ldots, t_m)$ is the kth elementary symmetric function in m variables, i.e.

$$\sigma_k = \sum_{\substack{I \subseteq \{1,\ldots,m\} \\ |I|=k}} (\prod_{i \in I} t_i).$$

Recall that

$$\binom{m}{k} = \frac{m!}{k!(m-k)!},$$

a *binomial coefficient* (with $0 \leq k \leq m$) where,

$$m! = m \cdot (m-1) \cdots 2 \cdot 1,$$

called *m-factorial*.

Given a polynomial curve $F\colon \mathbb{A} \to \mathbb{A}^n$ of degree m

$$\begin{aligned} x_1(t) &= F_1(t), \\ \ldots &= \ldots \\ x_n(t) &= F_n(t), \end{aligned}$$

where $F_1(t), \ldots, F_n(t)$ are polynomials of degree at most m, the map $F\colon \mathbb{A} \to \mathbb{A}^n$ arises from a unique symmetric multiaffine map $f\colon \mathbb{A}^m \to \mathbb{A}^n$, the *polar form of* F, such that

$$F(t) = f(\underbrace{t, \ldots, t}_{m}),$$

for all $t \in \mathbb{A}$.

For example, consider the plane cubic defined as follows:

$$F_1(t) = \frac{3}{4}t^2 - \frac{3}{2}t - \frac{9}{4}, \qquad F_2(t) = \frac{3}{4}t^3 - \frac{3}{2}t^2 - \frac{9}{4}t.$$

We get the polar forms

$$\begin{aligned} f_1(t_1, t_2, t_3) &= \frac{1}{4}(t_1t_2 + t_1t_3 + t_2t_3) - \frac{1}{2}(t_1 + t_2 + t_3) - \frac{9}{4} \\ f_2(t_1, t_2, t_3) &= \frac{3}{4}t_1t_2t_3 - \frac{1}{2}(t_1t_2 + t_1t_3 + t_2t_3) - \frac{3}{4}(t_1 + t_2 + t_3). \end{aligned}$$

Also, for $r \neq s$, the map $f\colon \mathbb{A}^m \to \mathbb{A}^n$ is determined by the $m+1$ *control points* $(b_0, \ldots, b_m)$, where

$$b_i = f(\underbrace{r, \ldots, r}_{m-i}, \underbrace{s, \ldots, s}_{i}),$$

since

$$f(t_1, \ldots, t_m) = \sum_{k=0}^{m} \sum_{\substack{I \cup J = \{1,\ldots,m\} \\ I \cap J = \emptyset,\, \mathrm{card}(J) = k}} \prod_{i \in I} \left(\frac{s - t_i}{s - r}\right) \prod_{j \in J} \left(\frac{t_j - r}{s - r}\right) f(\underbrace{r, \ldots, r}_{m-k}, \underbrace{s, \ldots, s}_{k}).$$

For example, with respect to the affine frame $r = -1$, $s = 3$, the coordinates of the control points of the cubic defined earlier are:

$$\begin{aligned} b_0 &= (0, 0) \\ b_1 &= (-4, 4) \\ b_2 &= (-4, -12) \\ b_3 &= (0, 0). \end{aligned}$$

Conversely, for every sequence of $m+1$ points $(b_0, \ldots, b_m)$, there is a unique symmetric multiaffine map f such that

$$b_i = f(\underbrace{r, \ldots, r}_{m-i}, \underbrace{s, \ldots, s}_{i}),$$

namely,

$$f(t_1, \ldots, t_m) = \sum_{k=0}^{m} \sum_{\substack{I \cup J = \{1,\ldots,m\} \\ I \cap J = \emptyset,\, \mathrm{card}(J) = k}} \prod_{i \in I} \left(\frac{s - t_i}{s - r}\right) \prod_{j \in J} \left(\frac{t_j - r}{s - r}\right) b_k.$$

Thus, there is a bijection between the set of polynomial curves of degree m and the set of sequences $(b_0, \ldots, b_m)$ of $m + 1$ control points.

The upshot of all this is that for algorithmic purposes, it is convenient to define polynomial curves in terms of polar forms.

Definition 3.1.1 A (parameterized) *polynomial curve in polar form of degree* m is an affine polynomial map $F: \mathbb{A} \to \mathcal{E}$ of polar degree m, defined by its m*-polar form*, which is some symmetric m-affine map $f: \mathbb{A}^m \to \mathcal{E}$, where $\mathbb{A}$ is the real affine line, and $\mathcal{E}$ is any affine space (of dimension at least 2).

Given any $r, s \in \mathbb{A}$, with $r < s$, a (parameterized) *polynomial curve segment* $F([r, s])$ *in polar form of degree* m is the restriction $F: [r, s] \to \mathcal{E}$ of an affine polynomial curve $F: \mathbb{A} \to \mathcal{E}$ in polar form of degree m.

We define the *trace of* F as $F(\mathbb{A})$, and the the *trace of* $F[r, s]$ as $F([r, s])$.

Typically, the affine space $\mathcal{E}$ is the real affine space $\mathbb{A}^3$ of dimension 3.

Remark: When defining polynomial curves, it is convenient to denote the polynomial map defining the curve by an upper-case letter, such as $F\colon \mathbb{A} \to \mathcal{E}$, and the polar form of F by the same, but lower-case letter, f.

It would then be confusing to denote the affine space which is the range of the maps F and f also as F, and thus, we denote it as $\mathcal{E}$ (or at least, we use a letter different from the letter used to denote the polynomial map defining the curve).

Also note that we defined a polynomial curve in polar form of degree at most m, rather than a polynomial curve in polar form of degree exactly m, because an affine polynomial map f of polar degree m may end up being degenerate, in the sense that it could be equivalent to a polynomial map of lower polar degree.

For convenience, we will allows ourselves the abuse of language where we abbreviate "polynomial curve in polar form" to "polynomial curve".

We summarize the relationship between control points and polynomial curves in the following lemma.

Lemma 3.1.2 *Given any sequence of* $m+1$ *points* $a_0, \ldots, a_m$ *in some affine space* $\mathcal{E}$*, there is a unique polynomial curve* $F\colon \mathbb{A} \to \mathcal{E}$ *of degree* m*, whose polar form* $f\colon \mathbb{A}^m \to \mathcal{E}$ *satisfies the conditions*

$$f(\underbrace{r, \ldots, r}_{m-k}, \underbrace{s, \ldots, s}_{k}) = a_k,$$

(where $r, s \in \mathbb{A}$*,* $r \neq s$*).*

Furthermore, the polar form f *of* F *is given by the formula*

$$f(t_1, \ldots, t_m) = \sum_{k=0}^{m} \sum_{\substack{I \cup J = \{1, \ldots, m\} \\ I \cap J = \emptyset,\, |J| = k}} \prod_{i \in I} \left(\frac{s - t_i}{s - r}\right) \prod_{j \in J} \left(\frac{t_j - r}{s - r}\right) a_k,$$

and $F(t)$ is given by the formula

$$F(t) = \sum_{k=0}^{m} B_k^m[r,s](t)\, a_k,$$

where the polynomials

$$B_k^m[r,s](t) = \binom{m}{k} (\frac{s-t}{s-r})^{m-k} (\frac{t-r}{s-r})^k$$

are the Bernstein polynomials of degree m over $[r,s]$.

Note that since the polar form f of a polynomial curve F of degree m is symmetric, the order of the arguments is irrelevant.

Often, when argument are repeated, we also omit commas between argument. For example, we abbreviate $f(\underbrace{r, \ldots, r}_{i}, \underbrace{s, \ldots, s}_{j})$ by $f(r^i s^j)$.

In the next section, we will abbreviate $f(\underbrace{t, \ldots, t}_{j}, \underbrace{r, \ldots, r}_{m-i-j}, \underbrace{s, \ldots, s}_{i})$ by $f(t^j r^{m-i-j} s^i)$.

3.2 The de Casteljau Algorithm

The definition of polynomial curves in terms of polar forms leads to a very nice algorithm known as the *de Casteljau algorithm*, to draw polynomial curves.

Using the de Casteljau algorithm, it is possible to determine any point $F(t)$ on the curve, by repeated affine interpolations (see Farin [**?**, **?**], Hoschek and Lasser [**?**], Risler [**?**], or Gallier [**?**]).

The example below shows $F(1/2)$.

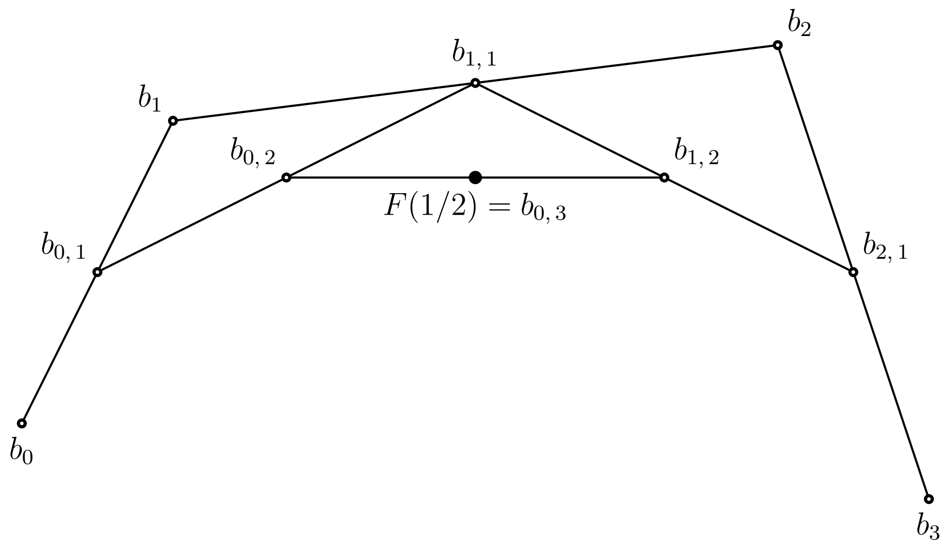


Figure 3.1: A de Casteljau diagram for $t = 1/2$

In the general case where a curve F is specified by $m+1$ control points $(b_0, \ldots, b_m)$ w.r.t. to an interval $[r, s]$, let us define the following points $b_{i,j}$ used during the computation of $F(t)$ (where f is the polar form of F):

$$b_{i,j} = \begin{cases} b_i & \text{if } j = 0,\ 0 \leq i \leq m, \\ f(t^j r^{m-i-j} s^i) & \text{if } 1 \leq j \leq m,\ 0 \leq i \leq m-j. \end{cases}$$

Then, we have the following equations:

$$\boxed{b_{i,j} = \left(\frac{s-t}{s-r}\right) b_{i,j-1} + \left(\frac{t-r}{s-r}\right) b_{i+1,j-1}.}$$

The result is $F(t) = b_{0,m}$.

The computation can be conveniently represented in the following triangular form:

$$
\begin{array}{ccccccccc}
0 & 1 & \ldots & j-1 & j & \ldots & m-k & \ldots & m \\
b_{0,0} & & & & & & & & \\
 & b_{0,1} & & & & & & & \\
b_{1,0} & & \ddots & & & & & & \\
 & & & b_{0,j-1} & & & & & \\
 & & & \vdots & b_{0,j} & & & & \\
 & & & b_{i,j-1} & \vdots & \ddots & & & \\
 & & & & b_{i,j} & & b_{0,m-k} & & \\
 & & & b_{i+1,j-1} & \vdots & & & & \\
 & & & \vdots & b_{m-k-j,j} & & \vdots & & b_{0,m} \\
 & & & b_{m-k-j+1,j-1} & & & & & \\
 & & & & \vdots & & b_{k,m-k} & & \\
 & b_{m-k-1,1} & & \vdots & & & & & \\
b_{m-k,0} & & & & b_{m-j,j} & & & & \\
 & \vdots & & b_{m-j+1,j-1} & & & & & \\
\vdots & & & & & & & & \\
b_{m-1,0} & & & & & & & & \\
 & b_{m-1,1} & & & & & & & \\
b_{m,0} & & & & & & & &
\end{array}
$$

When $r \leq t \leq s$, each interpolation step computes a convex combination, and $b_{i,j}$ lies between $b_{i,j-1}$ and $b_{i+1,j-1}$.

In this case, geometrically, the algorithm consists of a diagram consisting of the m polylines

$$
\begin{gathered}
(b_{0,0},\, b_{1,0}),\, (b_{1,0},\, b_{2,0}),\, (b_{2,0},\, b_{3,0}),\, (b_{3,0},\, b_{4,0}),\, \ldots,(b_{m-1,0},\, b_{m,0})\\
(b_{0,1},\, b_{1,1}),\, (b_{1,1},\, b_{2,1}),\, (b_{2,1},\, b_{3,1}),\, \ldots,(b_{m-2,1},\, b_{m-1,1})\\
(b_{0,2},\, b_{1,2}),\, (b_{1,2},\, b_{2,2}),\, \ldots,(b_{m-3,2},\, b_{m-2,2})\\
\ldots\\
(b_{0,m-2},\, b_{1,m-2}),\, (b_{1,m-2},\, b_{2,m-2})\\
(b_{0,m-1},\, b_{1,m-1})
\end{gathered}
$$

called *shells*, and with the point $b_{0,m}$, they form the *de Casteljau diagram*.

Note that the shells are nested nicely. The polyline

$$(b_0,\, b_1),\, (b_1,\, b_2),\, (b_2,\, b_3),\, (b_3,\, b_4),\, \ldots,(b_{m-1},\, b_m)$$

is also called a *control polygon* of the curve. When t is outside $[r, s]$, we still obtain m shells and a de Casteljau diagram, but the shells are not nicely nested.

One of the best features of the de Casteljau algorithm is that it lends itself very well to recursion.

Indeed, going back to the case of a cubic curve, it is easy to show that the sequences of points $(b_0, b_{0,1}, b_{0,2}, b_{0,3})$ and $(b_{0,3}, b_{1,2}, b_{2,1}, b_3)$ are also control polygons for the exact same curve.

Thus, we can compute the points corresponding to $t = 1/2$ with respect to the control polygons

$$(b_0, b_{0,1}, b_{0,2}, b_{0,3}) \quad \text{and} \quad (b_{0,3}, b_{1,2}, b_{2,1}, b_3),$$

and this yields a recursive method for approximating the curve. This method called the *subdivision method* applies to polynomial curves of any degree and can be used to render efficiently a curve segment F over $[r, s]$.

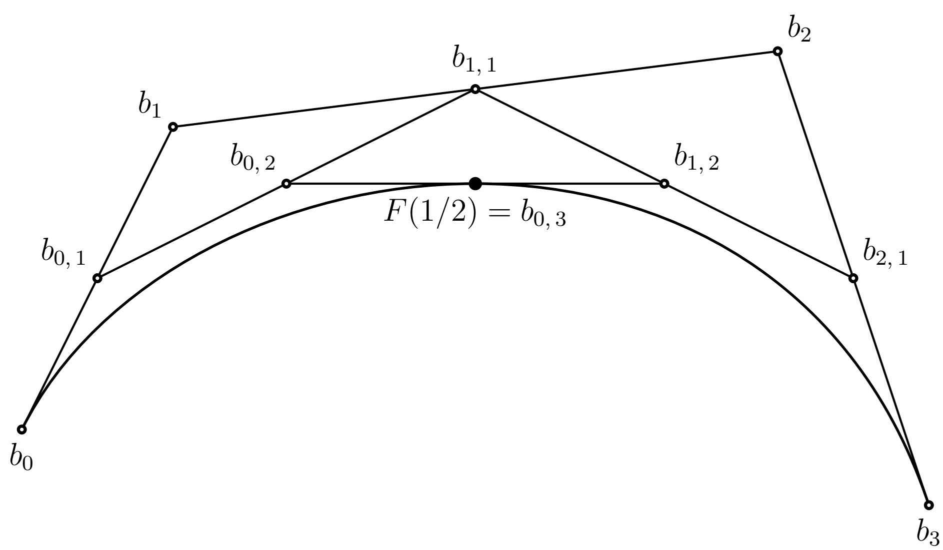


Figure 3.2: Approximating a curve using subdivision

Chapter 4

Polynomial Surfaces

4.1 Polar Forms

The purpose of this short chapter is to show how polynomial surfaces are handled in terms of control points.

As in Chapter 3, this chapter is just a brief introduction

The deep reason why polynomial surfaces can be effectively handled in terms of control points is that multivariate polynomials arise from multiaffine symmetric maps.

Denoting the affine plane $\mathbb{A}^2$ as $\mathcal{P}$, traditionally, a *polynomial surface* in $\mathbb{A}^n$ is a function $F\colon \mathcal{P} \to \mathbb{A}^n$, defined such that

$$\begin{aligned} x_1 &= F_1(u, v), \\ \ldots &= \ldots \\ x_n &= F_n(u, v), \end{aligned}$$

for all $(u, v) \in \mathbb{A}^2$, where $F_1(U, V), \ldots, F_n(U, V)$ are polynomials in $\mathbb{R}[U, V]$.

There are two natural ways to polarize the polynomials defining F.

The first way is to polarize *separately* in u and v.

If p is the highest degree in u and q is the highest degree in v, we get a unique multiaffine map

$$f\colon (\mathbb{A})^p \times (\mathbb{A})^q \to \mathbb{A}^n$$

of degree $(p + q)$ which is symmetric in its first p arguments and symmetric in its last q arguments, such that

$$F(u, v) = f(\underbrace{u, \ldots, u}_{p}; \underbrace{v, \ldots, v}_{q}).$$

We get what is traditionally called a *tensor product surface*, or as we prefer to call it, a *bipolynomial surface of bidegree* $\langle p, q\rangle$ (or a *rectangular surface patch*).

We also say that the multiaffine maps arising in polarizing separately in u and v are $\langle p, q\rangle$-*symmetric*.

The second way to polarize is to treat the variables u and v *as a whole*.

This way, if F is a polynomial surface such that the maximum total degree of the monomials is m, we get a unique symmetric degree m multiaffine map

$$f\colon (\mathbb{A}^2)^m \to \mathbb{A}^n,$$

such that

$$F(u,v) = f(\underbrace{(u,v),\ldots,(u,v)}_{m}).$$

We get what is called a *total degree surface* (or a *triangular surface patch*).

Let us go back to the first case. Using linearity, it is clear that all we have to do is to polarize a monomial $u^h v^k$.

It is easily verified that the unique $\langle p, q\rangle$-symmetric multiaffine polar form of degree $p+q$

$$f_{h,k}^{p,q}(u_1, \ldots, u_p; v_1, \ldots, v_q)$$

of the monomial $u^h v^k$ is given by

$$f_{h,k}^{p,q}(u_1, \ldots, u_p; v_1, \ldots, v_q) =$$

$$\frac{1}{\binom{p}{h}\binom{q}{k}} \sum_{\substack{I\subseteq\{1,\ldots,p\},|I|=h\\ J\subseteq\{1,\ldots,q\},|J|=k}} \left(\prod_{i\in I} u_i\right)\left(\prod_{j\in J} v_j\right).$$

The denominator

$$\binom{p}{h}\binom{q}{k}$$

is the number of terms in the above sum.

Recall that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

a *binomial coefficient*, where

$$n! = n\cdot(n-1)\cdots 2\cdot 1,$$

called n *factorial*.

It is also easily verified that in the second case, the unique symmetric multiaffine polar form of degree m

$$f^m_{h,k}((u_1, v_1), \ldots, (u_m, v_m))$$

of the monomial $u^h v^k$ is given by

$$f^m_{h,k}((u_1, v_1), \ldots, (u_m, v_m)) =$$

$$\frac{1}{\binom{m}{h}\binom{m-h}{k}} \sum_{\substack{I \cup J \subseteq \{1,\ldots,m\} \\ |I|=h, |J|=k, I \cap J = \emptyset}} \left(\prod_{i \in I} u_i\right)\left(\prod_{j \in J} v_j\right).$$

The denominator

$$\binom{m}{h}\binom{m-h}{k} = \binom{m}{h\; k\; (m-h-k)}$$

is the number of terms in the above sum.

As an example, consider the following surface known as Enneper's surface:

$$\begin{aligned} F_1(U,V) &= U - \frac{U^3}{3} + UV^2 \\ F_2(U,V) &= V - \frac{V^3}{3} + U^2V \\ F_3(U,V) &= U^2 - V^2. \end{aligned}$$

We get the polar forms

$$\begin{aligned} f_1((U_1,V_1),(U_2,V_2),(U_3,V_3)) &= \frac{U_1+U_2+U_3}{3} - \frac{U_1U_2U_3}{3} \\ &\quad + \frac{U_1V_2V_3 + U_2V_1V_3 + U_3V_1V_2}{3} \\ f_2((U_1,V_1),(U_2,V_2),(U_3,V_3)) &= \frac{V_1+V_2+V_3}{3} - \frac{V_1V_2V_3}{3} \\ &\quad + \frac{U_1U_2V_3 + U_1U_3V_2 + U_2U_3V_1}{3} \\ f_3((U_1,V_1),(U_2,V_2),(U_3,V_3)) &= \frac{U_1U_2 + U_1U_3 + U_2U_3}{3} - \frac{V_1V_2 + V_1V_3 + V_2V_3}{3}. \end{aligned}$$

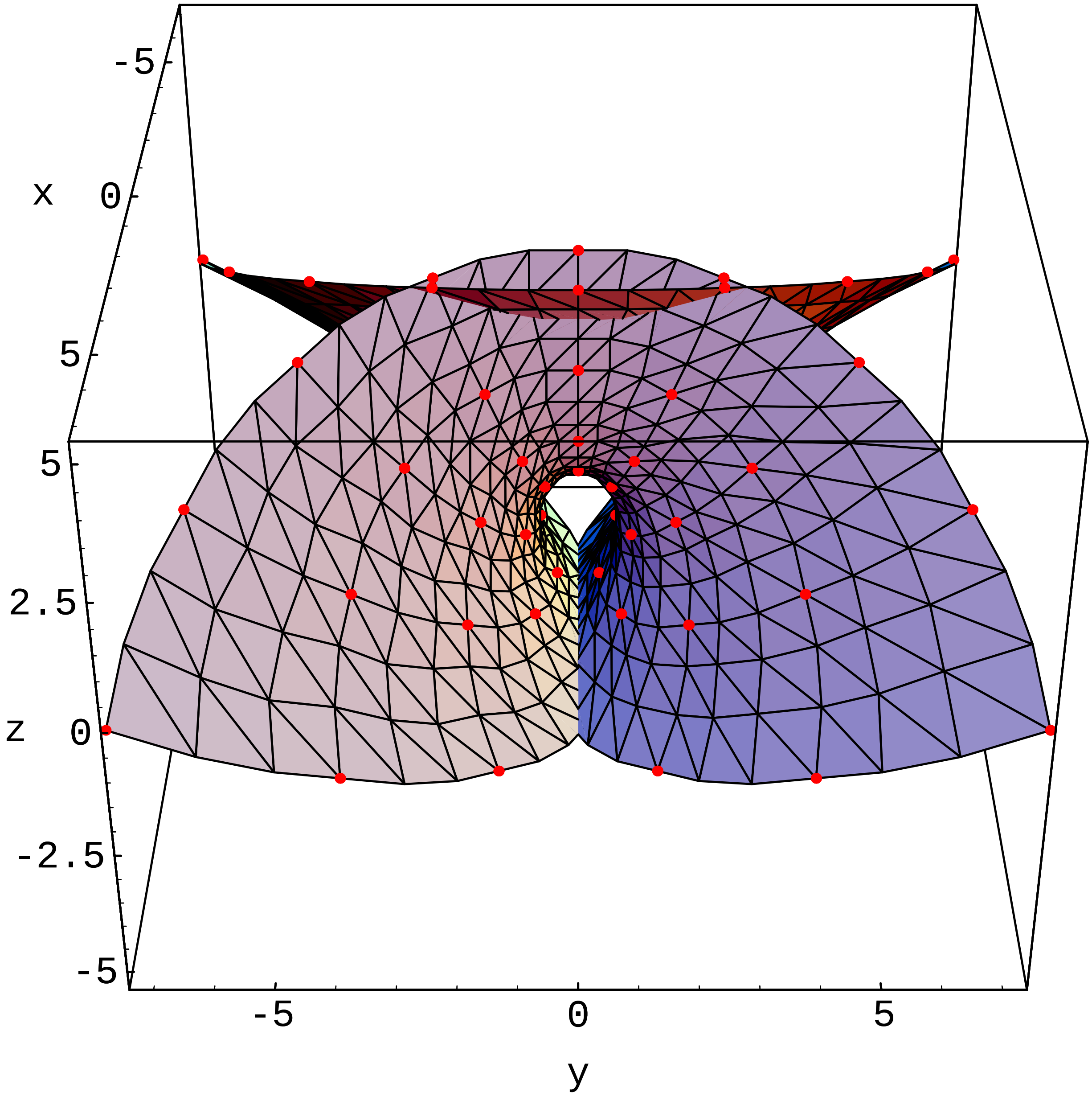


Figure 4.1: The Enneper surface

4.2 Control Points For Triangular Surfaces

Given an affine frame Δrst in the plane (where $r, s, t \in \mathcal{P}$ are affinely independent points), it turns out that any symmetric multiaffine map $f\colon \mathcal{P}^m \to \mathcal{E}$ is uniquely determined by a family of $\frac{(m+1)(m+2)}{2}$ points (where $\mathcal{E}$ is any affine space, say $\mathbb{A}^n$). Let

$$\Delta_m = \{(i,\, j,\, k) \in \mathbb{N}^3 \mid i + j + k = m\}.$$

The following lemma is easily shown (see Ramshaw [**?**] or Gallier [**?**]).

Lemma 4.2.1 *Given a reference triangle Δrst in the affine plane $\mathcal{P}$, given any family $(b_{i,\,j,\,k})_{(i,j,k)\in\Delta_m}$ of $\frac{(m+1)(m+2)}{2}$ points in $\mathcal{E}$, there is a unique surface $F\colon \mathcal{P} \to \mathcal{E}$ of total degree m, defined by a symmetric m-affine polar form $f\colon \mathcal{P}^m \to \mathcal{E}$, such that*

$$f(\underbrace{r,\ldots,r}_{i},\underbrace{s,\ldots,s}_{j},\underbrace{t,\ldots,t}_{k}) = b_{i,\,j,\,k},$$

for all $(i,j,k) \in \Delta_m$. Furthermore, f is given by the expression

$$f(a_1,\ldots,a_m) = \sum_{\substack{I\cup J\cup K=\{1,\ldots,m\}\\ I,J,K \text{ pairwise disjoint}}} \Big(\prod_{i\in I}\lambda_i\Big)\Big(\prod_{j\in J}\mu_j\Big)\Big(\prod_{k\in K}\nu_k\Big)\, f(\underbrace{r,\ldots,r}_{|I|},\underbrace{s,\ldots,s}_{|J|},\underbrace{t,\ldots,t}_{|K|}),$$

where $a_i = \lambda_i r + \mu_i s + \nu_i t$, with $\lambda_i + \mu_i + \nu_i = 1$, and $1 \leq i \leq m$.

A point $F(a)$ on the surface F can be expressed in terms of the Bernstein polynomials

$$B^m_{i,j,k}(U,V,T) = \frac{m!}{i!j!k!}\,U^iV^jT^k,$$

as

$$F(a) = f(\underbrace{a,\ldots,a}_{m}) =$$
$$\sum_{(i,j,k)\in\Delta_m} B^m_{i,j,k}(\lambda,\mu,\nu)\, f(\underbrace{r,\ldots,r}_{i},\underbrace{s,\ldots,s}_{j},\underbrace{t,\ldots,t}_{k}),$$

where $a = \lambda r + \mu s + \nu t$, with $\lambda + \mu + \nu = 1$.

For example, with respect to the standard frame $\Delta rst = ((1,0,0),(0,1,0),(0,0,1))$, we obtain the following 10 control points for the Enneper surface:

$$
\begin{array}{ccccccc}
 & & & \begin{array}{c} f(r,r,r) \\ \left(\frac{2}{3},0,1\right) \end{array} & & & \\
 & & \begin{array}{c} f(r,r,t) \\ \left(\frac{2}{3},0,\frac{1}{3}\right) \end{array} & & \begin{array}{c} f(r,r,s) \\ \left(\frac{2}{3},\frac{2}{3},\frac{1}{3}\right) \end{array} & & \\
 & \begin{array}{c} f(r,t,t) \\ \left(\frac{1}{3},0,0\right) \end{array} & & \begin{array}{c} f(r,s,t) \\ \left(\frac{1}{3},\frac{1}{3},0\right) \end{array} & & \begin{array}{c} f(r,s,s) \\ \left(\frac{2}{3},\frac{2}{3},-\frac{1}{3}\right) \end{array} & \\
\begin{array}{c} f(t,t,t) \\ (0,0,0) \end{array} & & \begin{array}{c} f(s,t,t) \\ \left(0,\frac{1}{3},0\right) \end{array} & & \begin{array}{c} f(s,s,t) \\ \left(0,\frac{2}{3},-\frac{1}{3}\right) \end{array} & & \begin{array}{c} f(s,s,s) \\ \left(0,\frac{2}{3},-1\right) \end{array}
\end{array}
$$

A family $\mathcal{N} = (b_{i,\,j,\,k})_{(i,j,k)\in\Delta_m}$ of $\frac{(m+1)(m+2)}{2}$ points in $\mathcal{E}$ is called a *(triangular) control net, or Bézier net*. Note that the points in

$$\Delta_m = \{(i,\,j,\,k) \in \mathbb{N}^3 \mid i + j + k = m\},$$

can be thought of as a triangular grid of points in $\mathcal{P}$. For example, when $m = 5$, we have the following grid of 21 points:

$$
\begin{array}{ccccccccccc}
 & & & & & 500 & & & & & \\
 & & & & 401 & & 410 & & & & \\
 & & & 302 & & 311 & & 320 & & & \\
 & & 203 & & 212 & & 221 & & 230 & & \\
 & 104 & & 113 & & 122 & & 131 & & 140 & \\
005 & & 014 & & 023 & & 032 & & 041 & & 050
\end{array}
$$

We intentionally let i be the row index, starting from the left lower corner, and j be the column index, also starting from the left lower corner. The control net $\mathcal{N} = (b_{i,\,j,\,k})_{(i,j,k)\in\Delta_m}$ can be viewed as an image of the triangular grid Δ_m in the affine space $\mathcal{E}$.

It follows from lemma 9.4.2 that there is a bijection between polynomial surfaces of degree m and control nets $\mathcal{N} = (b_{i,\,j,\,k})_{(i,j,k)\in\Delta_m}$.

4.3 Control Points For Rectangular Surfaces

Given any two affine frames $(\overline{r}_1, \overline{s}_1)$ and $(\overline{r}_2, \overline{s}_2)$ for the affine line $\mathbb{A}$, it turns out that a $\langle p, q\rangle$-symmetric multiaffine map

$$f\colon (\mathbb{A})^p \times (\mathbb{A})^q \to \mathcal{E}$$

is completely determined by the family of $(p+1)(q+1)$ points in $\mathcal{E}$

$$b_{i,\,j} = f(\underbrace{\overline{r}_1, \ldots, \overline{r}_1}_{p-i}, \underbrace{\overline{s}_1, \ldots, \overline{s}_1}_{i}; \underbrace{\overline{r}_2, \ldots, \overline{r}_2}_{q-j}, \underbrace{\overline{s}_2, \ldots, \overline{s}_2}_{j}),$$

where $0 \leq i \leq p$ and $0 \leq j \leq q$. The following lemma is easily shown (see Ramshaw [**?**] or Gallier [**?**]).

Lemma 4.3.1 *Let $(\overline{r}_1, \overline{s}_1)$ and $(\overline{r}_2, \overline{s}_2)$ be any two affine frames for the affine line $\mathbb{A}$, and let $\mathcal{E}$ be an affine space (of finite dimension $n \geq 3$). For any natural numbers p, q, for any family $(b_{i,\,j})_{0\leq i\leq p,\, 0\leq j\leq q}$ of $(p+1)(q+1)$ points in $\mathcal{E}$, there is a unique bipolynomial surface $F\colon \mathbb{A}\times\mathbb{A} \to \mathcal{E}$ of degree $\langle p, q\rangle$, with polar form the $(p+q)$-multiaffine $\langle p, q\rangle$-symmetric map*

$$f\colon (\mathbb{A})^p \times (\mathbb{A})^q \to \mathcal{E},$$

such that

$$f(\underbrace{\overline{r}_1, \ldots, \overline{r}_1}_{p-i}, \underbrace{\overline{s}_1, \ldots, \overline{s}_1}_{i}; \underbrace{\overline{r}_2, \ldots, \overline{r}_2}_{q-j}, \underbrace{\overline{s}_2, \ldots, \overline{s}_2}_{j}) = b_{i,\,j},$$

for all $i, 1 \leq i \leq p$ and all $j, 1 \leq j \leq q$. Furthermore, f is given by the expression

$$\begin{aligned} & f(\overline{u}_1, \ldots, \overline{u}_p; \overline{v}_1, \ldots, \overline{v}_q) \\ & \quad = \sum_{\substack{I\cap J=\emptyset\\ I\cup J=\{1,\ldots,p\}\\ K\cap L=\emptyset\\ K\cup L=\{1,\ldots,q\}}} \prod_{i\in I}\Big(\frac{s_1-u_i}{s_1-r_1}\Big) \prod_{j\in J}\Big(\frac{u_j-r_1}{s_1-r_1}\Big) \\ & \qquad\qquad \prod_{k\in K}\Big(\frac{s_2-v_k}{s_2-r_2}\Big) \prod_{l\in L}\Big(\frac{v_l-r_2}{s_2-r_2}\Big)\, b_{|J|,\,|L|}. \end{aligned}$$

A point $F(\overline{u}, \overline{v})$ on the surface F can be expressed in terms of the Bernstein polynomials $B_i^p[r_1, s_1](u)$ and $B_j^q[r_2, s_2](v)$, as

$$F(\overline{u}, \overline{v}) = \sum_{\substack{0 \leq i \leq p \\ 0 \leq j \leq q}} B_i^p[r_1, s_1](u)\, B_j^q[r_2, s_2](v)$$

$$f(\underbrace{\overline{r}_1, \ldots, \overline{r}_1}_{p-i}, \underbrace{\overline{s}_1, \ldots, \overline{s}_1}_{i}; \underbrace{\overline{r}_2, \ldots, \overline{r}_2}_{q-j}, \underbrace{\overline{s}_2, \ldots, \overline{s}_2}_{j}).$$

A family $\mathcal{N} = (b_{i,\,j})_{0\leq i\leq p,\,0\leq j\leq q}$ of $(p+1)(q+1)$ points in $\mathcal{E}$, is often called a *(rectangular) control net, or Bézier net*. Note that we can view the set of pairs

$$\square_{p,q} = \{(i,j) \in \mathbb{N}^2 \mid 0 \leq i \leq p,\, 0 \leq j \leq q\},$$

as a rectangular grid of $(p+1)(q+1)$ points in $\mathbb{A} \times \mathbb{A}$. The control net $\mathcal{N} = (b_{i,\,j})_{(i,j)\in\square_{p,q}}$, can be viewed as an image of the rectangular grid $\square_{p,q}$ in the affine space $\mathcal{E}$. The portion of the surface F corresponding to the points $F(\overline{u}, \overline{v})$ for which the parameters $\overline{u}, \overline{v}$ satisfy the inequalities $r_1 \leq u \leq s_1$ and $r_2 \leq v \leq s_2$, is called a *rectangular (surface) patch, or rectangular Bézier patch*, and $F([\overline{r}_1, \overline{s}_1], [\overline{r}_2, \overline{s}_2])$ is the *trace of the rectangular patch*.

As an example, the *monkey saddle* is the surface defined by the equation

$$z = x^3 - 3xy^2.$$

It is easily shown that the monkey saddle is specified by the following rectangular control net of degree $(3, 2)$ over $[0, 1] \times [0, 1]$:

```
sqmonknet1 =  {{0, 0, 0}, {0, 1/2, 0}, {0, 1, 0}, {1/3, 0, 0},
   {1/3, 1/2, 0}, {1/3, 1, -1}, {2/3, 0, 0}, {2/3, 1/2, 0},
   {2/3, 1, -2}, {1, 0, 1}, {1, 1/2, 1}, {1, 1, -2}}
```

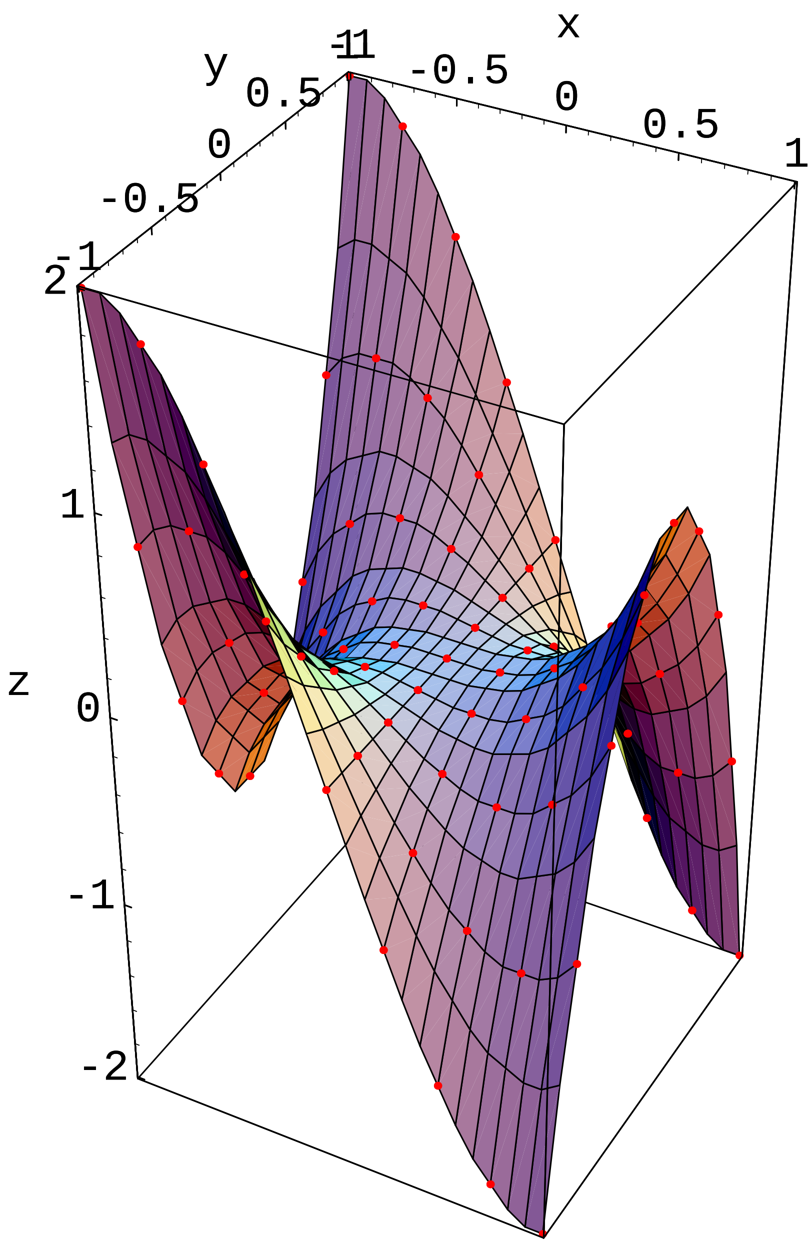


Figure 4.2: A monkey saddle, rectangular subdivision

Chapter 5

Affine Geometry

5.1 Affine Spaces

For simplicity, it is assumed that all vector spaces under consideration are defined over the field $\mathbb{R}$ of real numbers.

It is also assumed that all families $(\lambda_i)_{i\in I}$ of scalars have finite support. Recall that a family $(\lambda_i)_{i\in I}$ of scalars has *finite support* if

$\lambda_i = 0$ for all $i \in I - J$,

where J is a finite subset of I.

Obviously, finite families of scalars have finite support, and for simplicity, the reader may assume that all families are finite.

Definition 5.1.1 An *affine space* is either the empty set or a triple $\langle E, \overrightarrow{E}, +\rangle$ consisting of a nonempty set E (of *points*), a vector space $\overrightarrow{E}$ (of *translations, or free vectors*), and an action $+\colon E \times \overrightarrow{E} \to E$, satisfying the following conditions:

(AF1) $a + \overrightarrow{0} = a$, for every $a \in E$;

(AF2) $(a + \overrightarrow{u}) + \overrightarrow{v} = a + (\overrightarrow{u} + \overrightarrow{v})$, for every $a \in E$, and every $\overrightarrow{u}, \overrightarrow{v} \in \overrightarrow{E}$;

(AF3) For any two points $a, b \in E$, there is a unique $\overrightarrow{u} \in \overrightarrow{E}$ such that $a + \overrightarrow{u} = b$. The unique vector $\overrightarrow{u} \in \overrightarrow{E}$ such that $a + \overrightarrow{u} = b$ is denoted as $\overrightarrow{ab}$, or sometimes as $b - a$. Thus, we also write

$$b = a + \overrightarrow{ab}$$

(or even $b = a + (b - a)$).

The *dimension of the affine space* $\langle E, \overrightarrow{E}, +\rangle$ is the dimension $\dim(\overrightarrow{E})$ of the vector space $\overrightarrow{E}$. For simplicity, it is denoted as $\dim(E)$.

Conditions (AF1) and (AF2) say that the (abelian) group $\overrightarrow{E}$ acts on E, and condition (AF3) says that $\overrightarrow{E}$ acts transitively and faithfully on E.

Note that

$$\overrightarrow{a(a+\overrightarrow{v})} = \overrightarrow{v}$$

for all $a \in E$ and all $\overrightarrow{v} \in \overrightarrow{E}$, since $\overrightarrow{a(a+\overrightarrow{v})}$ is the unique vector such that $a+\overrightarrow{v} = a + \overrightarrow{a(a+\overrightarrow{v})}$.

Thus, $b = a + \overrightarrow{v}$ is equivalent to $\overrightarrow{ab} = \overrightarrow{v}$.

It is natural to think of all vectors as having the same origin, the null vector.

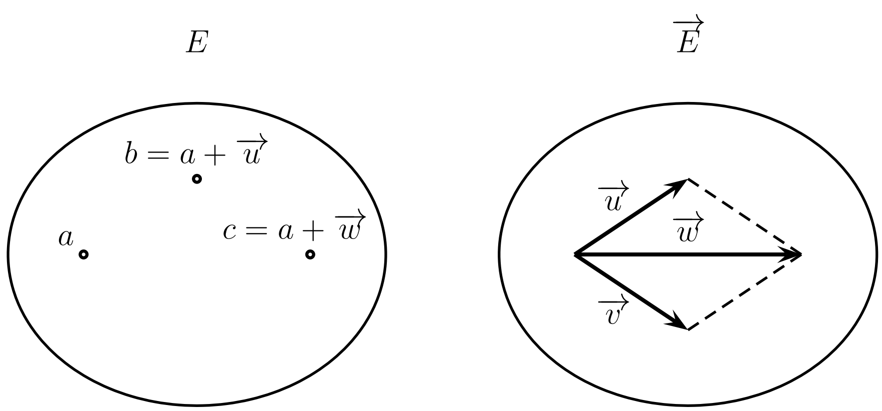


Figure 5.1: Intuitive picture of an affine space

For every $a \in E$, consider the mapping from $\overrightarrow{E}$ to E:

$$\overrightarrow{u} \mapsto a + \overrightarrow{u},$$

where $\overrightarrow{u} \in \overrightarrow{E}$, and consider the mapping from E to $\overrightarrow{E}$:

$$b \mapsto \overrightarrow{ab},$$

where $b \in E$.

The composition of the first mapping with the second is

$$\overrightarrow{u} \mapsto a + \overrightarrow{u} \mapsto \overrightarrow{a(a + \overrightarrow{u})},$$

which, in view of (AF3), yields $\overrightarrow{u}$.

The composition of the second with the first mapping is

$$b \mapsto \overrightarrow{ab} \mapsto a + \overrightarrow{ab},$$

which, in view of (AF3), yields b.

Thus, these compositions are the identity from $\overrightarrow{E}$ to $\overrightarrow{E}$ and the identity from E to E, and the mappings are both bijections.

When we identify E to $\overrightarrow{E}$ via the mapping $b \mapsto \overrightarrow{ab}$, we say that we consider E as the vector space obtained *by taking a as the origin in E*, and we denote it as E_a. Thus, an affine space $\langle E, \overrightarrow{E}, +\rangle$ is a way of defining a vector space structure on a set of points E, without making a commitment to a **fixed** origin in E.

For notational simplicity, we will often denote an affine space $\langle E, \overrightarrow{E}, +\rangle$ as $(E, \overrightarrow{E})$, or even as E. The vector space $\overrightarrow{E}$ is called the *vector space associated with E*.

One should be careful about the overloading of the addition symbol $+$. Addition is well-defined on vectors, as in $\overrightarrow{u} + \overrightarrow{v}$, the translate $a + \overrightarrow{u}$ of a point $a \in E$ by a vector $\overrightarrow{u} \in \overrightarrow{E}$ is also well-defined, but addition of points $a + b$ **does not make sense**.

In this respect, the notation $b - a$ for the unique vector $\overrightarrow{u}$ such that $b = a + \overrightarrow{u}$, is somewhat confusing, since it suggests that points can be substracted (but not added!).

Any vector space $\overrightarrow{E}$ has an affine space structure specified by choosing $E = \overrightarrow{E}$, and letting $+$ be addition in the vector space $\overrightarrow{E}$. We will refer to this affine space $\langle \overrightarrow{E}, \overrightarrow{E}, + \rangle$ as the *canonical (or natural) affine structure on* $\overrightarrow{E}$.

In particular, the vector space $\mathbb{R}^n$ can be viewed as an affine space $\langle \mathbb{R}^n, \mathbb{R}^n, + \rangle$ denoted as $\mathbb{A}^n$. In order to distinguish between the double role played by members of $\mathbb{R}^n$, points and vectors, we will denote points as row vectors, and vectors as column vectors. Thus, the action of the vector space $\mathbb{R}^n$ over the set $\mathbb{R}^n$ simply viewed as a set of points, is given by

$$(a_1, \ldots, a_n) + \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = (a_1 + u_1, \ldots, a_n + u_n).$$

The affine space $\mathbb{A}^n$ is called the *real affine space of dimension* n. In most cases, we will consider $n = 1, 2, 3$.

For a slightly wilder example, consider the subset P of $\mathbb{A}^3$ consisting of all points (x, y, z) satisfying the equation

$$x^2 + y^2 - z = 0.$$

The set P is paraboloid of revolution, with axis Oz.

The surface P can be made into an official affine space by defining the action

$$+\colon P \times \mathbb{R}^2 \to P$$

of $\mathbb{R}^2$ on P defined such that for every point (x, y, x^2+y^2) on P and any $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$,

$$(x, y, x^2+y^2)+\begin{pmatrix} u \\ v \end{pmatrix} = (x+u, y+v, (x+u)^2+(y+v)^2).$$

Affine spaces not already equipped with an obvious vector space structure arise in projective geometry.

Given any three points $a, b, c \in E$, since $c = a + \overrightarrow{ac}$, $b = a + \overrightarrow{ab}$, and $c = b + \overrightarrow{bc}$, we get

$$c = b + \overrightarrow{bc} = (a + \overrightarrow{ab}) + \overrightarrow{bc} = a + (\overrightarrow{ab} + \overrightarrow{bc})$$

by (AF2), and thus, by (AF3),

$$\overrightarrow{ab} + \overrightarrow{bc} = \overrightarrow{ac},$$

which is known as *Chasles' identity*.

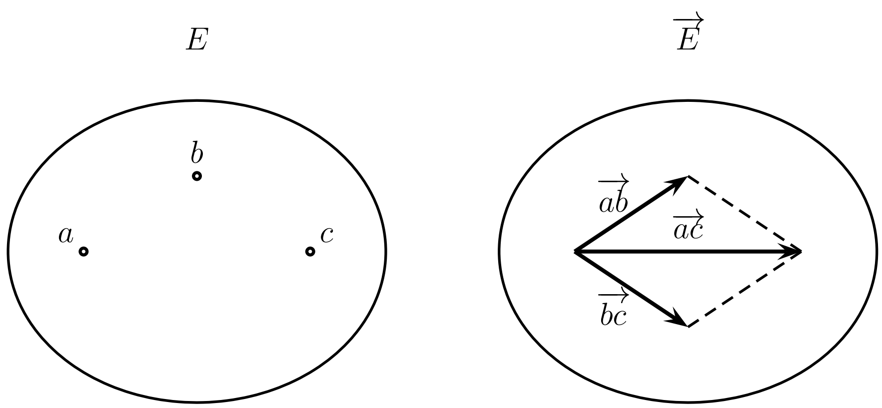


Figure 5.2: Points and corresponding vectors in affine geometry

5.2 Affine Combinations, Barycenters

A fundamental concept in linear algebra is that of a linear combination. The corresponding concept in affine geometry is that of an affine combination, also called a barycenter.

However, there is a problem with the naive approach involving a coordinate system. The problem is that the sum $a + b$ may correspond to two different points depending on which coordinate system is used for its computation!

Thus, some extra condition is needed in order for affine combinations to make sense. It turns out that if the scalars sum up to 1, the definition is intrinsic, as the following lemma shows.

Lemma 5.2.1 *Given an affine space E, let $(a_i)_{i\in I}$ be a family of points in E, and let $(\lambda_i)_{i\in I}$ be a family of scalars. For any two points $a, b \in E$, the following properties hold:*

(1) If $\sum_{i\in I} \lambda_i = 1$, then

$$a + \sum_{i\in I} \lambda_i \overrightarrow{aa_i} = b + \sum_{i\in I} \lambda_i \overrightarrow{ba_i}.$$

(2) If $\sum_{i\in I} \lambda_i = 0$, then

$$\sum_{i\in I} \lambda_i \overrightarrow{aa_i} = \sum_{i\in I} \lambda_i \overrightarrow{ba_i}.$$

Thus, by lemma 5.2.1, for any family of points $(a_i)_{i\in I}$ in E, for any family $(\lambda_i)_{i\in I}$ of scalars such that $\sum_{i\in I} \lambda_i = 1$, the point

$$x = a + \sum_{i\in I} \lambda_i \overrightarrow{aa_i}$$

is independent of the choice of the origin $a \in E$.

The unique point x is called the *barycenter (or barycentric combination, or affine combination) of the points* a_i *assigned the weights* λ_i. and it is denoted as

$$\sum_{i \in I} \lambda_i a_i.$$

In dealing with barycenters, it is convenient to introduce the notion of a *weighted point*, which is just a pair (a, λ), where $a \in E$ is a point, and $\lambda \in \mathbb{R}$ is a scalar.

Then, given a family of weighted points $((a_i, \lambda_i))_{i \in I}$, where $\sum_{i \in I} \lambda_i = 1$, we also say that the point

$$\sum_{i \in I} \lambda_i a_i$$

is the *barycenter of the family of weighted points* $((a_i, \lambda_i))_{i \in I}$.

Note that the barycenter x of the family of weighted points $((a_i, \lambda_i))_{i \in I}$ is also the unique point such that

$$\overrightarrow{ax} = \sum_{i \in I} \lambda_i \overrightarrow{aa_i} \quad \text{for every } a \in E,$$

and setting $a = x$, the point x is the unique point such that

$$\sum_{i \in I} \lambda_i \overrightarrow{xa_i} = \overrightarrow{0}.$$

In physical terms, the barycenter is the *center of mass* of the family of weighted points $((a_i, \lambda_i))_{i \in I}$ (where the masses have been normalized, so that $\sum_{i \in I} \lambda_i = 1$, and negative masses are allowed).

The figure below illustrates the geometric construction of the barycenters g_1 and g_2 of the weighted points $\left(a, \frac{1}{4}\right)$, $\left(b, \frac{1}{4}\right)$, and $\left(c, \frac{1}{2}\right)$, and $(a, -1)$, $(b, 1)$, and $(c, 1)$.

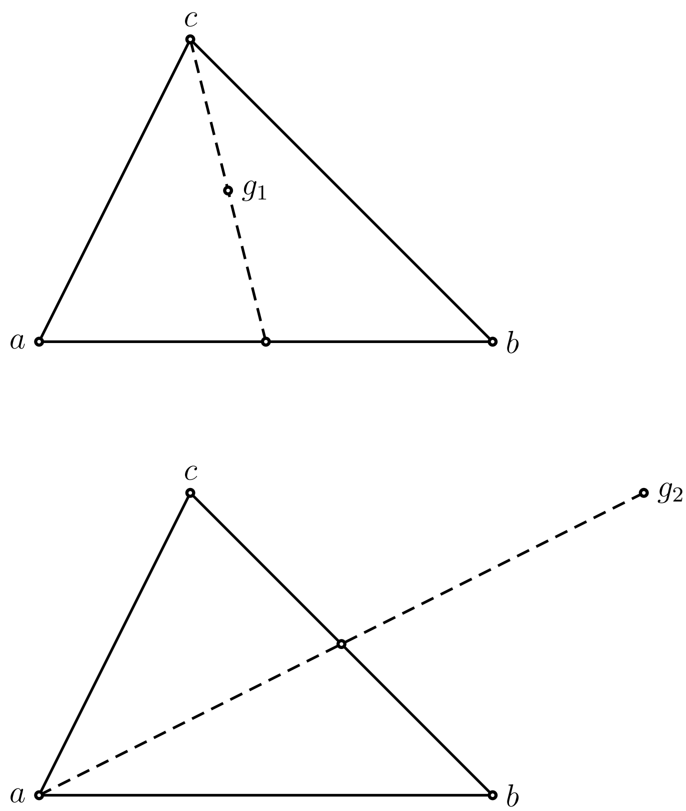


Figure 5.3: Barycenters, $g_1 = \frac{1}{4}a + \frac{1}{4}b + \frac{1}{2}c, \quad g_2 = -a + b + c$.

5.3 Affine Subspaces

In linear algebra, a (linear) subspace can be characterized as a nonempty subset of a vector space closed under linear combinations. In affine spaces, the notion corresponding to the notion of (linear) subspace is the notion of affine subspace.

It is natural to define an affine subspace as a subset of an affine space closed under affine combinations.

Definition 5.3.1 Given an affine space $\langle E, \overrightarrow{E}, +\rangle$, a subset V of E is an *affine subspace* if for every family of points $(a_i)_{i\in I}$ in V, for any family $(\lambda_i)_{i\in I}$ of scalars such that $\sum_{i\in I}\lambda_i = 1$, the barycenter $\sum_{i\in I}\lambda_i a_i$ belongs to V.

An affine subspace is also called a *flat* by some authors.

According to definition 5.3.1, the empty set is trivially an affine subspace, and every intersection of affine subspaces is an affine subspace.

As an example, consider the subset U of $\mathbb{A}^2$ defined by

$$U = \{(x, y) \in \mathbb{R}^2 \mid ax + by = c\},$$

i.e. the set of solutions of the equation

$$ax + by = c,$$

where it is assumed that $a \neq 0$ or $b \neq 0$.

Given any m points $(x_i, y_i) \in U$ and any m scalars λ_i such that $\lambda_1 + \cdots + \lambda_m = 1$, we claim that

$$\sum_{i=1}^{m} \lambda_i (x_i, y_i) \in U.$$

Thus, U is an affine subspace of $\mathbb{A}^2$. In fact, it is just a usual line in $\mathbb{A}^2$.

It turns out that U is closely related to the subset of $\mathbb{R}^2$ defined by

$$\overrightarrow{U} = \{(x, y) \in \mathbb{R}^2 \mid ax + by = 0\},$$

i.e. the set of solution of the homogeneous equation

$$ax + by = 0$$

obtained by setting the right-hand side of $ax + by = c$ to zero.

Indeed, for any m scalars λ_i, the same calculation as above yields that

$$\sum_{i=1}^{m} \lambda_i(x_i, y_i) \in \overrightarrow{U},$$

this time **without any restriction on the** λ_i, since the right-hand side of the equation is null.

Thus, $\overrightarrow{U}$ is a subspace of $\mathbb{R}^2$. In fact, $\overrightarrow{U}$ is one-dimensional, and it is just a usual line in $\mathbb{R}^2$.

This line can be identified with a line passing through the origin of $\mathbb{A}^2$, line which is parallel to the line U of equation $ax + by = c$.

Now, if (x_0, y_0) is any point in U, we claim that

$$U = (x_0, y_0) + \overrightarrow{U},$$

where

$$(x_0, y_0) + \overrightarrow{U} = \{(x_0 + u_1, y_0 + u_2) \mid (u_1, u_2) \in \overrightarrow{U}\}.$$

The above example shows that the affine line U defined by the equation

$$ax + by = c$$

is obtained by "translating" the parallel line $\overrightarrow{U}$ of equation

$$ax + by = 0$$

passing through the origin.

In fact, given any point $(x_0, y_0) \in U$,

$$U = (x_0, y_0) + \overrightarrow{U}.$$

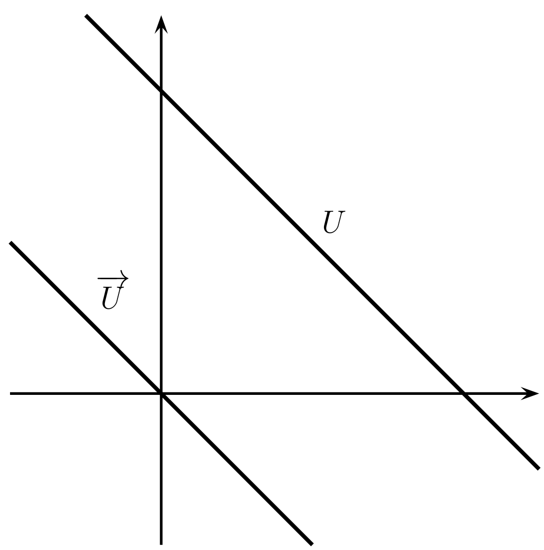


Figure 5.4: An affine line U and its direction

More generally, it is easy to prove the following fact. Given any $m \times n$ matrix A and any vector $c \in \mathbb{R}^m$, the subset U of $\mathbb{A}^n$ defined by

$$U = \{x \in \mathbb{R}^n \mid Ax = c\}$$

is an affine subspace of $\mathbb{A}^n$.

Furthermore, if we consider the corresponding homogeneous equation $Ax = 0$, the set

$$\overrightarrow{U} = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

is a subspace of $\mathbb{R}^n$, and for any $x_0 \in U$, we have

$$U = x_0 + \overrightarrow{U}.$$

This is a general situation. Affine subspaces can also be characterized in terms of subspaces of $\overrightarrow{E}$.

Given any point $a \in E$ and any subspace $\overrightarrow{V}$ of $\overrightarrow{E}$, let $a + \overrightarrow{V}$ denote the following subset of E:

$$a + \overrightarrow{V} = \{a + \overrightarrow{v} \mid \overrightarrow{v} \in \overrightarrow{V}\}.$$

Lemma 5.3.2 *Let* $\langle E, \overrightarrow{E}, +\rangle$ *be an affine space.*

(1) A nonempty subset V *of* E *is an affine subspace iff, for every point* $a \in V$*, the set*

$$\overrightarrow{V_a} = \{\overrightarrow{ax} \mid x \in V\}$$

is a subspace of $\overrightarrow{E}$*. Consequently,* $V = a + \overrightarrow{V_a}$*. Furthermore,*

$$\overrightarrow{V} = \{\overrightarrow{xy} \mid x, y \in V\}$$

is a subspace of $\overrightarrow{E}$ *and* $\overrightarrow{V_a} = \overrightarrow{V}$ *for all* $a \in E$*. Thus,* $V = a + \overrightarrow{V}$*.*

(2) For any subspace $\overrightarrow{V}$ *of* $\overrightarrow{E}$*, for any* $a \in E$*, the set* $V = a + \overrightarrow{V}$ *is an affine subspace.*

The subspace $\overrightarrow{V}$ associated with an affine subspace V is called the *direction of* V.

It is clear that the map $+: V \times \overrightarrow{V} \to V$ induced by $+: E \times \overrightarrow{E} \to E$ confers to $\langle V, \overrightarrow{V}, +\rangle$ an affine structure.

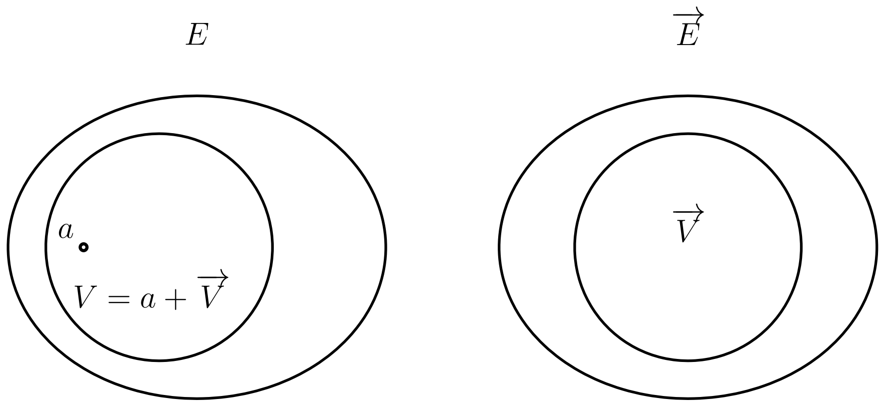


Figure 5.5: An affine subspace V and its direction $\overrightarrow{V}$

By the dimension of the subspace V, we mean the dimension of $\overrightarrow{V}$.

An affine subspace of dimension 1 is called a *line*, and an affine subspace of dimension 2 is called a *plane*.

An affine subspace of codimension 1 is called an *hyperplane*.

We say that two affine subspaces U and V are *parallel* if their directions are identical. Equivalently, since $\overrightarrow{U} = \overrightarrow{V}$, we have $U = a + \overrightarrow{U}$, and $V = b + \overrightarrow{U}$, for any $a \in U$ and any $b \in V$, and thus, V is obtained from U by the translation $\overrightarrow{ab}$.

In general, when we talk about n points $a_1, \ldots, a_n$, we mean the sequence $(a_1, \ldots, a_n)$, and not the set $\{a_1, \ldots, a_n\}$ (the a_i's need not be distinct).

We say that three points a, b, c are *collinear*, if the vectors $\overrightarrow{ab}$ and $\overrightarrow{ac}$ are linearly dependent.

If two of the points a, b, c are distinct, say $a \neq b$, then there is a unique $\lambda \in \mathbb{R}$, such that $\overrightarrow{ac} = \lambda\overrightarrow{ab}$, and we define the ratio $\frac{\overrightarrow{ac}}{\overrightarrow{ab}} = \lambda$.

We say that four points a, b, c, d are *coplanar*, if the vectors $\overrightarrow{ab}, \overrightarrow{ac}$, and $\overrightarrow{ad}$, are linearly dependent.

Lemma 5.3.3 *Given an affine space $\langle E, \overrightarrow{E}, +\rangle$, for any family $(a_i)_{i\in I}$ of points in E, the set V of barycenters $\sum_{i\in I} \lambda_i a_i$ (where $\sum_{i\in I} \lambda_i = 1$) is the smallest affine subspace containing $(a_i)_{i\in I}$.*

Given a nonempty subset S of E, the smallest affine subspace of E generated by S is often denoted as $\langle S\rangle$. For example, a line specified by two distinct points a and b is denoted as $\langle a, b\rangle$, or even (a, b), and similarly for planes, etc.

Remarks: Since it can be shown that the barycenter of n weighted points can be obtained by repeated computations of barycenters of two weighted points, a nonempty subset V of E is an affine subspace iff for every two points $a, b \in V$, the set V contains all barycentric combinations of a and b.

If V contains at least two points, V is an affine subspace iff for any two distinct points $a, b \in V$, the set V contains the line determined by a and b, that is, the set of all points $(1-\lambda)a + \lambda b$, $\lambda \in \mathbb{R}$.

5.4 Affine Independence and Affine Frames

Corresponding to the notion of linear independence in vector spaces, we have the notion of affine independence.

Given a family $(a_i)_{i\in I}$ of points in an affine space E, we will reduce the notion of (affine) independence of these points to the (linear) independence of the families $(\overrightarrow{a_i a_j})_{j\in(I-\{i\})}$ of vectors obtained by choosing any a_i as an origin.

First, the following lemma shows that it sufficient to consider only one of these families.

Lemma 5.4.1 *Given an affine space $\langle E, \overrightarrow{E}, +\rangle$, let $(a_i)_{i\in I}$ be a family of points in E. If the family $(\overrightarrow{a_i a_j})_{j\in(I-\{i\})}$ is linearly independent for some $i \in I$, then $(\overrightarrow{a_i a_j})_{j\in(I-\{i\})}$ is linearly independent for every $i \in I$.*

Definition 5.4.2 Given an affine space $\langle E, \overrightarrow{E}, +\rangle$, a family $(a_i)_{i\in I}$ of points in E is *affinely independent* if the family $(\overrightarrow{a_ia_j})_{j\in(I-\{i\})}$ is linearly independent for some $i \in I$.

Definition 5.4.2 is reasonable, since by Lemma 5.4.1, the independence of the family $(\overrightarrow{a_ia_j})_{j\in(I-\{i\})}$ does not depend on the choice of a_i.

A crucial property of linearly independent vectors $(\overrightarrow{u_1}, \ldots, \overrightarrow{u_m})$ is that if a vector $\overrightarrow{v}$ is a linear combination

$$\overrightarrow{v} = \sum_{i=1}^{m} \lambda_i \overrightarrow{u_i}$$

of the $\overrightarrow{u_i}$, then the λ_i are unique. A similar result holds for affinely independent points.

Lemma 5.4.3 *Given an affine space $\langle E, \overrightarrow{E}, +\rangle$, let $(a_0, \ldots, a_m)$ be a family of $m+1$ points in E. Let $x \in E$, and assume that $x = \sum_{i=0}^{m} \lambda_i a_i$, where $\sum_{i=0}^{m} \lambda_i = 1$. Then, the family $(\lambda_0, \ldots, \lambda_m)$ such that $x = \sum_{i=0}^{m} \lambda_i a_i$ is unique iff the family $(\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m})$ is linearly independent.*

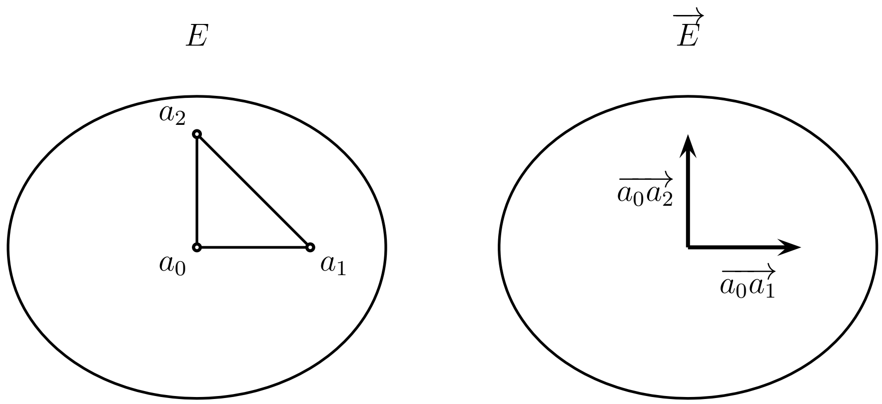


Figure 5.6: Affine independence and linear independence

Lemma 5.4.3 suggests the notion of affine frame.

Let $\langle E, \overrightarrow{E}, +\rangle$ be a nonempty affine space, and let $(a_0, \ldots, a_m)$ be a family of $m+1$ points in E. The family $(a_0, \ldots, a_m)$ determines the family of m vectors $(\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m})$ in $\overrightarrow{E}$.

Conversely, given a point a_0 in E and a family of m vectors $(\overrightarrow{u_1}, \ldots, \overrightarrow{u_m})$ in $\overrightarrow{E}$, we obtain the family of $m+1$ points $(a_0, \ldots, a_m)$ in E, where $a_i = a_0 + \overrightarrow{u_i}$, $1 \leq i \leq m$.

Thus, for any $m \geq 1$, it is equivalent to consider a family of $m+1$ points $(a_0, \ldots, a_m)$ in E, and a pair $(a_0, (\overrightarrow{u_1}, \ldots, \overrightarrow{u_m}))$, where the $\overrightarrow{u_i}$ are vectors in $\overrightarrow{E}$.

When $(\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m})$ is a basis of $\overrightarrow{E}$, then, for every $x \in E$, since $x = a_0 + \overrightarrow{a_0x}$, there is a unique family $(x_1, \ldots, x_m)$ of scalars, such that

$$x = a_0 + x_1\overrightarrow{a_0a_1} + \cdots + x_m\overrightarrow{a_0a_m}.$$

The scalars $(x_1, \ldots, x_m)$ are coordinates with respect to $(a_0, (\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m}))$. Since

$$x = a_0 + \sum_{i=1}^{m} x_i\overrightarrow{a_0a_i} \quad \text{iff} \quad x = (1 - \sum_{i=1}^{m} x_i)a_0 + \sum_{i=1}^{m} x_i a_i,$$

$x \in E$ can also be expressed uniquely as

$$x = \sum_{i=0}^{m} \lambda_i a_i$$

with $\sum_{i=0}^{m} \lambda_i = 1$, and where $\lambda_0 = 1 - \sum_{i=1}^{m} x_i$, and $\lambda_i = x_i$ for $1 \leq i \leq m$.

The scalars $(\lambda_0, \ldots, \lambda_m)$ are also certain kinds of coordinates with respect to $(a_0, \ldots, a_m)$.

Definition 5.4.4 Given an affine space $\langle E, \overrightarrow{E}, +\rangle$, an *affine frame with origin* a_0 is a family $(a_0, \ldots, a_m)$ of $m+1$ points in E such that $(\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m})$ is a basis of $\overrightarrow{E}$. The pair $(a_0, (\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m}))$ is also called an *affine frame with origin* a_0.

Then, every $x \in E$ can be expressed as

$$x = a_0 + x_1\overrightarrow{a_0a_1} + \cdots + x_m\overrightarrow{a_0a_m}$$

for a unique family $(x_1, \ldots, x_m)$ of scalars, called the *coordinates of* x *w.r.t. the affine frame* $(a_0, (\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m}))$.

Furthermore, every $x \in E$ can be written as

$$x = \lambda_0 a_0 + \cdots + \lambda_m a_m$$

for some unique family $(\lambda_0, \ldots, \lambda_m)$ of scalars such that $\lambda_0 + \cdots + \lambda_m = 1$ called the *barycentric coordinates of* x *with respect to the affine frame* $(a_0, \ldots, a_m)$.

The coordinates $(x_1, \ldots, x_m)$ and the barycentric coordinates $(\lambda_0, \ldots, \lambda_m)$ are related by the equations $\lambda_0 = 1 - \sum_{i=1}^{m} x_i$ and $\lambda_i = x_i$, for $1 \leq i \leq m$.

An affine frame is called an *affine basis* by some authors. The figure below shows affine frames for $|I| = 0, 1, 2, 3$.

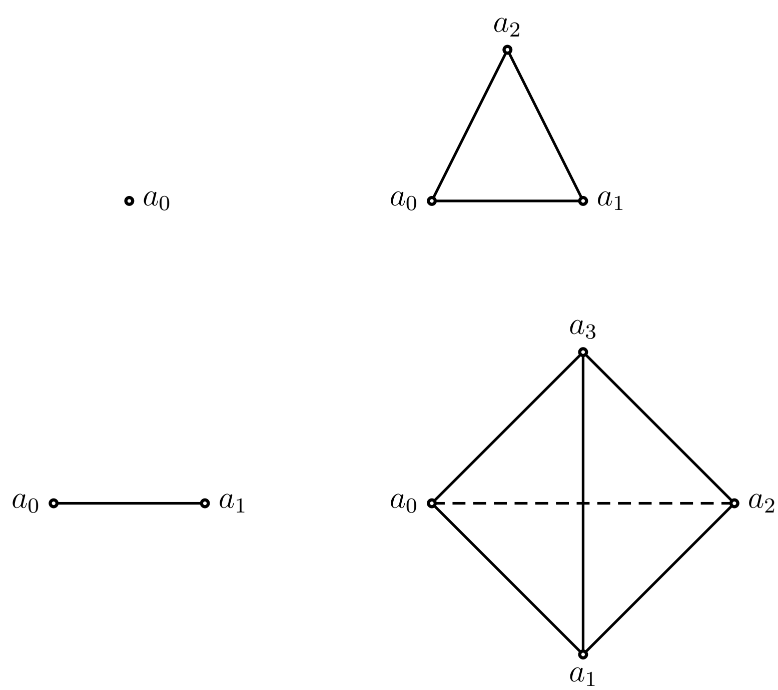


Figure 5.7: Examples of affine frames.

A family of two points (a, b) in E is affinely independent iff $\overrightarrow{ab} \neq \overrightarrow{0}$, iff $a \neq b$. If $a \neq b$, the affine subspace generated by a and b is the set of all points $(1-\lambda)a+\lambda b$, which is the unique line passing through a and b.

A family of three points (a, b, c) in E is affinely independent iff $\overrightarrow{ab}$ and $\overrightarrow{ac}$ are linearly independent, which means that a, b, and c are not on a same line (they are not collinear). In this case, the affine subspace generated by (a, b, c) is the set of all points $(1 - \lambda - \mu)a + \lambda b + \mu c$, which is the unique plane containing a, b, and c.

A family of four points (a, b, c, d) in E is affinely independent iff $\overrightarrow{ab}$, $\overrightarrow{ac}$, and $\overrightarrow{ad}$ are linearly independent, which means that a, b, c, and d are not in a same plane (they are not coplanar). In this case, a, b, c, and d, are the vertices of a tetrahedron.

Given $n+1$ affinely independent points $(a_0, \ldots, a_n)$ in E, we can consider the set of points $\lambda_0 a_0 + \cdots + \lambda_n a_n$, where $\lambda_0 + \cdots + \lambda_n = 1$ and $\lambda_i \geq 0$, $\lambda_i \in \mathbb{R}$. Such affine combinations are called *convex combinations*. This set is called the *convex hull* of $(a_0, \ldots, a_n)$ (or *n-simplex spanned by* $(a_0, \ldots, a_n)$).

When $n = 1$, we get the segment between a_0 and a_1, including a_0 and a_1.

When $n = 2$, we get the interior of the triangle whose vertices are a_0, a_1, a_2, including boundary points (the edges).

When $n = 3$, we get the interior of the tetrahedron whose vertices are a_0, a_1, a_2, a_3, including boundary points (faces and edges).

The set

$$\{a_0 + \lambda_1 \overrightarrow{a_0 a_1} + \cdots + \lambda_n \overrightarrow{a_0 a_n} \mid \text{where } 0 \leq \lambda_i \leq 1 \ (\lambda_i \in \mathbb{R})\},$$

is called the *parallelotope spanned by* $(a_0, \ldots, a_n)$. When E has dimension 2, a parallelotope is also called a *parallelogram*, and when E has dimension 3, a *parallelepiped*.

A parallelotope is shown in figure 5.8: it consists of the points inside of the parallelogram (a_0, a_1, a_2, d), including its boundary.

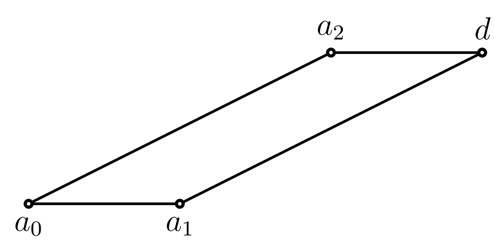


Figure 5.8: A parallelotope

More generally, we say that a subset V of E is *convex*, if for any two points $a, b \in V$, we have $c \in V$ for every point $c = (1 - \lambda)a + \lambda b$, with $0 \leq \lambda \leq 1$ $(\lambda \in \mathbb{R})$.

5.5 Affine Maps

Corresponding to linear maps, we have the notion of an affine map.

Definition 5.5.1 Given two affine spaces $\langle E, \overrightarrow{E}, +\rangle$ and $\langle E', \overrightarrow{E'}, +'\rangle$, a function $f\colon E \to E'$ is an *affine map* iff for every family $(a_i)_{i\in I}$ of points in E, for every family $(\lambda_i)_{i\in I}$ of scalars such that $\sum_{i\in I} \lambda_i = 1$, we have

$$f\left(\sum_{i\in I} \lambda_i a_i\right) = \sum_{i\in I} \lambda_i f(a_i).$$

In other words, f preserves affine combinations (barycenters).

Affine maps can be obtained from linear maps as follows. For simplicity of notation, the same symbol $+$ is used for both affine spaces (instead of using both $+$ and $+'$).

Given any point $a \in E$, any point $b \in E'$, and any linear map $h\colon \overrightarrow{E} \to \overrightarrow{E'}$, the map $f\colon E \to E'$ defined such that

$$f(a + \overrightarrow{v}) = b + h(\overrightarrow{v})$$

is an affine map.

As a more concrete example, the map

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

defines an affine map in $\mathbb{A}^2$. It is a "shear" followed by a translation. The effect of this shear on the square (a, b, c, d) is shown in figure 5.9. The image of the square (a, b, c, d) is the parallelogram (a', b', c', d').

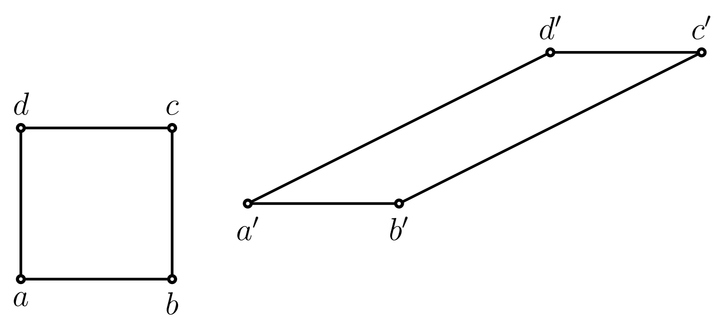


Figure 5.9: The effect of a shear

Let us consider one more example.

The map

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

is an affine map.

Since we can write

$$\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} = \sqrt{2} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

this affine map is the composition of a shear, followed by a rotation of angle $\pi/4$, followed by a magnification of ratio $\sqrt{2}$, followed by a translation. The effect of this map on the square (a, b, c, d) is shown in figure 5.10. The image of the square (a, b, c, d) is the parallelogram (a', b', c', d').

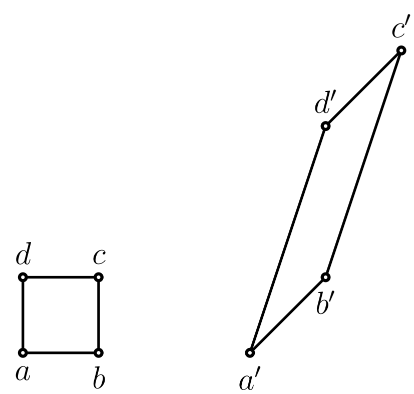


Figure 5.10: The effect of an affine map

The following lemma shows the converse of what we just showed. Every affine map is determined by the image of any point and a linear map.

Lemma 5.5.2 *Given an affine map $f \colon E \to E'$, there is a unique linear map $\overrightarrow{f} \colon \overrightarrow{E} \to \overrightarrow{E'}$, such that*

$$f(a + \overrightarrow{v}) = f(a) + \overrightarrow{f}(\overrightarrow{v}),$$

for every $a \in E$ and every $\overrightarrow{v} \in \overrightarrow{E}$.

The unique linear map $\overrightarrow{f} \colon \overrightarrow{E} \to \overrightarrow{E'}$ given by lemma 5.5.2 is the *linear map associated with the affine map f.*

Note that the condition

$$f(a + \overrightarrow{v}) = f(a) + \overrightarrow{f}(\overrightarrow{v}),$$

for every $a \in E$ and every $\overrightarrow{v} \in \overrightarrow{E}$, can be stated equivalently as

$$f(x) = f(a) + \overrightarrow{f}(\overrightarrow{ax}), \quad \text{or} \quad \overrightarrow{f(a)f(x)} = \overrightarrow{f}(\overrightarrow{ax}),$$

for all $a, x \in E$.

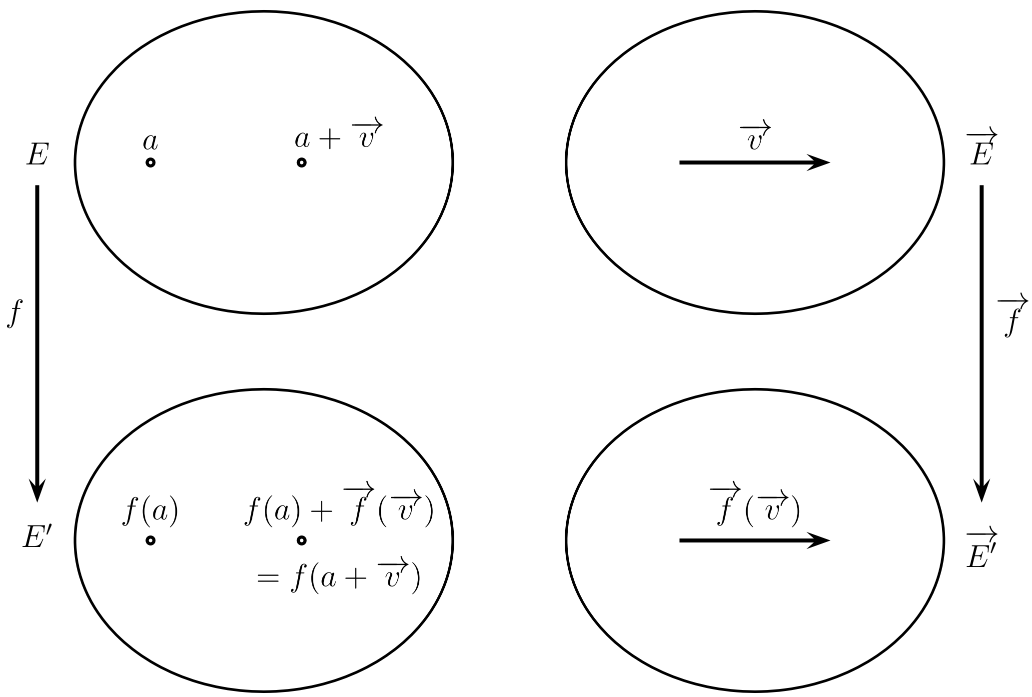


Figure 5.11: An affine map f and its associated linear map $\overrightarrow{f}$

Lemma 5.5.2 shows that for any affine map $f\colon E \to E'$, there are points $a \in E$, $b \in E'$, and a unique linear map $\overrightarrow{f} : \overrightarrow{E} \to \overrightarrow{E'}$, such that

$$f(a + \overrightarrow{v}) = b + \overrightarrow{f}(\overrightarrow{v}),$$

for all $\overrightarrow{v} \in \overrightarrow{E}$ (just let $b = f(a)$, for any $a \in E$).

Since an affine map preserves barycenters, and since an affine subspace V is closed under barycentric combinations, the image $f(V)$ of V is an affine subspace in E'.

So, for example, the image of a line is a point or a line, the image of a plane is either a point, a line, or a plane.

Affine maps for which $\overrightarrow{f}$ is the identity map are called *translations*. Indeed, if $\overrightarrow{f} = \mathrm{id}$, it is easy to show that for any two points $a, x \in E$,

$$f(x) = x + \overrightarrow{af(a)}.$$

It is easily verified that the composition of two affine maps is an affine map.

Also, given affine maps $f: E \to E'$ and $g: E' \to E''$, we have

$$g(f(a+\overrightarrow{v})) = g(f(a)+\overrightarrow{f}(\overrightarrow{v})) = g(f(a))+\overrightarrow{g}(\overrightarrow{f}(\overrightarrow{v})),$$

which shows that $\overrightarrow{(g \circ f)} = \overrightarrow{g} \circ \overrightarrow{f}$.

It is easy to show that an affine map $f: E \to E'$ is injective iff $\overrightarrow{f} : \overrightarrow{E} \to \overrightarrow{E'}$ is injective, and that $f: E \to E'$ is surjective iff $\overrightarrow{f} : \overrightarrow{E} \to \overrightarrow{E'}$ is surjective.

An affine map $f: E \to E'$ is constant iff $\overrightarrow{f} : \overrightarrow{E} \to \overrightarrow{E'}$ is the null (constant) linear map equal to $\overrightarrow{0}$ for all $\overrightarrow{v} \in \overrightarrow{E}$.

If E is an affine space of dimension m, and $(a_0, a_1, \ldots, a_m)$ is an affine frame for E, for any other affine space F, for any sequence $(b_0, b_1, \ldots, b_m)$ of $m+1$ points in F, there is a unique affine map $f\colon E \to F$ such that $f(a_i) = b_i$, for $0 \leq i \leq m$.

The following diagram illustrates the above result when $m = 2$.

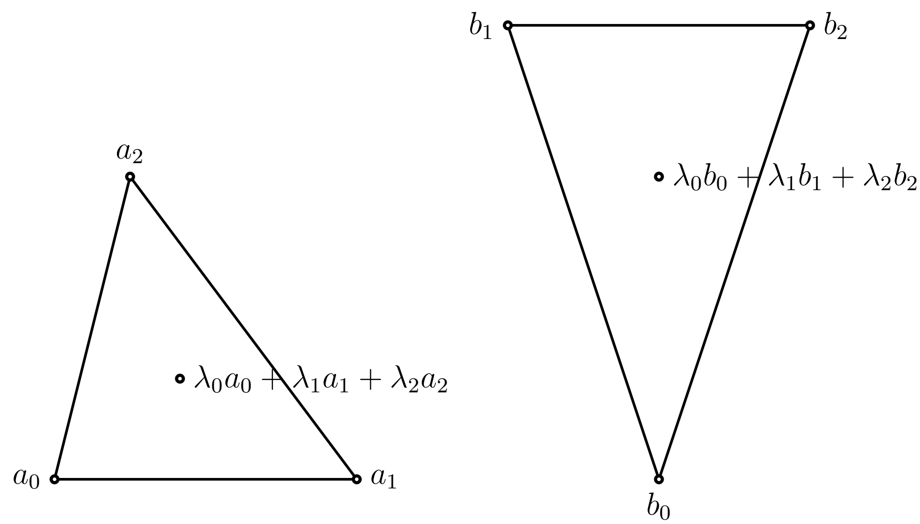


Figure 5.12: An affine map mapping a_0, a_1, a_2 to b_0, b_1, b_2.

Using affine frames, affine maps can be represented in terms of matrices.

We explain how an affine map $f\colon E \to E$ is represented with respect to a frame $(a_0, \ldots, a_n)$ in E.

Since

$$f(a_0 + \overrightarrow{x}) = f(a_0) + \overrightarrow{f}(\overrightarrow{x})$$

for all $\overrightarrow{x} \in \overrightarrow{E}$, we have

$$\overrightarrow{a_0 f(a_0 + \overrightarrow{x})} = \overrightarrow{a_0 f(a_0)} + \overrightarrow{f}(\overrightarrow{x}).$$

Since $\overrightarrow{x}$, $\overrightarrow{a_0 f(a_0)}$, and $\overrightarrow{a_0 f(a_0 + \overrightarrow{x})}$, can be expressed as

$$\begin{aligned}
\overrightarrow{x} &= x_1 \overrightarrow{a_0 a_1} + \cdots + x_n \overrightarrow{a_0 a_n}, \\
\overrightarrow{a_0 f(a_0)} &= b_1 \overrightarrow{a_0 a_1} + \cdots + b_n \overrightarrow{a_0 a_n}, \\
\overrightarrow{a_0 f(a_0 + \overrightarrow{x})} &= y_1 \overrightarrow{a_0 a_1} + \cdots + y_n \overrightarrow{a_0 a_n},
\end{aligned}$$

if $A = (a_{i\,j})$ is the $n \times n$-matrix of the linear map $\overrightarrow{f}$ over the basis $(\overrightarrow{a_0 a_1}, \ldots, \overrightarrow{a_0 a_n})$, letting x, y, and b denote the column vectors of components $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$, and $(b_1, \ldots, b_n)$,

$$\overrightarrow{a_0 f(a_0 + \overrightarrow{x})} = \overrightarrow{a_0 f(a_0)} + \overrightarrow{f}(\overrightarrow{x})$$

is equivalent to

$$y = Ax + b.$$

Note that $b \neq 0$ unless $f(a_0) = a_0$. Thus, f is generally not a linear transformation, unless it has a *fixed point*, i.e., there is a point a_0 such that $f(a_0) = a_0$. The vector b is the "translation part" of the affine map.

Affine maps do not always have a fixed point. Obviously, nonnull translations have no fixed point. A less trivial example is given by the affine map

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This map is a reflection about the x-axis followed by a translation along the x-axis.

The affine map

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -\sqrt{3} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

can also be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which shows that it is the composition of a rotation of angle $\pi/3$, followed by a stretch (by a factor of 2 along the x-axis, and by a factor of $1/2$ along the y-axis), followed by a translation. It is easy to show that this affine map has a unique fixed point.

On the other hand, the affine map

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} \frac{8}{5} & -\frac{6}{5} \\ \frac{3}{10} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

has no fixed point, even though

$$\begin{pmatrix} \frac{8}{5} & -\frac{6}{5} \\ \frac{3}{10} & \frac{2}{5} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix},$$

and the second matrix is a rotation of angle θ such that $\cos\theta = \frac{4}{5}$ and $\sin\theta = \frac{3}{5}$.

There is a useful trick to convert the equation $y = Ax + b$ into what looks like a linear equation. The trick is to consider an $(n+1) \times (n+1)$-matrix. We add 1 as the $(n+1)$th component to the vectors x, y, and b, and form the $(n+1) \times (n+1)$-matrix

$$\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}$$

so that $y = Ax + b$ is equivalent to

$$\begin{pmatrix} y \\ 1 \end{pmatrix} = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

This trick is very useful in kinematics and dynamics, where A is a rotation matrix. Such affine maps are called *rigid motions*.

If $f\colon E \to E'$ is a bijective affine map, given any three collinear points a, b, c in E, with $a \neq b$, where say, $c = (1-\lambda)a + \lambda b$, since f preserves barycenters, we have $f(c) = (1-\lambda)f(a) + \lambda f(b)$, which shows that $f(a), f(b), f(c)$ are collinear in E'.

There is a converse to this property, which is simpler to state when the ground field is $K = \mathbb{R}$.

The converse states that given any bijective function $f\colon E \to E'$ between two real affine spaces of the same dimension $n \geq 2$, if f maps any three collinear points to collinear points, then f is affine. The proof is rather long (see Berger [**?**] or Samuel [**?**]).

Given three collinear points where a, b, c, where $a \neq c$, we have $b = (1-\beta)a + \beta c$ for some unique β, and we define the *ratio of the sequence* $a, b,$ c, as

$$\mathrm{ratio}(a, b, c) = \frac{\beta}{(1-\beta)} = \frac{\overrightarrow{ab}}{\overrightarrow{bc}},$$

provided that $\beta \neq 1$, i.e. that $b \neq c$. When $b = c$, we agree that $\mathrm{ratio}(a, b, c) = \infty$.

We warn our readers that other authors define the ratio of a, b, c as $-\mathrm{ratio}(a, b, c) = \frac{\overrightarrow{ba}}{\overrightarrow{bc}}$. Since affine maps preserves barycenters, it is clear that affine maps preserve the ratio of three points.

5.6 Affine Groups

We now take a quick look at the bijective affine maps.

Given an affine space E, the set of affine bijections $f\colon E \to E$ is clearly a group, called the *affine group of* E, and denoted as $\mathrm{GA}(E)$.

Recall that the group of bijective linear maps of the vector space $\overrightarrow{E}$ is denoted as $\mathrm{GL}(\overrightarrow{E})$. Then, the map $f \mapsto \overrightarrow{f}$ defines a group homomorphism $L\colon \mathrm{GA}(E) \to \mathrm{GL}(\overrightarrow{E})$. The kernel of this map is the set of translations on E.

The subset of all linear maps of the form $\lambda\,\mathrm{id}_E$, where $\lambda \in \mathbb{R} - \{0\}$, is a subgroup of $\mathrm{GL}(\overrightarrow{E})$, and is denoted as $\mathbb{R}^*\mathrm{id}_E$.

The subgroup $\mathrm{DIL}(E) = L^{-1}(\mathbb{R}^*\mathrm{id}_E)$ of $\mathrm{GA}(E)$ is particularly interesting. It turns out that it is the disjoint union of the translations and of the dilatations of ratio $\lambda \neq 1$.

The elements of $\mathrm{DIL}(E)$ are called *affine dilatations (or dilations)*.

Given any point $a \in E$, and any scalar $\lambda \in \mathbb{R}$, a *dilatation (or central dilatation, or magnification, or homothety) of center a and ratio λ*, is a map $H_{a,\lambda}$ defined such that

$$H_{a,\lambda}(x) = a + \lambda\overrightarrow{ax},$$

for every $x \in E$.

Observe that $H_{a,\lambda}(a) = a$, and when $\lambda \neq 0$ and $x \neq a$, $H_{a,\lambda}(x)$ is on the line defined by a and x, and is obtained by "scaling" $\overrightarrow{ax}$ by λ. When $\lambda = 1$, $H_{a,1}$ is the identity.

Note that $\overrightarrow{H_{a,\lambda}} = \lambda\,\mathrm{id}_E$. When $\lambda \neq 0$, it is clear that $H_{a,\lambda}$ is an affine bijection.

It is immediately verified that

$$H_{a,\lambda} \circ H_{a,\mu} = H_{a,\lambda\mu}.$$

We have the following useful result.

Lemma 5.6.1 *Given any affine space E, for any affine bijection $f \in GA(E)$, if $\overrightarrow{f} = \lambda\,\mathrm{id}_E$, for some $\lambda \in \mathbb{R}^*$ with $\lambda \neq 1$, then there is a unique point $c \in E$ such that $f = H_{c,\lambda}$.*

Clearly, if $\overrightarrow{f} = \mathrm{id}_E$, the affine map f is a translation.

Thus, the group of affine dilatations $\mathrm{DIL}(E)$ is the disjoint union of the translations and of the dilatations of ratio $\lambda \neq 0, 1$. Affine dilatations can be given a purely geometric characterization.

5.7 Affine Hyperplanes

In section 5.3, we observed that the set L of solutions of an equation

$$ax + by = c$$

is an affine subspace of $\mathbb{A}^2$ of dimension 1, in fact a line (provided that a and b are not both null).

It would be equally easy to show that the set P of solutions of an equation

$$ax + by + cz = d$$

is an affine subspace of $\mathbb{A}^3$ of dimension 2, in fact a plane (provided that a, b, c are not all null).

More generally, the set H of solutions of an equation

$$\lambda_1 x_1 + \cdots + \lambda_m x_m = \mu$$

is an affine subspace of $\mathbb{A}^m$, and if $\lambda_1, \ldots, \lambda_m$ are not all null, it turns out that it is a subspace of dimension $m - 1$ called a *hyperplane*.

We can interpret the equation

$$\lambda_1 x_1 + \cdots + \lambda_m x_m = \mu$$

in terms of the map $f\colon \mathbb{R}^m \to \mathbb{R}$ defined such that

$$f(x_1, \ldots, x_m) = \lambda_1 x_1 + \cdots + \lambda_m x_m - \mu$$

for all $(x_1, \ldots, x_m) \in \mathbb{R}^m$.

It is immediately verified that this map is affine, and the set H of solutions of the equation

$$\lambda_1 x_1 + \cdots + \lambda_m x_m = \mu$$

is the *null set, or kernel,* of the affine map $f\colon \mathbb{A}^m \to \mathbb{R}$, in the sense that

$$H = f^{-1}(0) = \{x \in \mathbb{A}^m \mid f(x) = 0\},$$

where $x = (x_1, \ldots, x_m)$.

Thus, it is interesting to consider *affine forms*, which are just affine maps $f\colon E \to \mathbb{R}$ from an affine space to $\mathbb{R}$.

Unlike linear forms f^*, for which $\mathrm{Ker}\, f^*$ is never empty (since it always contains the vector $\overrightarrow{0}$), it is possible that $f^{-1}(0) = \emptyset$, for an affine form f.

Recall the characterization of hyperplanes in terms of linear forms.

Given a vector space E, a linear map $f\colon E \to \mathbb{R}$ is called a *linear form*. The set of all linear forms $f\colon E \to \mathbb{R}$ is a vector space called the *dual space of* E, and denoted as E^*.

Hyperplanes are precisely the Kernels of nonnull linear forms.

Lemma 5.7.1 *Let E be a vector space. The following properties hold:*

(a) Given any nonnull linear form $f \in E^*$, its kernel $H = \operatorname{Ker} f$ is a hyperplane.

(b) For any hyperplane H in E, there is a (nonnull) linear form $f \in E^*$ such that $H = \operatorname{Ker} f$.

(c) Given any hyperplane H in E and any (nonnull) linear form $f \in E^*$ such that $H = \operatorname{Ker} f$, for every linear form $g \in E^*$, $H = \operatorname{Ker} g$ iff $g = \lambda f$ for some $\lambda \neq 0$ in $\mathbb{R}$.

Going back to an affine space E, given an affine map $f\colon E \to \mathbb{R}$, we also denote $f^{-1}(0)$ as $\operatorname{Ker} f$, and we call it the *kernel* of f. Recall that an (affine) hyperplane is an affine subspace of codimension 1.

Affine hyperplanes are precisely the Kernels of nonconstant affine forms.

Lemma 5.7.2 *Let E be an affine space. The following properties hold:*

(a) Given any nonconstant affine form $f: E \to \mathbb{R}$, its kernel $H = \operatorname{Ker} f$ is a hyperplane.

(b) For any hyperplane H in E, there is a nonconstant affine form $f: E \to \mathbb{R}$ such that $H = \operatorname{Ker} f$. For any other affine form $g: E \to \mathbb{R}$ such that $H = \operatorname{Ker} g$, there is some $\lambda \in \mathbb{R}$ such that $g = \lambda f$ (with $\lambda \neq 0$).

(c) Given any hyperplane H in E and any (nonconstant) affine form $f: E \to \mathbb{R}$ such that $H = \operatorname{Ker} f$, every hyperplane H' parallel to H is defined by a nonconstant affine form g such that $g(a) = f(a) - \lambda$, for all $a \in E$, for some $\lambda \in \mathbb{R}$.

Chapter 6

Multiaffine Maps and Polar Forms

6.1 Multiaffine Maps

Let $E_1, \ldots, E_m$, and F, be vector spaces over $\mathbb{R}$, where $m \geq 1$.

Definition 6.1.1 A function $f \colon E_1 \times \ldots \times E_m \to F$ is a *multilinear map (or an m-linear map)*, iff it is linear in each argument, holding the others fixed.

Having reviewed the definition of a multilinear map, we define multiaffine maps. Let $E_1, \ldots, E_m$, and F, be affine spaces over $\mathbb{R}$, where $m \geq 1$ (you may assume that the E_i's and F are of the form $\mathbb{A}^n$, for some $n \geq 0$. Also, we use the notation $\overrightarrow{E}$ for the vector space $\mathbb{R}^n$ associated with the affine space $E = \mathbb{A}^n$.)

Definition 6.1.2 A function $f\colon E_1 \times \ldots \times E_m \to F$ is a *multiaffine map (or an m-affine map)*, iff it is affine in each argument, that is, for every i, $1 \leq i \leq m$, for all $a_1 \in E_1, \ldots, a_{i-1} \in E_{i-1}, a_{i+1} \in E_{i+1}, \ldots, a_m \in E_m, a \in E_i$, the map

$$a \mapsto f(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_m)$$

is an affine map, i.e. iff it preserves barycentric combinations.

An arbitrary function $f\colon E^m \to F$ is symmetric (where E and F are arbitrary sets, not just vector spaces or affine spaces), iff

$$f(x_{\pi(1)}, \ldots, x_{\pi(m)}) = f(x_1, \ldots, x_m),$$

for every permutation $\pi\colon \{1, \ldots, m\} \to \{1, \ldots, m\}$.

It is immediately verified that a multilinear map is also a multiaffine map (viewing a vector space as an affine space).

A good example of n-affine forms is the elementary symmetric functions. Given n variables $x_1, \ldots, x_n$, for each k, $0 \leq k \leq n$, we define the k-th *elementary symmetric function* $\sigma_k(x_1, \ldots, x_n)$, for short, σ_k, as follows:

$$\sigma_0 = 1;$$

$$\sigma_1 = x_1 + \cdots + x_n;$$

$$\sigma_2 = x_1x_2 + x_1x_3 + \cdots + x_1x_n + x_2x_3 + \cdots + x_{n-1}x_n;$$

$$\sigma_k = \sum_{1 \leq i_1 < \ldots < i_k \leq n} x_{i_1} \cdots x_{i_k};$$

$$\sigma_n = x_1x_2 \cdots x_n.$$

A concise way to express σ_k is as follows:

$$\sigma_k = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I|=k}} \left(\prod_{i \in I} x_i \right).$$

Note that σ_k consists of a sum of $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ terms of the form $x_{i_1} \cdots x_{i_k}$. As a consequence,

$$\sigma_k(x, x, \ldots, x) = \binom{n}{k} x^k.$$

Clearly, each σ_k is symmetric.

Multiaffine maps can be characterized in terms of multilinear maps. This is a generalization of lemma 5.5.2,

The proof is more complicated than we originally expected. It uses an adaptation of Cartan's use of "successive differences."

We will not present this result here, and instead, give a special version later that will be enough for our purposes (see Lemma 6.2.3).

6.2 Affine Polynomials, and Polar Forms

The beauty and usefulness of symmetric affine maps lies is the fact that these maps can be used to define the notion of a polynomial function from an affine (or vector) space E of any dimension to an affine (or vector) space F of any dimension.

Definition 6.2.1 Given two affine spaces E and F, an *affine polynomial function of polar degree m, for short, an affine polynomial of polar degree m*, is a map $h: E \to F$, such that there is some symmetric m-affine map $f: E^m \to F$, called the *m-polar form of h*, with

$$h(a) = f(\underbrace{a, \ldots, a}_{m}),$$

for all $a \in E$.

Remark: Note that Definition 6.2.1 only asks for the *existence* of a symmetric multiaffine map f. Thus, it is a priori possible that there exists *distinct* symmetric multiaffine maps f and g so that

$$f(a, \cdots, a) = g(a, \cdots, a)$$

for all $a \in E$. In fact, this is not so! We will prove *uniqueness* of the symmetric multiaffine map f defining an affine polynomial map h.

A *homogeneous polynomial function of degree* m, is a map $h\colon \overrightarrow{E} \to \overrightarrow{F}$, such that there is some nonnull symmetric m-linear map $f\colon \overrightarrow{E}^m \to \overrightarrow{F}$, called the *polar form of* h, with

$$h(\overrightarrow{v}) = f(\underbrace{\overrightarrow{v}, \ldots, \overrightarrow{v}}_{m}),$$

for all $\overrightarrow{v} \in \overrightarrow{E}$.

A *polynomial function of polar degree* m, is a map $h\colon \overrightarrow{E} \to \overrightarrow{F}$, such that there are m symmetric k-linear map $f_k\colon \overrightarrow{E}^k \to \overrightarrow{F}$, $1 \leq k \leq m$, and some $f_0 \in \overrightarrow{F}$, with

$$h(\overrightarrow{v}) = f_m(\underbrace{\overrightarrow{v}, \ldots, \overrightarrow{v}}_{m}) + f_{m-1}(\underbrace{\overrightarrow{v}, \ldots, \overrightarrow{v}}_{m-1}) + \cdots + f_1(\overrightarrow{v}) + f_0,$$

for all $\overrightarrow{v} \in \overrightarrow{E}$.

Instead of defining polynomial maps of degree exactly m (as Cartan does), we define polynomial maps of degree at most m. For example, if $\overrightarrow{E} = \mathbb{R}^n$ and $\overrightarrow{F} = \mathbb{R}$, we have the bilinear map $f\colon (\mathbb{R}^n)^2 \to \mathbb{R}$ (inner product), defined such that

$$f((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

The corresponding polynomial $h\colon \mathbb{R}^n \to \mathbb{R}$, such that

$$h(x_1, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2,$$

is a polynomial of total degree 2 in n variables.

However the triaffine map $f\colon \mathbb{R}^3 \to \mathbb{R}$ defined such that

$$f(x, y, z) = xy + yz + xz,$$

induces the polynomial $h\colon \mathbb{R} \to \mathbb{R}$ such that

$$h(x) = 3x^2,$$

which is of polar degree 3, but a polynomial of degree 2 in x.

Let us see what are the homogeneous polynomials of degree m, when $\overrightarrow{E}$ is a vector space of finite dimension n, and $\overrightarrow{F}$ is a vector space (readers who are nervous, may assume for simplicity that $\overrightarrow{F} = \mathbb{R}$).

Let $(\overrightarrow{e_1}, \ldots, \overrightarrow{e_n})$ be a basis of $\overrightarrow{E}$.

Lemma 6.2.2 *Given any vector space $\overrightarrow{E}$ of finite dimension n, and any vector space $\overrightarrow{F}$, for any basis $(\overrightarrow{e_1}, \ldots, \overrightarrow{e_n})$ of $\overrightarrow{E}$, for any symmetric multilinear map $f\colon \overrightarrow{E}^m \to \overrightarrow{F}$, for any m vectors*

$$\overrightarrow{v_j} = v_{1,\,j}\,\overrightarrow{e_1} + \cdots + v_{n,\,j}\,\overrightarrow{e_n} \in \overrightarrow{E},$$

we have

$$f(\overrightarrow{v_1}, \ldots, \overrightarrow{v_m}) = \sum_{\substack{I_1 \cup \ldots \cup I_n = \{1,\ldots,m\} \\ I_i \cap I_j = \emptyset,\, i \neq j \\ 1 \leq i,j \leq n}} \left(\prod_{i_1 \in I_1} v_{1,\,i_1} \right) \cdots \left(\prod_{i_n \in I_n} v_{n,\,i_n} \right) f(\underbrace{\overrightarrow{e_1}, \ldots, \overrightarrow{e_1}}_{|I_1|}, \ldots, \underbrace{\overrightarrow{e_n}, \ldots, \overrightarrow{e_n}}_{|I_n|}),$$

and for any $\overrightarrow{v} \in \overrightarrow{E}$, *the homogeneous polynomial function* h *associated with* f *is given by*

$$h(\overrightarrow{v}) = \sum_{\substack{k_1+\cdots+k_n=m\\ 0\leq k_i,\, 1\leq i\leq n}} \binom{m}{k_1,\ldots,k_n} v_1^{k_1}\cdots v_n^{k_n}$$

$$f(\underbrace{\overrightarrow{e_1},\ldots,\overrightarrow{e_1}}_{k_1},\ldots,\underbrace{\overrightarrow{e_n},\ldots,\overrightarrow{e_n}}_{k_n}).$$

Thus, lemma 6.2.2 shows that we can write $h(\overrightarrow{v})$ as

$$h(\overrightarrow{v}) = \sum_{\substack{k_1+\cdots+k_n=m \\ 0\leq k_i,\, 1\leq i\leq n}} v_1^{k_1} \cdots v_n^{k_n}\, c_{k_1,\ldots,k_n},$$

for some "coefficients" $c_{k_1,\ldots,k_n} \in \overrightarrow{F}$, which are vectors. When $\overrightarrow{F} = \mathbb{R}$, the homogeneous polynomial function h of degree m in n arguments $v_1, \ldots, v_n$ agrees with the notion of polynomial function defined by a homogeneous polynomial. Indeed, h is the homogeneous polynomial function induced by the homogeneous polynomial of degree m in the variables $X_1, \ldots, X_n$,

$$\sum_{\substack{(k_1,\ldots,k_n),\, k_j\geq 0 \\ k_1+\cdots+k_n=m}} c_{k_1,\ldots,k_n} X_1^{k_1} \cdots X_n^{k_n}.$$

Thus, when $\overrightarrow{E} = \mathbb{R}^n$ and $\overrightarrow{F} = \mathbb{R}$, the notion of (affine) polynomial of polar degree m in n arguments, agrees with the notion of polynomial function induced by a polynomial of degree $\leq m$ in n variables $(X_1, \ldots, X_n)$.

Lemma 6.2.3 *Given any affine space E of finite dimension n, and any affine space F, for any basis $(\overrightarrow{e_1}, \ldots, \overrightarrow{e_n})$ of $\overrightarrow{E}$, for any symmetric multiaffine map $f\colon E^m \to F$, for any m vectors*

$$\overrightarrow{v_j} = v_{1,\,j}\,\overrightarrow{e_1} + \cdots + v_{n,\,j}\,\overrightarrow{e_n} \in \overrightarrow{E},$$

for any points $a_1, \ldots, a_m \in E$, we have

$$f(a_1 + \overrightarrow{v_1}, \ldots, a_m + \overrightarrow{v_m}) = b + \sum_{1\leq p\leq m} \sum_{\substack{I_1\cup\ldots\cup I_n=\{1,\ldots,p\}\\ I_i\cap I_j=\emptyset,\, i\neq j\\ 1\leq i,j\leq n}} \left(\prod_{i_1\in I_1} v_{1,\,i_1}\right)\cdots\left(\prod_{i_n\in I_n} v_{n,\,i_n}\right)\, \overrightarrow{w}_{|I_1|,\ldots,|I_n|},$$

for some $b \in F$, and some $\overrightarrow{w}_{|I_1|,\ldots,|I_n|} \in \overrightarrow{F}$, and for any $a \in E$, and $\overrightarrow{v} \in \overrightarrow{E}$, the affine polynomial function h associated with f is given by

$$h(a + \overrightarrow{v}) = b + \sum_{1\leq p\leq m} \sum_{\substack{k_1+\cdots+k_n=p\\ 0\leq k_i,\, 1\leq i\leq n}} v_1^{k_1}\cdots v_n^{k_n}\,\overrightarrow{w}_{k_1,\ldots,k_n},$$

for some $b \in F$, and some $\overrightarrow{w}_{k_1,\ldots,k_n} \in \overrightarrow{F}$.

Lemma 6.2.3 shows the crucial role played by homogeneous polynomials. We could have taken the form of an affine map given by this lemma as a definition, when E is of finite dimension.

Recall that the canonical affine space associated with the field $\mathbb{R}$ is denoted as $\mathbb{A}$.

Definition 6.2.4 A (parameterized) *polynomial curve in polar form of degree* m is an affine polynomial map $F\colon \mathbb{A} \to \mathcal{E}$ of polar degree m, defined by its *polar form* which is some symmetric m-affine map $f\colon \mathbb{A}^m \to \mathcal{E}$, where $\mathbb{A}$ is the real affine line, and $\mathcal{E}$ is any affine space (of dimension at least 2).

Given any $r, s \in \mathbb{A}$, with $r < s$, a (parameterized) *polynomial curve segment* $F([r, s])$ *in polar form of degree* m is the restriction $F\colon [r, s] \to \mathcal{E}$ of an affine polynomial curve $F\colon \mathbb{A} \to \mathcal{E}$ in polar form of degree m. We define the *trace of* F as $F(\mathbb{A})$, and the the *trace of* $F[r, s]$ as $F([r, s])$.

Lemma 6.2.5 *Given any sequence of $m+1$ points $a_0, \ldots, a_m$ in some affine space $\mathcal{E}$, there is a unique polynomial curve $F\colon \mathbb{A} \to \mathcal{E}$ of degree m, whose polar form $f\colon \mathbb{A}^m \to \mathcal{E}$ satisfies the conditions*

$$f(\underbrace{r, \ldots, r}_{m-k}, \underbrace{s, \ldots, s}_{k}) = a_k,$$

(where $r, s \in \mathbb{A}$, $r \neq s$). Furthermore, the polar form f of F is given by the formula

$$f(t_1, \ldots, t_m) = \sum_{k=0}^{k=m} \sum_{\substack{I \cup J = \{1,\ldots,m\} \\ I \cap J = \emptyset, |J| = k}} \prod_{i \in I} \left(\frac{s - t_i}{s - r}\right) \prod_{j \in J} \left(\frac{t_j - r}{s - r}\right) a_k,$$

and $F(t)$ is given by the formula

$$F(t) = \sum_{k=0}^{k=m} B_k^m[r, s](t)\, a_k,$$

where the polynomials

$$B_k^m[r, s](t) = \binom{m}{k} \left(\frac{s - t}{s - r}\right)^{m-k} \left(\frac{t - r}{s - r}\right)^{k}$$

are the Bernstein polynomials of degree m over $[r, s]$.

Remarkably, we can prove in full generality, that the polar form f defining an affine polynomial h of degree m is unique. All the ingredients to prove this result are in Bourbaki [**?**] (chapter A.I, section §8.2, proposition 2), and [**?**] (chapter A.IV, section §5.4, proposition 3), but they are deeply buried!

Before plunging into the proof of lemma 6.2.6, you may want to verify that for a polynomial $h(X)$ of degree 2, the polar form is given by the identity

$$f(x_1, x_2) = \frac{1}{2}\left[4h\left(\frac{x_1 + x_2}{2}\right) - h(x_1) - h(x_2)\right].$$

You may also want to try working out on your own, a formula giving the polar form for a polynomial $h(X)$ of degree 3. Note that when $h(X)$ is a homogeneous polynomial of degree 2, the above identity reduces to the (perhaps more familiar) identity

$$f(x_1, x_2) = \frac{1}{2}\left[h(x_1 + x_2) - h(x_1) - h(x_2)\right],$$

used for passing from a quadratic form to a bilinear form.

Lemma 6.2.6 *Given two affine spaces E and F, for any polynomial function h of degree m, the polar form $f\colon E^m \to F$ of h is unique, and is given by the following expression:*

$$f(a_1, \ldots, a_m) = \frac{1}{m!} \left[\sum_{\substack{H \subseteq \{1,\ldots,m\} \\ k=|H|,\, k \geq 1}} (-1)^{m-k}\, k^m\, h\left(\frac{\sum_{i \in H} a_i}{k} \right) \right].$$

It should be noted that lemma 6.2.6 is very general, since it applies to arbitrary affine spaces, even of infinite dimension (for example, Hilbert spaces). The expression of lemma 6.2.6 is far from being economical, since it contains $2^m - 1$ terms. In particular cases, it is often possible to reduce the number of terms.

We now use lemma 6.2.6 to show that polynomials in one or several variables are uniquely defined by polar forms which are multiaffine maps.

Lemma 6.2.7 *The following properties hold.*

(1) For every polynomial $p(X) \in \mathbb{R}[X]$, of degree $\leq m$, there is a symmetric m-affine form $f \colon \mathbb{R}^m \to \mathbb{R}$, such that

$$p(x) = f(x, x, \ldots, x)$$

for all $x \in \mathbb{R}$. If $p(X) \in \mathbb{R}[X]$ is a homogeneous polynomial of degree exactly m, then the symmetric m-affine form f is multilinear.

(2) For every polynomial $p(X_1, \ldots, X_n) \in \mathbb{R}[X_1, \ldots, X_n]$, of total degree $\leq m$, there is a symmetric m-affine form $f \colon (\mathbb{R}^n)^m \to \mathbb{R}$, such that

$$p(x_1, \ldots, x_n) = f(x, x, \ldots, x),$$

for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. If $p(X_1, \ldots, X_n) \in \mathbb{R}[X_1, \ldots, X_n]$ is a homogeneous polynomial of total degree exactly m, then the symmetric m-affine form f is multilinear.

Proof. (1) It is enough to prove it for a monomial of the form X^k, $k \leq m$. Clearly,

$$f(x_1, \ldots, x_m) = \frac{k!(m-k)!}{m!}\sigma_k$$

is a symmetric m-affine form satisfying the lemma (where σ_k is the k-th elementary symmetric function, which consists of $\binom{m}{k} = \frac{m!}{k!(m-k)!}$ terms), and when $k = m$, we get a multilinear map.

(2) It is enough to prove it for a homogeneous monomial of the form $X_1^{k_1} \cdots X_n^{k_n}$, where $k_i \geq 0$, and $k_1 + \cdots + k_n = d \leq m$. Let

$$f((x_{1,1}, \ldots, x_{n,1}), \ldots, (x_{1,m}, \ldots, x_{n,m})) =$$
$$\frac{k_1! \cdots k_n!(m-d)!}{m!} \sum_{\substack{I_1 \cup \ldots \cup I_n \subseteq \{1,\ldots,m\} \\ I_i \cap I_j = \emptyset,\, i \neq j,\, |I_j| = k_j}} \left(\prod_{i_1 \in I_1} x_{1,i_1}\right) \cdots \left(\prod_{i_n \in I_n} x_{n,i_n}\right).$$

The idea is to split any subset of $\{1, \ldots, m\}$ consisting of $d \leq m$ elements into n disjoint subsets $I_1, \ldots, I_n$, where I_j is of size k_j (and with $k_1 + \cdots + k_n = d$).

As an example, if

$$p(X) = X^3 + 3X^2 + 5X - 1,$$

we get

$$f(x_1, x_2, x_3) = \\ x_1x_2x_3 + x_1x_2 + x_1x_3 + x_2x_3 + \frac{5}{3}(x_1 + x_2 + x_3) - 1.$$

When $n = 2$, which corresponds to the case of surfaces, we can give an expression which is easier to understand. Writing $U = X_1$ and $V = X_2$, to minimize the number of subscripts, given the monomial U^hV^k, with $h + k = d \leq m$, we get

$$f((u_1, v_1), \ldots, (u_m, v_m)) = \\ \frac{h!k!(m - (h + k))!}{m!} \sum_{\substack{I \cup J \subseteq \{1,\ldots,m\} \\ I \cap J = \emptyset \\ |I|=h,\, |J|=k}} \left(\prod_{i \in I} u_i\right)\left(\prod_{j \in J} v_j\right).$$

For a concrete example involving two variables, if

$$p(U, V) = UV + U^2 + V^2,$$

we get

$$f((u_1, v_1),\ (u_2, v_2)) = \frac{u_1 v_2 + u_2 v_1}{2} + u_1 u_2 + v_1 v_2.$$

Theorem 6.2.8 *There is an equivalence between polynomials $p(X_1, \ldots, X_n) \in \mathbb{R}[X_1, \ldots, X_n]$, of total degree $\leq m$, and symmetric m-affine maps $f\colon (\mathbb{R}^n)^m \to \mathbb{R}$, in the following sense:*

(1) If $f\colon (\mathbb{R}^n)^m \to \mathbb{R}$ is a symmetric m-affine map, then the function $p\colon \mathbb{R}^n \to \mathbb{R}$ defined such that

$$p(x_1, \ldots, x_n) = f(x, x, \ldots, x)$$

for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, is a polynomial (function) corresponding to a unique polynomial $p(X_1, \ldots, X_n) \in \mathbb{R}[X_1, \ldots, X_n]$ of total degree $\leq m$.

(2) For every polynomial $p(X_1, \ldots, X_n) \in \mathbb{R}[X_1, \ldots, X_n]$, *of total degree* $\leq m$, *there is a unique symmetric* m*-affine map* $f\colon (\mathbb{R}^n)^m \to \mathbb{R}$, *such that*

$$p(x_1, \ldots, x_n) = f(x, x, \ldots, x)$$

for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Furthermore, when $p(X_1, \ldots, X_n) \in \mathbb{R}[X_1, \ldots, X_n]$ *is a homogeneous polynomial of total degree exactly* m, *the symmetric* m*-affine map* f *is multilinear, and conversely.*

We conclude this section by proving that the Bernstein polynomials $B_0^m(t), \ldots, B_m^m(t)$ also form a basis of the polynomials of degree $\leq m$. For this, we express each t^i, $0 \leq i \leq m$, in terms of the Bernstein polynomials $B_j^m(t)$ (over $[0, 1]$).

$$t^i = \sum_{j=0}^{j=m-i} \frac{\binom{i+j}{i}}{\binom{m}{i}} B_{i+j}^m(t).$$

Chapter 7

Polynomial Curves as Bézier Curves

7.1 The de Casteljau Algorithm For Polynomial Curves

An affine polynomial curve $F\colon \mathbb{A} \to \mathcal{E}$ of degree m, defined by its m-polar form $f\colon \mathbb{A}^m \to \mathcal{E}$, is completely determined by the sequence of $m+1$ points $b_k = f(r^{m-k}\, s^k)$, where $r, s \in \mathbb{A}$, $r \neq s$, $0 \leq k \leq m$, and we showed that

$$f(t_1, \ldots, t_m) = \sum_{k=0}^{k=m} p_k(t_1, \ldots, t_m)\, f(r^{m-k}\, s^k),$$

where the coefficient

$$p_k(t_1, \ldots, t_m) = \sum_{\substack{I \cup J = \{1,\ldots,m\} \\ I \cap J = \emptyset,\, \mathrm{card}(J) = k}} \prod_{i \in I} \left(\frac{s - t_i}{s - r}\right) \prod_{j \in J} \left(\frac{t_j - r}{s - r}\right)$$

of $f(r^{m-k}\, s^k)$, is a symmetric m-affine function.

The de Casteljau algorithm gives a geometric iterative method for determining any point $F(t) = f(t, \ldots, t)$ on the curve F specified by the sequence of control points $b_0, b_1, \ldots, b_m$, where $t \in \mathbb{A}$.

What's good about the algorithm is that it does not assume any prior knowledge of the curve. All that is given is the sequence $b_0, b_1, \ldots, b_m$ of $m+1$ control points, and the idea is that we are trying to approximate the shape of the polygonal line consisting of the m line segments $(b_0,\, b_1)$, $(b_1,\, b_2)$, …, $(b_{m-1},\, b_m)$.

Let us review the case of polynomial cubic curves.

As we observed, the computation of the point $F(t)$ on a polynomial cubic curve F can be arranged in a triangular array, as shown below:

$$
\begin{array}{cccc}
 & 1 & 2 & 3 \\
f(r,r,r) & & & \\
 & f(r,r,t) & & \\
f(r,r,s) & & f(t,t,r) & \\
 & f(r,t,s) & & f(t,t,t) \\
f(r,s,s) & & f(t,t,s) & \\
 & f(t,s,s) & & \\
f(s,s,s) & & &
\end{array}
$$

The following diagram shows an example of the de Casteljau algorithm for computing the point $F(t)$ on a cubic, where r, s, and t, are arbitrary:

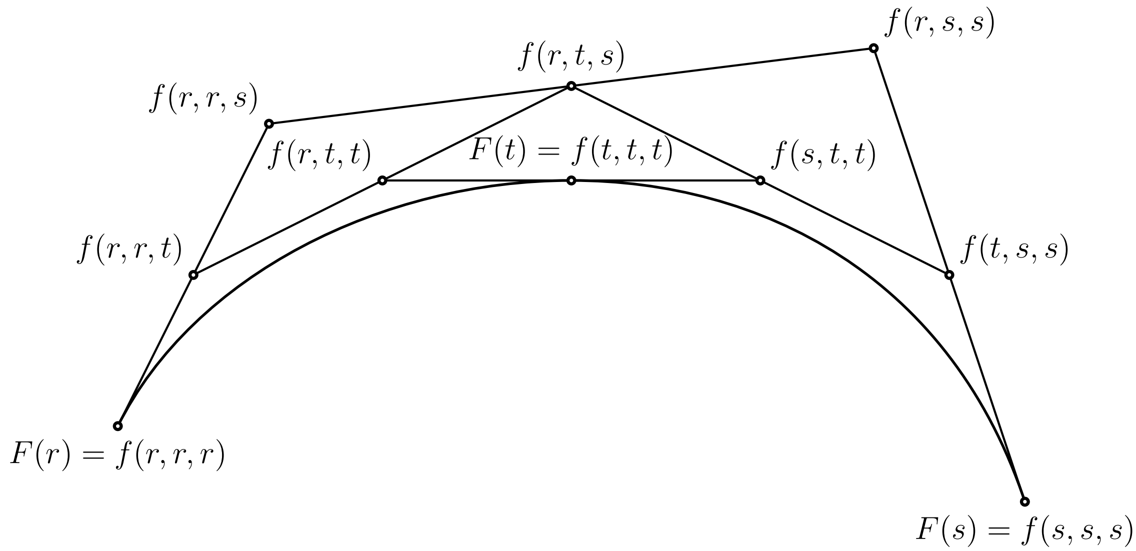


Figure 7.1: The de Casteljau algorithm

The general case for computing the point $F(t)$ on the curve F determined by the sequence of control points $b_0, \ldots, b_m$, where $b_k = f(r^{m-k}\, s^k)$, is shown below. We will abbreviate

$$f(\underbrace{t, \ldots, t}_{i}, \underbrace{r, \ldots, r}_{j}, \underbrace{s, \ldots, s}_{k}),$$

as $f(t^i r^j s^k)$, where $i + j + k = m$.

The point $f(t^j r^{m-i-j} s^i)$ is obtained at step i of phase j, for $1 \leq j \leq m$, $0 \leq i \leq m-j$, by the interpolation step

$$f(t^j r^{m-i-j} s^i) = \left(\frac{s-t}{s-r}\right) f(t^{j-1} r^{m-i-j+1} s^i) + \left(\frac{t-r}{s-r}\right) f(t^{j-1} r^{m-i-j} s^{i+1}).$$

In order to make the triangular array a bit more readable, let us define the following points $b_{i,j}$, used during the computation:

$$b_{i,j} = \begin{cases} b_i & \text{if } j = 0,\ 0 \leq i \leq m, \\ f(t^j r^{m-i-j} s^i) & \text{if } 1 \leq j \leq m,\ 0 \leq i \leq m-j. \end{cases}$$

Then, we have the following equations:

$$b_{i,j} = \left(\frac{s-t}{s-r}\right) b_{i,j-1} + \left(\frac{t-r}{s-r}\right) b_{i+1,j-1}.$$

By lemma 6.2.5, we have an explicit formula giving any point $F(t)$ associated with a parameter $t \in \mathbb{A}$, on the unique polynomial curve $F\colon \mathbb{A} \to \mathcal{E}$ of degree m determined by the sequence of control points $b_0, \ldots, b_m$. The point $F(t)$ is given by the formula

$$F(t) = \sum_{k=0}^{k=m} B_k^m[r, s](t)\, b_k,$$

where the polynomials

$$B_k^m[r, s](t) = \binom{m}{k} \left(\frac{s-t}{s-r}\right)^{m-k} \left(\frac{t-r}{s-r}\right)^{k}$$

are the *Bernstein polynomials of degree m over* $[r, s]$.

Thus, the de Casteljau algorithm provides an iterative method for computing $F(t)$, without actually using the Bernstein polynomials. This can be advantageous for numerical stability.

The de Casteljau algorithm is very easy to implement, and we give below several versions in *Mathematica*.

The function **badecas** simply computes the point $F(t)$ on a polynomial curve F specified by a control polygon **cpoly** (over $[r, s]$). The result is the point $F(t)$.

The function **decas** computes the point $F(t)$ on a polynomial curve F specified by a control polygon **cpoly** (over $[r, s]$), but also the shells of the de Casteljau diagram. The output is a list consisting of two sublists, the first one being the shells of the de Casteljau diagram, and the second one being $F(t)$ itself.

```
(* Performs general affine interpolation
   between two points p1, p2  *)
(* w.r.t. affine basis [r, s], and
   interpolating value t   *)

 lerp[p_List,q_List,r_,s_,t_] :=
 (s - t)/(s - r)  p + (t - r)/(s - r) q;
```

```
(*  computes a point F(t) on a curve using
    the de Casteljau algorithm  *)
(*  this is the simplest version of de Casteljau  *)
(*  the auxiliary points involved in the algorithm
    are not computed  *)

 badecas[{cpoly__}, r_, s_, t_] :=
 Block[
 {bb = {cpoly}, b = {}, m, i, j},
 (m = Length[bb] - 1;
 Do[
    Do[
       b = Append[b, lerp[bb[[i]], bb[[i+1]], r, s, t]],
       {i, 1, m - j + 1}
      ]; bb = b; b = {}, {j, 1, m}
   ];
 bb[[1]]
 )
 ];
```

```
(*  computes the point F(t) and the line segments involved in
    computing F(t) using the de Casteljau algorithm  *)

 decas[{cpoly__}, r_, s_, t_] :=
 Block[
 {bb = {cpoly}, b = {},
 m, i, j, lseg = {}, res},
 (m = Length[bb] - 1;
 Do[
    Do[
       b = Append[b, lerp[bb[[i]], bb[[i+1]], r, s, t]];
       If[i > 1, lseg = Append[lseg, {b[[i - 1]], b[[i]]}]]
       , {i, 1, m - j + 1}
      ]; bb = b; b = {}, {j, 1, m}
   ];
 res := Append[lseg, bb[[1]]];
 res
 )
 ];
```

The following function `pdecas` creates a list consisting of *Mathematica* line segments and of the point of the curve, ready for display.

```
(* this function calls decas, and computes the line segments
    in Mathematica, with colors   *)

pdecas[{cpoly__}, r_, s_, t_] :=
 Block[
 {bb = {cpoly}, pt, ll, res, i, l1, edge, lt, rt},
 res = decas[bb, r, s, t];
 pt = Last[res]; res = Drop[res, -1];
 l1 = Length[res];
 ll = {};
 Do[
    edge = res[[i]]; lt = edge[[1]];
    rt = edge[[2]]; edge = {lt, rt};
    ll = Append[ll, Line[edge]], {i, 1, l1}
   ];
 res = Append[ll, {RGBColor[1,0,0], PointSize[0.01], Point[pt]}];
 res
 ];
```

7.2 Subdivision Algorithms for Polynomial Curves

We now consider the subdivision method. As we will see, subdivision can be used to approximate a curve segment using a polygon, and the convergence is very fast. Given a sequence of control points $b_0, \ldots, b_m$, and an interval $[r, s]$, for every $t \in \mathbb{A}$, we saw how the de Casteljau algorithm gives a way of computing the point $b_{0,m}$ on the Bézier curve, and the computation can be conveniently represented in triangular form:

Let us now assume that $r < t < s$. Observe that the two diagonals

$$b_{0,0},\ b_{0,1}, \ldots, b_{0,j}, \ldots, b_{0,m},$$

and

$$b_{0,m},\ b_{1,m-1}, \ldots, b_{m-j,j}, \ldots, b_{m,0},$$

consist of $m + 1$ points.

We claim that these two sequences of control points specify the original curve.

Indeed, if f is the polar form associated with the Bézier curve specified by the sequence of control points b_0, ..., b_m over $[r, s]$, g is the polar form associated with the Bézier curve specified by the sequence of control points $b_{0,0}$, ..., $b_{0,j}$, ..., $b_{0,m}$ over $[r, t]$, and h is the polar form associated with the Bézier curve specified by the sequence of control points $b_{0,m}$, ..., $b_{m-j,j}$, ..., $b_{m,0}$ over $[t, s]$, since f and g agree on the sequence of $m + 1$ points

$$b_{0,0}, \ldots, b_{0,j}, \ldots, b_{0,m},$$

and f and h agree on the sequence of $m + 1$ points

$$b_{0,m}, \ldots, b_{m-j,j}, \ldots, b_{m,0},$$

by lemma 6.2.5, we have $f = g = h$.

The subdivision method can easily be implemented. Given a polynomial curve F defined by a control polygon $\mathcal{B} = (b_0, \ldots, b_m)$ over an affine frame $[r, s]$, for every $t \in \mathbb{A}$, we denote as $\mathcal{B}_{[r,t]}$ the control polygon

$$b_{0,0},\ b_{0,1}, \ldots, b_{0,j}, \ldots, b_{0,m},$$

and as $\mathcal{B}_{[t,s]}$ the control polygon

$$b_{0,m},\ b_{1,m-1}, \ldots, b_{m-j,j}, \ldots, b_{m,0}.$$

The following *Mathematica* function returns a pair consisting of $\mathcal{B}_{[r,t]}$ and $\mathcal{B}_{[t,s]}$, from an input control polygon `cpoly`.

```
(*  Performs a single subdivision step
    using the de Casteljau algorithm  *)
(*  Returns the control poly (f(r,...,r, t, ..., t)) and  *)
(*  (f(t, ..., t, s, ..., s))                             *)


 subdecas[{cpoly__}, r_, s_, t_] :=
 Block[
 {bb = {cpoly}, b = {}, ud = {}, ld = {},
 m, i, j, res},
 (m = Length[bb] - 1; ud = {bb[[1]]}; ld = {bb[[m + 1]]};
 Do[
    Do[
       b = Append[b, lerp[bb[[i]], bb[[i+1]], r, s, t]],
        {i, 1, m - j + 1}
      ];
      ud = Append[ud, b[[1]]];
      ld = Prepend[ld, b[[m - j + 1]]];
      bb = b; b = {}, {j, 1, m}
   ];
 res := Join[{ud},{ld}];
 res
 )
 ];
```

In order to approximate the curve segment over $[r, s]$, we recursively apply subdivision to a list consisting originally of a single control polygon. The function **subdivstep** subdivides each control polygon in a list of control polygons. The function **subdiv** performs **n** calls to **subdivstep**. Finally, in order to display the resulting curve, the function **makedgelis** makes a list of *Mathematica* line segments from a list of control polygons.

```
(* subdivides each control polygon in a list
  of control polygons  *)
(* using  subdecas. Uses t = (r + s)/2   *)

 subdivstep[{poly__}, r_, s_] :=
 Block[
 {cpoly = {poly}, lpoly = {},  t, l, i},
 (l = Length[cpoly]; t = (r + s)/2;
 Do[
    lpoly = Join[lpoly, subdecas[cpoly[[i]], r, s, t]] ,
    {i, 1, l}
   ];
 lpoly
 )
 ];
```

```
(*  calls subdivstep n times    *)

 subdiv[{poly__}, r_, s_, n_] :=
 Block[
 {pol1 = {poly}, newp = {}, i},
 (
  newp = {pol1};
  Do[
    newp = subdivstep[newp, r, s], {i, 1, n}
   ];
  newp
 )
 ];


(* To create a list of line segments from a list
   of control polygons *)

 makedgelis[{poly__}] :=
 Block[
 {res, sl,  newsl = {poly},
  i, j, l1, l2},
  (l1 = Length[newsl];  res = {};
  Do[
    sl = newsl[[i]]; l2 = Length[sl];
    Do[If[j > 1, res = Append[res, Line[{sl[[j-1]], sl[[j]]}]]],
       {j, 1, l2}
      ], {i, 1, l1}
   ];
  res
 )
 ];
```

The subdivision method is illustrated by the following example of a curve of degree 4 given by the control polygon

$$cpoly = ((0,\ -4),\ (10,\ 30),\ (5,\ -20),\ (0,\ 30),\ (10,\ -4)).$$

The following 6 pictures show polygonal approximations of the curve segment over $[0, 1]$ using `subdiv`, for $n = 1, 2, 3, 4, 5, 6$.

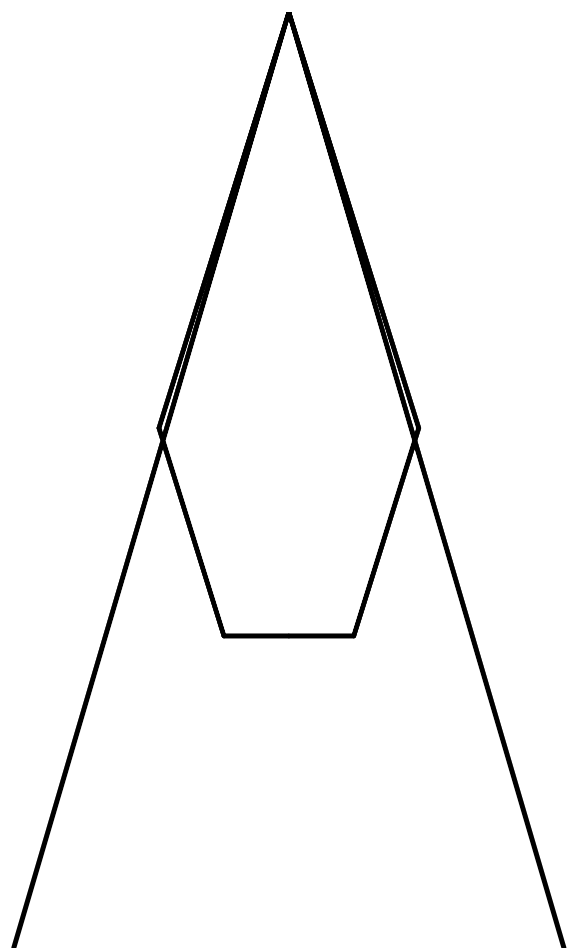


Figure 7.2: Subdivision, 1 iteration

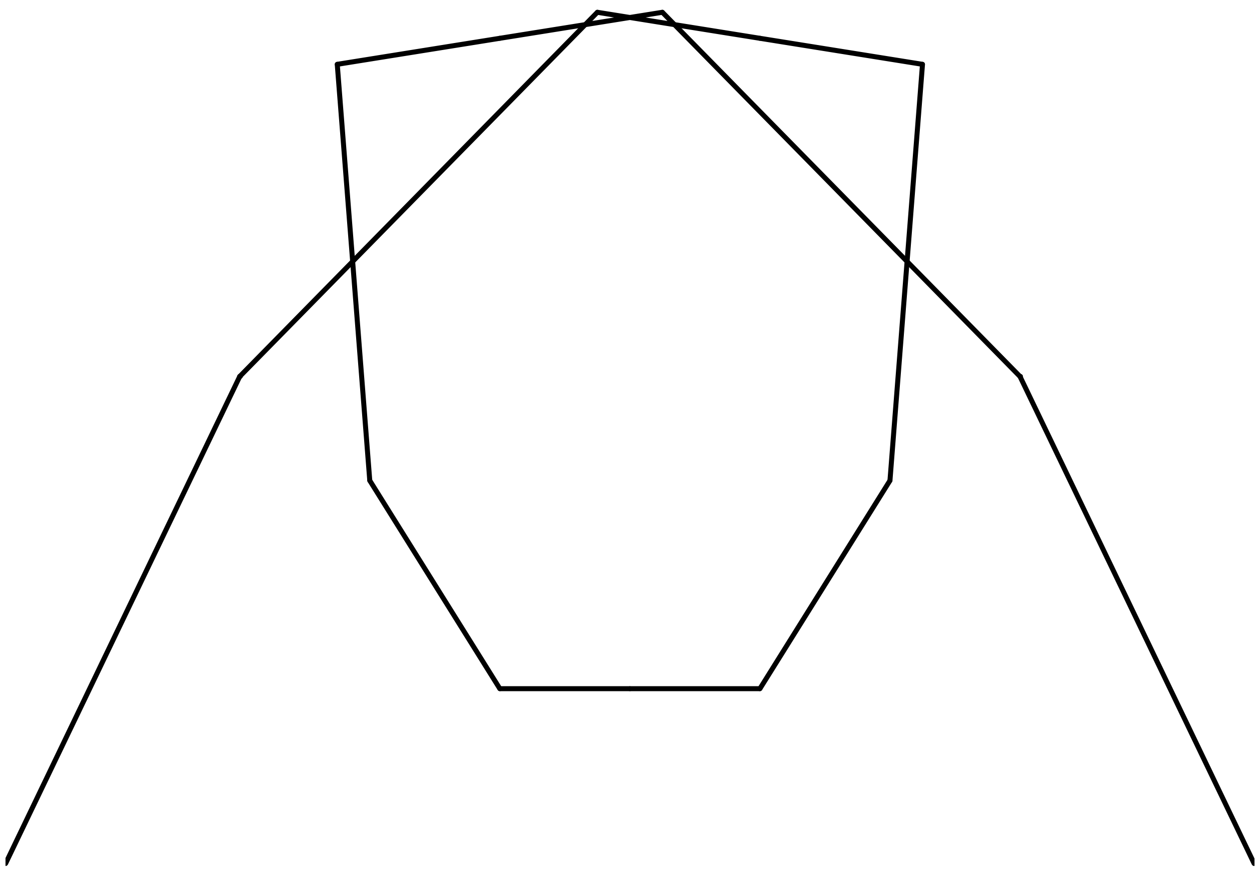


Figure 7.3: Subdivision, 2 iterations

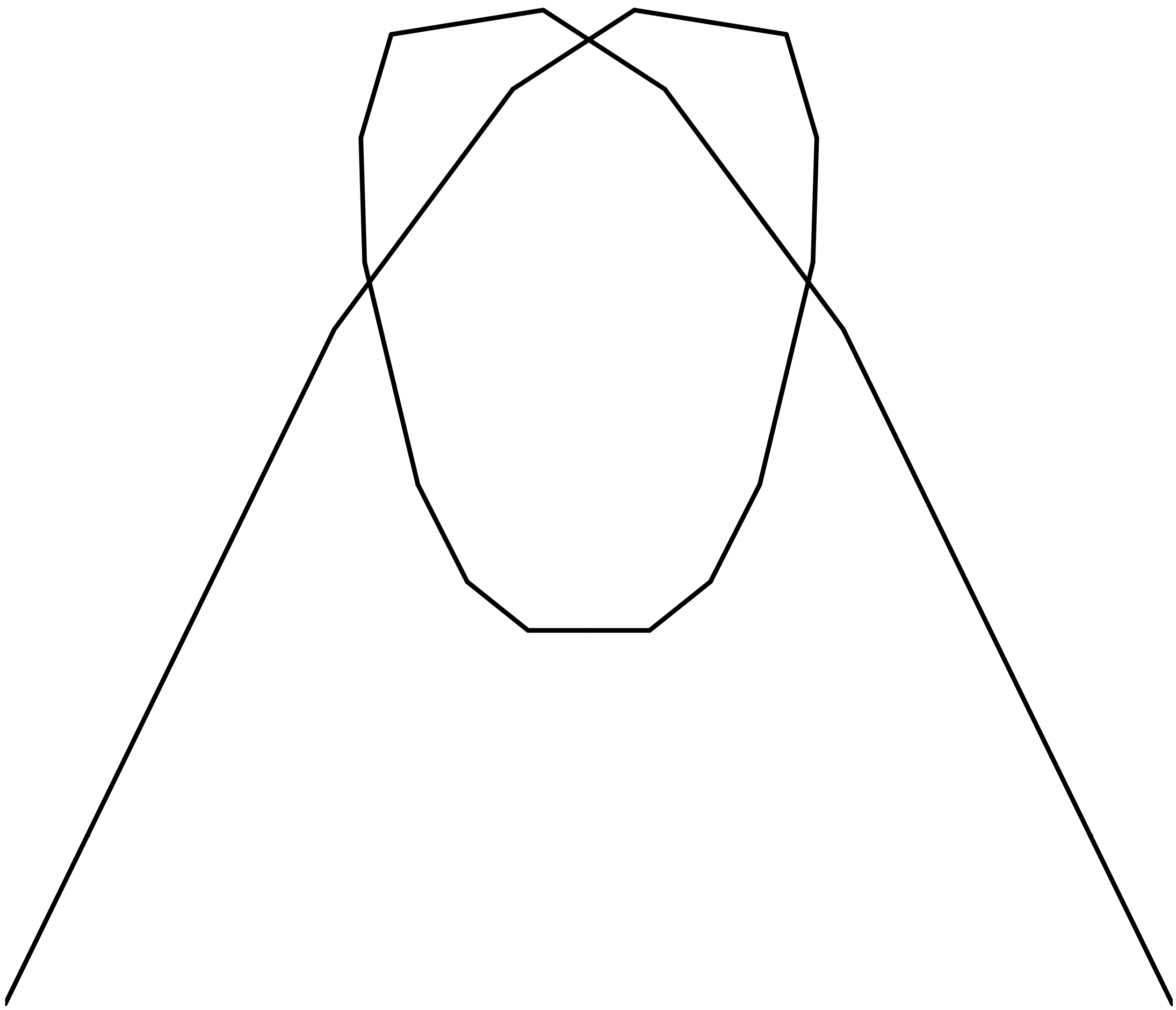


Figure 7.4: Subdivision, 3 iterations

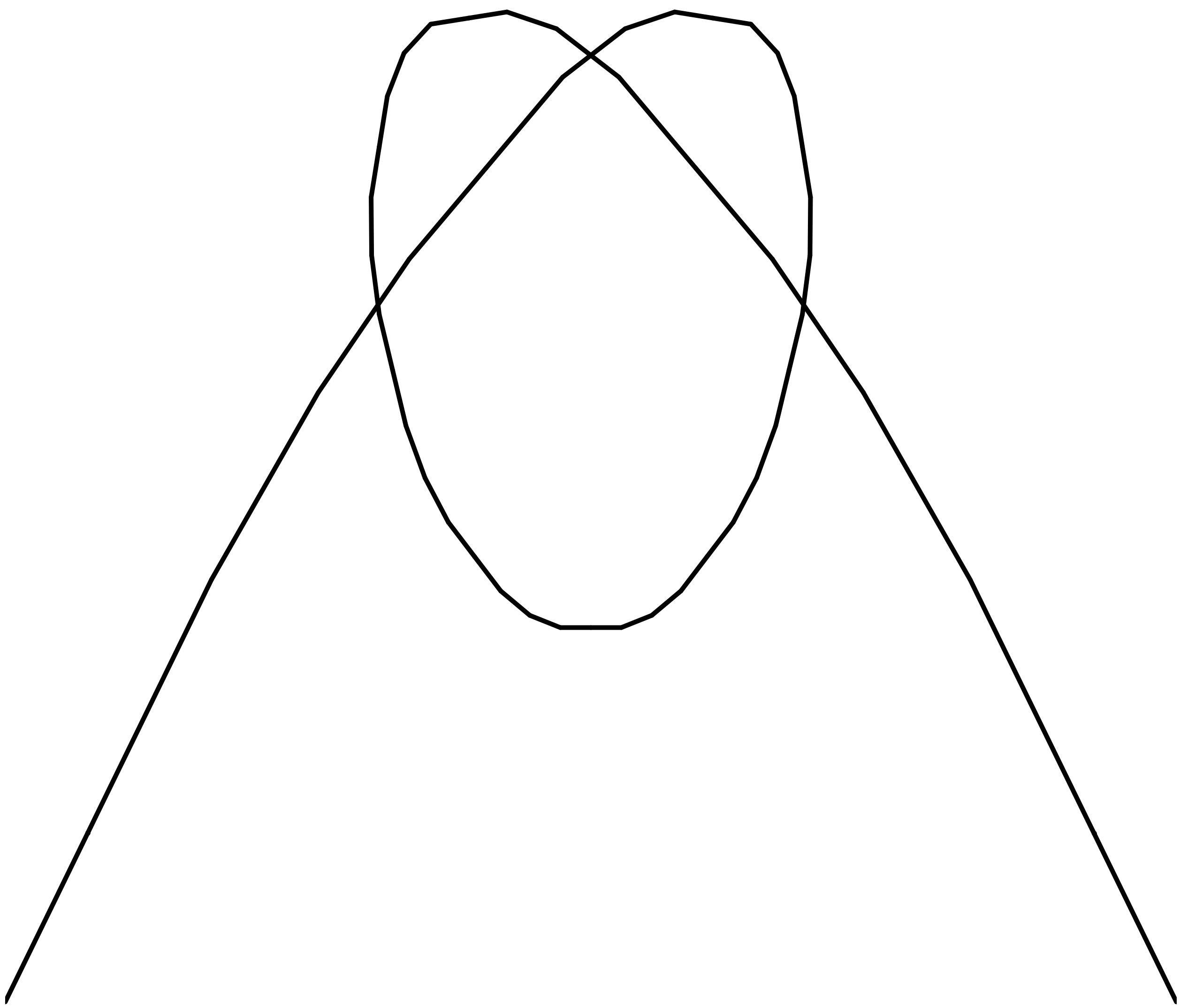


Figure 7.5: Subdivision, 4 iterations

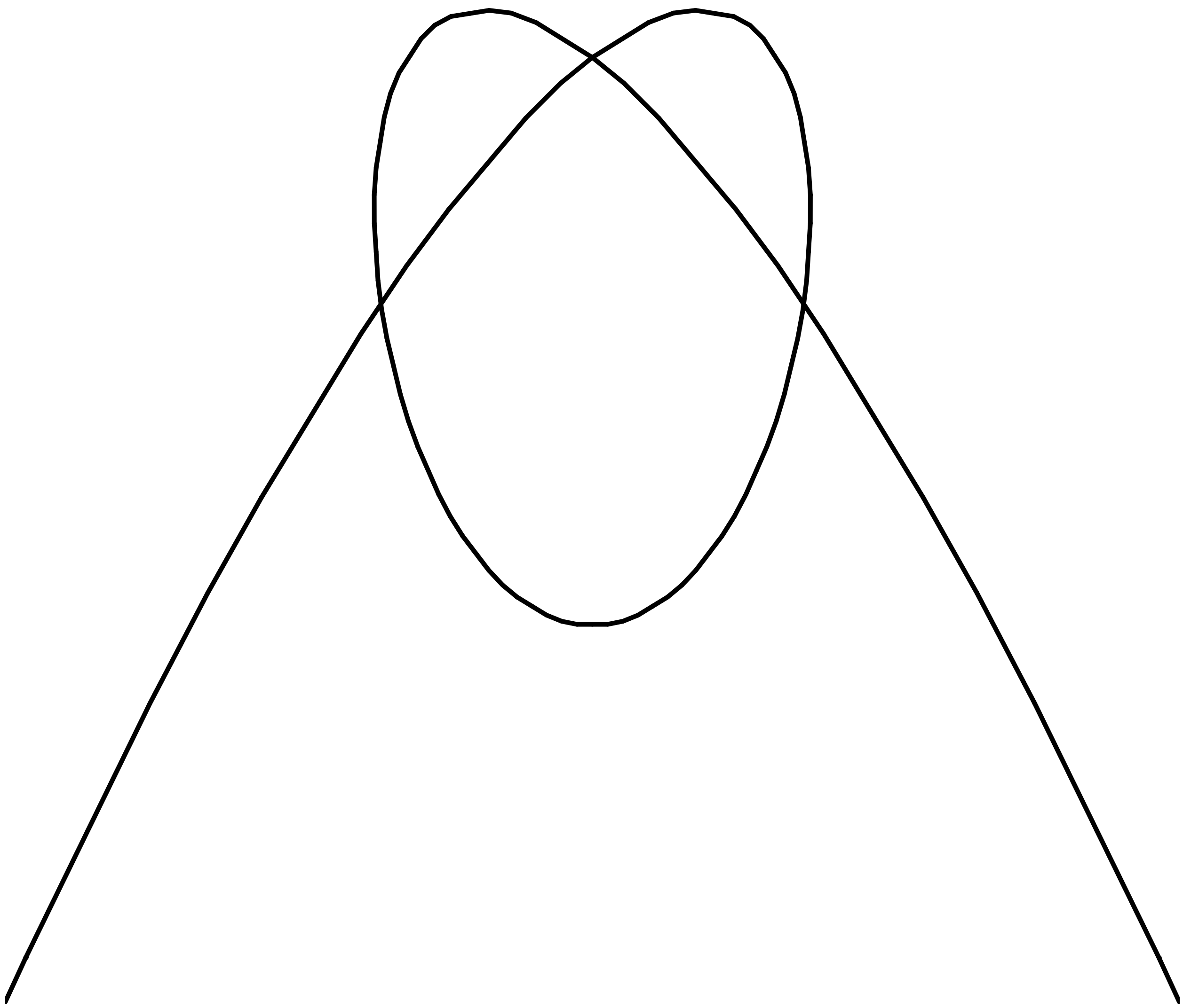


Figure 7.6: Subdivision, 5 iterations

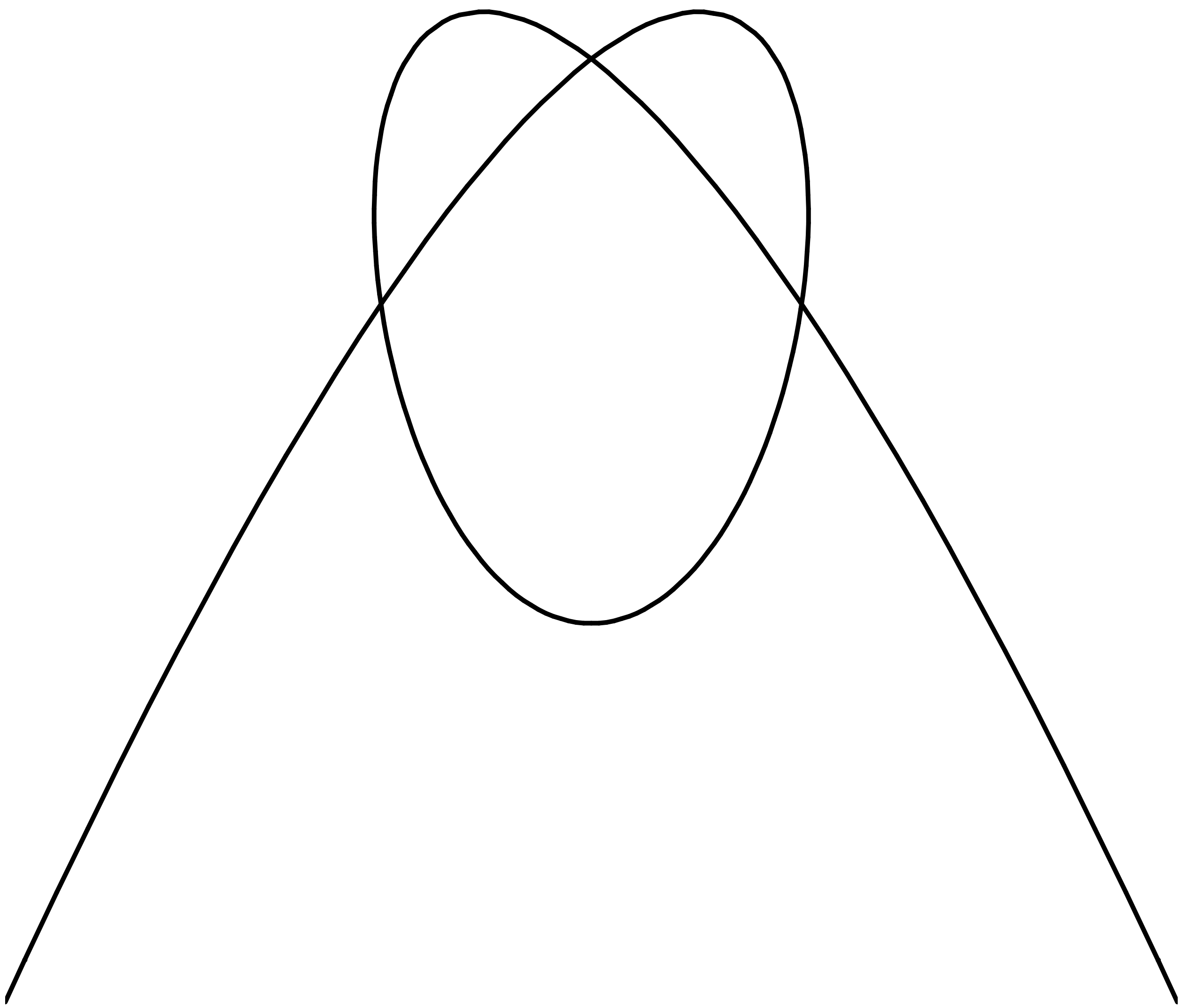


Figure 7.7: Subdivision, 6 iterations

Another nice application of the subdivision method is that we can compute very cheaply the control polygon $\mathcal{B}_{[a,b]}$ over a new affine frame $[a, b]$ of a polynomial curve given by a control polygon $\mathcal{B}$ over $[r, s]$. Indeed, assuming $a \neq r$, by subdividing once w.r.t. $[r, s]$ using the parameter a, we get the control polygon $\mathcal{B}_{[r,a]}$, and then we reverse this control polygon and subdivide again w.r.t. $[a, r]$ using b, to get $\mathcal{B}_{[a,b]}$. When $r = a$, we subdivide w.r.t. $[r, s]$, using b.

```
(*  Computes the control polygon wrt new affine frame (a, b) *)
(*  Assumes that a = (1 - lambda) r + lambda t and          *)
(*  that b = (1 - mu) r + mu t, wrt original frame (s, t     *)
(*  Returns control poly  (f(a, ..., a, b, ..., b))          *)

 newcpoly[{cpoly__}, r_, s_, a_, b_] :=
 Block[
 {poly = {cpoly}, m, i, pol1, pola, pol2, npoly, pt},
 (
  If[a =!= r, pol1 = subdecas[poly, r, s, a];
             pola = pol1[[1]]; pol2 = {};
             m = Length[pola];
             Do[
                pt = pola[[i]];
                pol2 = Prepend[pol2, pt], {i, 1, m}
               ];
             npoly = subdecas[pol2, a, r, b],
      (*              Print[" npoly: ", npoly]   *)
             npoly = subdecas[poly, r, s, b]
    ];
  npoly[[1]]
 )
 ];
```

The above function can be used to render curve segments over intervals $[a, b]$ different from the interval $[r, s]$ over which the original control polygon is defined.

We consider one more property of Bézier curves, *degree raising*. Given a Bézier curve F of polar degree m, and specified by a sequence of $m+1$ control points $b_0, \ldots, b_m$, it is sometimes necessary to view F as a curve of polar degree $m + 1$. For example, certain algorithms can only be applied to curve segments of the same degree. Or a system may only accept curves of a specified degree, say 3, and thus, in order to use such a system on a curve of lower degree, for example, a curve of degree 2, it may be necessary to raise the (polar) degree of the curve.

Indeed, if F is defined by the polar form $f\colon \mathbb{A}^m \to \mathcal{E}$, the polar form $g\colon \mathbb{A}^{m+1} \to \mathcal{E}$ that will yield the same curve F, in the sense that

$$g(\underbrace{t, \ldots, t}_{m+1}) = f(\underbrace{t, \ldots, t}_{m}) = F(t),$$

is necessarily

$$g(t_1, \ldots, t_{m+1}) = \frac{1}{m+1} \left(\sum_{1 \leq i_1 < \ldots < i_m \leq m+1} f(t_{i_1}, \ldots, t_{i_m}) \right).$$

Instead of the above notation, the following notation is often used,

$$g(t_1, \ldots, t_{m+1}) = \frac{1}{m+1} \sum_{i=1}^{i=m+1} f(t_1, \ldots, \widehat{t_i}, \ldots, t_{m+1}),$$

where the hat over the argument $\widehat{t_i}$ indicates that this argument is omitted.

For example, if f is biaffine, we have

$$g(t_1, t_2, t_3) = \frac{f(t_1, t_2) + f(t_1, t_3) + f(t_2, t_3)}{3}.$$

If F (and thus f) is specified by the $m+1$ control points $b_0, \ldots, b_m$, then F considered of degree $m+1$ (and thus g), is specified by $m+2$ control points $b_0^1, \ldots, b_{m+1}^1$, and it is an easy exercise to show that the points b_j^1 are given in terms of the original points b_i, by the equations:

$$b_i^1 = \frac{i}{m+1}\, b_{i-1} + \frac{m+1-i}{m+1}\, b_i,$$

where $1 \leq i \leq m$, with $b_0^1 = b_0$, and $b_{m+1}^1 = b_m$.

One can also raise the degree again, and so on. It can be shown that the control polygons obtained by successive degree raising, converge to the original curve segment. However, this convergence is much slower than the convergence obtained by subdivision, and it is not useful in practice.

7.3 The Progressive Version of the de Casteljau Algorithm (the de Boor Algorithm)

When dealing with splines, it is convenient to consider control points not just of the form $f(r^{m-i}\, s^i)$, but of the form $f(u_{k+1}, \ldots, u_{k+m})$, where the u_i are real numbers taken from a sequence $\langle u_1, \ldots, u_{2m} \rangle$ of length $2m$, satisfying certain inequality conditions. Let us begin with the case $m = 3$.

Given a sequence $\langle u_1, u_2, u_3, u_4, u_5, u_6 \rangle$, we say that this sequence is *progressive* iff the inequalities indicated in the following array hold:

$$\begin{array}{cccc}
u_1 & \neq & & \\
u_2 & \neq & \neq & \\
u_3 & \neq & \neq & \neq \\
 & u_4 & u_5 & u_6
\end{array}$$

Then, we consider the following four control points:

$$f(u_1, u_2, u_3),\ f(u_2, u_3, u_4),\ f(u_3, u_4, u_5),\ f(u_4, u_5, u_6).$$

Observe that these points are obtained from the sequence $\langle u_1, u_2, u_3, u_4, u_5, u_6 \rangle$, by sliding a window of length 3 over the sequence, from left to right.

We can compute any polar value $f(t_1, t_2, t_3)$ from the above control points, using the following triangular array obtained using the de Casteljau algorithm:

$$
\begin{array}{cccc}
 & 1 & 2 & 3 \\
f(u_1, u_2, u_3) & & & \\
 & f(t_1, u_2, u_3) & & \\
f(u_2, u_3, u_4) & & f(t_1, t_2, u_3) & \\
 & f(t_1, u_3, u_4) & & f(t_1, t_2, t_3) \\
f(u_3, u_4, u_5) & & f(t_1, t_2, u_4) & \\
 & f(t_1, u_4, u_5) & & \\
f(u_4, u_5, u_6) & & &
\end{array}
$$

The following diagram shows the computation of the polar value $f(t_1, t_2, t_3)$, given the progressive sequence

$$\langle u_1,\ u_2,\ u_3,\ u_4,\ u_5,\ u_6 \rangle :$$

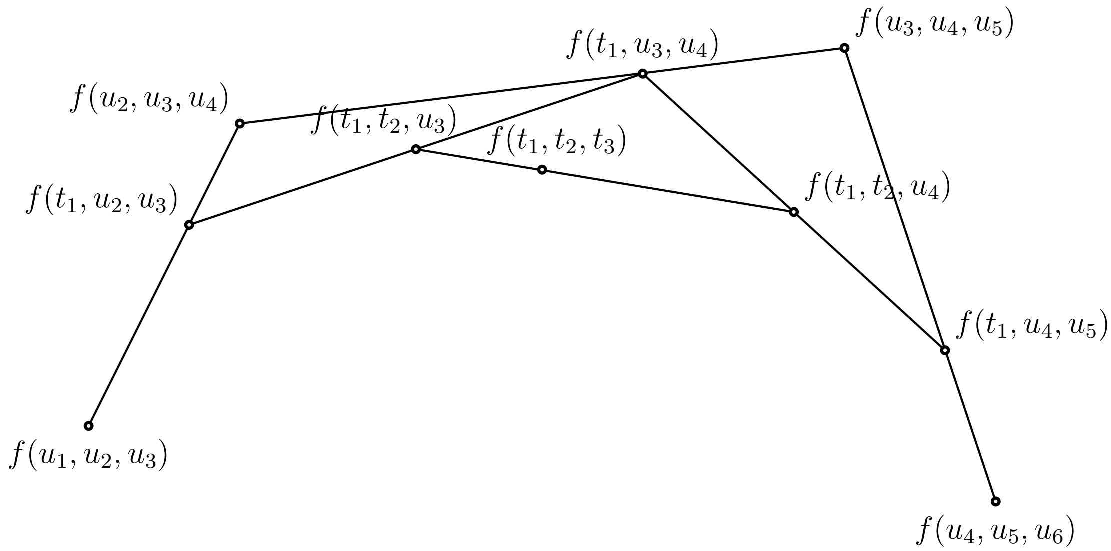


Figure 7.8: The de Casteljau algorithm, progressive case

In the general case, we have a sequence $\langle u_1, \ldots, u_{2m} \rangle$ of numbers $u_i \in \mathbb{R}$.

Definition 7.3.1 A sequence $\langle u_1, \ldots, u_{2m} \rangle$ of numbers $u_i \in \mathbb{R}$ is *progressive* iff $u_j \neq u_{m+i}$, for all j, and all i, $1 \leq i \leq j \leq m$. These $\frac{m(m+1)}{2}$ conditions correspond to the lower triangular part of the following array:

$$
\begin{array}{cccccccccc}
u_1 & \neq & & & & & & & & \\
u_2 & \neq & \neq & & & & & & & \\
\vdots & & & & & & & & & \\
u_j & \neq & \neq & \ldots & \neq & & & & & \\
\vdots & & & \ldots & & \ldots & & & & \\
\vdots & & & \ldots & & \ldots & \neq & & & \\
\vdots & & & \ldots & & \ldots & & \ldots & & \\
u_{m-1} & \neq & \neq & \ldots & \neq & \ldots & \neq & \ldots & \neq & \\
u_m & \neq & \neq & \ldots & \neq & \ldots & \neq & \ldots & \neq & \neq \\
 & u_{m+1} & u_{m+2} & \ldots & u_{m+j} & \ldots & u_{2m-j+1} & \ldots & u_{2m-1} & u_{2m}
\end{array}
$$

The point $f(t_1 \ldots t_j u_{i+j+1} \ldots u_{m+i})$ is obtained at step i of phase j, for $1 \leq j \leq m$, $0 \leq i \leq m-j$, by the interpolation step

$$
\begin{aligned}
f(t_1 \ldots t_j u_{i+j+1} \ldots u_{m+i}) = \\
\left(\frac{u_{m+i+1} - t_j}{u_{m+i+1} - u_{i+j}} \right) f(t_1 \ldots t_{j-1} u_{i+j} \ldots u_{m+i}) \\
+ \left(\frac{t_j - u_{i+j}}{u_{m+i+1} - u_{i+j}} \right) f(t_1 \ldots t_{j-1} u_{i+j+1} \ldots u_{m+i+1}).
\end{aligned}
$$

In order to make the above triangular array a bit more readable, let us define the following points $b_{i,j}$, used during the computation:

$$b_{i,j} = f(t_1 \ldots t_j u_{i+j+1} \ldots u_{m+i}),$$

for $1 \leq j \leq m$, $0 \leq i \leq m-j$, with

$$b_{i,0} = f(u_{i+1}, \ldots, u_{m+i}),$$

for $0 \leq i \leq m$. Then, we have the following equations:

$$b_{i,j} = \left(\frac{u_{m+i+1} - t_j}{u_{m+i+1} - u_{i+j}} \right) b_{i,j-1} + \left(\frac{t_j - u_{i+j}}{u_{m+i+1} - u_{i+j}} \right) b_{i+1,j-1}.$$

The progressive version of the de Casteljau algorithm is also called the *de Boor algorithm*. It is the major algorithm used in dealing with splines.

One may wonder whether it is possible to give a closed form for $f(t_1, \ldots, t_m)$, as computed by the progressive case of the de Casteljau algorithm, and come up with a version of lemma 6.2.5. This turns out to be difficult, as the case $m = 2$ already reveals!

We can still prove the following theorem generalizing lemma 6.2.5 to the progressive case. The easy half follows from the progressive version of the de Casteljau algorithm, and the converse will be proved later.

Theorem 7.3.2 *Let $\langle u_1, \ldots, u_{2m} \rangle$ be a progressive sequence of numbers $u_i \in \mathbb{R}$. Given any sequence of $m+1$ points $b_0, \ldots, b_m$ in some affine space $\mathcal{E}$, there is a unique polynomial curve $F\colon \mathbb{A} \to \mathcal{E}$ of degree m, whose polar form $f\colon \mathbb{A}^m \to \mathcal{E}$ satisfies the conditions*

$$f(u_{k+1}, \ldots, u_{m+k}) = b_k,$$

for every k, $0 \leq k \leq m$.

There are at least two ways of proving the existence of a curve satisfying the conditions of theorem 7.3.2. One proof is fairly computational, and requires computing a certain determinant, which turns out to be nonzero precisely because the sequence is progressive. The other proof, due to Ramshaw, is more elegant and conceptual, but it uses the more sophisticated concept of symmetric tensor product.

7.4 Derivatives of Polynomial Curves

In this section, it is assumed that $\mathcal{E}$ is some affine space $\mathbb{A}^n$, with $n \geq 2$. Our intention is to give the formulae for the derivatives of polynomial curves $F: \mathbb{A} \to \mathcal{E}$ in terms of control points.

This characterization will be used in the next section dealing with the conditions for joining polynomial curves with C^k-continuity.

In this section, following Ramshaw, it will be convenient to denote a point in $\mathbb{A}$ as $\overline{a}$, to distinguish it from the vector $a \in \mathbb{R}$.

The unit vector $1 \in \mathbb{R}$ is denoted as δ. When dealing with derivatives, it is also more convenient to denote the vector $\overrightarrow{ab}$ as $b - a$.

Given a polynomial curve $F\colon \mathbb{A} \to \mathcal{E}$, for any $\overline{a} \in \mathbb{A}$, recall that the derivative $\mathrm{D}F(\overline{a})$ is the limit

$$\lim_{t\to 0,\, t\neq 0} \frac{F(\overline{a} + t\delta) - F(\overline{a})}{t},$$

if it exists.

Recall that since $F\colon \mathbb{A} \to \mathcal{E}$, where $\mathcal{E}$ is an affine space, the derivative $\mathrm{D}F(\overline{a})$ of F at $\overline{a}$ is a *vector* in $\overrightarrow{\mathcal{E}}$, and not a point in $\mathcal{E}$.

Since coefficients of the form $m(m-1)\cdots(m-k+1)$ occur a lot when taking derivatives, following Knuth, it is useful to introduce the *falling power* notation. We define the *falling power* $m^{\underline{k}}$, as

$$m^{\underline{k}} = m(m-1)\cdots(m-k+1),$$

for $0 \leq k \leq m$, with $m^{\underline{0}} = 1$, and with the convention that $m^{\underline{k}} = 0$ when $k > m$.

The following lemma giving the k-th derivative $\mathrm{D}^k F(\overline{r})$ of F at $\overline{r}$ in terms of polar values, can be shown.

Lemma 7.4.1 *Given an affine polynomial function $F\colon \mathbb{A} \to \mathcal{E}$ of polar degree m, for any $\overline{r}, \overline{s} \in \mathbb{A}$, with $r \neq s$, the k-th derivative $\mathrm{D}^k F(\overline{r})$ can be computed from the polar form f of F as follows, where $1 \leq k \leq m$:*

$$\mathrm{D}^k F(\overline{r}) = \frac{m^{\underline{k}}}{(s-r)^k} \sum_{i=0}^{i=k} \binom{k}{i} (-1)^{k-i} f(\underbrace{\overline{r}, \ldots, \overline{r}}_{m-i}, \underbrace{\overline{s}, \ldots, \overline{s}}_{i}).$$

A proof is given in section 12.2. It is also possible to obtain this formula by expressing $F(\overline{r})$ in terms of the Bernstein polynomials and computing their derivatives.

If F is specified by the sequence of $m+1$ control points $b_i = f(\overline{r}^{\,m-i}\, \overline{s}^{\,i})$, $0 \leq i \leq m$, the above lemma shows that the k-th derivative $\mathrm{D}^k F(\overline{r})$ of F at $\overline{r}$, depends only on the $k+1$ control points $b_0, \ldots, b_k$

In terms of the control points $b_0, \ldots, b_k$, the formula of lemma 7.4.1 reads as follows:

$$\mathrm{D}^k F(\overline{r}) = \frac{m^{\underline{k}}}{(s-r)^k} \sum_{i=0}^{i=k} \binom{k}{i} (-1)^{k-i}\, b_i.$$

In particular, if $b_0 \neq b_1$, then $\mathrm{D}F(\overline{r})$ is the velocity vector of F at b_0, and it is given by

$$\mathrm{D}F(\overline{r}) = \frac{m}{s-r}\,\overrightarrow{b_0 b_1} = \frac{m}{s-r}\,(b_1 - b_0).$$

This shows that when b_0 and b_1 are distinct, the tangent to the Bézier curve at the point b_0 is the line determined by b_0 and b_1.

Similarly, the tangent at the point b_m is the line determined by b_{m-1} and b_m (provided that these points are distinct).

More generally, the tangent at the current point $F(\overline{t})$ defined by the parameter $\overline{t}$, is determined by the two points

$$b_{0,\,m-1} = f(\underbrace{\overline{t}, \ldots, \overline{t}}_{m-1}, \overline{r}) \quad \text{and} \quad b_{1,\,m-1} = f(\underbrace{\overline{t}, \ldots, \overline{t}}_{m-1}, \overline{s}),$$

given by the de Casteljau algorithm. It can be shown that

$$\mathrm{D}F(\overline{t}) = \frac{m}{s-r}(b_{1,m-1} - b_{0,m-1}).$$

The acceleration vector $\mathrm{D}^2F(\overline{r})$ is given by

$$\begin{aligned} \mathrm{D}^2F(\overline{r}) = \frac{m(m-1)}{(s-r)^2}\,(\overrightarrow{b_0b_2} - 2\overrightarrow{b_0b_1}) \\ = \frac{m(m-1)}{(s-r)^2}\,(b_2 - 2b_1 + b_0). \end{aligned}$$

More generally, if $b_0 = b_1 = \ldots = b_k$, and $b_k \neq b_{k+1}$, it can be shown that the tangent at the point b_0 is determined by the points b_0 and b_{k+1}.

7.5 Joining Affine Polynomial Functions

When dealing with splines, we have several curve segments that need to be joined with certain required continuity conditions ensuring smoothness.

The weakest condition is no condition at all, called C^{-1}-continuity. This means that we don't even care whether $F(\overline{q}) = G(\overline{q})$, that is, there could be a discontinuity at $\overline{q}$. In this case, we say that $\overline{q}$ is a *discontinuity knot*. The next weakest condition, called C^0-continuity, is that $F(\overline{q}) = G(\overline{q})$. In other words, we impose continuity at $\overline{q}$, but no conditions on the derivatives.

Definition 7.5.1 Two curve segments $F([\overline{p}, \overline{q}])$ and $G[\overline{q}, \overline{r}])$ of polar degree m are said to *join with C^k-continuity at $\overline{q}$*, where $0 \leq k \leq m$, iff

$$\mathrm{D}^i F(\overline{q}) = \mathrm{D}^i G(\overline{q}),$$

for all i, $0 \leq i \leq k$, where by convention,
$\mathrm{D}^0 F(\overline{q}) = F(\overline{q})$, and $\mathrm{D}^0 G(\overline{q}) = G(\overline{q})$.

As we will see, for curve segments F and G of polar degree m, C^m-continuity imposes that $F = G$, which is too strong, and thus, we usually consider C^k-continuity, where $0 \leq k \leq m - 1$ (or even $k = -1$, as mentioned above). The continuity conditions of definition 7.5.1 are ususally referred to as *parametric continuity*. There are other (more) useful kinds of continuity, for example *geometric continuity*.

We can characterize C^k-continuity of joins of curve segments very conveniently in terms of polar forms. A more conceptual proof of a slightly more general lemma, will be given in section **??**, using symmetric tensor products (see lemma **??**).

Lemma 7.5.2 *Given two intervals $[\overline{p}, \overline{q}]$ and $[\overline{q}, \overline{r}]$, where $\overline{p}, \overline{q}, \overline{r} \in \mathbb{A}$, with $p < q < r$, and two affine curve segments $F\colon [\overline{p}, \overline{q}] \to \mathcal{E}$ and $G\colon [\overline{q}, \overline{r}] \to \mathcal{E}$, of polar degree m, the curve segments $F([\overline{p}, \overline{q}])$ and $G[\overline{q}, \overline{r}])$ join with continuity C^k at $\overline{q}$, where $0 \leq k \leq m$, iff their polar forms $f\colon \mathbb{A}^m \to \mathcal{E}$ and $g\colon \mathbb{A}^m \to \mathcal{E}$ agree on all multisets of points that contain at most k points distinct from $\overline{q}$, i.e.,*

$$f(\overline{u_1}, \ldots, \overline{u_k}, \underbrace{\overline{q}, \ldots, \overline{q}}_{m-k}) = g(\overline{u_1}, \ldots, \overline{u_k}, \underbrace{\overline{q}, \ldots, \overline{q}}_{m-k}),$$

for all $\overline{u_1}, \ldots, \overline{u_k} \in \mathbb{A}$.

Another way to state lemma 7.5.2 is to say that the curve segments $F([\overline{p}, \overline{q}])$ and $G[\overline{q}, \overline{r}])$ join with continuity C^k at $\overline{q}$, where $0 \leq k \leq m$, iff their polar forms $f\colon \mathbb{A}^m \to \mathcal{E}$ and $g\colon \mathbb{A}^m \to \mathcal{E}$ agree on all multisets of points that contain at least $m - k$ copies of the argument $\overline{q}$.

Thus, the number k is the number of arguments that can be varied away from $\overline{q}$ without disturbing the values of the polar forms f and g.

When $k = 0$, we can't change any of the arguments, and this means that f and g agree on the multiset

$$\underbrace{\overline{q}, \ldots, \overline{q}}_{m},$$

i.e., the curve segments F and G simply join at $\overline{q}$, without any further conditions.

On the other hand, for $k = m - 1$, we can vary $m - 1$ arguments away from $\overline{q}$ without changing the value of the polar forms, which means that the curve segments F and G join with a high degre of smoothness (C^{m-1}-continuity).

In the extreme case where $k = m$ (C^m-continuity), the polar forms f and g must agree when all arguments vary, and thus $f = g$, i.e. F and G coincide. We will see that lemma 7.5.2 yields a very pleasant treatment of parametric continuity for splines.

The following diagrams illustrate the geometric conditions that must hold so that two segments of cubic curves $F\colon \mathbb{A} \to \mathcal{E}$ and $G\colon \mathbb{A} \to \mathcal{E}$ defined on the intervals $[\overline{p}, \overline{q}]$ and $[\overline{q}, \overline{r}]$, join at $\overline{q}$ with C^k-continuity, for $k = 0, 1, 2, 3$. Let f and g denote the polar forms of F and G.

The curve segments F and G join at $\overline{q}$ with C^0-continuity iff the polar forms f and g agree on the (multiset) triplet $\overline{q}, \overline{q}, \overline{q}$.

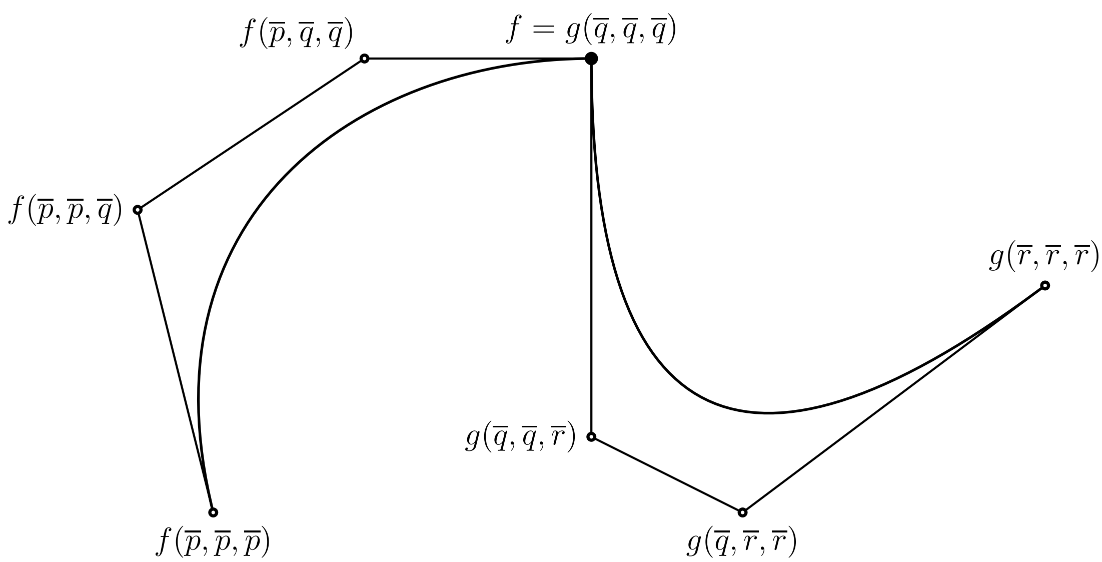


Figure 7.9: Cubic curves joining at $\overline{q}$ with C^0-continuity

The curve segments F and G join at $\overline{q}$ with C^1-continuity iff the polar forms f and g agree on all (multiset) triplets including two copies of the argument $\overline{q}$.

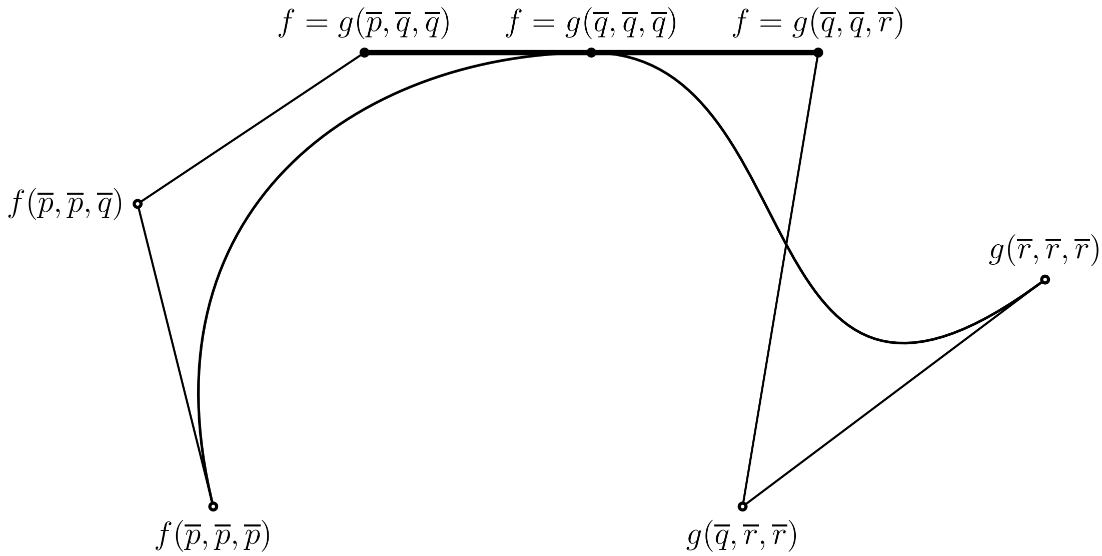


Figure 7.10: Cubic curves joining at $\overline{q}$ with C^1-continuity

The curve segments F and G join at $\overline{q}$ with C^2-continuity iff the polar forms f and g agree on all (multiset) triplets including the argument $\overline{q}$.

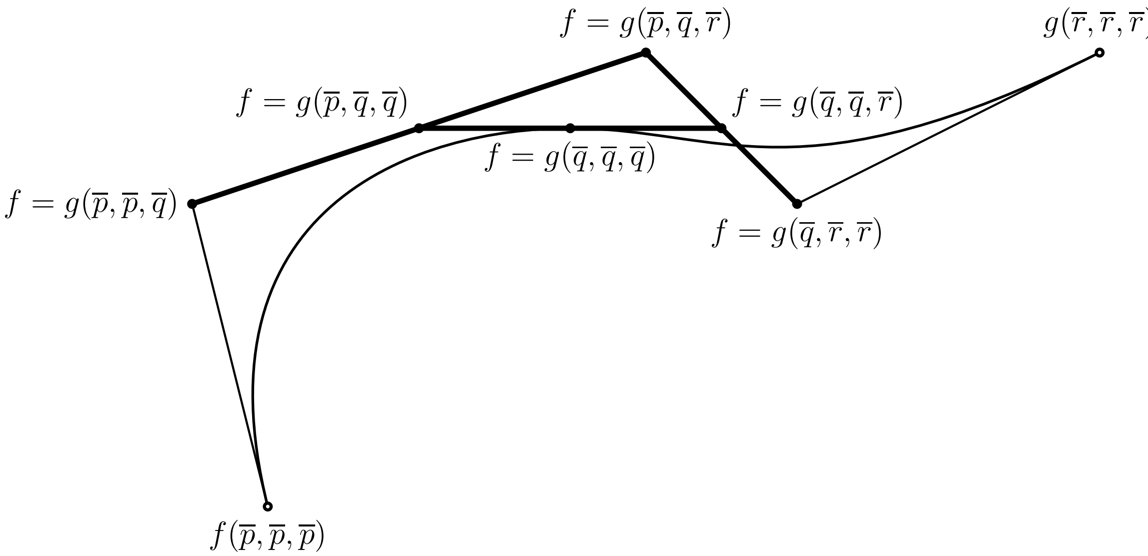


Figure 7.11: Cubic curves joining at $\overline{q}$ with C^2-continuity

The curve segments F and G join at $\overline{q}$ with C^3-continuity iff the polar forms f and g agree on all (multiset) triplets, i.e., iff $f = g$.

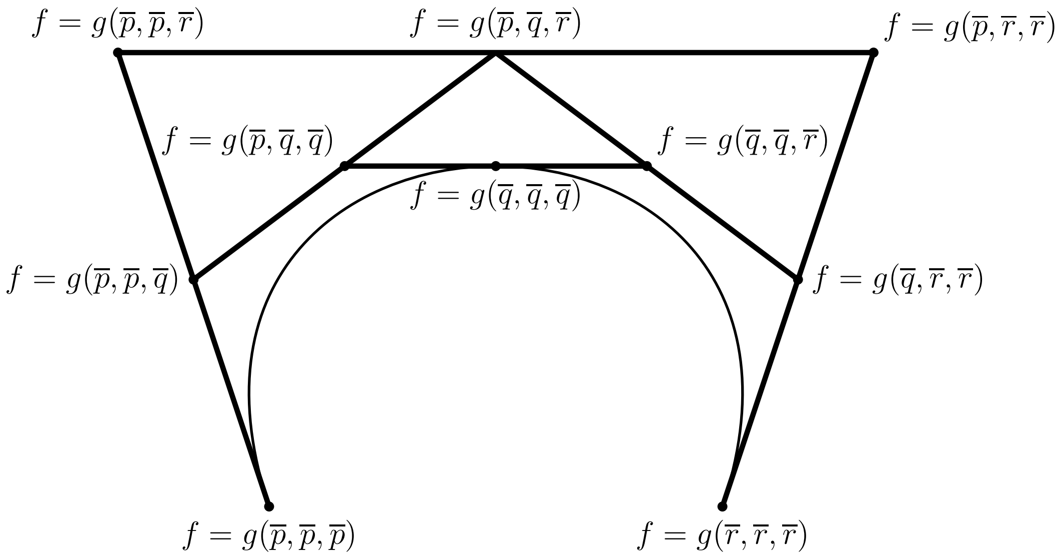


Figure 7.12: Cubic curves joining at $\overline{q}$ with C^3-continuity

The above examples show that the points corresponding to the common values

$$f(\overline{p}^{\,i}, \overline{r}^{\,j}, \overline{q}^{\,3-i-j}) = g(\overline{p}^{\,i}, \overline{r}^{\,j}, \overline{q}^{\,3-i-j})$$

of the polar forms f and g, where $i + j \leq k \leq 3$, constitute a de Casteljau diagram with k shells, where k is the degree of continuity required. These de Casteljau diagrams are represented in bold.

This is a general fact. When two polynomial curves F and G of degree m join at $\overline{q}$ with C^k-continuity (where $0 \leq k \leq m$), then

$$f(\overline{p}^{\,i}, \overline{r}^{\,j}, \overline{q}^{\,m-i-j}) = g(\overline{p}^{\,i}, \overline{r}^{\,j}, \overline{q}^{\,m-i-j})$$

for all i, j with $i + j \leq k \leq m$, and these points form a de Casteljau diagram with k shells.

Chapter 8

Spline Curves (B-Spline Curves)

8.1 Introduction: Knot Sequences, de Boor Control Points

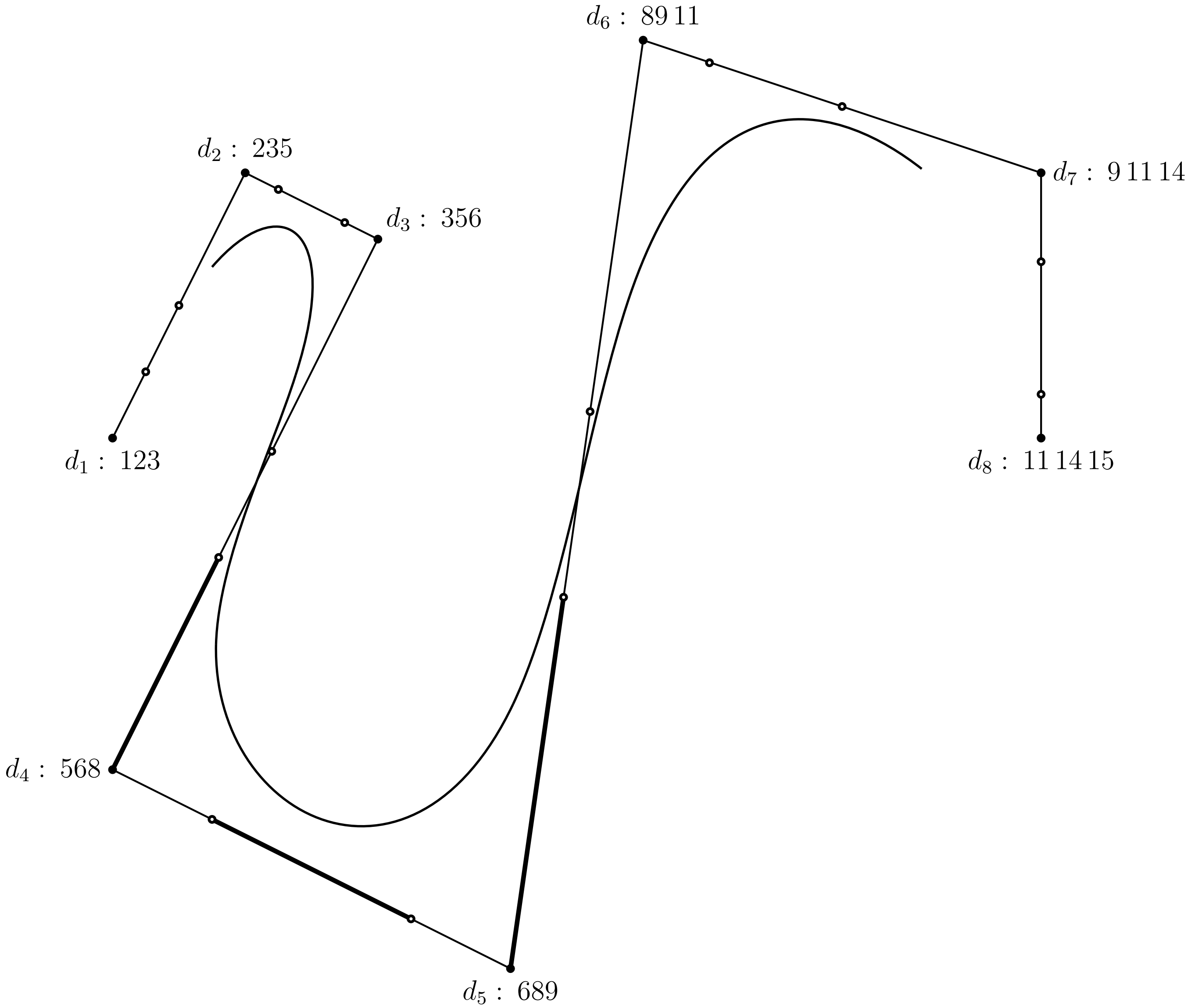


Figure 8.1: Part of a cubic spline with knot sequence $\ldots, 1, 2, 3, 5, 6, 8, 9, 11, 14, 15, \ldots$. Thick segments are images of $[6, 8]$.

Given a sequence of $2m$ knots

$$\langle u_{k+1},\, u_{k+2},\, \ldots,\, u_{k+2m}\rangle,$$

for any parameter value in the middle interval $t \in [u_{k+m},\, u_{k+m+1}]$, a point on the curve segment specified by the $m+1$ control points $d_i,\, d_{i+1}, \ldots,\, d_{i+m}$ (where d_i is mapped onto u_{k+1}), is computed by repeated affine interpolation, as follows:

Using the mapping

$$u \mapsto \frac{u_{k+m+j+1} - u}{u_{k+m+j+1} - u_{k+j+1}} d_{i+j} + \frac{u - u_{k+j+1}}{u_{k+m+j+1} - u_{k+j+1}} d_{i+j+1},$$

mapping the interval $[u_{k+j+1},\, u_{k+m+j+1}]$ onto the line segment $(d_{i+j},\, d_{i+j+1})$, where $0 \leq j \leq m-1$, we map $t \in [u_{k+m},\, u_{k+m+1}]$ onto the line segment $(d_{i+j},\, d_{i+j+1})$, which gives us a point $d_{j,\,1}$.

Then, we consider the new control polygon determined by the m points

$$d_{0,\,1},\; d_{1,\,1},\; \ldots,\; d_{m-1,\,1},$$

and we map affinely each of the $m-1$ intervals $[u_{k+j+2},\, u_{k+m+j+1}]$ onto the line segment $(d_{j,\,1},\, d_{j+1,\,1})$, where $0 \leq j \leq m-2$, and for $t \in [u_{k+m},\, u_{k+m+1}]$, we get a point $d_{j,\,2}$ on $(d_{j,\,1},\, d_{j+1,\,1})$.

Note that each interval $[u_{k+j+2},\, u_{k+m+j+1}]$ now consists of $m-1$ consecutive subintervals, and that the leftmost interval $[u_{k+2},\, u_{k+m+1}]$ starts at knot u_{k+2}, the immediate successor of the starting knot u_{k+1} of the leftmost interval used at the previous stage.

The above round gives us a new control polygon determined by the $m-1$ points

$$d_{0,\,2},\; d_{1,\,2},\; \ldots,\; d_{m-2,\,2},$$

and we repeat the procedure.

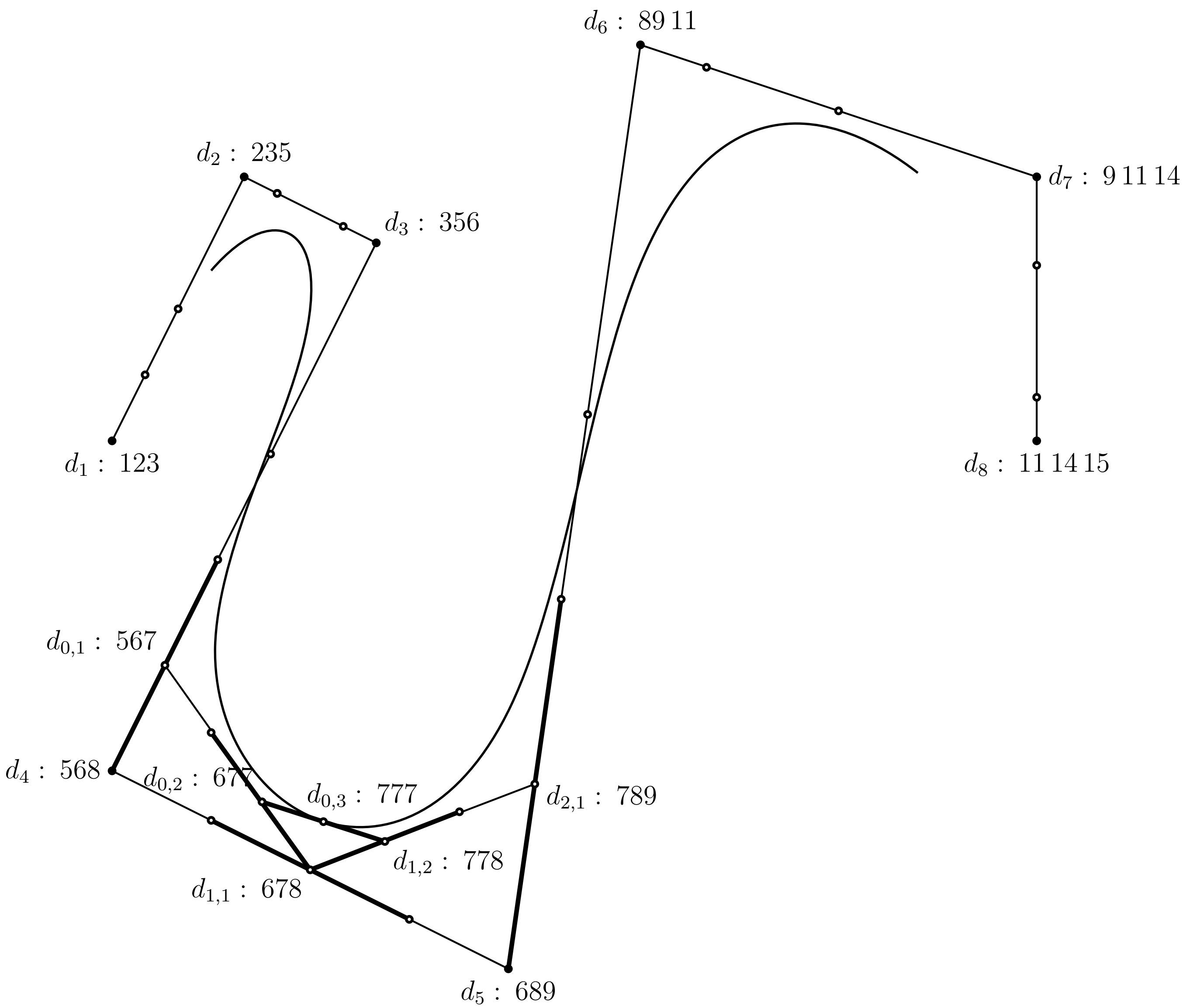


Figure 8.2: Part of a cubic spline with knot sequence $\ldots, 1, 2, 3, 5, 6, 8, 9, 11, 14, 15, \ldots$, and construction of the point corresponding to $t = 7$. Thick segments are images of $[6, 8]$.

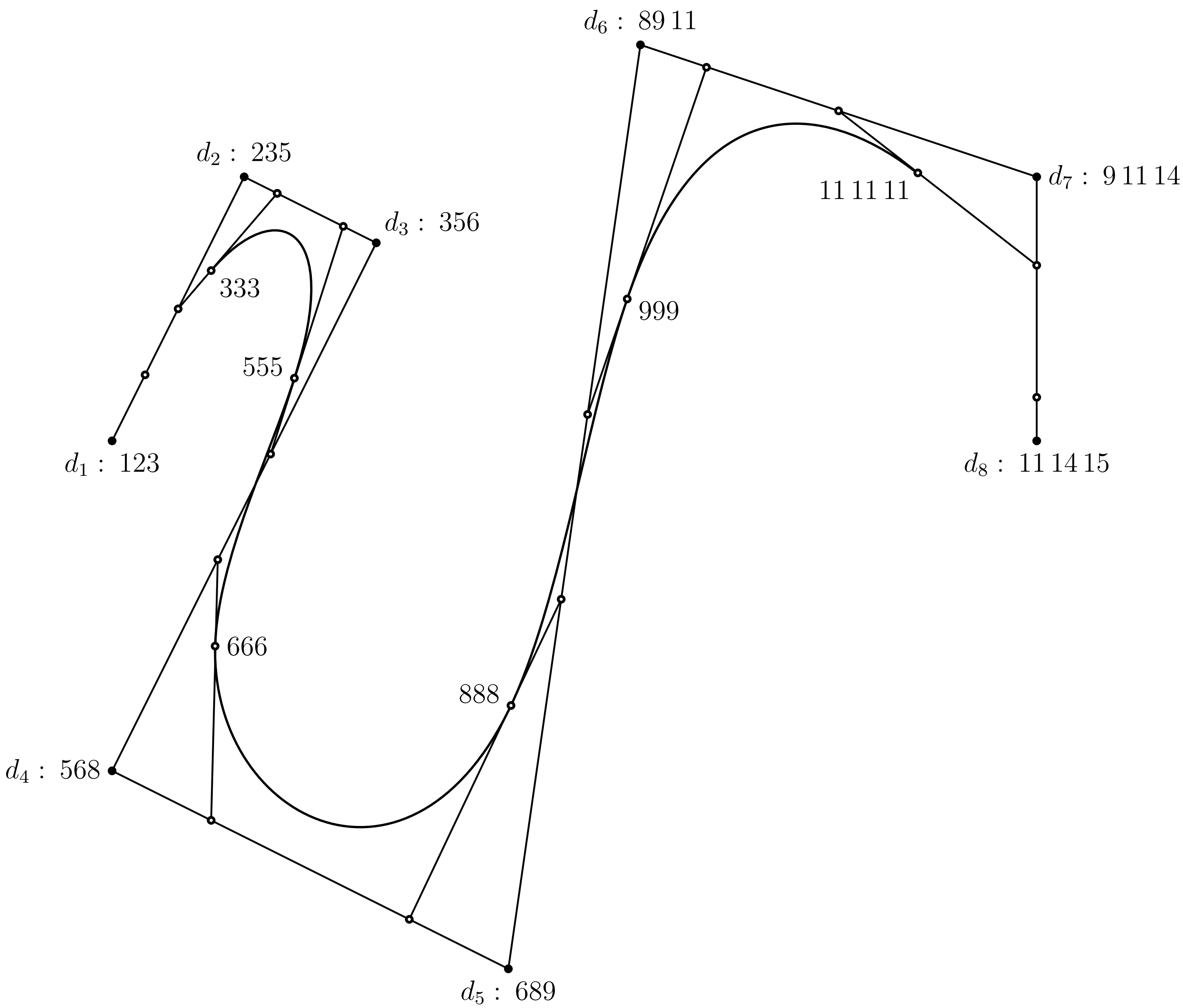


Figure 8.3: Part of a cubic spline with knot sequence $\ldots, 1, 2, 3, 5, 6, 8, 9, 11, 14, 15, \ldots$, and some of its Bézier control points

8.2 Infinite Knot Sequences, Open B-Spline Curves

We begin with knot sequences.

Definition 8.2.1 A *knot sequence* is a bi-infinite non-decreasing sequence $\langle \overline{u}_k \rangle_{k\in\mathbb{Z}}$ of points $\overline{u}_k \in \mathbb{A}$ (i.e. $\overline{u}_k \leq \overline{u}_{k+1}$ for all $k \in \mathbb{Z}$), such that every knot in the sequence has finitely many occurrences. A knot $\overline{u}_k$ in a knot sequence $\langle \overline{u}_k \rangle_{k\in\mathbb{Z}}$ has *multiplicity* n ($n \geq 1$) iff it occurs exactly n (consecutive) times in the knot sequence. Given any natural number $m \geq 1$, a knot sequence has *degree of multiplicity at most* $m+1$ iff every knot has multiplicity at most $m+1$, i.e. there are at most $m+1$ occurrences of identical knots in the sequence. Thus, for a knot sequence of degree of multiplicity at most $m+1$, we must have $\overline{u}_k \leq \overline{u}_{k+1}$ for all $k \in \mathbb{Z}$, and for every $k \in \mathbb{Z}$, if

$$\overline{u}_{k+1} = \overline{u}_{k+2} = \ldots = \overline{u}_{k+n},$$

then $1 \leq n \leq m+1$. A knot $\overline{u}_k$ of multiplicity $m+1$ is called a *discontinuity (knot)*. A knot of multiplicity 1 is called a *simple knot*. A knot sequence $\langle \overline{u}_k \rangle_{k\in\mathbb{Z}}$ is *uniform* iff $\overline{u}_{k+1} = \overline{u}_k + h$, for some fixed $h \in \mathbb{R}_+$.

We can now define spline (B-spline) curves.

Definition 8.2.2 Given any natural number $m \geq 1$, given any knot sequence $\langle \overline{u}_k \rangle_{k \in \mathbb{Z}}$ of degree of multiplicity at most $m + 1$, a *piecewise polynomial curve of (polar) degree m based on the knot sequence* $\langle \overline{u}_k \rangle_{k \in \mathbb{Z}}$, is a function $F \colon \mathbb{A} \to \mathcal{E}$, where $\mathcal{E}$ is some affine space (of dimension at least 2), such that, for any two consecutive distinct knots $\overline{u}_i < \overline{u}_{i+1}$, if $\overline{u}_{i+1}$ is a knot of multiplicity n, the next distinct knot being $\overline{u}_{i+n+1}$ (since we must have $\overline{u}_{i+1} = \ldots = \overline{u}_{i+n} < \overline{u}_{i+n+1}$), then the following condition hold:

1. The restriction of F to $[\overline{u}_i, \overline{u}_{i+1}[$ agrees with a polynomial curve F_i of polar degree m, with associated polar form f_i.

A *spline curve F of (polar) degree m based on the knot sequence* $\langle \overline{u}_k \rangle_{k \in \mathbb{Z}}$, is a piecewise polynomial curve $F\colon \mathbb{A} \to \mathcal{E}$, such that, for every two consecutive distinct knots $\overline{u}_i < \overline{u}_{i+1}$, the following condition holds:

2. The curve segments F_i and F_{i+n} join with continuity (at least) C^{m-n} at $\overline{u}_{i+1}$, in the sense of definition 7.5.1, where n is the multiplicity of the knot $\overline{u}_{i+1}$ $(1 \leq n \leq m + 1)$.

Thus, in particular, if $\overline{u}_{i+1}$ is a discontinuity knot, that is, a knot of multiplicity $m+1$, then we have C^{-1}-continuity, and $F_i(\overline{u}_{i+1})$ and $F_{i+n}(\overline{u}_{i+1})$ may differ. The set $F(\mathbb{A})$ is called the *trace of the spline F*.

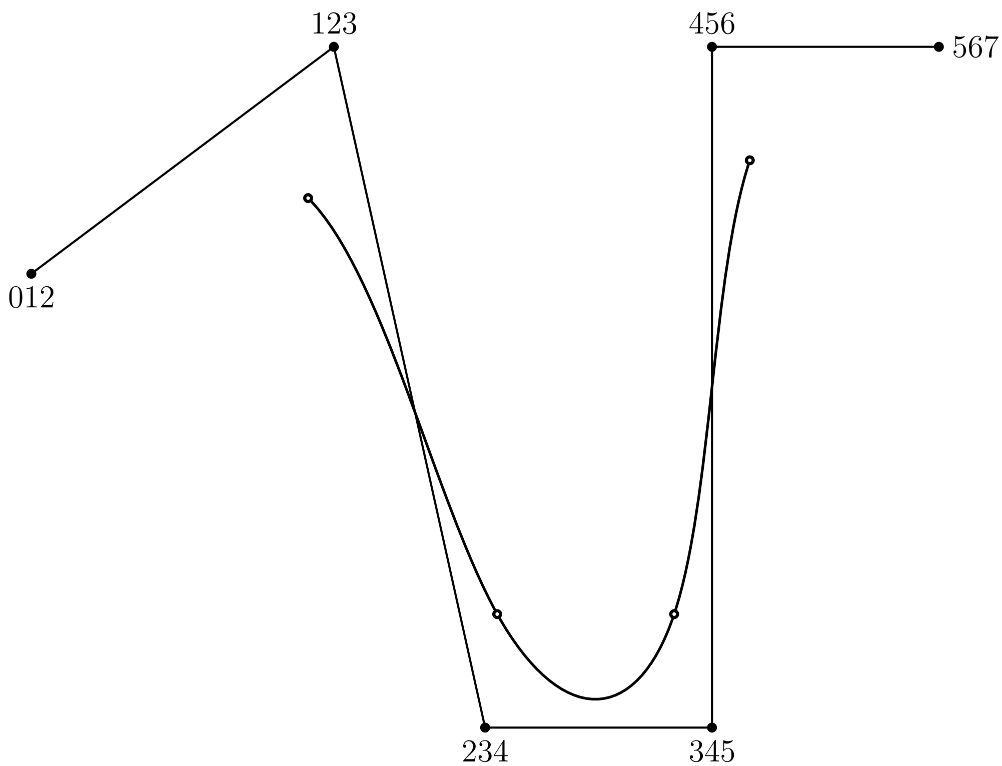


Figure 8.4: Part of a cubic spline with knot sequence $\ldots, 0, 1, 2, 3, 4, 5, 6, 7, \ldots$

The next figure shows the construction of the control points for the three Bézier curve segments constituting this part of the spline.

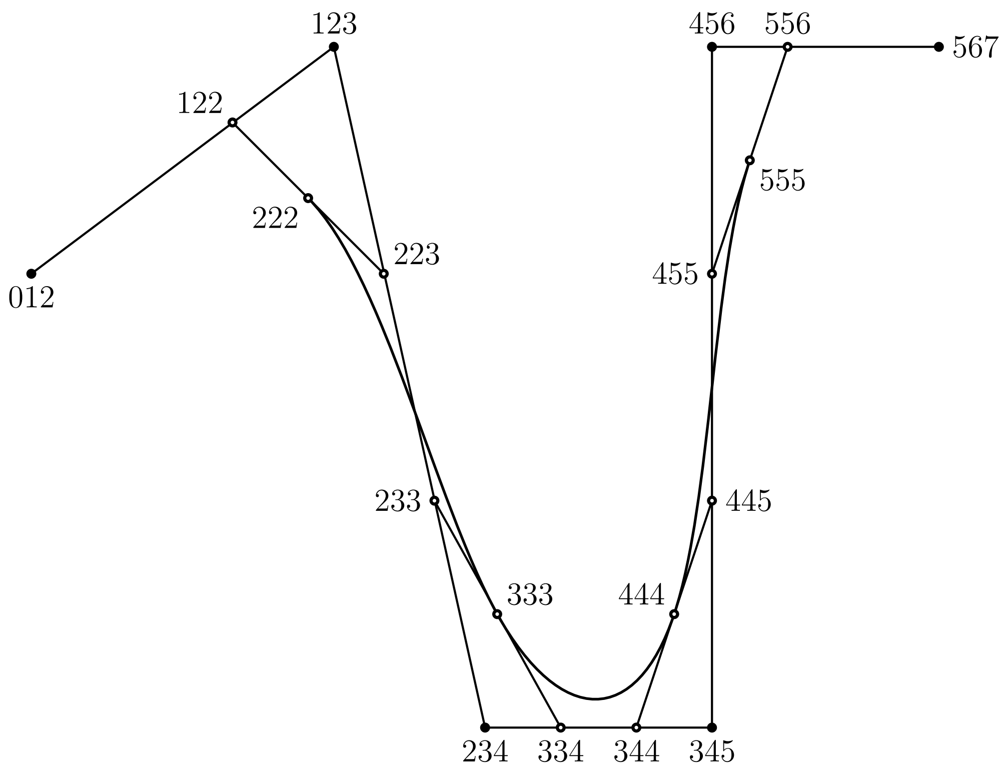


Figure 8.5: Construction of part of a cubic spline with knot sequence $\ldots, 0, 1, 2, 3, 4, 5, 6, 7, \ldots$

Lemma 8.2.3 *Given any $m \geq 1$, given any knot sequence $\langle \overline{u}_k \rangle_{k \in \mathbb{Z}}$ of degree of multiplicity at most $m+1$, for any piecewise polynomial curve F of (polar) degree m based on the knot sequence $\langle \overline{u}_k \rangle_{k \in \mathbb{Z}}$, the curve F is a spline iff the following conditions holds:*

For all i, j, with $i < j \leq i + m$, $\overline{u}_i < \overline{u}_{i+1}$ and $\overline{u}_j < \overline{u}_{j+1}$, the polar forms f_i and f_j agree on all multisets of m elements from $\mathbb{A}$ (supermultisets) containing the multiset of intervening knots

$$\{\overline{u}_{i+1}, \overline{u}_{i+2}, \ldots, \overline{u}_j\}.$$

The following figure shows part of a cubic spline corresponding to the knot sequence

$$\ldots, \overline{r}, \overline{s}, \overline{u}, \overline{v}, \overline{t}, \ldots$$

with C^2-continuity at the knots $\overline{s}, \overline{u}, \overline{v}$.

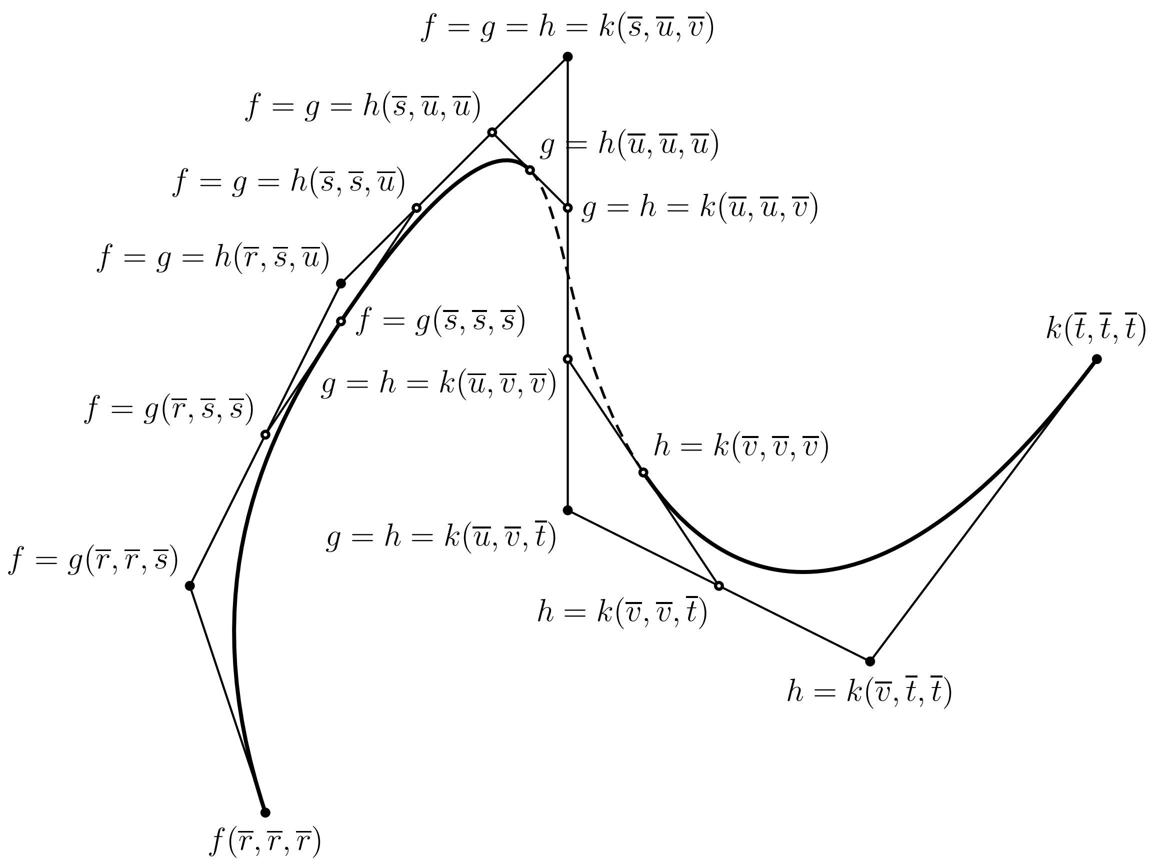


Figure 8.6: Part of a cubic spline with C^2-continuity at the knots $\overline{s}$, $\overline{u}$, $\overline{v}$

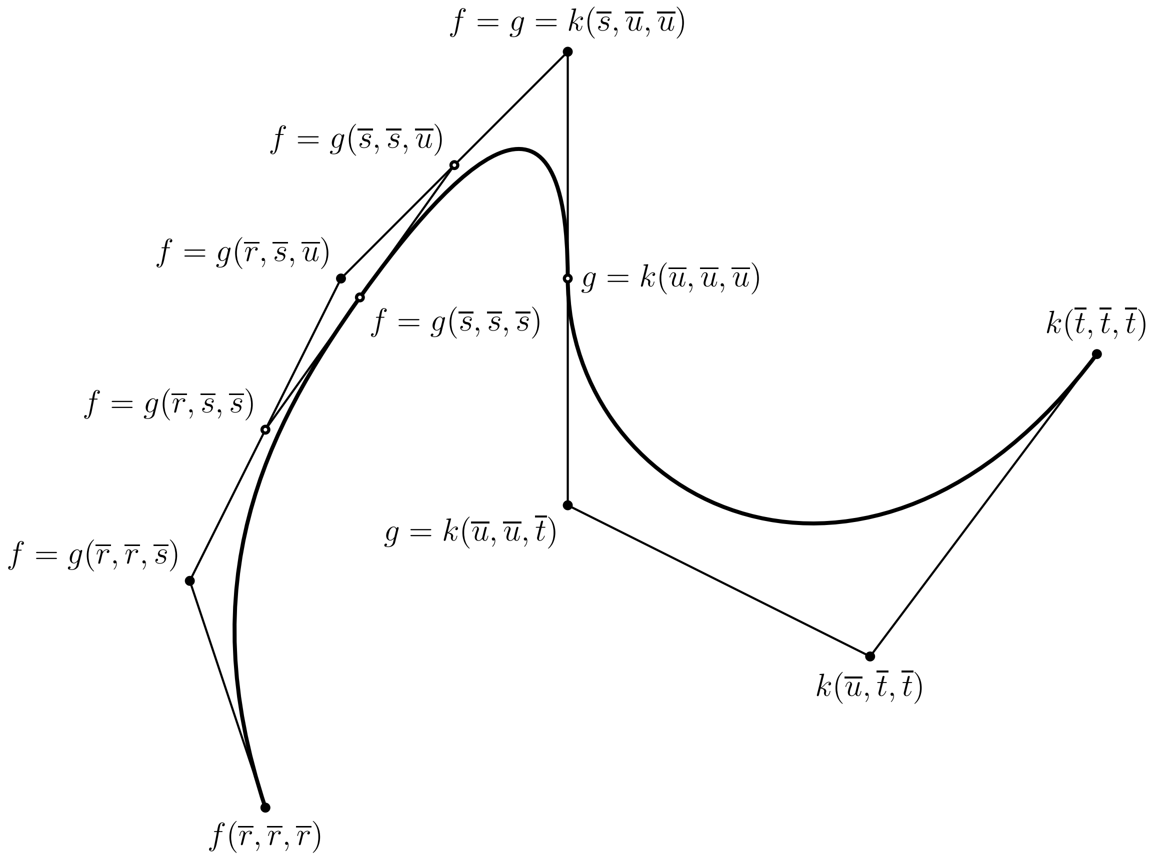


Figure 8.7: Part of a cubic spline with C^1-continuity at the knot $\overline{u}$

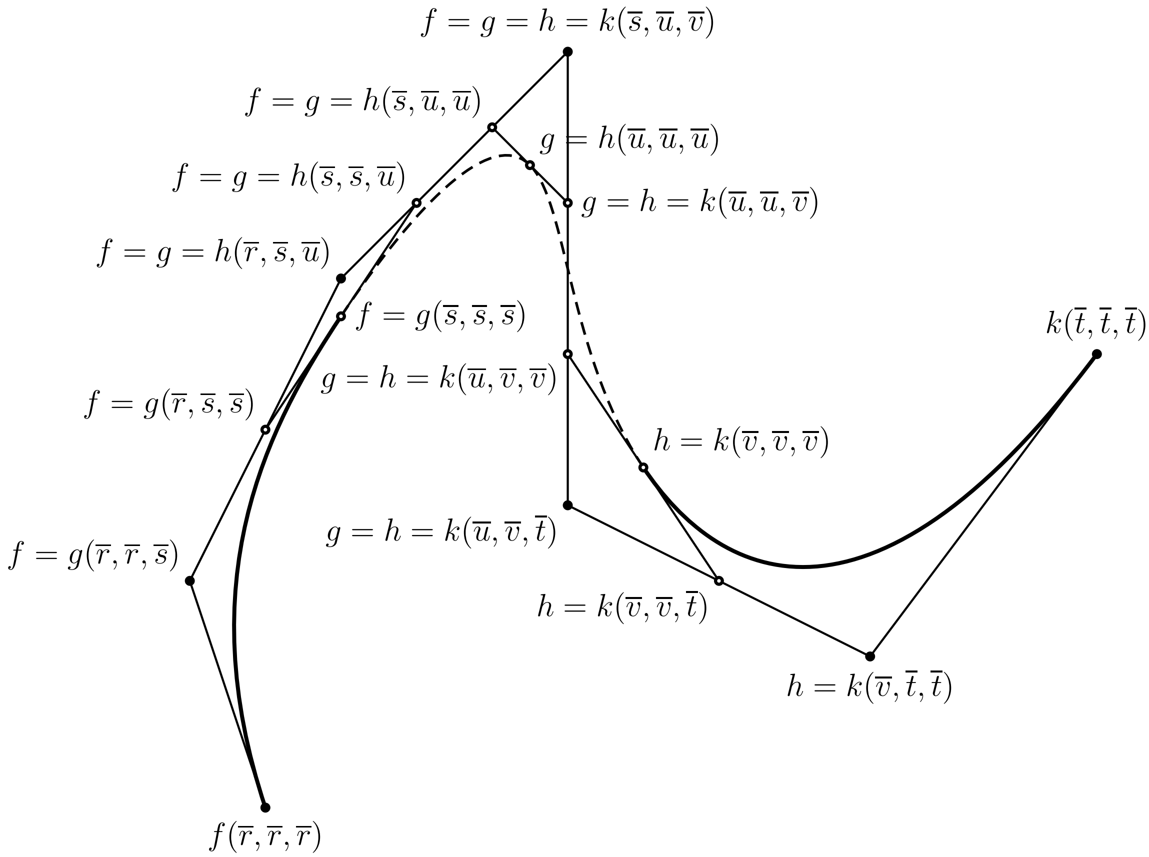


Figure 8.8: Part of a cubic spline with C^2-continuity at the knots $\overline{s}$, $\overline{u}$, $\overline{v}$

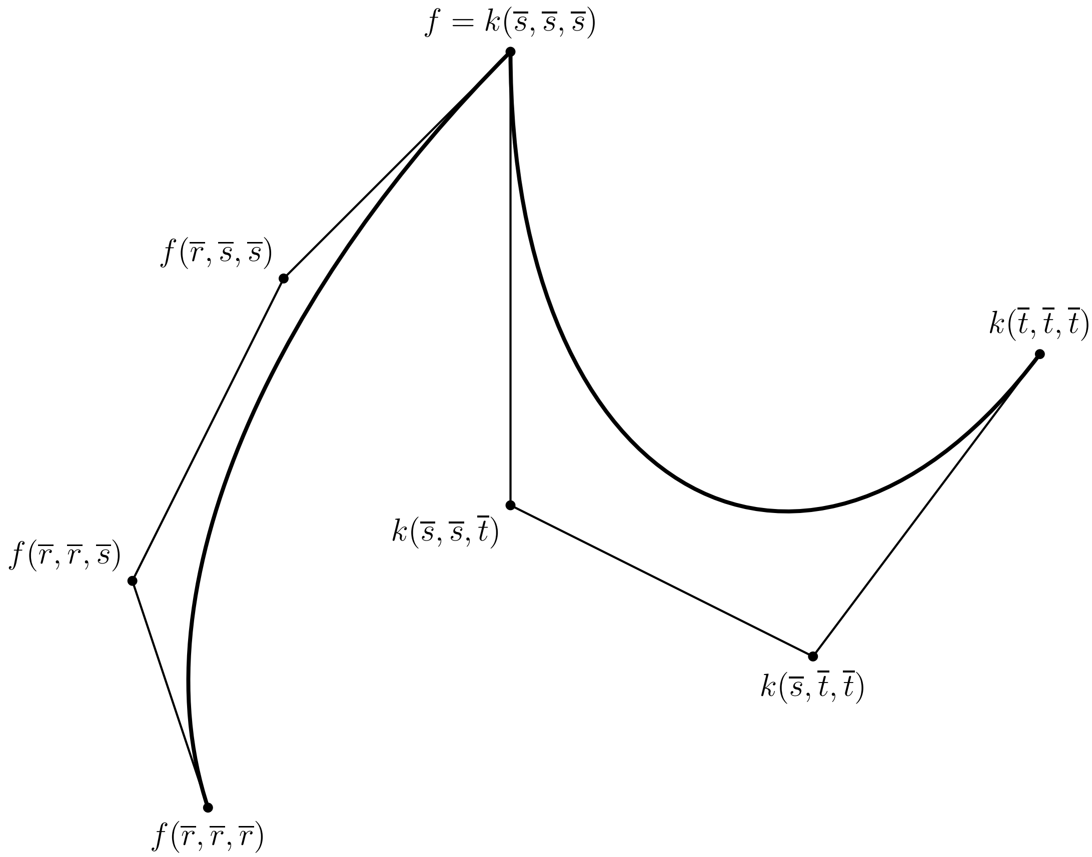


Figure 8.9: Part of a cubic spline with C^0-continuity at the triple knot $\overline{s}$

Theorem 8.2.4 *Given any $m \geq 1$, given any knot sequence $\langle \overline{u}_k \rangle_{k\in\mathbb{Z}}$ of degree of multiplicity at most $m+1$, for any bi-infinite sequence $\langle d_k \rangle_{k\in\mathbb{Z}}$ of points in some affine space $\mathcal{E}$, there exists a unique spline curve $F\colon \mathbb{A} \to \mathcal{E}$, such that the following conditions hold:*

$$d_k = f_i(\overline{u}_{k+1}, \ldots, \overline{u}_{k+m}),$$

for all k, i, where $\overline{u}_i < \overline{u}_{i+1}$ and $k \leq i \leq k+m$.

Given a knot $\overline{u}_i$ in the knot sequence, such that $\overline{u}_i < \overline{u}_{i+1}$, the inequality $k \leq i \leq k+m$ can be interpreted in two ways. If we think of k as fixed, the theorem tells us which curve segments F_i of the spline F are influenced by the specific de Boor point d_k: the de Boor point d_k influences at most $m+1$ curve segments. This is achieved when all the knots are simple.

On the other hand, we can consider i as fixed, and think of the inequalities as $i-m \leq k \leq i$. In this case, the theorem tells us which de Boor points influence the specific curve segment F_i: there are $m+1$ de Boor points that influence the curve segment F_i. This does not depend on the knot multiplicity.

8.3 Finite Knot Sequences, Finite B-Spline Curves

In the case of a finite knot sequence, we have to deal with the two end knots. A reasonable method is to assume that the end knots have multiplicity $m+1$. This way the first curve segment is unconstrained at its left end, and the last curve segment is unconstrained at its right end.

Actually, multiplicity m will give the same results, but multiplicity $m+1$ allows us to view a finite spline curve as a fragment of an infinite spline curve delimited by two discontinuity knots.

Definition 8.3.1 Given any natural numbers $m \geq 1$ and $N \geq 0$, a *finite knot sequence of degree of multiplicity at most $m+1$ with N intervening knots* is any finite nondecreasing sequence $\langle \overline{u}_k \rangle_{-m \leq k \leq N+m+1}$, such that $\overline{u}_{-m} < \overline{u}_{N+m+1}$, and every knot $\overline{u}_k$ has multiplicty at most $m+1$. A knot $\overline{u}_k$ of multiplicity $m+1$ is called a *discontinuity (knot)*. A knot of multiplicity 1 is called a *simple knot*.

Given a finite knot sequence $\langle \overline{u}_k \rangle_{-m \leq k \leq N+m+1}$, of degree of multiplicity at most $m+1$ and with N intervening knots, we now define *the number L of subintervals* in the knot sequence. If $N = 0$, the knot sequence $\langle \overline{u}_k \rangle_{-m \leq k \leq m+1}$ consists of $2(m+1)$ knots, where $\overline{u}_{-m}$ and $\overline{u}_{m+1}$ are distinct and of multiplicity $m+1$, and we let $L = 1$. If $N \geq 1$, then we let $L - 1 \geq 1$ be the number of distinct knots in the sequence $\langle \overline{u}_1, \ldots, \overline{u}_N \rangle$. If the multiplicities of the $L-1$ distinct knots in the sequence $\langle \overline{u}_1, \ldots, \overline{u}_N \rangle$ are $n_1, \ldots, n_{L-1}$ (where $1 \leq n_i \leq m+1$, and $1 \leq i \leq L-1$), then

$$N = n_1 + \cdots + n_{L-1},$$

and the knot sequence $\langle \overline{u}_k \rangle_{-m \leq k \leq N+m+1}$ consists of $2(m+1) + N = 2(m+1) + n_1 + \cdots + n_{L-1}$ knots, with $L+1$ of them distinct.

The picture below gives a clearer idea of the knot multiplicities.

$$\begin{array}{cccccc}
\overline{u}_{-m}, & \overline{u}_1, & \overline{u}_{n_1+1}, & \ldots, & \overline{u}_{N-n_{L-1}+1}, & \overline{u}_{N+1}, \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
\overline{u}_0 & \overline{u}_{n_1} & \overline{u}_{n_1+n_2} & \ldots & \overline{u}_N & \overline{u}_{N+m+1} \\
m+1 & n_1 & n_2 & \ldots & n_{L-1} & m+1
\end{array}$$

Definition 8.3.2 Given any natural numbers $m \geq 1$ and $N \geq 0$, given any finite knot sequence $\langle \overline{u}_k \rangle_{-m \leq k \leq N+m+1}$ of degree of multiplicity at most $m + 1$ and with N intervening knots, a *piecewise polynomial curve of degree m based on the finite knot sequence* $\langle \overline{u}_k \rangle_{-m \leq k \leq N+m+1}$ is a function $F\colon [\overline{u}_0, \overline{u}_{N+1}] \to \mathcal{E}$, where $\mathcal{E}$ is some affine space (of dimension at least 2), such that the following condition hold:

1. If $N = 0$, then $F\colon [\overline{u}_0, \overline{u}_{m+1}] \to \mathcal{E}$ agrees with a polynomial curve F_0 of polar degree m, with associated polar form f_0. When $N \geq 1$, then for any two consecutive distinct knots $\overline{u}_i < \overline{u}_{i+1}$, if $0 \leq i \leq N - n_{L-1}$, then the restriction of F to $[\overline{u}_i, \overline{u}_{i+1}[$ agrees with a polynomial curve F_i of polar degree m, with associated polar form f_i, and if $i = N$, then the restriction of F to $[\overline{u}_N, \overline{u}_{N+1}]$ agrees with a polynomial curve F_N of polar degree m, with associated polar form f_N.

A *spline curve* F *of degree* m *based on the finite knot sequence* $\langle \overline{u}_k \rangle_{-m \leq k \leq N+m+1}$ or for short, a *finite* B*-spline*, is a piecewise polynomial curve $F\colon [\overline{u}_0, \overline{u}_{N+1}] \to \mathcal{E}$, such that, when $N \geq 1$ and $L \geq 2$, for every two consecutive distinct knots $\overline{u}_i < \overline{u}_{i+1}$ (where $0 \leq i \leq N - n_{L-1}$), if $\overline{u}_{i+1}$ has multiplicity $n \leq m+1$, the following condition holds:

2. The curve segments F_i and F_{i+n} join with continuity (at least) C^{m-n} at $\overline{u}_{i+1}$, in the sense of definition 7.5.1.

The set $F([\overline{u}_0, \overline{u}_{N+1}])$ is called the *trace of the finite spline* F.

Remark: The remarks about discontinuities made after definition 8.2.2 also apply. However, we also want the last curve segment F_N to be defined at $\overline{u}_{N+1}$. Note that if we assume that $\overline{u}_0$ and $\overline{u}_{N+1}$ have multiplicity m, then we get the same curve. However, using multiplicity $m+1$ allows us to view a finite spline as a fragment of an infinite spline.

Note that a spline curve defined on the finite knot sequence

$$\begin{array}{cccccc}
\overline{u}_{-m}, & \overline{u}_1, & \overline{u}_{n_1+1}, & \ldots, & \overline{u}_{N-n_{L-1}+1}, & \overline{u}_{N+1}, \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
\overline{u}_0 & \overline{u}_{n_1} & \overline{u}_{n_1+n_2} & \ldots & \overline{u}_N & \overline{u}_{N+m+1} \\
m+1 & n_1 & n_2 & \ldots & n_{L-1} & m+1
\end{array}$$

where

$$N = n_1 + \cdots + n_{L-1},$$

consists of L curve segments,
F_0, $F_{n_1}, \ldots$, $F_{n_1+\cdots+n_{L-1}} = F_N$.

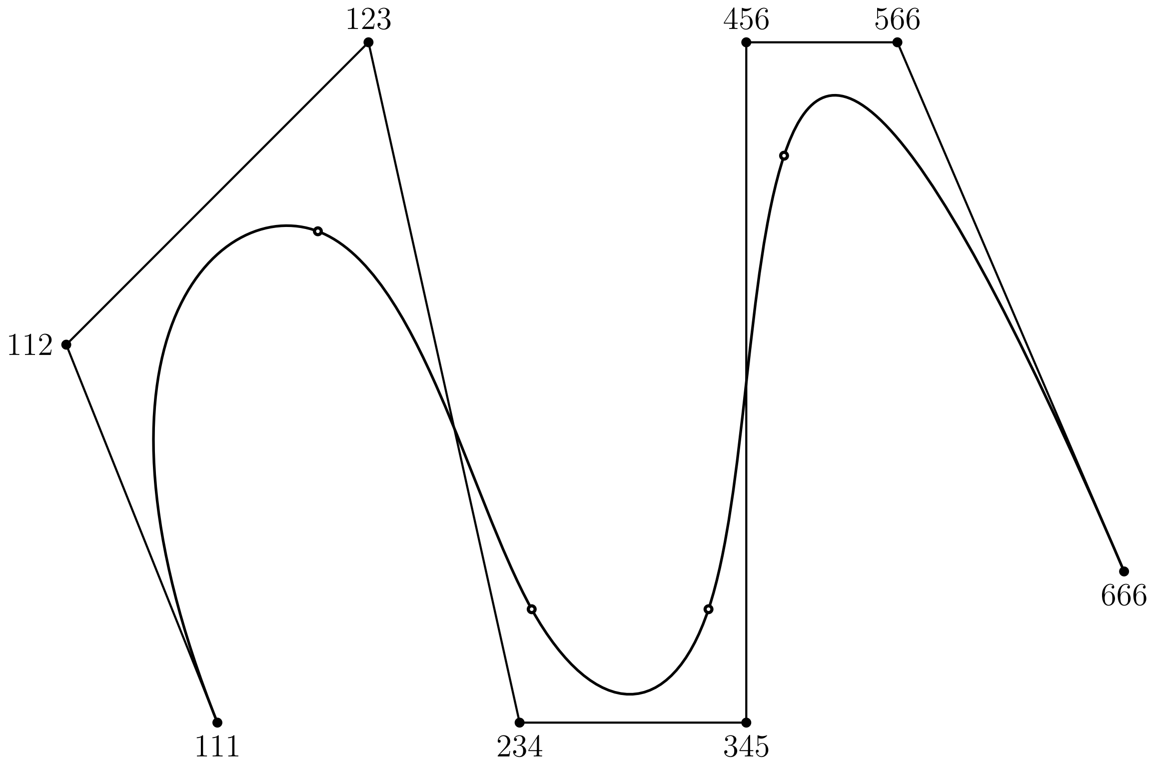


Figure 8.10: A cubic spline with knot sequence $1, 1, 1, 1, 2, 3, 4, 5, 6, 6, 6, 6$

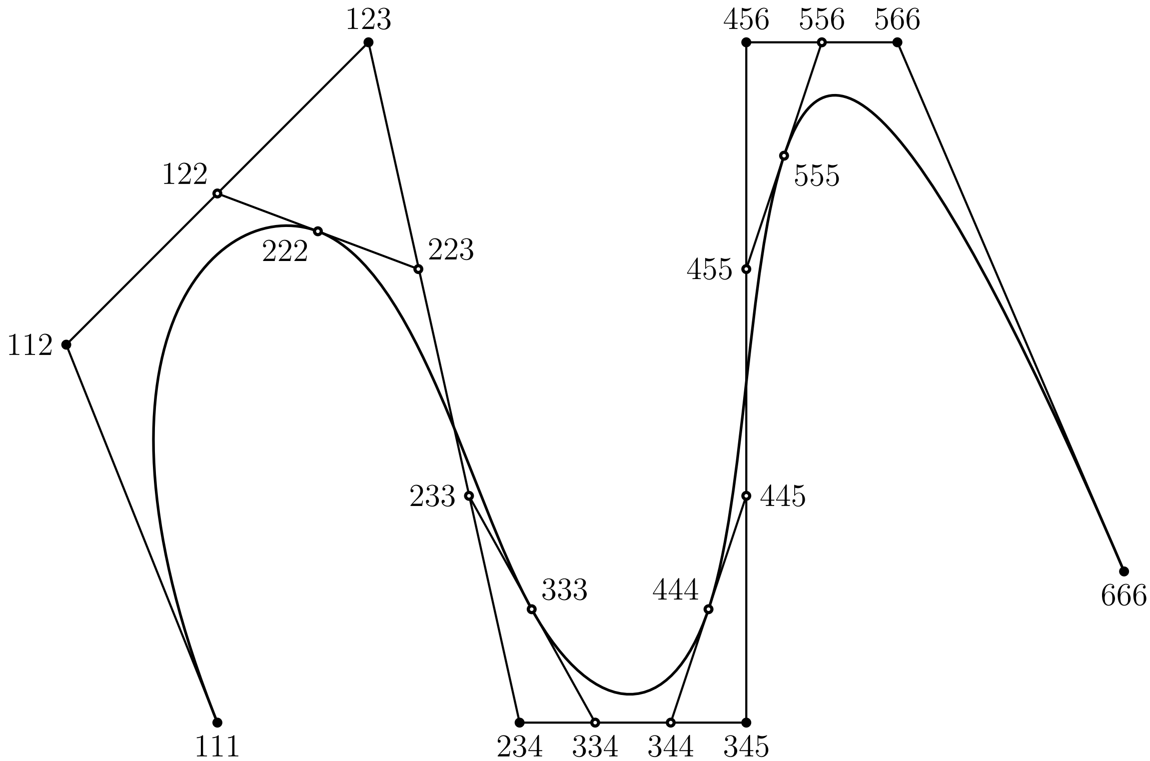


Figure 8.11: Construction of a cubic spline with knot sequence $1, 1, 1, 1, 2, 3, 4, 5, 6, 6, 6, 6$

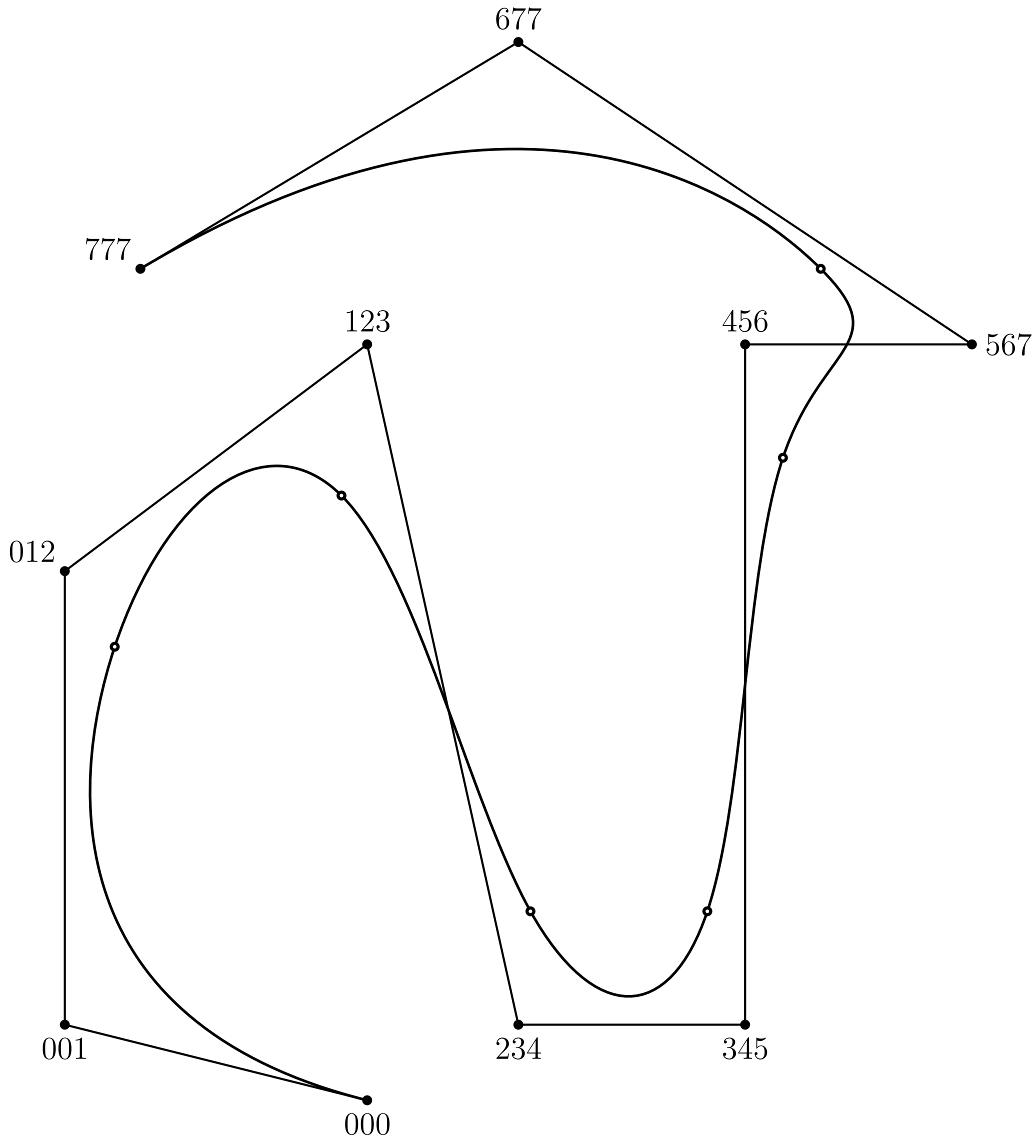


Figure 8.12: A cubic spline with knot sequence $0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 7, 7, 7$

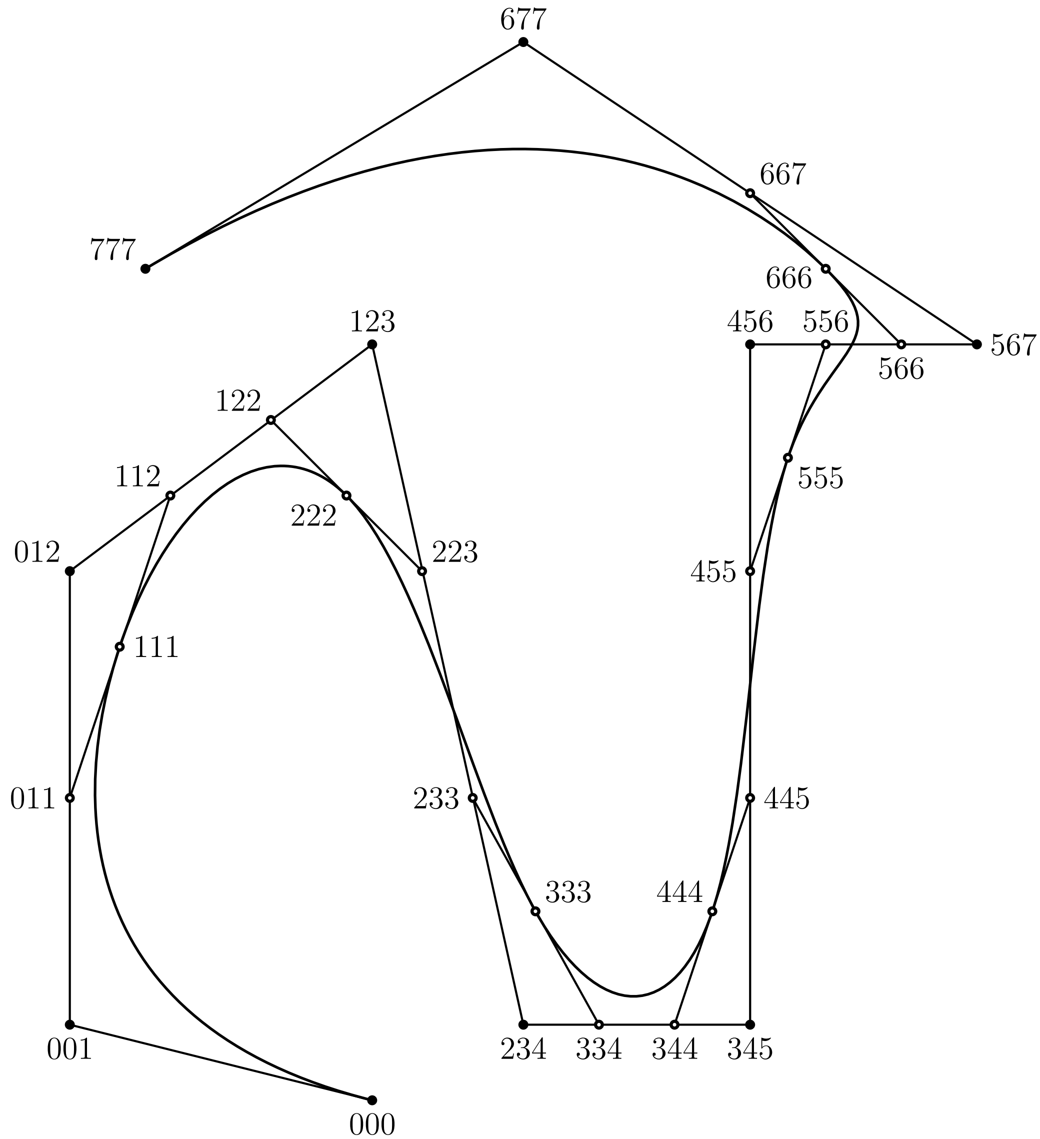


Figure 8.13: Construction of a cubic spline with knot sequence $0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 7, 7, 7$

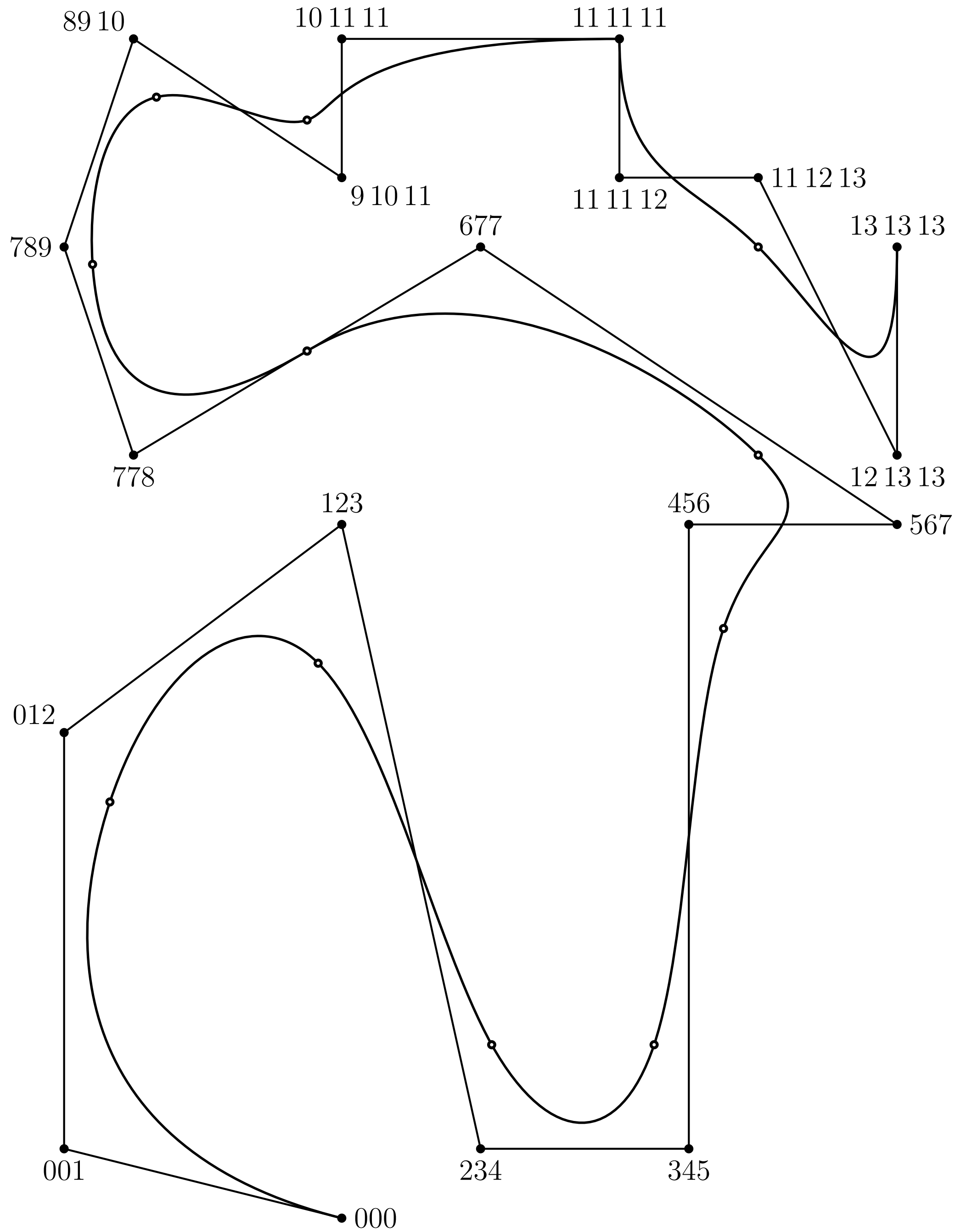


Figure 8.14: A cubic spline with knot sequence
0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 7, 8, 9, 10, 11, 11, 11, 12, 13, 13, 13, 13

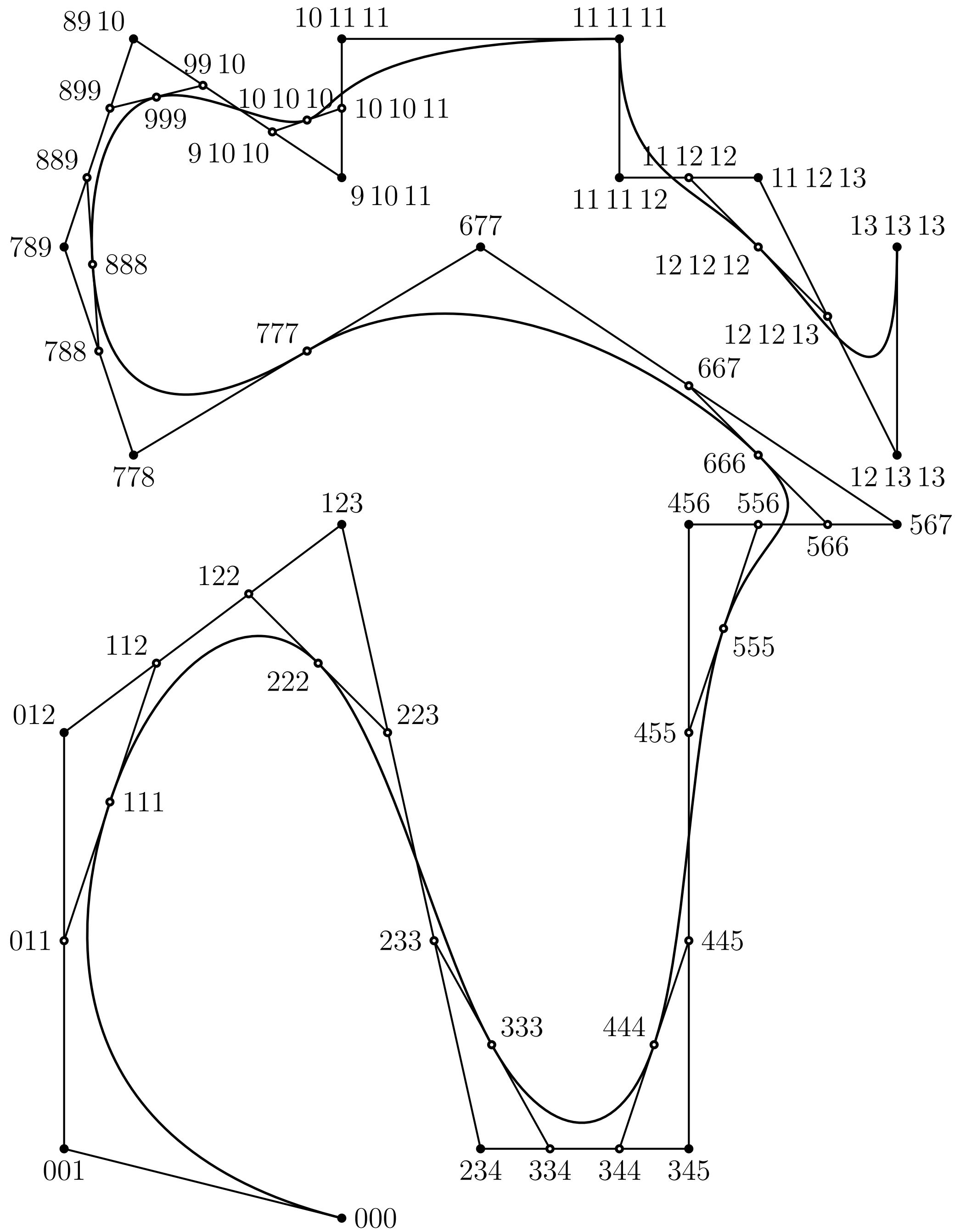


Figure 8.15: Construction of a cubic spline with knot sequence
$0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 7, 8, 9, 10, 11, 11, 11, 12, 13, 13, 13, 13$

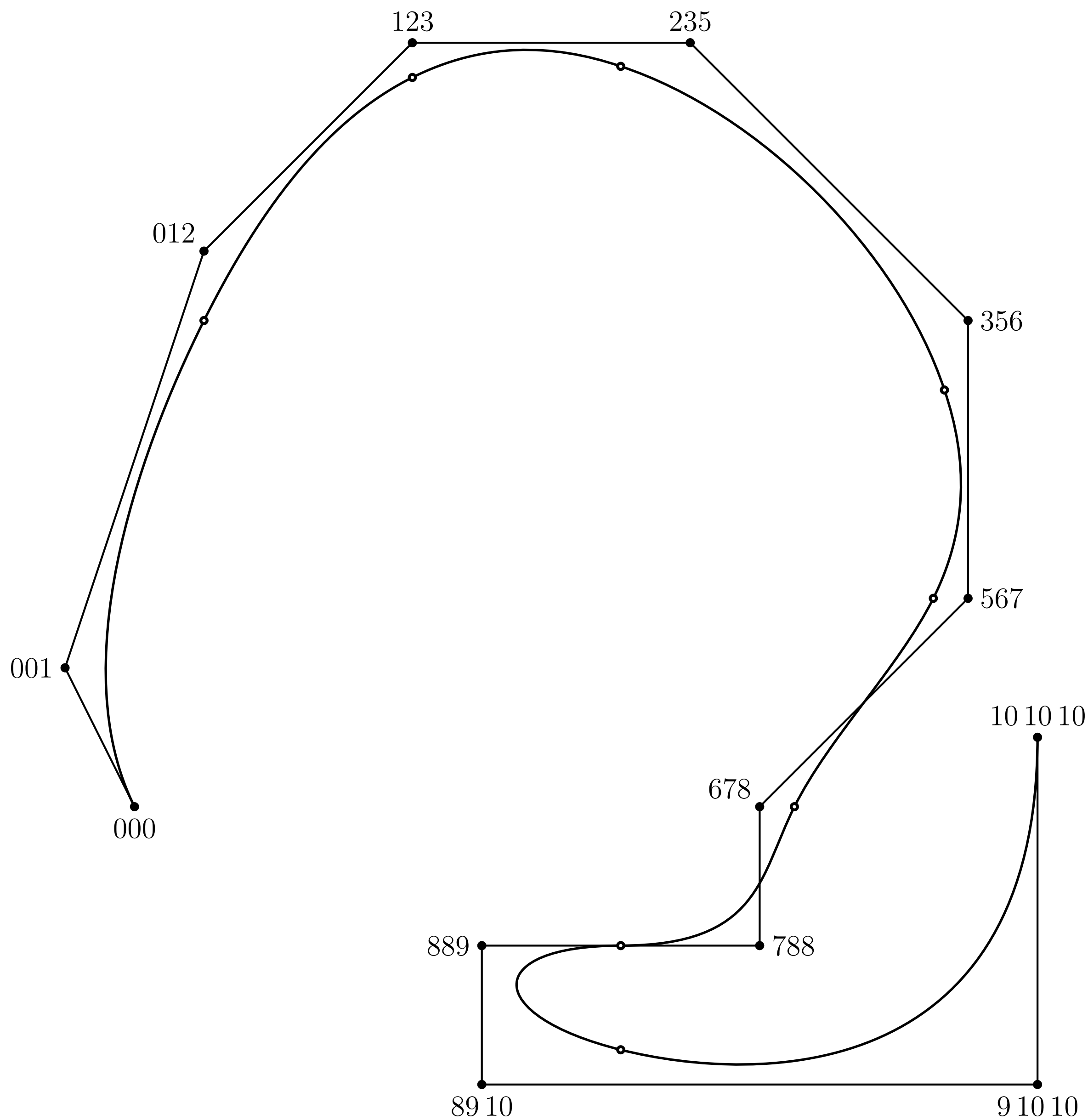


Figure 8.16: A cubic spline with non-uniform knot sequence
$0, 0, 0, 0, 1, 2, 3, 5, 6, 7, 8, 8, 9, 10, 10, 10, 10$

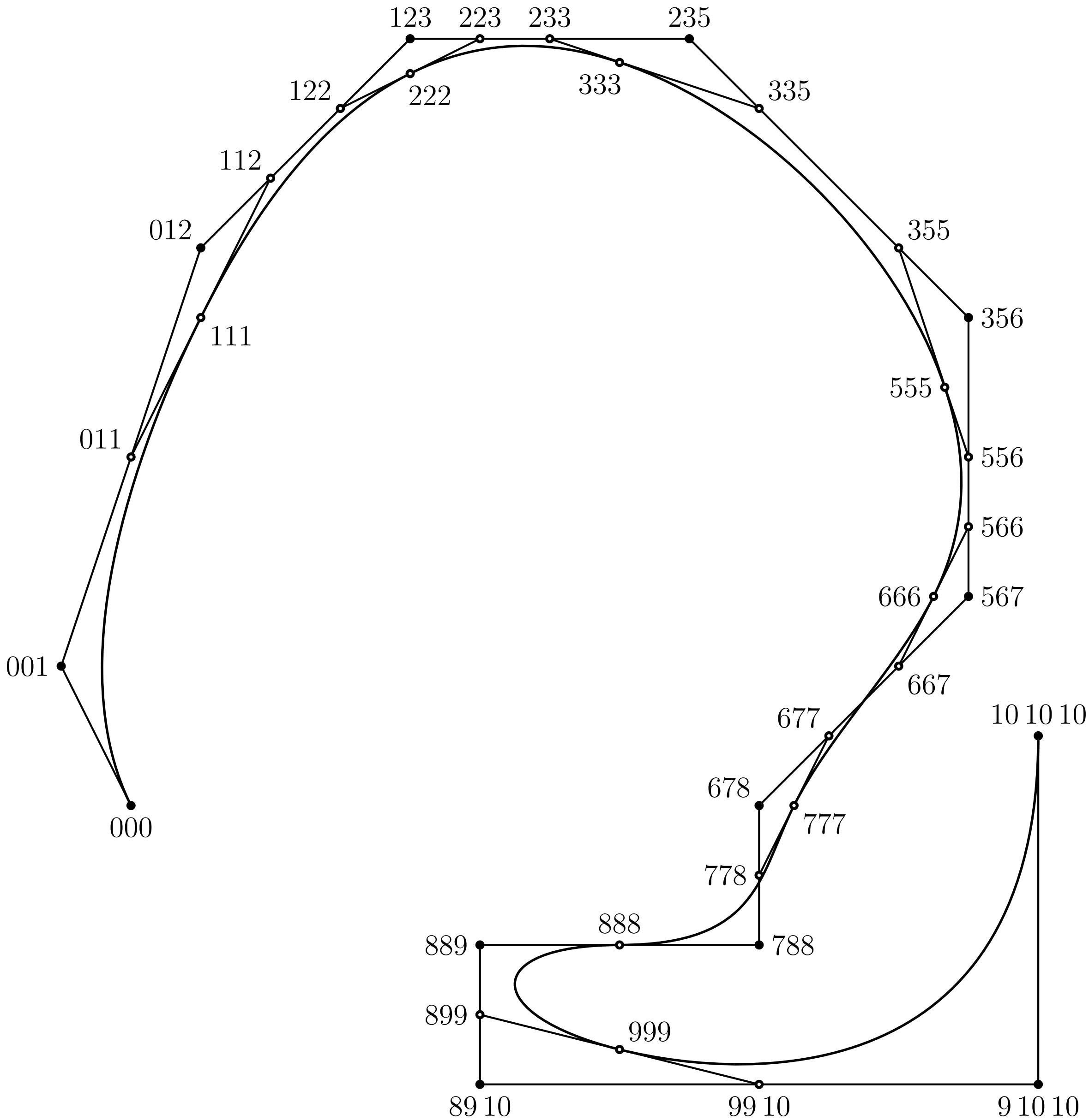


Figure 8.17: Construction of a cubic spline with non-uniform knot sequence
$0, 0, 0, 0, 1, 2, 3, 5, 6, 7, 8, 8, 9, 10, 10, 10, 10$

8.4 Cyclic Knot Sequences, Closed B-Spline Curves

Definition 8.4.1 A *cyclic knot sequence of period L, cycle length N, and period size T*, is any bi-infinite nondecreasing sequence $\langle \overline{u}_k \rangle_{k\in\mathbb{Z}}$ of points $\overline{u}_k \in \mathbb{A}$ (i.e. $\overline{u}_k \leq \overline{u}_{k+1}$ for all $k \in \mathbb{Z}$), where $L, N, T \in \mathbb{N}$, $L \geq 2$, and $N \geq L$, such that there is some subsequence

$$\langle \overline{u}_{j+1}, \ldots, \overline{u}_{j+N} \rangle$$

of N consecutive knots containing exactly L distinct knots, with multiplicities $n_1, \ldots, n_L$, $\overline{u}_{j+N} < \overline{u}_{j+N+1}$, and $\overline{u}_{k+N} = \overline{u}_k + T$, for every $k \in \mathbb{Z}$. Note that we must have $N = n_1 + \cdots + n_L$ (and $n_i \geq 1$). Given any natural number $m \geq 1$, a cyclic knot sequence of period L, cycle length N, and period size T, has *degree of multiplicity at most m*, iff every knot has multiplicity at most m.

As before, a knot sequence (finite, or cyclic) is *uniform* iff $\overline{u}_{k+1} = \overline{u}_k + h$, for some fixed $h \in \mathbb{R}_+$.

A cyclic knot sequence of period L, cycle length N, and period size T, is completely determined by a sequence of $N+1$ consecutive knots, which looks as follows (assuming for simplicity that the index of the starting knot of the cycle that we are looking at is $k = 1$):

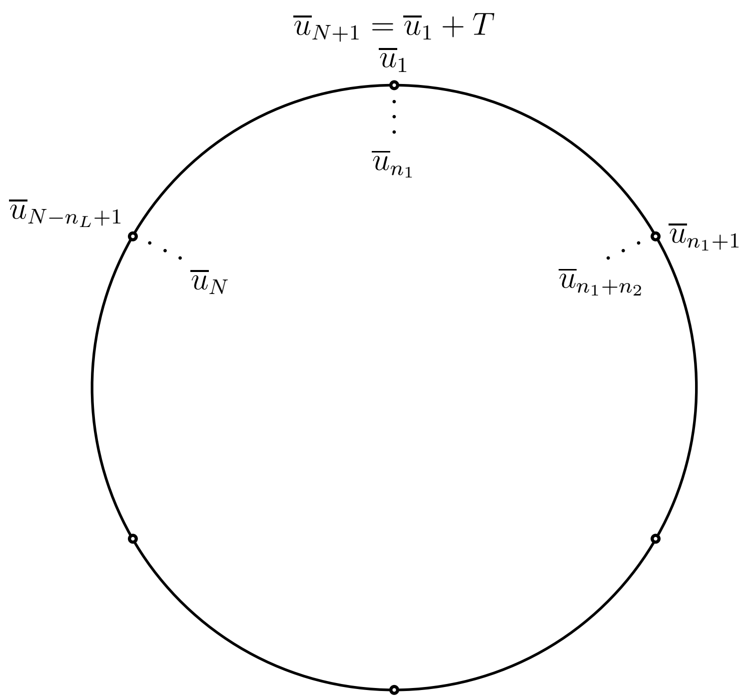


Figure 8.18: A cyclic knot sequence

Closed B-spline curves are defined as follows.

Definition 8.4.2 Given any natural number $m \geq 1$, given any cyclic knot sequence $\langle \overline{u}_k \rangle_{k \in \mathbb{Z}}$ of period L, cycle length N, period size T, and of degree of multiplicity at most m, a *closed piecewise polynomial curve of (polar) degree m based on the cyclic knot sequence* $\langle \overline{u}_k \rangle_{k \in \mathbb{Z}}$, is a function $F \colon \mathbb{A} \to \mathcal{E}$, where $\mathcal{E}$ is some affine space (of dimension at least 2), such that, for any two consecutive distinct knots $\overline{u}_i < \overline{u}_{i+1}$, if $\overline{u}_{i+1}$ is a knot of multiplicity n, the next distinct knot being $\overline{u}_{i+n+1}$, then the following condition hold:

1. The restriction of F to $[\overline{u}_i, \overline{u}_{i+1}]$ agrees with a polynomial curve F_i of polar degree m, with associated polar form f_i, and $F_{i+N}(\overline{t}+T) = F_i(\overline{t})$, for all $\overline{t} \in [\overline{u}_i, \overline{u}_{i+1}]$.

A *closed spline curve F of (polar) degree m based on the cyclic knot sequence* $\langle \overline{u}_k \rangle_{k \in \mathbb{Z}}$ of period L, cycle length N, and period size T, is a closed piecewise polynomial curve $F \colon \mathbb{A} \to \mathcal{E}$, such that, for every two consecutive distinct knots $\overline{u}_i < \overline{u}_{i+1}$, the following condition holds:

2. The curve segments F_i and F_{i+n} join with continuity (at least) C^{m-n} at $\overline{u}_{i+1}$, in the sense of definition 7.5.1, where n is the multiplicity of the knot $\overline{u}_{i+1}$ $(1 \leq n \leq m)$.

The set $F(\mathbb{A})$ is called the *trace of the closed spline F*.

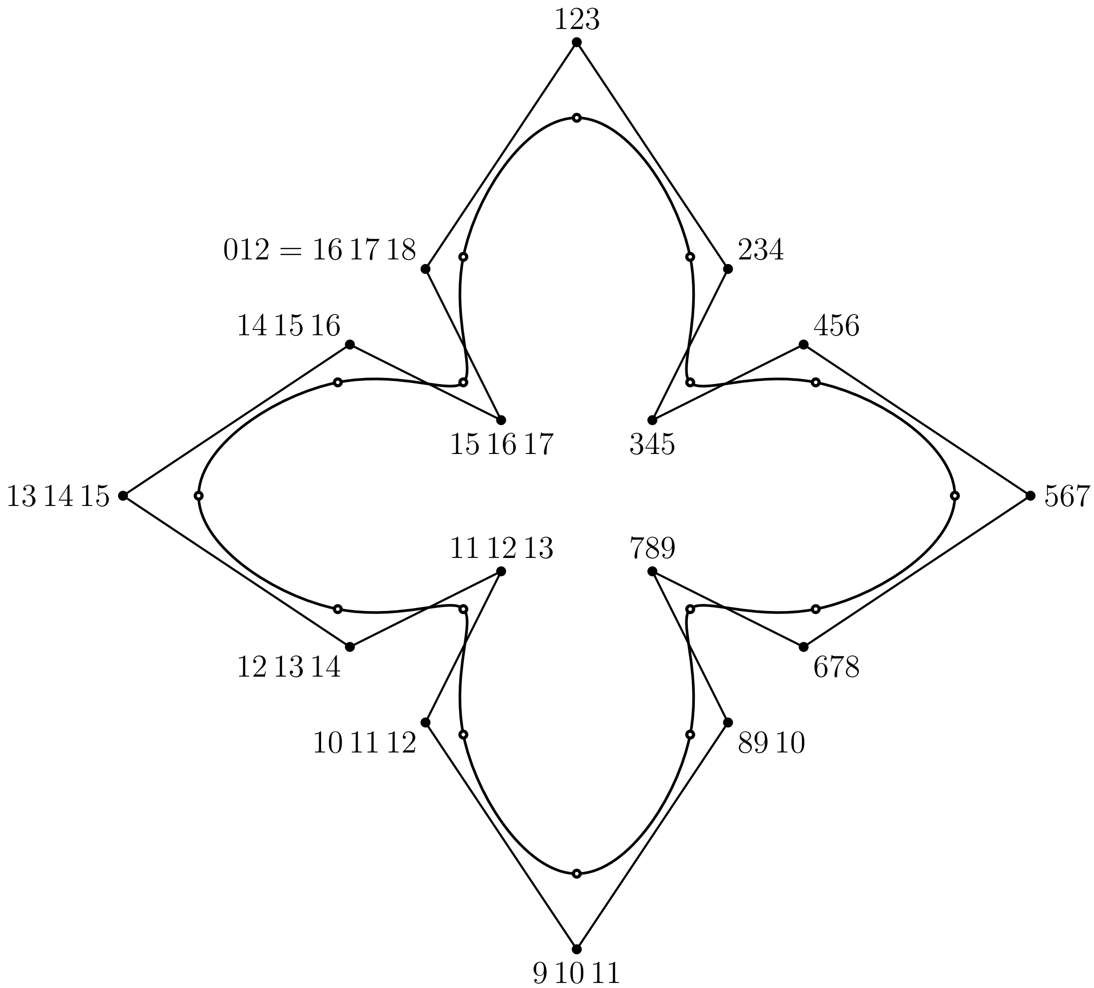


Figure 8.19: A closed cubic spline with cyclic knot sequence of period 16, cycle length 16, and period size 16: $\ldots, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, \ldots$

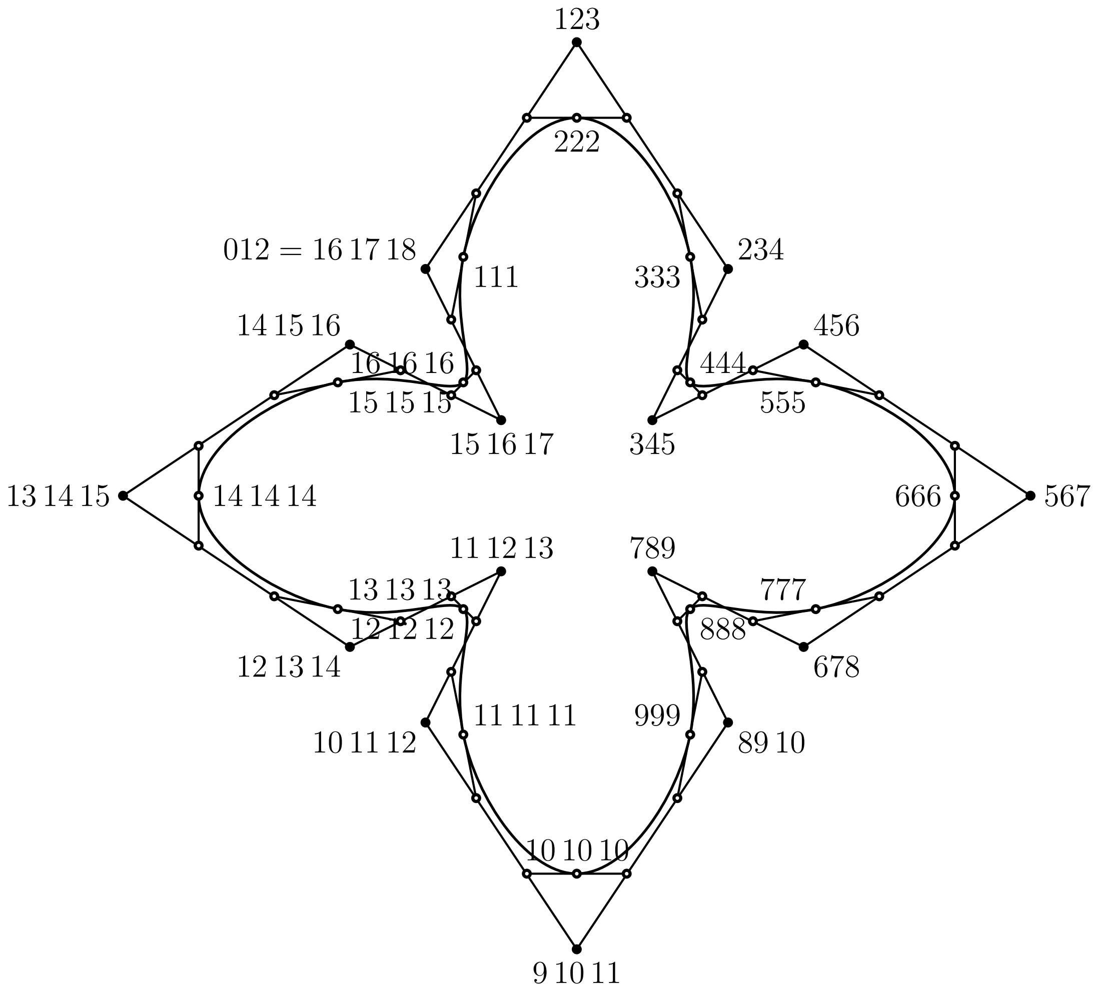


Figure 8.20: Construction of a closed cubic spline with cyclic knot sequence of period 16, cycle length 16, and period size 16: $\ldots, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, \ldots$

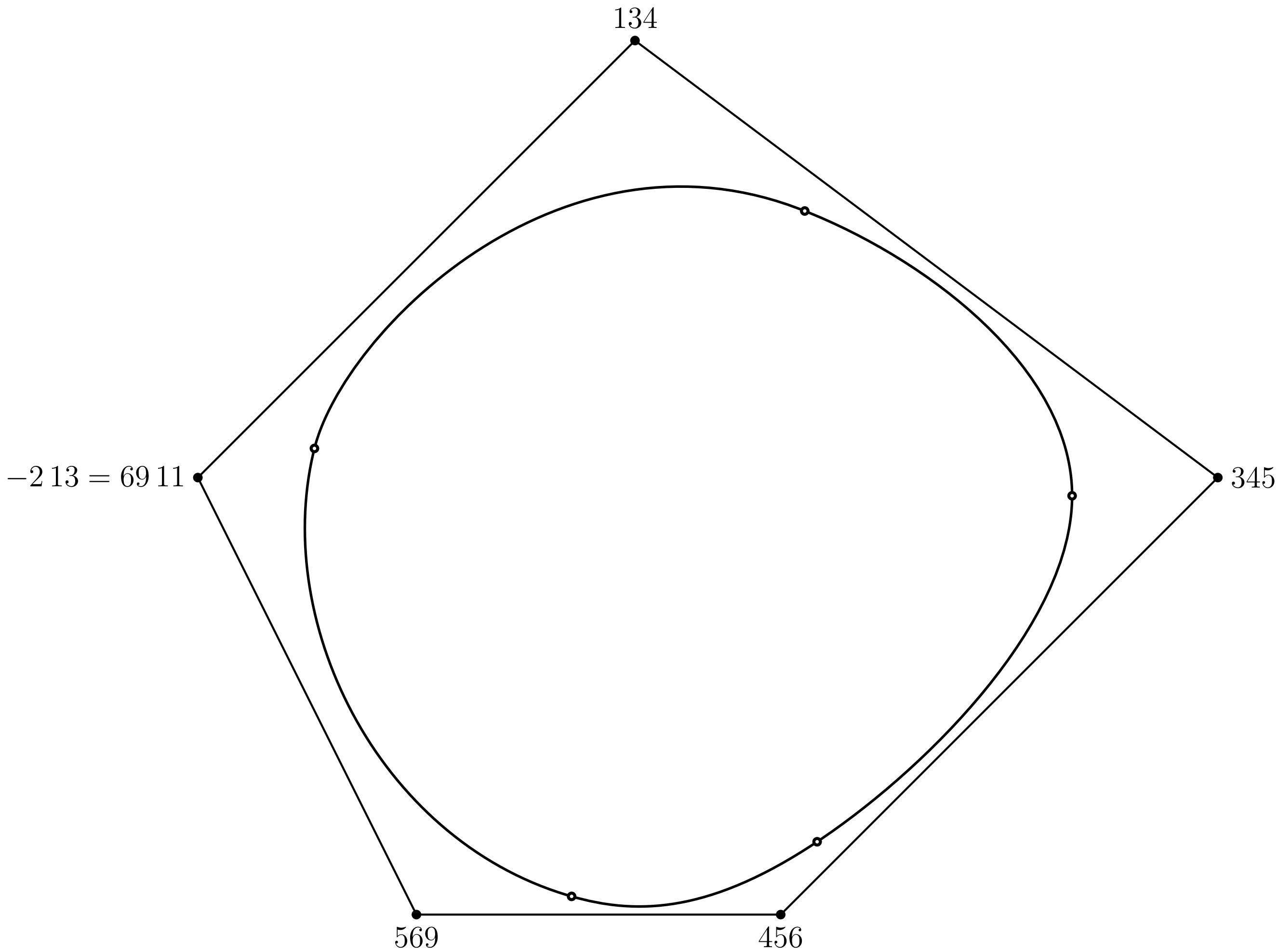


Figure 8.21: A closed cubic spline with cyclic knot sequence $\ldots, -3, -2, 1, 3, 4, 5, 6, 9, 11, 12, \ldots$, with $L = N = 5, T = 8$

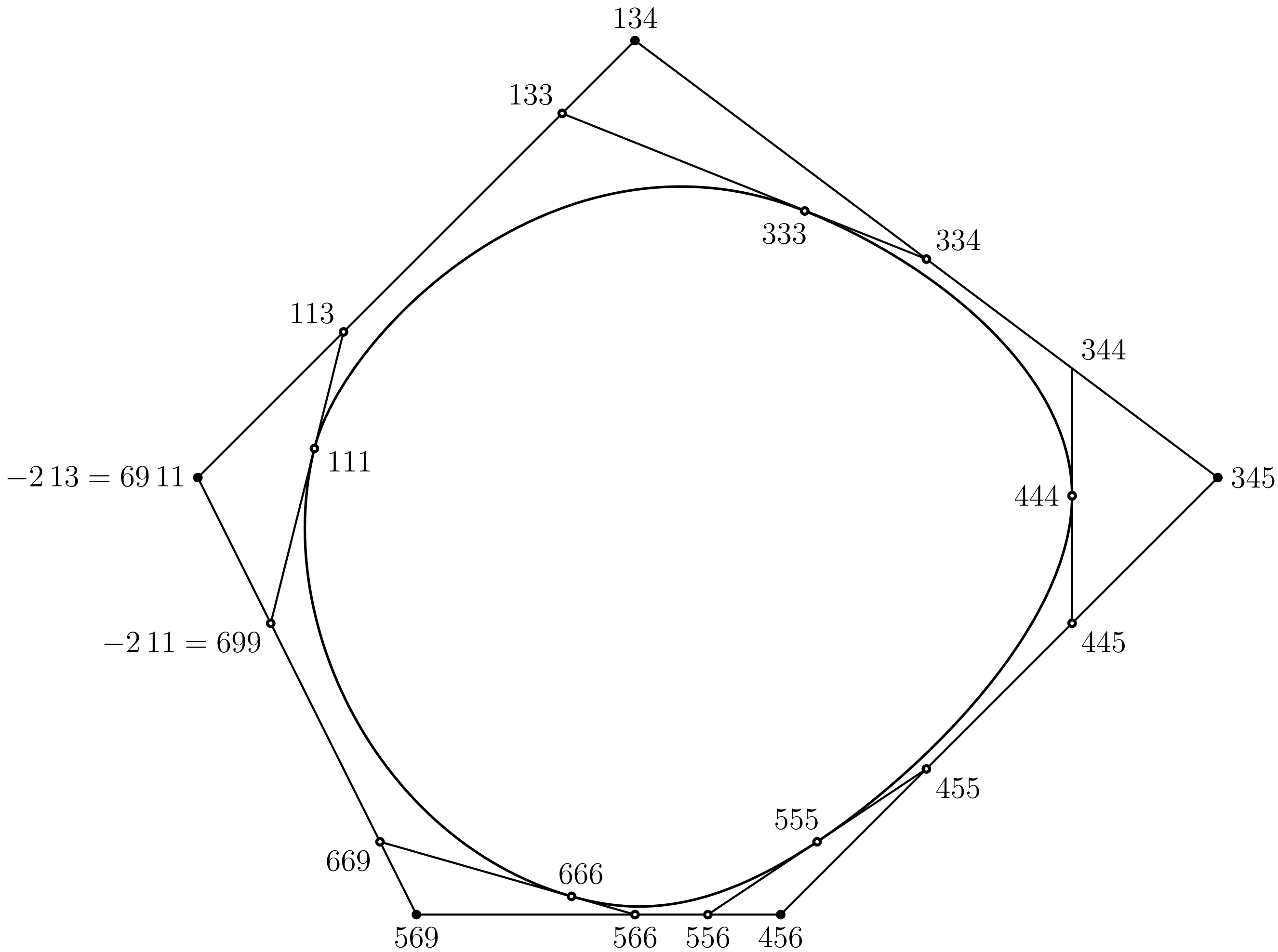


Figure 8.22: Construction of a closed cubic spline with cyclic knot sequence $\ldots, -3, -2, 1, 3, 4, 5, 6, 9, 11, 12, \ldots$, with $L = N = 5, T = 8$

Theorem 8.4.3 *(1) Given any $m \geq 1$, given any finite knot sequence $\langle \overline{u}_k \rangle_{-m \leq k \leq N+m+1}$ of degree of multiplicity at most $m+1$, for any sequence $\langle d_{-m}, \ldots, d_N \rangle$ of $N+m+1$ points in some affine space $\mathcal{E}$, there exists a unique spline curve $F\colon [\overline{u}_0, \overline{u}_{N+1}] \to \mathcal{E}$, such that the following conditions hold:*

$$d_k = f_i(\overline{u}_{k+1}, \ldots, \overline{u}_{k+m}),$$

for all k, i, where $-m \leq k \leq N$, $\overline{u}_i < \overline{u}_{i+1}$, and $k \leq i \leq k+m$.

(2) Given any $m \geq 1$, given any finite cyclic knot sequence $\langle \overline{u}_k \rangle_{k \in \mathbb{Z}}$ of period L, cycle length N, period size T, and of degree of multiplicity at most m, for any bi-infinite periodic sequence $\langle d_k \rangle_{k \in \mathbb{Z}}$ of period N of points in some affine space $\mathcal{E}$, i.e., sequence such that $d_{k+N} = d_k$ for all $k \in \mathbb{Z}$, there exists a unique closed spline curve $F\colon \mathbb{A} \to \mathcal{E}$, such that the following conditions hold:

$$d_k = f_i(\overline{u}_{k+1}, \ldots, \overline{u}_{k+m}),$$

for all k, i, where $\overline{u}_i < \overline{u}_{i+1}$ and $k \leq i \leq k+m$.

8.5 The de Boor Algorithm, and Knot Insertion

Given a knot sequence $\langle \overline{u}_k \rangle$ (infinite, finite, of cyclic), and a sequence $\langle d_k \rangle$ of control points (corresponding to the nature of the knot sequence), given any parameter $\overline{t} \in \mathbb{A}$ (where $\overline{t} \in [\overline{u}_0, \overline{u}_{N+1}]$, in case of a finite spline), in order to compute the point $F(\overline{t})$ on the spline curve F determined by $\langle \overline{u}_k \rangle$ and $\langle d_k \rangle$, we just have to find the interval $[\overline{u}_I, \overline{u}_{I+1}]$ for which $\overline{u}_I \leq \overline{t} < \overline{u}_{I+1}$, and then to apply the progressive version of the de Casteljau algorithm, starting from the $m+1$ control points indexed by the sequences $\langle \overline{u}_{I-m+k}, \ldots, \overline{u}_{I+k-1} \rangle$, where
$1 \leq k \leq m+1$.

As in section 7.3, let us assume for simplicity that $I = m$, since the indexing will be a bit more convenient. Indeed, in this case $[\overline{u}_m, \overline{u}_{m+1}]$ is the middle of the sequence $\langle \overline{u}_1, \ldots, \overline{u}_{2m} \rangle$ of length $2m$. For the general case, we translate all knots by $I - m$.

Recall that $F(\overline{t}) = f(\overline{t}, \ldots, \overline{t})$ is computed by iteration, as the point $b_{0,m}$ determined by the inductive computation

$$b_{k,j} = \left(\frac{u_{m+k+1} - t}{u_{m+k+1} - u_{k+j}}\right) b_{k,j-1} + \left(\frac{t - u_{k+j}}{u_{m+k+1} - u_{k+j}}\right) b_{k+1,j-1},$$

where

$$b_{k,j} = f(\overline{t}^{\,j}\,\overline{u}_{k+j+1} \ldots \overline{u}_{m+k}),$$

for $1 \leq j \leq m$, $0 \leq k \leq m - j$, and with
$b_{k,0} = f(\overline{u}_{k+1}, \ldots, \overline{u}_{m+k}) = d_k$, for $0 \leq k \leq m$.

The computation proceeds by rounds, and during round j, the points $b_{0,j}$, $b_{1,j}$, ..., $b_{m-j,j}$ are computed.

If $\overline{t} = \overline{u}_m$, and the knot $\overline{u}_m$ has multiplicity r $(1 \leq r \leq m)$, we notice that

$$b_{0,m-r} = b_{0,m-r+1} = \ldots = b_{0,m},$$

because $b_{0,m-r} = f(\overline{t}^{\,m-r}\overline{u}_{m-r+1} \ldots \overline{u}_m) = F(\overline{t})$, since $\overline{u}_m$ is of multiplicity r, and $\overline{t} = \overline{u}_{m-r+1} = \ldots = \overline{u}_m$.

Thus, in this case, we only need to start with the $m-r+1$ control points $b_{0,0}, \ldots, b_{m-r,0}$, and we only need to construct the part of the triangle above the ascending diagonal

$$b_{m-r,0},\; b_{m-r-1,1},\; \ldots,\; b_{0,m-r}.$$

It turns out that it is convenient in order to present the de Boor algorithm, to index the points $b_{k,j}$ differently. First, we will label the starting control points as $d_{1,0}, \ldots, d_{m+1,0}$, and second, for every round j, rather than indexing the points on the j-th column with an index k always starting from 0, and running up to $m-j$, it will be convenient to index the points in the j-th column with an index i starting at $j+1$, and always ending at $m+1$.

Thus, at round j, the points

$$b_{0,j},\, b_{1,j},\, \ldots,\, b_{k,j},\, \ldots,\, b_{m-j,j}$$

indexed using our original indexing, will correspond the to the points

$$d_{j+1,j},\, d_{j+2,j},\, \ldots,\, d_{k+j+1,j},\, \ldots,\, d_{m+1,j},$$

under our new indexing.

As we can easily see, the inductive relation giving $d_{k+j+1,j}$ in terms of $d_{k+j+1,j-1}$ and $d_{k+j,j-1}$, is given by the equation:

$$
\begin{aligned}
d_{k+j+1,j} = (\frac{u_{m+k+1} - t}{u_{m+k+1} - u_{k+j}})d_{k+j,j-1} \\
+ (\frac{t - u_{k+j}}{u_{m+k+1} - u_{k+j}})d_{k+j+1,j-1},
\end{aligned}
$$

where $1 \leq j \leq m - r$, $0 \leq k \leq m - r - j$, and with $d_{k+1,0} = d_k$, when $0 \leq k \leq m - r$, where r is the multiplicity of $\overline{u}_m$ when $\overline{t} = \overline{u}_m$, and $r = 0$ otherwise.

Letting $i = k + j + 1$, the above equation becomes

$$d_{i,j} = (\frac{u_{m+i-j} - t}{u_{m+i-j} - u_{i-1}})d_{i-1,j-1} + (\frac{t - u_{i-1}}{u_{m+i-j} - u_{i-1}})d_{i,j-1},$$

where $1 \leq j \leq m - r$, $j + 1 \leq i \leq m + 1 - r$, and with $d_{i,0} = d_{i-1}$, when $1 \leq i \leq m + 1 - r$. The point $F(\overline{t})$ on the spline curve is $d_{m+1-r,m-r}$.

Finally, in order to deal with the general case where $\overline{t} \in [\overline{u}_I, \overline{u}_{I+1}[$, we translate all the knot indices by $I - m$, which does not change differences of indices, and we get the equation

$$d_{i,j} = (\frac{u_{m+i-j} - t}{u_{m+i-j} - u_{i-1}})d_{i-1,j-1} + (\frac{t - u_{i-1}}{u_{m+i-j} - u_{i-1}})d_{i,j-1},$$

where $1 \leq j \leq m - r$, $I - m + j + 1 \leq i \leq I + 1 - r$, and with $d_{i,0} = d_{i-1}$, when $I - m + 1 \leq i \leq I + 1 - r$, where r is the multiplicity of the knot $\overline{u}_I$ when $\overline{t} = \overline{u}_I$, and $r = 0$ when $\overline{u}_I < \overline{t} < \overline{u}_{I+1}$ $(1 \leq r \leq m)$.

The point $F(\bar{t})$ on the spline curve is $d_{I+1-r,m-r}$. This is the de Boor algorithm. Note that other books often use a superscript for the "round index" j, and write our $d_{i,j}$ as d_i^j. The de Boor algorithm can be described as follows in "pseudo-code":

begin
  $I = \max\{k \mid \overline{u}_k \leq \bar{t} < \overline{u}_{k+1}\}$;
  if $\bar{t} = \overline{u}_I$ **then** $r := multiplicity(\overline{u}_I)$ **else** $r := 0$ **endif**;
  for $i := I - m + 1$ **to** $I + 1 - r$ **do**
    $d_{i,0} := d_{i-1}$
  endfor;
  for $j := 1$ **to** $m - r$ **do**
    for $i := I - m + j + 1$ **to** $I + 1 - r$ **do**
      $d_{i,j} := \left(\frac{u_{m+i-j} - t}{u_{m+i-j} - u_{i-1}}\right) d_{i-1,j-1} + \left(\frac{t - u_{i-1}}{u_{m+i-j} - u_{i-1}}\right) d_{i,j-1}$
    endfor
  endfor;
  $F(\bar{t}) := d_{I+1-r,m-r}$
end

The process of knot insertion consists of inserting a knot $\overline{w}$ into a given knot sequence, without altering the spline curve. The knot $\overline{w}$ may be new or may coincide with some existing knot of multiplicity $r < m$, and in the latter case, the effect will be to increase the degree of multiplicity of $\overline{w}$ by 1.

Knot insertion can be used either to construct new control points, Bézier control points associated with the curve segments forming a spline curve, and even for computing a point on a spline curve.

If I is the largest knot index such that $\overline{u}_I \leq \overline{w} < \overline{u}_{I+1}$, inserting the knot $\overline{w}$ will affect the $m-1-r$ control points $f(\overline{u}_{I-m+k+1}, \ldots, \overline{u}_{I+k})$ associated with the sequences $\langle \overline{u}_{I-m+k+1}, \ldots, \overline{u}_{I+k} \rangle$ containing the subinterval $[\overline{u}_I, \overline{u}_{I+1}]$, where $1 \leq k \leq m-1-r$, and where r is the multiplicity of $\overline{u}_I$ if $\overline{w} = \overline{u}_I$ (with $1 \leq r < m$), and $r = 0$ if $\overline{u}_I < \overline{w}$.

Let, $\overline{v}_k = \overline{u}_k$, for all $k \leq I$, $\overline{v}_{I+1} = \overline{w}$, and $\overline{v}_{k+1} = \overline{u}_k$, for all $k \geq I + 1$.

We need to compute the $m - r$ new control points

$$f(\overline{v}_{I-m+k+1}, \ldots, \overline{v}_{I+1}, \ldots, \overline{v}_{I+k}),$$

which are just the polar values corresponding to the $m-r$ subsequences of $m - 1$ consecutive subintervals

$$\langle \overline{v}_{I-m+k+1}, \ldots, \overline{v}_{I+1}, \ldots, \overline{v}_{I+k} \rangle,$$

one of which containing $\overline{w} = \overline{v}_{I+1}$, where $1 \leq k \leq m-r$. We can use the de Boor algorithm to compute the new $m - r$ control points.

In fact, note that these points constitute the first column obtained during the first round of the de Boor algorithm. Thus, we can describe knot insertion in "pseudo-code", as follows:

begin
  $I = \max\{k \mid \overline{u}_k \leq \overline{w} < \overline{u}_{k+1}\}$;
  if $\overline{w} = \overline{u}_I$ **then** $r := multiplicity(\overline{u}_I)$ **else** $r := 0$ **endif**;
  for $i := I - m + 1$ **to** $I + 1 - r$ **do**
    $d_{i,0} := d_{i-1}$
  endfor;
  for $i := I - m + 2$ **to** $I + 1 - r$ **do**
    $d_{i,1} := \left(\frac{u_{m+i-1} - w}{u_{m+i-1} - u_{i-1}}\right) d_{i-1,0} + \left(\frac{w - u_{i-1}}{u_{m+i-1} - u_{i-1}}\right) d_{i,0}$
  endfor
  return $\langle d_{I-m+2,1}, \ldots, d_{I+1-r,1} \rangle$
end

Note that evaluation of a point $F(\overline{t})$ on the spline curve amounts to repeated knot insertions: we perform $m - r$ rounds of knot insertions, to raise the original multiplicity r of the knot $\overline{t}$ to m (again, $r = 0$ if $\overline{t}$ is distinct from all existing knots).

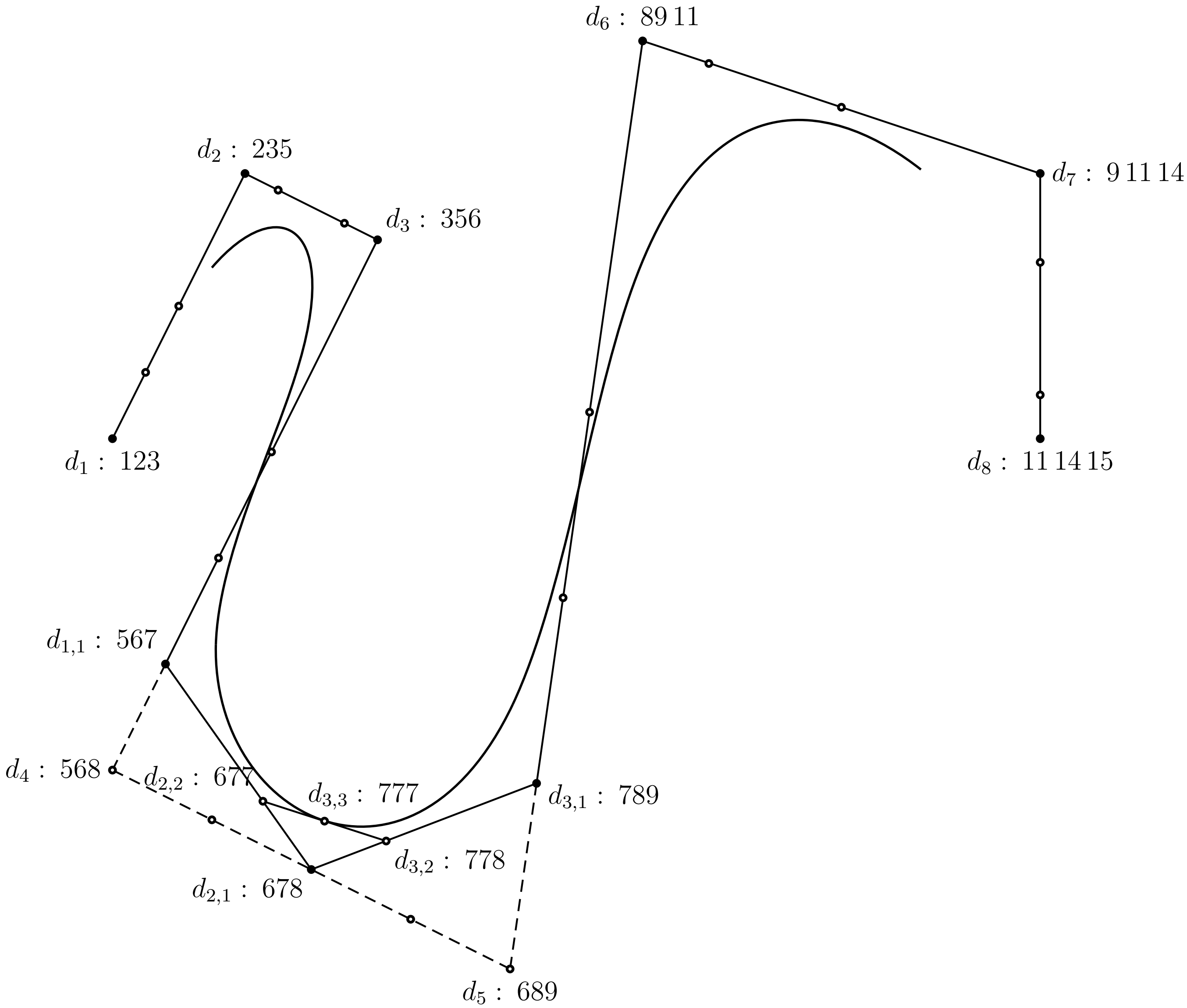


Figure 8.23: Part of a cubic spline with knot sequence $\ldots, 1, 2, 3, 5, 6, 8, 9, 11, 14, 15, \ldots$, insertion of the knot $\bar{t} = 7$

Evaluation of the point $F(7)$ on the spline curve above, consists in inserting the knot $\bar{t} = 7$ three times.

8.6 Cubic Spline Interpolation

We now consider the problem of interpolation by smooth curves. Unlike the problem of approximating a shape by a smooth curve, interpolation problems require finding curves passing through some given data points, and possibly satisfying some extra constraints.

Problem 1: Given $N+1$ data points $x_0, \ldots, x_N$, and a sequence of $N+1$ knots $\overline{u}_0, \ldots, \overline{u}_N$, with $\overline{u}_i < \overline{u}_{i+1}$ for all i, $0 \leq i \leq N-1$, find a C^2 cubic spline curve F, such that $F(\overline{u}_i) = x_i$, for all i, $0 \leq i \leq N$.

In order to solve the above problem, we can try to find the de Boor control points of a C^2 cubic spline curve F based on the finite knot sequence

$$\overline{u}_0, \overline{u}_0, \overline{u}_0, \overline{u}_1, \overline{u}_2, \ldots, \overline{u}_{N-2}, \overline{u}_{N-1}, \overline{u}_N, \overline{u}_N, \overline{u}_N.$$

We note that we are looking for a total of $N + 3$ de Boor control points $d_{-1}, \ldots, d_{N+1}$. Actually, since the first control point d_{-1} coincides with x_0, and the last control point d_{N+1} coincides with x_N, we are looking for $N+1$ de Boor control points $d_0, \ldots, d_N$. However, using the de Boor evaluation algorithm, we only come up with $N-1$ equations expressing $x_1, \ldots, x_{N-1}$ in terms of the $N+1$ unknown variables $d_0, \ldots, d_N$.

The figure below shows $N + 1 = 7 + 1 = 8$ data points, and a C^2 cubic spline curve F passing throught these points, for a uniform knot sequence. The control points d_0 and $d_7 = d_N$ were chosen arbitrarily.

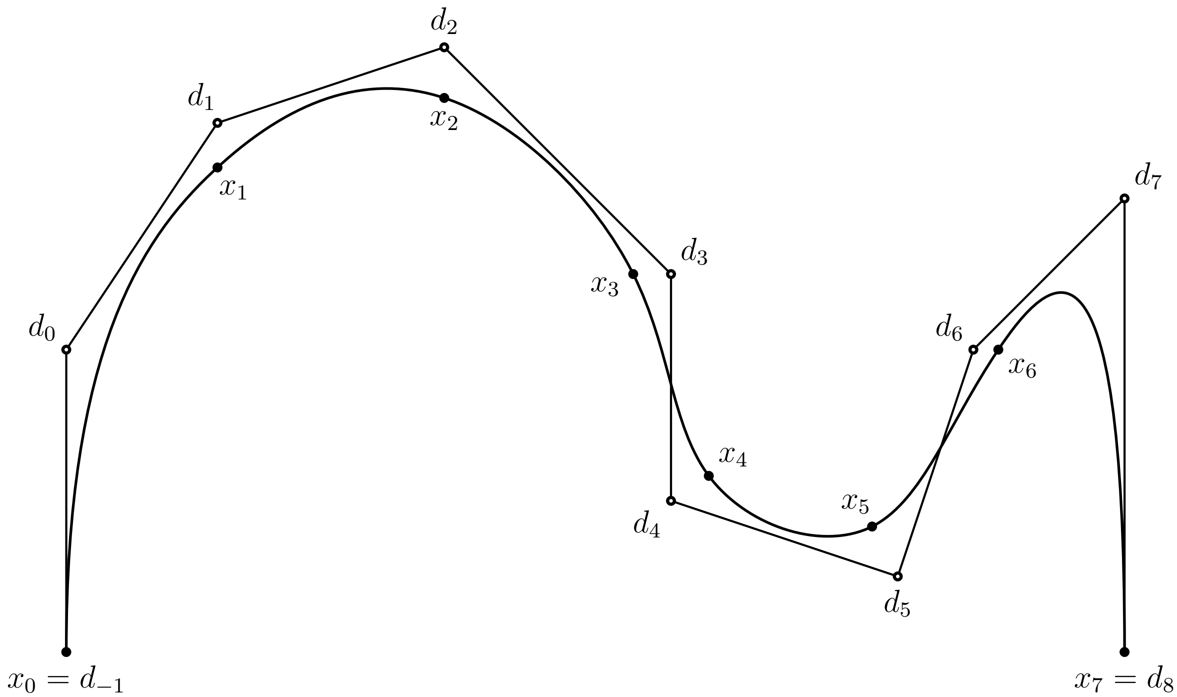


Figure 8.24: A C^2 cubic interpolation spline curve passing through the points $x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7$

Thus, the above problem has two degrees of freedom, and it is under-determined. To remove these degrees of freedom, we can add various "end conditions", which amounts to the assumption that the de Boor control points d_0 and d_N are known.

For example, we can specify that the tangent vectors at x_0 and x_N be equal to some desired value. We now have a system of $N-1$ linear equations in the $N-1$ variables $d_1, \ldots, d_{N-1}$.

In order to derive this system of linear equations, we use the de Boor evaluation algorithm. Note that for all i, with $1 \leq i \leq N-1$, the de Boor control point d_i corresponds to the polar label $\overline{u}_{i-1}\,\overline{u}_i\,\overline{u}_{i+1}$, x_i corresponds to the polar label $\overline{u}_i\,\overline{u}_i\,\overline{u}_i$, and d_{-1}, d_0, d_N and d_{N+1}, correspond respectively to $\overline{u}_0\,\overline{u}_0\,\overline{u}_0$, $\overline{u}_0\,\overline{u}_0\,\overline{u}_1$, $\overline{u}_{N-1}\,\overline{u}_N\,\overline{u}_N$, and $\overline{u}_N\,\overline{u}_N\,\overline{u}_N$.

For every i, with $1 \leq i \leq N-1$, x_i can be computed from d_{i-1}, d_i, d_{i+1}, using the following diagram representing the de Boor algorithm:

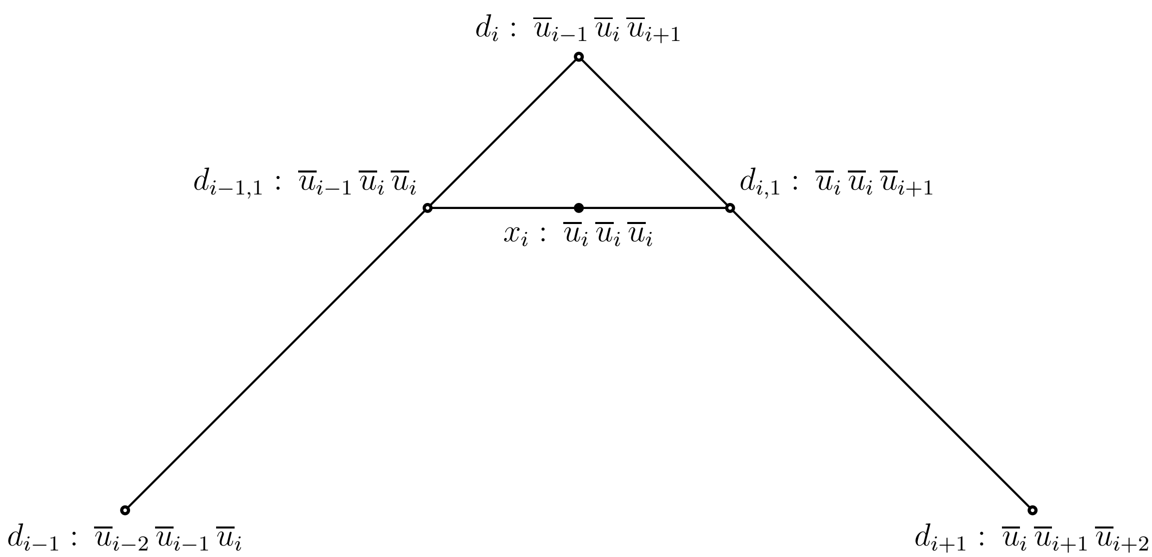


Figure 8.25: Computation of x_i from d_{i-1}, d_i, d_{i+1}

Letting

$$r_i = (u_{i+1} - u_{i-1})\, x_i,$$

for all i, $1 \leq i \leq N-1$, and r_0 and r_N be arbitrary points, we obtain the following $(N+1) \times (N+1)$ system of linear equations in the unknowns $d_0, \ldots, d_N$:

$$\begin{pmatrix}
1 & & & & & & & \\
\alpha_1 & \beta_1 & \gamma_1 & & & & & \\
& \alpha_2 & \beta_2 & \gamma_2 & & 0 & & \\
& & & \ddots & & & & \\
& & 0 & & \alpha_{N-2} & \beta_{N-2} & \gamma_{N-2} & \\
& & & & & \alpha_{N-1} & \beta_{N-1} & \gamma_{N-1} \\
& & & & & & & 1
\end{pmatrix}
\begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{N-2} \\ d_{N-1} \\ d_N \end{pmatrix}
=
\begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \\ r_{N-2} \\ r_{N-1} \\ r_N \end{pmatrix}$$

The matrix of the system of linear equations is tridiagonal, and it is clear that $\alpha_i, \beta_i, \gamma_i \geq 0$. If

$$\alpha_i + \gamma_i < \beta_i,$$

for all i, $1 \leq i \leq N-1$, which means that the matrix is diagonally dominant, then it can be shown that the matrix is invertible. In particular, this is the case for a uniform knot sequence.

There are methods for solving diagonally dominant systems of linear equations very efficiently, for example, using an LU-decomposition.

In the case of a uniform knot sequence, it is an easy exercise to show that the linear system can be written as

$$
\begin{pmatrix}
1 & & & & & & & \\
\frac{3}{2} & \frac{7}{2} & 1 & & & & & \\
 & 1 & 4 & 1 & & 0 & & \\
 & & & & \ddots & & & \\
 & & 0 & & & 1 & 4 & 1 \\
 & & & & & & 1 & \frac{7}{2} & \frac{3}{2} \\
 & & & & & & & & 1
\end{pmatrix}
\begin{pmatrix}
d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{N-2} \\ d_{N-1} \\ d_N
\end{pmatrix}
=
\begin{pmatrix}
r_0 \\ 6x_1 \\ 6x_2 \\ \vdots \\ 6x_{N-2} \\ 6x_{N-1} \\ r_N
\end{pmatrix}
$$

It can also be shown that the general system of linear equations has a unique solution when the knot sequence is strictly increasing, that is, when $u_i < u_{i+1}$ for all i, $0 \leq i \leq N-1$.

For example, this can be shown by expressing each spline segment in terms of the Hermite polynomials. Writing the C^2 conditions leads to a tridiagonal system, which is diagonally dominant when the knot sequence is strictly increasing. For details, see Farin [**?**].

We can also solve the problem of finding a closed interpolating spline curve, formulated as follows.

Problem 2: Given N data points $x_0, \ldots, x_{N-1}$, and a sequence of $N+1$ knots $\overline{u}_0, \ldots, \overline{u}_N$, with $\overline{u}_i < \overline{u}_{i+1}$ for all i, $0 \leq i \leq N-1$, find a C^2 closed cubic spline curve F, such that $F(\overline{u}_i) = x_i$, for all i, $0 \leq i \leq N$, where we let $x_N = x_0$.

This time, we consider the cyclic knot sequence determined by the $N+1$ knots $\overline{u}_0, \ldots, \overline{u}_N$, which means that we consider the infinite cyclic knot sequence $\langle \overline{u}_k \rangle_{k \in \mathbb{Z}}$ which agrees with $\overline{u}_0, \ldots, \overline{u}_N$ for $i = 0, \ldots, N$, and such that,

$$\overline{u}_{k+N} = \overline{u}_k + u_N - u_0,$$

for all $k \in \mathbb{Z}$, and we observe that we are now looking for N de Boor control points $d_0, \ldots, d_{N-1}$, since the condition $x_0 = x_N$ implies that $d_0 = d_N$, so that we can write a system of N linear equations in the N unknowns $d_0, \ldots, d_{N-1}$. The following system of linear equations is easily obtained:

$$\begin{pmatrix} \beta_0 & \gamma_0 & & & & & & \alpha_0 \\ \alpha_1 & \beta_1 & \gamma_1 & & & & & \\ & \alpha_2 & \beta_2 & \gamma_2 & & 0 & & \\ & & & \ddots & & & & \\ & & 0 & & \alpha_{N-3} & \beta_{N-3} & \gamma_{N-3} & \\ & & & & & \alpha_{N-2} & \beta_{N-2} & \gamma_{N-2} \\ \gamma_{N-1} & & & & & & \alpha_{N-1} & \beta_{N-1} \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{N-3} \\ d_{N-2} \\ d_{N-1} \end{pmatrix} = \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \\ r_{N-3} \\ r_{N-2} \\ r_{N-1} \end{pmatrix}$$

The system is no longer tridiagonal, but it can still be solved efficiently.

The coefficients $\alpha_i, \beta_i, \gamma_i$ can be written in a uniform fashion for both the open and the closed interpolating C^2 cubic spline curves, if we let $\Delta_i = u_{i+1} - u_i$. It is immediately verified that we have

$$\begin{aligned}
\alpha_i &= \frac{\Delta_i^2}{\Delta_{i-2} + \Delta_{i-1} + \Delta_i}, \\
\beta_i &= \frac{\Delta_i(\Delta_{i-2} + \Delta_{i-1})}{\Delta_{i-2} + \Delta_{i-1} + \Delta_i} + \frac{\Delta_{i-1}(\Delta_i + \Delta_{i+1})}{\Delta_{i-1} + \Delta_i + \Delta_{i+1}}, \\
\gamma_i &= \frac{\Delta_{i-1}^2}{\Delta_{i-1} + \Delta_i + \Delta_{i+1}},
\end{aligned}$$

where in the case of an open spline curve, $\Delta_{-1} = \Delta_N = 0$, and in the case of a closed spline curve, $\Delta_{-1} = \Delta_{N-1}$, $\Delta_{-2} = \Delta_{N-2}$.

In the case of an open C^2 cubic spline interpolant, several end conditions have been proposed to determine d_0 and d_N, and we quickly review these conditions.

(a) The first method consists in specifying the tangent vectors m_0 and m_N at x_0 and x_N, usually called the *clamped condition* method. Since the tangent vector at x_0 is given by

$$\mathrm{D}F(\overline{u_0}) = \frac{3}{u_1 - u_0}\,(d_0 - x_0),$$

we get

$$d_0 = x_0 + \frac{u_1 - u_0}{3}\,m_0,$$

and similarly

$$d_N = x_N - \frac{u_N - u_{N-1}}{3}\,m_N.$$

One specific method is the *Bessel end condition*. If we consider the parabola interpolating the first three data points x_0, x_1, x_2, the method consists in picking the tangent vector to this parabola at x_0. A similar selection is made using the parabola interpolating the last three points x_{N-2}, x_{N-1}, x_N.

(b) Another method is the *quadratic end condition*. In this method, we require that

$$\mathrm{D}^2F(\overline{u}_0) = \mathrm{D}^2F(\overline{u}_1)$$

and

$$\mathrm{D}^2F(\overline{u}_{N-1}) = \mathrm{D}^2F(\overline{u}_N).$$

(c) Another method is the *natural end condition*. In this method, we require that

$$\mathrm{D}^2F(\overline{u}_0) = \mathrm{D}^2F(\overline{u}_N) = \overrightarrow{0}.$$

(d) Finally, we have the *not-a-knot condition*, which forces the first two cubic segments to merge into a single cubic segment and similarly for the last two cubic segments.

In practice, when attempting to solve an interpolation problem, the knot sequence $\overline{u}_0, \ldots, \overline{u}_N$ is not given. Thus, it is necessary to find knot sequences that produce reasonable results. We now briefly survey methods for producing knot sequences.

The simplest method consists in choosing a *uniform* knot sequence. Although simple, this method may produce bad results when the data points are heavily clustered in some areas.

Another popular method is to use a *chord length* knot sequence. In this method, after choosing $\overline{u}_0$ and $\overline{u}_N$, we determine the other knots in such a way that

$$\frac{u_{i+1} - u_i}{u_{i+2} - u_{i+1}} = \frac{\|x_{i+1} - x_i\|}{\|x_{i+2} - x_{i+1}\|},$$

where $\|x_{i+1} - x_i\|$ is the length of the chord between x_i and x_{i+1}. This method usually works quite well.

Another method is the so-called *centripedal* method, derived from physical heuristics, where we set

$$\frac{u_{i+1} - u_i}{u_{i+2} - u_{i+1}} = \Big(\frac{\|x_{i+1} - x_i\|}{\|x_{i+2} - x_{i+1}\|}\Big)^{1/2}.$$

There are other methods, in particular due to Foley. For details, the reader is referred to Farin [**?**].

Chapter 9

Polynomial Surfaces

9.1 Polarizing Polynomial Surfaces

We begin with the traditional definition of polynomial surfaces. As we shall see, there are two natural ways to polarize a polynomial surface. Intuitively, this depends on whether we decide to tile the parameter plane with rectangles, or with triangles.

We also denote the affine plane $\mathbb{A}^2$ as $\mathcal{P}$. We assume that some fixed affine frame $(O, (\overrightarrow{i_1}, \overrightarrow{i_2}))$ for $\mathcal{P}$ is chosen, typically, the canonical affine frame where $O = (0, 0)$, $\overrightarrow{i_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $\overrightarrow{i_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Let $\mathcal{E}$ be some affine space of finite dimension $n \geq 3$, and let $(\Omega_1, (\overrightarrow{e_1}, \ldots, \overrightarrow{e_n}))$ be an affine frame for $\mathcal{E}$.

Definition 9.1.1 A *polynomial surface* is a function $F\colon \mathcal{P} \to \mathcal{E}$, such that, for all $u, v \in \mathbb{R}$, we have

$$F(O + u\overrightarrow{i_1} + v\overrightarrow{i_2}) = \Omega_1 + F_1(u,v)\overrightarrow{e_1} + \cdots + F_n(u,v)\overrightarrow{e_n},$$

where $F_1(U,V)$, ..., $F_n(U,V)$ are polynomials in $\mathbb{R}[U,V]$. Given natural numbers p, q, and m, if each polynomial $F_i(U,V)$ has total degree $\leq m$, we say that F is a *polynomial surface of total degree* m. If the maximum degree of U in all the $F_i(U,V)$ is $\leq p$, and the maximum degree of V in all the $F_i(U,V)$ is $\leq q$, we say that F is a *bipolynomial surface of degree* $\langle p, q\rangle$. The *trace of the surface* F is the set $F(\mathcal{P})$.

For simplicity, we denote $F(O + u\overrightarrow{i_1} + v\overrightarrow{i_2})$ as $F(u,v)$.

The following polynomials define a polynomial surface of total degree 2 in $\mathbb{A}^3$:

$$
\begin{aligned}
F_1(U, V) &= U^2 + V^2 + UV + 2U + V - 1 \\
F_2(U, V) &= U - V + 1 \\
F_3(U, V) &= UV + U + V + 1.
\end{aligned}
$$

The above is also a bipolynomial surface of degree $\langle 2, 2\rangle$.

Another example known as *Enneper's surface* is as follows:

$$
\begin{aligned}
F_1(U,V) &= U - \frac{U^3}{3} + UV^2\\
F_2(U,V) &= V - \frac{V^3}{3} + U^2V\\
F_3(U,V) &= U^2 - V^2.
\end{aligned}
$$

As defined above, Enneper's surface is a surface of total degree 3, and a bipolynomial surface of degree $\langle 3, 3\rangle$.

Given a polynomial surface $F\colon \mathcal{P} \to \mathcal{E}$, there are two natural ways to polarize.

The first way to polarize, is to treat the variables u and v separately, and polarize separately in u and v.

This way, if p and q are such that F is a bipolynomial surface of degree $\langle p, q\rangle$, we get a $(p+q)$-multiaffine map

$$f\colon (\mathbb{A})^p \times (\mathbb{A})^q \to \mathcal{E},$$

which is symmetric separately in its first p arguments and in its last q arguments, but not symmetric in all its arguments.

We get what are traditionally called *tensor product surfaces*. Note that in this case, since

$$F(u, v) = f(\underbrace{u, \ldots, u}_{p}, \underbrace{v, \ldots, v}_{q}),$$

the surface F is really a map $F\colon \mathbb{A} \times \mathbb{A} \to \mathcal{E}$. However, since $\mathbb{A} \times \mathbb{A}$ is isomorphic to $\mathcal{P}$, we can view F as a polynomial surface $F\colon \mathcal{P} \to \mathcal{E}$.

The second way to polarize, is to treat the variables u and v as a whole, namely as the coordinates of a point (u, v) in $\mathcal{P}$, and to polarize the polynomials in both variables simultaneously.

This way, if m is such that F is a polynomial surface of total degree m, we get an m-multiaffine map

$$f: \mathcal{P}^m \to \mathcal{E},$$

which is symmetric in all of its m arguments. Since

$$F(u, v) = f(\underbrace{(u, v), \ldots, (u, v)}_{m}),$$

the surface F is a map $F: \mathcal{P} \to \mathcal{E}$.

We begin with the first method for polarizing, in which we polarize separately in u and v. Using linearity, it is enough to explain how to polarize a monomial $F(u, v)$ of the form $u^h v^k$ with respect to the bidegree $\langle p, q\rangle$, where $h \le p$ and $k \le q$.

$$
\begin{aligned}
&f(u_1, \ldots, u_p, v_1, \ldots, v_q) \\
&\qquad = \frac{1}{\binom{p}{h}\binom{q}{k}} \sum_{\substack{I \subseteq \{1,\ldots,p\},\, |I|=h \\ J \subseteq \{1,\ldots,q\},\, |J|=k}} \left(\prod_{i\in I} u_i\right)\left(\prod_{j\in J} v_j\right).
\end{aligned}
$$

Example 1.

Consider the following surface viewed as a bipolynomial surface of degree $\langle 2, 2\rangle$:

$$
\begin{aligned}
F_1(U,V) &= U^2 + V^2 + UV + 2U + V - 1\\
F_2(U,V) &= U - V + 1\\
F_3(U,V) &= UV + U + V + 1,
\end{aligned}
$$

In order to find the polar form $f(U_1, U_2, V_1, V_2)$ of F, viewed as a bipolynomial surface of degree $\langle 2, 2\rangle$, we polarize each of the $F_i(U, V)$ separately in U and V. It is quite obvious that the same result is obtained if we first polarize with respect to U, and then with respect to V, or conversely.

After polarizing, we have

$$
\begin{aligned}
f_1(U_1, U_2, V_1, V_2) &= U_1U_2 + V_1V_2 + \frac{(U_1 + U_2)(V_1 + V_2)}{4} + U_1 + U_2 + \frac{V_1 + V_2}{2} - 1\\
f_2(U_1, U_2, V_1, V_2) &= \frac{U_1 + U_2}{2} - \frac{V_1 + V_2}{2} + 1\\
f_3(U_1, U_2, V_1, V_2) &= \frac{(U_1 + U_2)(V_1 + V_2)}{4} + \frac{U_1 + U_2}{2} + \frac{V_1 + V_2}{2} + 1.
\end{aligned}
$$

The nine control points $b_{i,j}$ have coordinates:

$b_{i,j}$ $\quad j$ i	0	1	2
0	$(-1, 1, 1)$	$(-\frac{1}{2}, \frac{1}{2}, \frac{3}{2})$	$(1, 0, 2)$
1	$(0, \frac{3}{2}, \frac{3}{2})$	$(\frac{3}{4}, 1, \frac{9}{4})$	$(\frac{5}{2}, \frac{1}{2}, 3)$
2	$(2, 2, 2)$	$(3, \frac{3}{2}, 3)$	$(5, 1, 4)$

Let us now review how to polarize a polynomial in two variables as a polynomial of total degree m. Using linearity, it is enough to deal with a single monomial. According to lemma 6.2.7, given the monomial U^hV^k, with $h+k=d\leq m$, we get the following polar form of degree m:

$$
\begin{aligned}
& f((u_1,v_1),\ldots,(u_m,v_m)) \\
& \quad = \frac{h!k!(m-(h+k))!}{m!} \sum_{\substack{I\cup J\subseteq\{1,\ldots,m\}\\ I\cap J=\emptyset\\ |I|=h,\,|J|=k}} \left(\prod_{i\in I} u_i\right)\left(\prod_{j\in J} v_j\right).
\end{aligned}
$$

Example 2.

Let us now polarize the surface of Example 1 as a surface of total degree 2. Starting from

$$
\begin{aligned}
F_1(U,V) &= U^2 + V^2 + UV + 2U + V - 1\\
F_2(U,V) &= U - V + 1\\
F_3(U,V) &= UV + U + V + 1,
\end{aligned}
$$

we get

$$
\begin{aligned}
f_1((U_1,V_1),(U_2,V_2)) &= U_1U_2 + V_1V_2 + \frac{U_1V_2 + U_2V_1}{2} + U_1 + U_2 + \frac{V_1+V_2}{2} - 1\\
f_2((U_1,V_1),(U_2,V_2)) &= \frac{U_1+U_2}{2} - \frac{V_1+V_2}{2} + 1\\
f_3((U_1,V_1),(U_2,V_2)) &= \frac{U_1V_2 + U_2V_1}{2} + \frac{U_1+U_2}{2} + \frac{V_1+V_2}{2} + 1.
\end{aligned}
$$

Control points:

$$
\begin{array}{ccccc}
 & & f(r,r) & & \\
 & & (2,2,2) & & \\
 & f(r,t) & & f(r,s) & \\
 & (0,\frac{3}{2},\frac{3}{2}) & & (1,1,\frac{5}{2}) & \\
f(t,t) & & f(s,t) & & f(s,s) \\
(-1,1,1) & & (-\frac{1}{2},\frac{1}{2},\frac{3}{2}) & & (1,0,2)
\end{array}
$$

Example 3.

Let us also find the polar forms of the Enneper's surface, considered as a total degree surface (of degree 3):

$$
\begin{aligned}
F_1(U,V) &= U - \frac{U^3}{3} + UV^2 \\
F_2(U,V) &= V - \frac{V^3}{3} + U^2V \\
F_3(U,V) &= U^2 - V^2.
\end{aligned}
$$

We get

$$
\begin{aligned}
f_1((U_1,V_1),(U_2,V_2),(U_3,V_3)) &= \frac{U_1+U_2+U_3}{3} - \frac{U_1U_2U_3}{3} \\
&\quad + \frac{U_1V_2V_3 + U_2V_1V_3 + U_3V_1V_2}{3} \\
f_2((U_1,V_1),(U_2,V_2),(U_3,V_3)) &= \frac{V_1+V_2+V_3}{3} - \frac{V_1V_2V_3}{3} \\
&\quad + \frac{U_1U_2V_3 + U_1U_3V_2 + U_2U_3V_1}{3} \\
f_3((U_1,V_1),(U_2,V_2),(U_3,V_3)) &= \frac{U_1U_2 + U_1U_3 + U_2U_3}{3} - \frac{V_1V_2 + V_1V_3 + V_2V_3}{3},
\end{aligned}
$$

and evaluating these polar forms for argument pairs (U_1, V_1), (U_2, V_2), and (U_3, V_3), ranging over $(1, 0)$, $(0, 1)$ and $(0, 0)$, we find the following 10 control points:

$$
\begin{array}{ccccccc}
 & & & f(r,r,r) & & & \\
 & & & (\frac{2}{3}, 0, 1) & & & \\
 & & f(r,r,t) & & f(r,r,s) & & \\
 & & (\frac{2}{3}, 0, \frac{1}{3}) & & (\frac{2}{3}, \frac{2}{3}, \frac{1}{3}) & & \\
 & f(r,t,t) & & f(r,s,t) & & f(r,s,s) & \\
 & (\frac{1}{3}, 0, 0) & & (\frac{1}{3}, \frac{1}{3}, 0) & & (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}) & \\
f(t,t,t) & & f(s,t,t) & & f(s,s,t) & & f(s,s,s) \\
(0,0,0) & & (0, \frac{1}{3}, 0) & & (0, \frac{2}{3}, -\frac{1}{3}) & & (0, \frac{2}{3}, -1)
\end{array}
$$

Let us consider two more examples.

Example 4.

Let F be the surface considered as a total degree surface, and defined such that

$$\begin{aligned}
F_1(U, V) &= U, \\
F_2(U, V) &= V, \\
F_3(U, V) &= U^2 - V^2.
\end{aligned}$$

The polar forms are:

$$\begin{aligned}
f_1((U_1, V_1), (U_2, V_2)) &= \frac{U_1 + U_2}{2}, \\
f_2((U_1, V_1), (U_2, V_2)) &= \frac{V_1 + V_2}{2}, \\
f_3((U_1, V_1), (U_2, V_2)) &= U_1 U_2 - V_1 V_2.
\end{aligned}$$

With respect to the barycentric affine frame $(r, s, t) = (O + \overrightarrow{i_1}, O + \overrightarrow{i_2}, O)$, the control net consists of the following six points, obtained by evaluating the polar forms f_1, f_2, f_3 on the (u, v) coordinates of (r, s, t), namely $(1, 0)$, $(0, 1)$, and $(0, 0)$:

$$
\begin{array}{ccccc}
 & & \begin{array}{c} f(r,r) \\ (1,0,1) \end{array} & & \\
 & \begin{array}{c} f(r,t) \\ (\frac{1}{2},0,0) \end{array} & & \begin{array}{c} f(r,s) \\ (\frac{1}{2},\frac{1}{2},0) \end{array} & \\
\begin{array}{c} f(t,t) \\ (0,0,0) \end{array} & & \begin{array}{c} f(s,t) \\ (0,\frac{1}{2},0) \end{array} & & \begin{array}{c} f(s,s) \\ (0,1,-1) \end{array}
\end{array}
$$

The resulting surface is an *hyperbolic paraboloid*, of implicit equation

$$z = x^2 - y^2.$$

Example 5.

Let F be the surface considered as a total degree surface, and defined such that

$$\begin{aligned} F_1(U,V) &= U, \\ F_2(U,V) &= V, \\ F_3(U,V) &= 2U^2 + V^2. \end{aligned}$$

The polar forms are:

$$\begin{aligned} f_1((U_1,V_1),(U_2,V_2)) &= \frac{U_1+U_2}{2}, \\ f_2((U_1,V_1),(U_2,V_2)) &= \frac{V_1+V_2}{2}, \\ f_3((U_1,V_1),(U_2,V_2)) &= 2U_1U_2 + V_1V_2. \end{aligned}$$

With respect to the barycentric affine frame $(r, s, t) = (O + \overrightarrow{i_1}, O + \overrightarrow{i_2}, O)$, the control net consists of the following six points, obtained by evaluating the polar forms f_1, f_2, f_3 on the (u, v) coordinates of (r, s, t), namely $(1, 0)$, $(0, 1)$, and $(0, 0)$:

$$
\begin{array}{ccccc}
 & & f(r,r) & & \\
 & & (1,0,2) & & \\
 & f(r,t) & & f(r,s) & \\
 & (\frac{1}{2},0,0) & & (\frac{1}{2},\frac{1}{2},0) & \\
f(t,t) & & f(s,t) & & f(s,s) \\
(0,0,0) & & (0,\frac{1}{2},0) & & (0,1,1)
\end{array}
$$

The resulting surface is an *elliptic paraboloid*. of implicit equation

$$z = 2x^2 + y^2.$$

Its general shape is that of a “boulder hat”.

9.2 Bipolynomial Surfaces in Polar Form

Given a bipolynomial surface $F\colon \mathcal{P} \to \mathcal{E}$ of degree $\langle p, q \rangle$, where $\mathcal{E}$ is of dimension n, applying lemma 6.2.7 to each polynomial $F_i(U, V)$ defining F, first with respect to U, and then with respect to V, we get polar forms

$$f_i\colon (\mathbb{A})^p \times (\mathbb{A})^q \to \mathbb{A},$$

which together, define a $(p+q)$-multiaffine map

$$f\colon (\mathbb{A})^p \times (\mathbb{A})^q \to \mathcal{E},$$

such that $f(U_1, \ldots, U_p; V_1, \ldots, V_q)$ is symmetric in its first p-arguments, and symmetric in its last q-arguments, and with

$$F(u, v) = f(\underbrace{u, \ldots, u}_{p}; \underbrace{v, \ldots, v}_{q}),$$

for all $u, v \in \mathbb{R}$.

By analogy with polynomial curves, it is natural to propose the following definition.

Definition 9.2.1 Given any affine space $\mathcal{E}$ of dimension ≥ 3, a *bipolynomial surface of degree* $\langle p, q\rangle$ *in polar form* is a map $F\colon \mathbb{A} \times \mathbb{A} \to \mathcal{E}$, such that there is some multiaffine map

$$f\colon (\mathbb{A})^p \times (\mathbb{A})^q \to \mathcal{E},$$

which is symmetric in its first p-arguments, and symmetric in its last q-arguments, and with

$$F(\overline{u}, \overline{v}) = f(\underbrace{\overline{u}, \ldots, \overline{u}}_{p}; \underbrace{\overline{v}, \ldots, \overline{v}}_{q}),$$

for all $\overline{u}, \overline{v} \in \mathbb{A}$. We also say that f is $\langle p, q\rangle$*-symmetric*. The *trace of the surface* F is the set $F(\mathbb{A}, \mathbb{A})$.

Let $(\overline{r}_1, \overline{s}_1)$ and $(\overline{r}_2, \overline{s}_2)$ be two affine frames for the affine line $\mathbb{A}$. Every point $\overline{u} \in \mathbb{A}$ can be written as

$$\overline{u} = \left(\frac{s_1 - u}{s_1 - r_1}\right)\,\overline{r}_1 + \left(\frac{u - s_1}{s_1 - r_1}\right)\,\overline{s}_1,$$

and similarly any point $\overline{v} \in \mathbb{A}$ can be written as

$$\overline{v} = \left(\frac{s_2 - v}{s_2 - r_2}\right)\,\overline{r}_2 + \left(\frac{v - s_2}{s_2 - r_2}\right)\,\overline{s}_2.$$

We can expand

$$f(\overline{u}_1, \ldots, \overline{u}_p; \overline{v}_1, \ldots, \overline{v}_q),$$

using multiaffineness.

Lemma 9.2.2 *Let $(\overline{r}_1, \overline{s}_1)$ and $(\overline{r}_2, \overline{s}_2)$ be any two affine frames for the affine line $\mathbb{A}$, and let $\mathcal{E}$ be an affine space (of finite dimension $n \geq 3$). For any natural numbers p, q, for any family $(b_{i,\,j})_{0\leq i\leq p,\,0\leq j\leq q}$ of $(p+1)(q+1)$ points in $\mathcal{E}$, there is a unique bipolynomial surface $F\colon \mathbb{A}\times\mathbb{A}\to\mathcal{E}$ of degree $\langle p, q\rangle$, with polar form the $(p+q)$-multiaffine $\langle p, q\rangle$-symmetric map*

$$f\colon (\mathbb{A})^p \times (\mathbb{A})^q \to \mathcal{E},$$

such that

$$f(\underbrace{\overline{r}_1,\ldots,\overline{r}_1}_{p-i},\underbrace{\overline{s}_1,\ldots,\overline{s}_1}_{i};\underbrace{\overline{r}_2,\ldots,\overline{r}_2}_{q-j},\underbrace{\overline{s}_2,\ldots,\overline{s}_2}_{j}) = b_{i,\,j},$$

for all i, $1 \leq i \leq p$, and all j, $1 \leq j \leq q$. Furthermore, f is given by the expression

$$f(\overline{u}_1,\ldots,\overline{u}_p;\overline{v}_1,\ldots,\overline{v}_q)$$
$$= \sum_{\substack{I\cap J=\emptyset\\ I\cup J=\{1,\ldots,p\}\\ K\cap L=\emptyset\\ K\cup L=\{1,\ldots,q\}}} \prod_{i\in I}\left(\frac{s_1-u_i}{s_1-r_1}\right)\prod_{j\in J}\left(\frac{u_j-r_1}{s_1-r_1}\right)\prod_{k\in K}\left(\frac{s_2-v_k}{s_2-r_2}\right)\prod_{l\in L}\left(\frac{v_l-r_2}{s_2-r_2}\right) b_{|J|,\,|L|}.$$

A point $F(\overline{u}, \overline{v})$ on the surface F can be expressed in terms of the Bernstein polynomials $B_i^p[r_1, s_1](u)$ and $B_j^q[r_2, s_2](v)$, as

$$F(\overline{u}, \overline{v}) = \sum_{\substack{0 \le i \le p \\ 0 \le j \le q}} B_i^p[r_1, s_1](u)\, B_j^q[r_2, s_2](v)\, f(\underbrace{\overline{r}_1, \ldots, \overline{r}_1}_{p-i}, \underbrace{\overline{s}_1, \ldots, \overline{s}_1}_{i}; \underbrace{\overline{r}_2, \ldots, \overline{r}_2}_{q-j}, \underbrace{\overline{s}_2, \ldots, \overline{s}_2}_{j}).$$

Thus, we see that the Bernstein polynomials show up again, and indeed, in traditional presentations of bipolynomial surfaces, they are used in the definition itself.

A family $\mathcal{N} = (b_{i,\,j})_{0\leq i\leq p,\, 0\leq j\leq q}$ of $(p+1)(q+1)$ points in $\mathcal{E}$, is often called a *(rectangular) control net, or Bézier net*.

Note that we can view the set of pairs

$$\square_{p,q} = \{(i,j) \in \mathbb{N}^2 \mid 0 \leq i \leq p,\, 0 \leq j \leq q\},$$

as a rectangular grid of $(p+1)(q+1)$ points in $\mathbb{A} \times \mathbb{A}$. The control net $\mathcal{N} = (b_{i,\,j})_{(i,j)\in\square_{p,q}}$, can be viewed as an image of the rectangular grid $\square_{p,q}$ in the affine space $\mathcal{E}$.

By lemma 9.2.2, such a control net $\mathcal{N}$ determines a unique bipolynomial surface F of degree $\langle p, q\rangle$.

The portion of the surface F corresponding to the points $F(\overline{u}, \overline{v})$ for which the parameters $\overline{u}, \overline{v}$ satisfy the inequalities $r_1 \leq u \leq s_1$ and $r_2 \leq v \leq s_2$, is called a *rectangular (surface) patch, or rectangular Bézier patch*, and $F([\overline{r}_1, \overline{s}_1], [\overline{r}_2, \overline{s}_2])$ is the *trace of the rectangular patch*.

The surface F (or rectangular patch) determined by a control net $\mathcal{N}$, contains the four control points $b_{0,\,0}, b_{0,\,q}, b_{p,\,0}$, and $b_{p,\,q}$, the *corners* of the surface patch.

Note that there is a natural way of connecting the points in a control net $\mathcal{N}$: every point $b_{i,\,j}$, where $0 \leq i \leq p-1$, and $0 \leq j \leq q-1$, is connected to the three points $b_{i+1,\,j}, b_{i,\,j+1}$, and $b_{i+1,\,j+1}$. Generally, pq quadrangles are obtained in this manner, and together, they form a polyhedron which gives a rough approximation of the surface patch.

The de Casteljau algorithm can be generalized very easily to bipolynomial surfaces.

9.3 The de Casteljau Algorithm for Rectangular Surface Patches

Given a rectangular control net $\mathcal{N} = (b_{i,\,j})_{(i,j)\in\square_{p,q}}$, we can first compute the points

$$b_{0*}, \ldots, b_{p*},$$

where b_{i*} is obtained by applying the de Casteljau algorithm to the Bézier control points

$$b_{i,\,0}, \ldots, b_{i,\,q},$$

with $0 \leq i \leq p$, and then compute b^p_{0*}, by applying the de Casteljau algorithm to the control points

$$b_{0*}, \ldots, b_{p*}.$$

For every i, with $0 \leq i \leq p$, we first compute the points $b^j_{i*,\,k}$, where $b^0_{i*,\,j} = b_{i,\,j}$, and

$$b^j_{i*,\,k} = \left(\frac{s_2 - v}{s_2 - r_2}\right) b^{j-1}_{i*,\,k} + \left(\frac{v - r_2}{s_2 - r_2}\right) b^{j-1}_{i*,\,k+1},$$

with $1 \leq j \leq q$ and $0 \leq k \leq q - j$, and we let $b_{i*} = b^q_{i*,\,0}$.

It is easily shown by induction that

$$b^j_{i*,\,k} = f(\underbrace{\overline{r}_1, \ldots, \overline{r}_1}_{p-i}, \underbrace{\overline{s}_1, \ldots, \overline{s}_1}_{i}; \underbrace{\overline{v}, \ldots, \overline{v}}_{j}, \underbrace{\overline{r}_2, \ldots, \overline{r}_2}_{q-j-k}, \underbrace{\overline{s}_2, \ldots, \overline{s}_2}_{k}),$$

and since $b_{i*} = b^q_{i*,\,0}$, we have

$$b_{i*} = f(\underbrace{\overline{r}_1, \ldots, \overline{r}_1}_{p-i}, \underbrace{\overline{s}_1, \ldots, \overline{s}_1}_{i}; \underbrace{\overline{v}, \ldots, \overline{v}}_{q}).$$

Next, we compute the points b^j_{i*}, where $b^0_{i*} = b_{i*}$, and

$$b^j_{i*} = \left(\frac{s_1 - u}{s_1 - r_1}\right) b^{j-1}_{i*} + \left(\frac{u - r_1}{s_1 - r_1}\right) b^{j-1}_{i+1*},$$

with $1 \leq j \leq p$ and $0 \leq i \leq p - j$, and we let
$F(\overline{u}, \overline{v}) = b^p_{0*}$.

It is easily shown by induction that

$$b^j_{i*} = f(\underbrace{\overline{u}, \ldots, \overline{u}}_{j}, \underbrace{\overline{r}_1, \ldots, \overline{r}_1}_{p-i-j}, \underbrace{\overline{s}_1, \ldots, \overline{s}_1}_{i}; \underbrace{\overline{v}, \ldots, \overline{v}}_{q}),$$

and thus,

$$F(\overline{u}, \overline{v}) = b^p_{0*} = f(\underbrace{\overline{u}, \ldots, \overline{u}}_{p}; \underbrace{\overline{v}, \ldots, \overline{v}}_{q}).$$

Alternatively, we can first compute the points

$$b_{*0}, \ldots, b_{*q},$$

where b_{*j} is obtained by applying the de Casteljau algorithm to the Bézier control points

$$b_{0,\,j}, \ldots, b_{p,\,j},$$

with $0 \leq j \leq q$, and then compute b_{*0}^{q}, by applying the de Casteljau algorithm to the control points

$$b_{*0}, \ldots, b_{*q}.$$

The same result $b_{0*}^{p} = b_{*0}^{q}$, is obtained.

We give, in pseudo code, the version of the algorithm in which we compute first the control points

$$b_{0*}, \ldots, b_{p*},$$

and then b_{0*}^{p}. We assume that the input is a control net $\mathcal{N} = (b_{i,\,j})_{(i,j)\in\square_{p,q}}$.

begin
  for $i := 0$ **to** p **do**
    for $j := 0$ **to** q **do**
      $b^{0}_{i*,\,j} := b_{i,\,j}$
    endfor;
    for $j := 1$ **to** q **do**
      for $k := 0$ **to** $q - j$ **do**
        $b^{j}_{i*,\,k} := \left(\frac{s_2 - v}{s_2 - r_2}\right)\, b^{j-1}_{i*,\,k} + \left(\frac{v - r_2}{s_2 - r_2}\right)\, b^{j-1}_{i*,\,k+1}$
      endfor
    endfor;
    $b_{i*} = b^{q}_{i*,0}$;
  endfor;
  for $i := 0$ **to** p **do**
    $b^{0}_{i*} = b_{i*}$
  endfor;
  for $j := 1$ **to** p **do**
    for $i := 0$ **to** $p - j$ **do**
      $b^{j}_{i*} := \left(\frac{s_1 - u}{s_1 - r_1}\right)\, b^{j-1}_{i*} + \left(\frac{u - r_1}{s_1 - r_1}\right)\, b^{j-1}_{i+1*}$
    endfor
  endfor;
  $F(\overline{u}, \overline{v}) := b^{p}_{0*}$
end

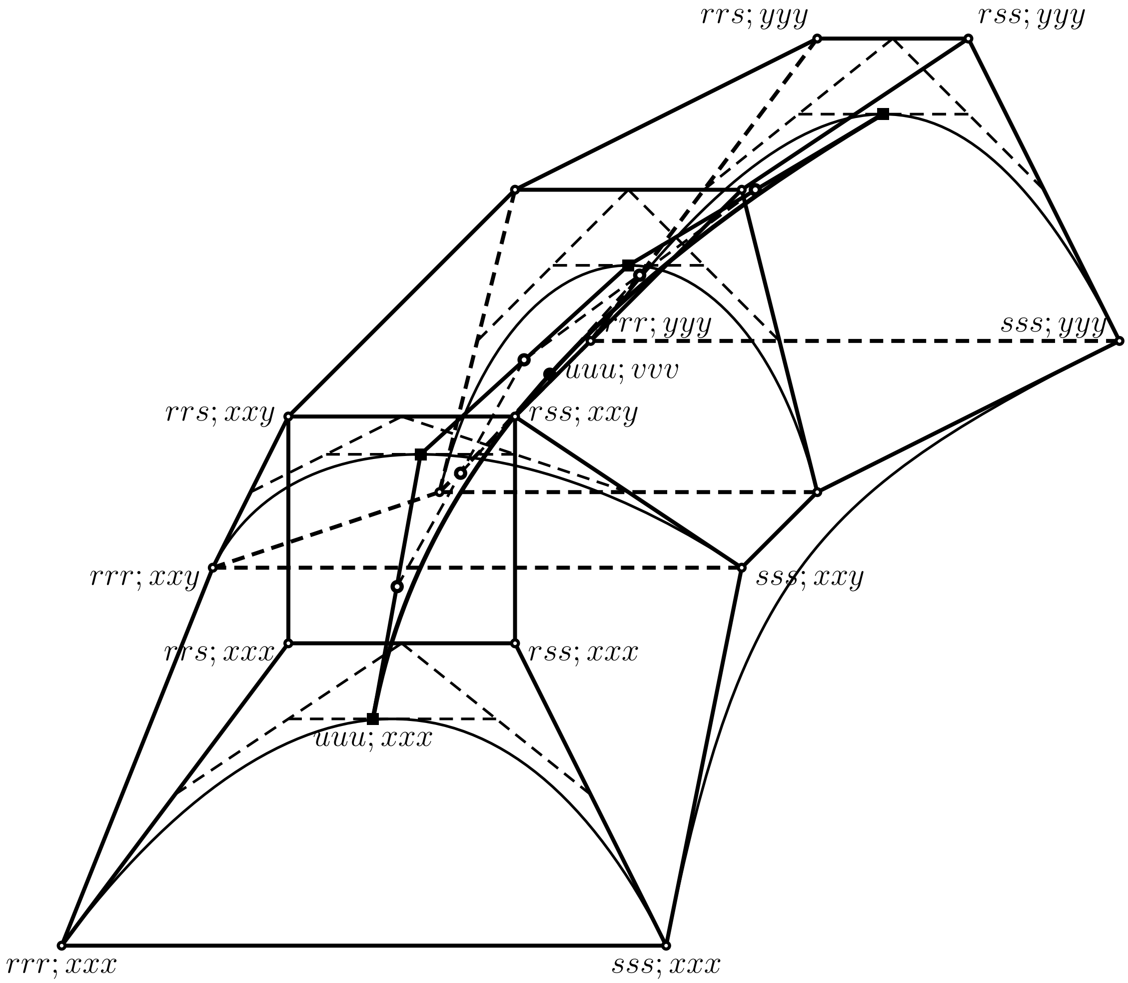


Figure 9.1: The de Casteljau algorithm for a bipolynomial surface of degree $\langle 3, 3\rangle$

9.4 Total Degree Surfaces in Polar Form

Given a surface $F\colon \mathcal{P} \to \mathcal{E}$ of total degree m, where $\mathcal{E}$ is of dimension n, applying lemma 6.2.7 to each polynomial $F_i(U, V)$ defining F, with respect to both U and V, we get polar forms

$$f_i\colon \mathcal{P}^m \to \mathbb{A},$$

which together, define an m-multiaffine and symmetric map

$$f\colon \mathcal{P}^m \to \mathcal{E},$$

such that

$$F(u, v) = f(\underbrace{(u, v), \ldots, (u, v)}_{m}).$$

Note that each $f((U_1, V_1), \ldots, (U_m, V_m))$ is multiaffine and symmetric in the pairs (U_i, V_i).

By analogy with polynomial curves, it is also natural to propose the following definition.

Definition 9.4.1 Given any affine space $\mathcal{E}$ of dimension ≥ 3, a *surface of total degree m in polar form*, is a map $F: \mathcal{P} \to \mathcal{E}$, such that there is some symmetric multiaffine map

$$f: \mathcal{P}^m \to \mathcal{E},$$

and with

$$F(a) = f(\underbrace{a, \ldots, a}_{m}),$$

for all $a \in \mathcal{P}$. The *trace of the surface F* is the set $F(\mathcal{P})$.

The polynomials in three variables U, V, T, defined such that

$$B^m_{i,j,k}(U, V, T) = \frac{m!}{i!j!k!}\, U^i V^j T^k,$$

where $i + j + k = m$, are also called *Bernstein polynomials*.

The points

$$f(\underbrace{r, \ldots, r}_{i}, \underbrace{s, \ldots, s}_{j}, \underbrace{t, \ldots, t}_{k}),$$

where $i + j + k = m$, can be viewed as control points. Let

$$\Delta_m = \{(i,\, j,\, k) \in \mathbb{N}^3 \mid i + j + k = m\}.$$

From now on, we will usually denote a barycentric affine frame (r, s, t) in the affine plane $\mathcal{P}$, as Δrst, and call it a *reference triangle*.

Lemma 9.4.2 *Given a reference triangle Δrst in the affine plane $\mathcal{P}$, given any family $(b_{i,\,j,\,k})_{(i,j,k)\in\Delta_m}$ of $\frac{(m+1)(m+2)}{2}$ points in $\mathcal{E}$, there is a unique surface $F\colon \mathcal{P} \to \mathcal{E}$ of total degree m, defined by a symmetric m-affine polar form $f\colon \mathcal{P}^m \to \mathcal{E}$, such that*

$$f(\underbrace{r,\ldots,r}_{i},\underbrace{s,\ldots,s}_{j},\underbrace{t,\ldots,t}_{k}) = b_{i,\,j,\,k},$$

for all $(i,j,k) \in \Delta_m$. Furthermore, f is given by the expression

$$f(a_1,\ldots,a_m) = \sum_{\substack{I\cup J\cup K=\{1,\ldots,m\}\\ I,J,K \text{ pairwise disjoint}}} \left(\prod_{i\in I}\lambda_i\right)\left(\prod_{j\in J}\mu_j\right)\left(\prod_{k\in K}\nu_k\right) f(\underbrace{r,\ldots,r}_{|I|},\underbrace{s,\ldots,s}_{|J|},\underbrace{t,\ldots,t}_{|K|}),$$

where $a_i = \lambda_i r + \mu_i s + \nu_i t$, with $\lambda_i + \mu_i + \nu_i = 1$, and $1 \le i \le m$.

A point $F(a)$ on the surface F can be expressed in terms of the Bernstein polynomials

$$B^m_{i,j,k}(U,V,T) = \frac{m!}{i!j!k!}\, U^i V^j T^k,$$

as

$$F(a) = f(\underbrace{a,\ldots,a}_{m}) = \sum_{(i,j,k)\in\Delta_m} B^m_{i,j,k}(\lambda,\mu,\nu)\, f(\underbrace{r,\ldots,r}_{i},\underbrace{s,\ldots,s}_{j},\underbrace{t,\ldots,t}_{k}),$$

where $a = \lambda r + \mu s + \nu t$, with $\lambda + \mu + \nu = 1$.

A family $\mathcal{N} = (b_{i,\,j,\,k})_{(i,j,k)\in\Delta_m}$ of $\frac{(m+1)(m+2)}{2}$ points in $\mathcal{E}$, is called a *(triangular) control net, or Bézier net*. Note that the points in

$$\Delta_m = \{(i,\, j,\, k) \in \mathbb{N}^3 \mid i + j + k = m\},$$

can be thought of as a triangular grid of points in $\mathcal{P}$. For example, when $m = 5$, we have the following grid of 21 points:

$$\begin{array}{ccccccccccc}
 & & & & & 500 & & & & & \\
 & & & & 401 & & 410 & & & & \\
 & & & 302 & & 311 & & 320 & & & \\
 & & 203 & & 212 & & 221 & & 230 & & \\
 & 104 & & 113 & & 122 & & 131 & & 140 & \\
005 & & 014 & & 023 & & 032 & & 041 & & 050
\end{array}$$

9.5 The de Casteljau Algorithm for Triangular Surface Patches

Given a reference triangle Δrst, given a triangular control net $\mathcal{N} = (b_{i,\,j,\,k})_{(i,j,k)\in\Delta_m}$, recall that in terms of the polar form $f\colon \mathcal{P}^m \to \mathcal{E}$ of the polynomial surface $F\colon \mathcal{P} \to \mathcal{E}$ defined by $\mathcal{N}$, for every $(i, j, k) \in \Delta_m$, we have

$$b_{i,\,j,\,k} = f(\underbrace{r, \ldots, r}_{i}, \underbrace{s, \ldots, s}_{j}, \underbrace{t, \ldots, t}_{k}).$$

Given $a = \lambda r + \mu s + \nu t$ in $\mathcal{P}$, where $\lambda + \mu + \nu = 1$, in order to compute $F(a) = f(a, \ldots, a)$, the computation builds a sort of tetrahedron consisting of $m + 1$ layers. The base layer consists of the original control points in $\mathcal{N}$, which are also denoted as $(b^0_{i,\,j,\,k})_{(i,j,k)\in\Delta_m}$. The other layers are computed in m stages, where at stage l, $1 \leq l \leq m$, the points $(b^l_{i,\,j,\,k})_{(i,j,k)\in\Delta_{m-l}}$ are computed such that

$$b^l_{i,\,j,\,k} = \lambda b^{l-1}_{i+1,\,j,\,k} + \mu b^{l-1}_{i,\,j+1,\,k} + \nu b^{l-1}_{i,\,j,\,k+1}.$$

During the last stage, the single point $b^m_{0,\,0,\,0}$ is computed. An easy induction shows that

$$b^l_{i,\,j,\,k} = f(\underbrace{a, \ldots, a}_{l}, \underbrace{r, \ldots, r}_{i}, \underbrace{s, \ldots, s}_{j}, \underbrace{t, \ldots, t}_{k}),$$

where $(i, j, k) \in \Delta_{m-l}$, and thus,

$$F(a) = b^m_{0,\,0,\,0}.$$

Similarly, given m points $a_1, \ldots, a_m$ in $\mathcal{P}$, where $a_l = \lambda_l r + \mu_l s + \nu_l t$, with $\lambda_l + \mu_l + \nu_l = 1$, we can compute the polar value $f(a_1, \ldots, a_m)$ as follows. Again, the base layer of the tetrahedron consists of the original control points in $\mathcal{N}$, which are also denoted as $(b^0_{i,\,j,\,k})_{(i,j,k)\in\Delta_m}$. At stage l, where $1 \leq l \leq m$, the points $(b^l_{i,\,j,\,k})_{(i,j,k)\in\Delta_{m-l}}$ are computed such that

$$b^l_{i,\,j,\,k} = \lambda_l b^{l-1}_{i+1,\,j,\,k} + \mu_l b^{l-1}_{i,\,j+1,\,k} + \nu_l b^{l-1}_{i,\,j,\,k+1}.$$

An easy induction shows that

$$b^l_{i,\,j,\,k} = f(a_1, \ldots, a_l, \underbrace{r, \ldots, r}_{i}, \underbrace{s, \ldots, s}_{j}, \underbrace{t, \ldots, t}_{k}),$$

where $(i, j, k) \in \Delta_{m-l}$, and thus,

$$f(a_1, \ldots, a_m) = b^m_{0,\,0,\,0}$$

In order to present the algorithm, it may be helpful to introduce some abbreviations. For example, a triple $(i, j, k) \in \Delta_m$ is denoted as $\mathbf{i}$, and we let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$, and $\mathbf{0} = (0, 0, 0)$. Let $a = \lambda r + \mu s + \nu t$, where $\lambda + \mu + \nu = 1$. We are assuming that we have initialized the family $(b^0_{\mathbf{i}})_{\mathbf{i} \in \Delta_m}$, such that $b^0_{\mathbf{i}} = b_{\mathbf{i}}$, for all $\mathbf{i} \in \Delta_m$. Then, we can describe the de Casteljau algorithm as follows.

begin
  for $l := 1$ **to** m **do**
    for $i := 0$ **to** $m - l$ **do**
      for $j := 0$ **to** $m - i - l$ **do**
        $k := m - i - j - l;$
        $\mathbf{i} := (i, j, k);$
        $b^l_{\mathbf{i}} := \lambda b^{l-1}_{\mathbf{i}+\mathbf{e}_1} + \mu b^{l-1}_{\mathbf{i}+\mathbf{e}_2} + \nu b^{l-1}_{\mathbf{i}+\mathbf{e}_3}$
      endfor
    endfor
  endfor;
  $F(a) := b^m_{\mathbf{0}}$
end

In order to compute the polar value $f(a_1, \ldots, a_m)$, for m points $a_1, \ldots, a_m$ in $\mathcal{P}$, where $a_l = \lambda_l r + \mu_l s + \nu_l t$, with $\lambda_l + \mu_l + \nu_l = 1$, we simply replace λ, μ, ν by λ_l, μ_l, ν_l.

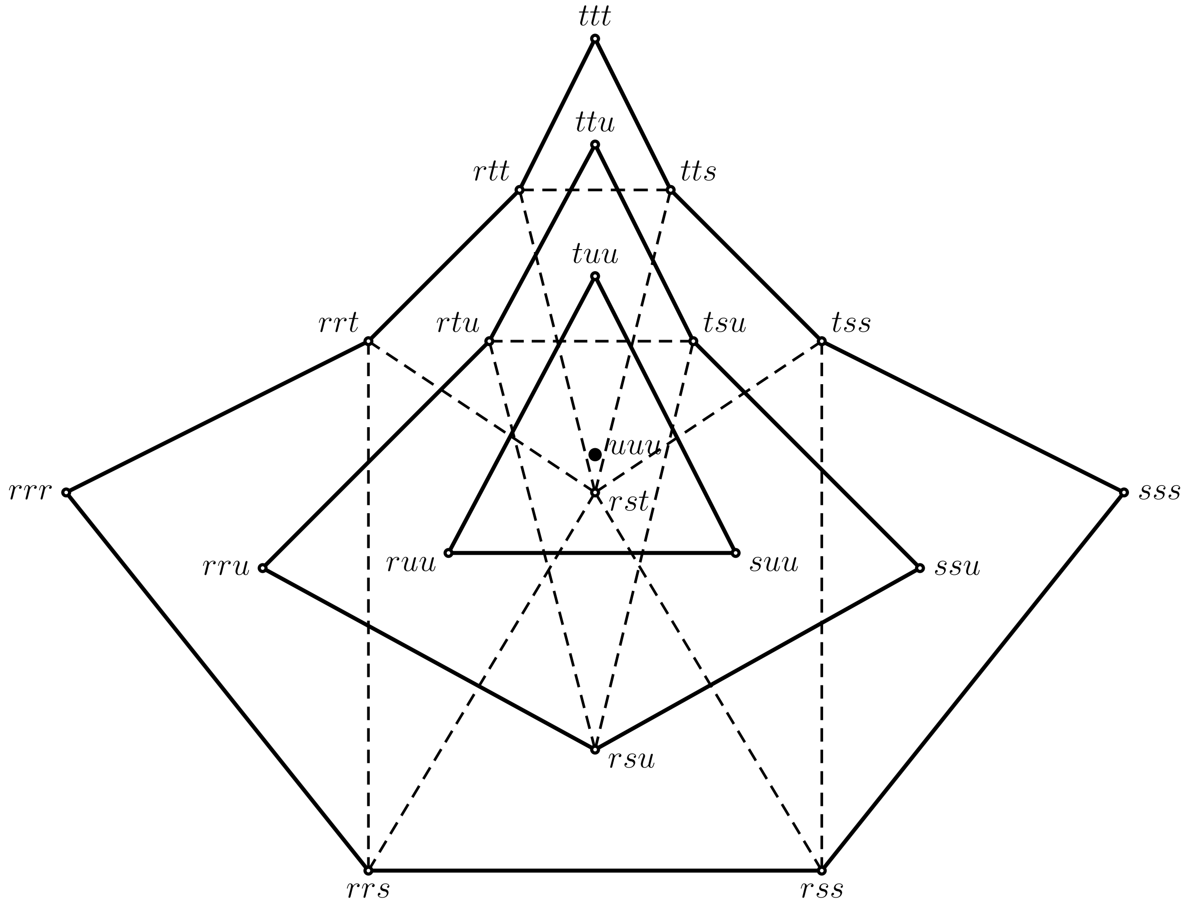


Figure 9.2: The de Casteljau algorithm for polynomial surfaces of total degree 3

It is interesting to note that the same polynomial surface F, when represented as a bipolynomial surface of degree $\langle p, q\rangle$, requires a control net of $(p+1)(q+1)$ control points, and when represented as a surface of total degree m, requires a control net of $\frac{(m+1)(m+2)}{2}$ points.

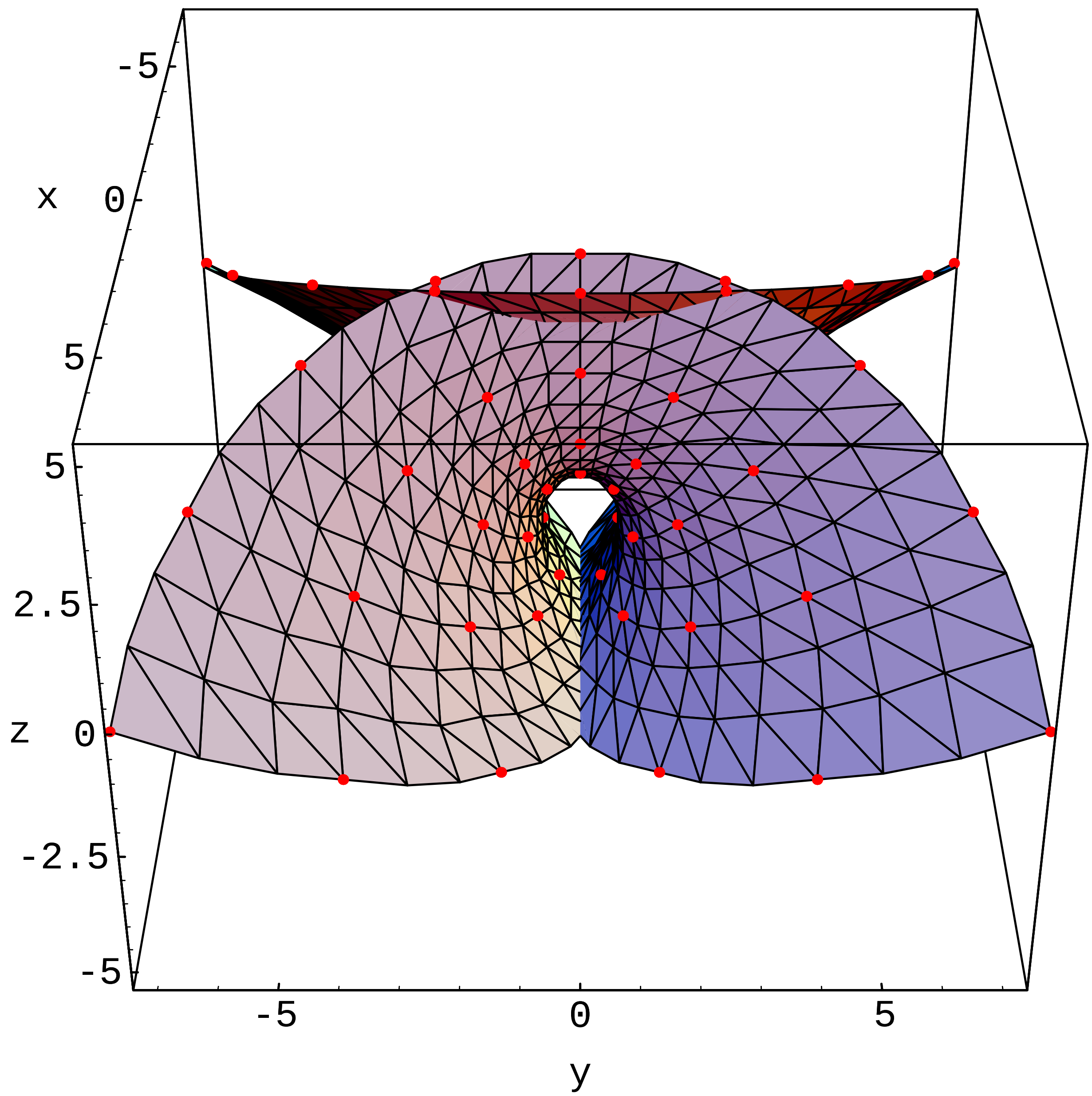


Figure 9.3: The Enneper surface

Chapter 10

Subdivision Algorithms for Polynomial Surfaces

10.1 Subdivision Algorithms for Triangular Patches

In this section, we explain in detail how the de Casteljau algorithm can be used to subdivide a triangular patch into subpatches, in order to obtain a triangulation of a surface patch using recursive subdivision.

Given a reference triangle Δrst, given a triangular control net $\mathcal{N} = (b_{i,\,j,\,k})_{(i,j,k)\in\Delta_m}$, recall that in terms of the polar form $f\colon \mathcal{P}^m \to \mathcal{E}$ of the polynomial surface $F\colon \mathcal{P} \to \mathcal{E}$ defined by $\mathcal{N}$, for every $(i, j, k) \in \Delta_m$, we have

$$b_{i,\,j,\,k} = f(\underbrace{r, \ldots, r}_{i}, \underbrace{s, \ldots, s}_{j}, \underbrace{t, \ldots, t}_{k}).$$

Given $a = \lambda r + \mu s + \nu t$ in $\mathcal{P}$, where $\lambda + \mu + \nu = 1$, in order to compute $F(a) = f(a, \ldots, a)$, the computation builds a sort of tetrahedron consisting of $m + 1$ layers.

The base layer consists of the original control points in $\mathcal{N}$, which are also denoted as $(b^0_{i,\,j,\,k})_{(i,j,k)\in\Delta_m}$. The other layers are computed in m stages, where at stage l, $1 \leq l \leq m$, the points $(b^l_{i,\,j,\,k})_{(i,j,k)\in\Delta_{m-l}}$ are computed such that

$$b^l_{i,\,j,\,k} = \lambda b^{l-1}_{i+1,\,j,\,k} + \mu b^{l-1}_{i,\,j+1,\,k} + \nu b^{l-1}_{i,\,j,\,k+1}.$$

During the last stage, the single point $b^m_{0,\,0,\,0}$ is computed.

An easy induction shows that

$$b^l_{i,\,j,\,k} = f(\underbrace{a,\ldots,a}_{l}, \underbrace{r,\ldots,r}_{i}, \underbrace{s,\ldots,s}_{j}, \underbrace{t,\ldots,t}_{k}),$$

where $(i, j, k) \in \Delta_{m-l}$, and thus,

$$F(a) = b^m_{0,\,0,\,0}.$$

Assuming that a is not on one of the edges of Δrst, the crux of the subdivision method is that the three other faces of the tetrahedron of polar values $b^l_{i,\,j,\,k}$ besides the face corresponding to the original control net, yield three control nets

$$\mathcal{N}ast = (b^l_{0,\,j,\,k})_{(l,j,k)\in\Delta_m},$$

corresponding to the base triangle Δast,

$$\mathcal{N}rat = (b^l_{i,\,0,\,k})_{(i,l,k)\in\Delta_m},$$

corresponding to the base triangle Δrat, and

$$\mathcal{N}rsa = (b^l_{i,\,j,\,0})_{(i,j,l)\in\Delta_m},$$

corresponding to the base triangle Δrsa.

From an implementation point of view, we found it convenient to assume that a triangular net $\mathcal{N} = (b_{i,\,j,\,k})_{(i,j,k)\in\Delta_m}$ is represented as the list consisting of the concatenation of the $m+1$ rows

$$b_{i,\,0,\,m-i},\ b_{i,\,1,\,m-i-1},\ \ldots,\ b_{i,\,m-i,\,0}.$$

As a triangle, the net $\mathcal{N}$ is listed (from top-down) as

$$\begin{array}{ccccc}
f(\underbrace{t,\ldots,t}_{m}) & f(\underbrace{t,\ldots,t}_{m-1},\,s) & \ldots & f(t,\,\underbrace{s,\ldots,s}_{m-1}) & f(\underbrace{s,\ldots,s}_{m}) \\
 & \ldots & & \ldots & \\
 & & \ldots & & \\
 & f(\underbrace{r,\ldots,r}_{m-1},\,t) & & f(\underbrace{r,\ldots,r}_{m-1},\,s) & \\
 & & f(\underbrace{r,\ldots,r}_{m}) & &
\end{array}$$

The main advantage of this representation is that we can view the net $\mathcal{N}$ as a two-dimensional array net, such that $net[i,j] = b_{i,\,j,\,k}$ (with $i+j+k=m$). In fact, only a triangular portion of this array is filled. This way of representing control nets fits well with the convention that the reference triangle Δrst is represented as follows:

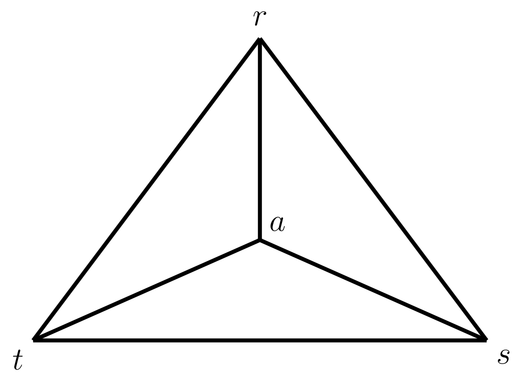


Figure 10.1: Reference triangle

Instead of simply computing $F(a) = b^m_{0,\,0,\,0}$, the de Casteljau algorithm can be easily adapted to output the three nets $\mathcal{N}ast$, $\mathcal{N}rat$, and $\mathcal{N}rsa$. We call this version of the de Casteljau algorithm the *subdivision algorithm*.

In implementing such a program, we found that it was convenient to compute the nets $\mathcal{N}ast$, $\mathcal{N}art$, and $\mathcal{N}ars$.

In order to compute $\mathcal{N}rat$ from $\mathcal{N}art$, we wrote a very simple function `transnetj`, and in order to compute $\mathcal{N}rsa$ from $\mathcal{N}ars$, we wrote a very simple function `transnetk`. We also have a function `convtomat` which converts a control net given as a list of rows, into a two-dimensional array.

We found it convenient to write three distinct functions `subdecas3ra`, `subdecas3sa`, and `subdecas3ta`, computing the control nets with respect to the reference triangles Δast, Δart, and Δars.

The subdivision strategy that we will follow is to divide the reference triangle Δrst into four subtriangles Δabt, Δbac, Δcrb, and Δsca, where $a = (0, 1/2, 1/2)$, $b = (1/2, 0, 1/2)$, and $c = (1/2, 1/2, 0)$, are the middle points of the sides st, rt and rs respectively, as shown in the diagram below:

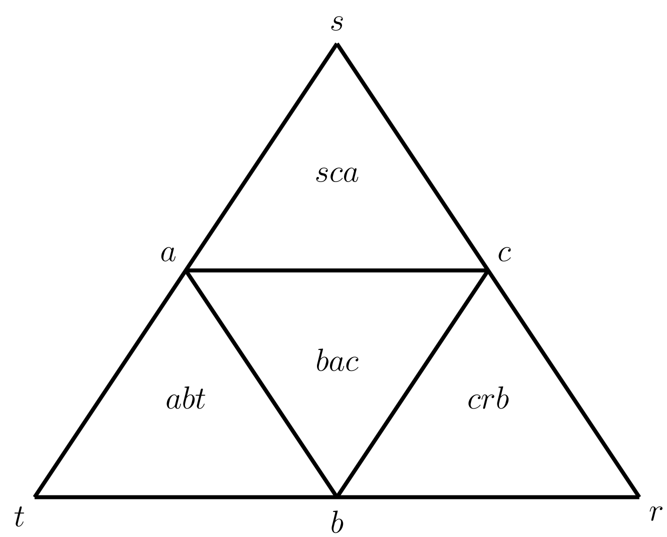


Figure 10.2: Subdividing a reference triangle Δrst

The first step is to compute the control net for the reference triangle Δbat. This can be done using two steps.

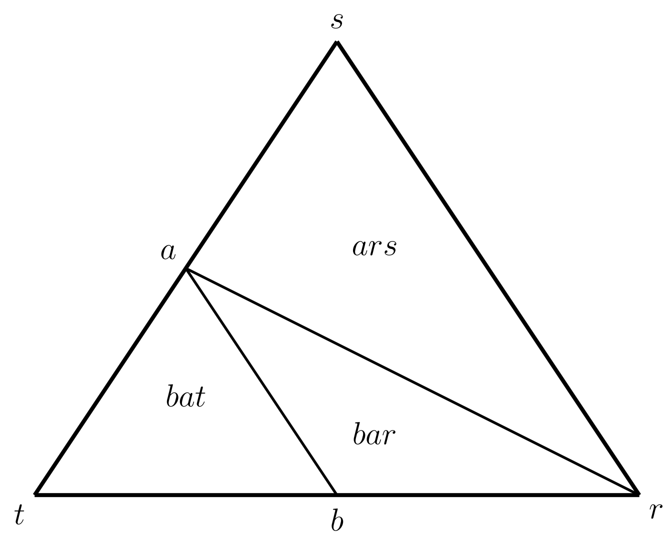


Figure 10.3: Computing the nets $\mathcal{N}bat$, $\mathcal{N}bar$ and $\mathcal{N}ars$ from $\mathcal{N}rst$

We will now compute the net $\mathcal{N}cas$ from the net $\mathcal{N}ars$.

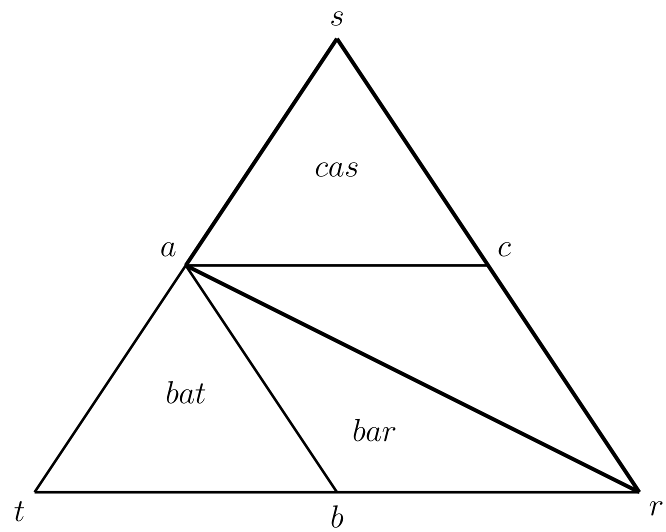


Figure 10.4: Computing the net $\mathcal{N}cas$ from $\mathcal{N}ars$

We can now compute the nets $\mathcal{N}cbr$ and $\mathcal{N}cba$ from the net $\mathcal{N}bar$.

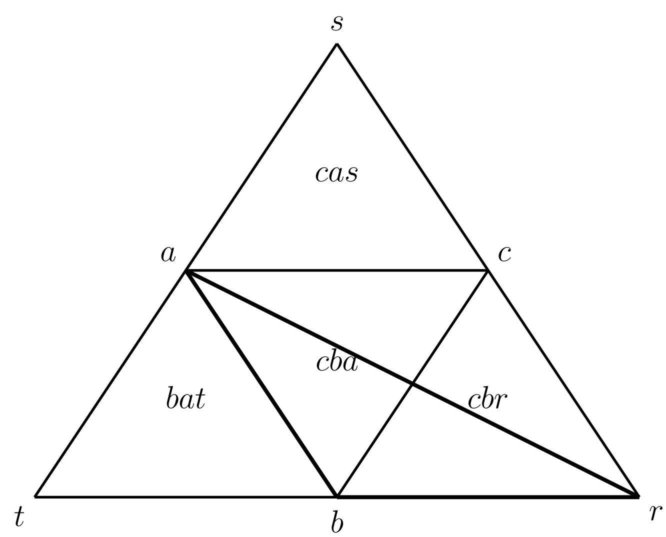


Figure 10.5: Computing the nets $\mathcal{N}cbr$ and $\mathcal{N}cba$ from $\mathcal{N}bar$

Finally, we apply some net permutations, and we get

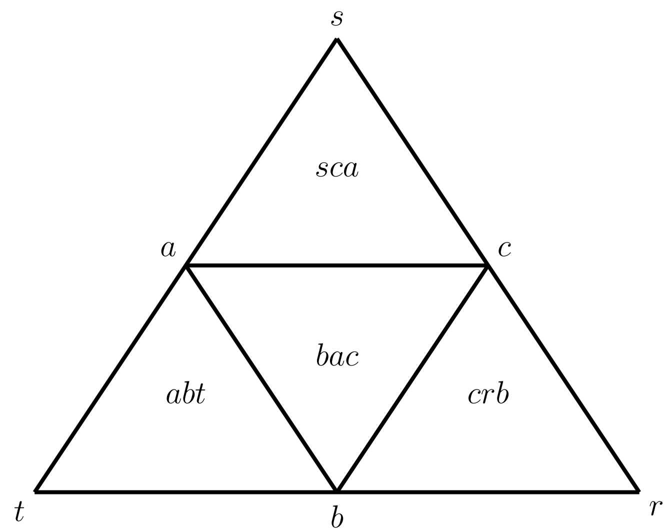


Figure 10.6: Subdividing Δrst into Δabt, Δbac, Δcrb, and Δsca

Using **mainsubdecas4**, starting from a list consisting of a single control net **net**, we can repeatedly subdivide the nets in a list of nets, in order to obtain a triangulation of the surface patch specified by the control net **net**.

The function **rsubdiv4** performs **n** recursive steps of subdivision, starting with an input control net **net**. The function **itersub4** takes a list of nets and subdivides each net in this list into four subnets.

The function `rsubdiv4` creates a list of nets, where each net is a list of points. In order to render the surface patch, it is necessary to triangulate each net, that is, to join the control points in a net by line segments. This can be done in a number of ways, and is left as an exercise.

The best thing to do is to use the `Polygon` construct of *Mathematica*. Indeed, polygons are considered non-transparent. and the rendering algorithm automatically removes hidden parts. It is also very easy to use the shading options of *Mathematica*, or color the polygons as desired. This is very crucial to understand complicated surfaces.

The subdivision method is illustrated by the following example of a cubic patch specified by the control net

```
net = {{0, 0, 0}, {2, 0, 2}, {4, 0, 2}, {6, 0, 0},
       {1, 2, 2}, {3, 2, 5}, {5, 2, 2},
       {2, 4, 2}, {4, 4, 2},  {3, 6, 0}};
```

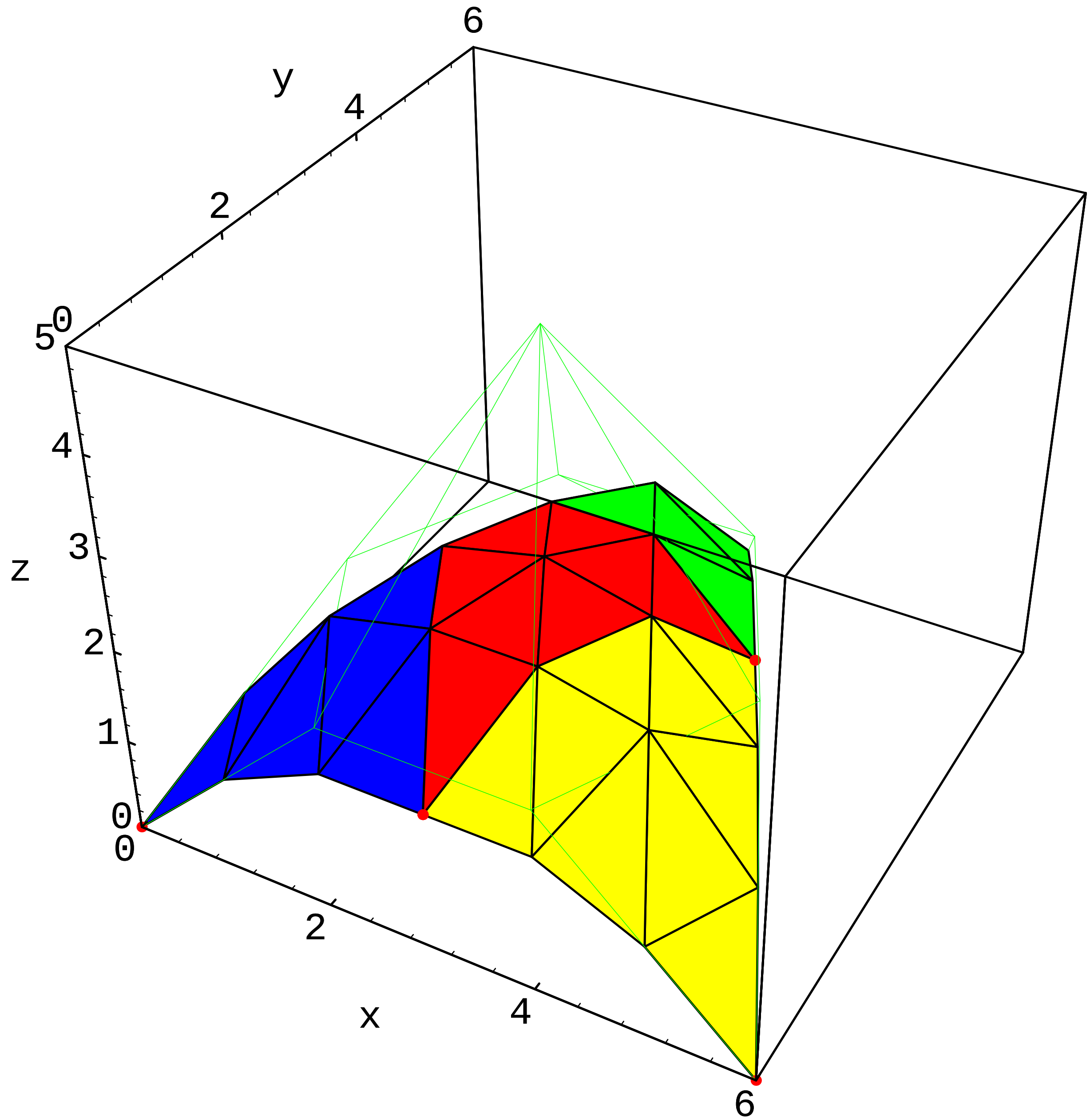


Figure 10.7: Subdivision, 1 iteration

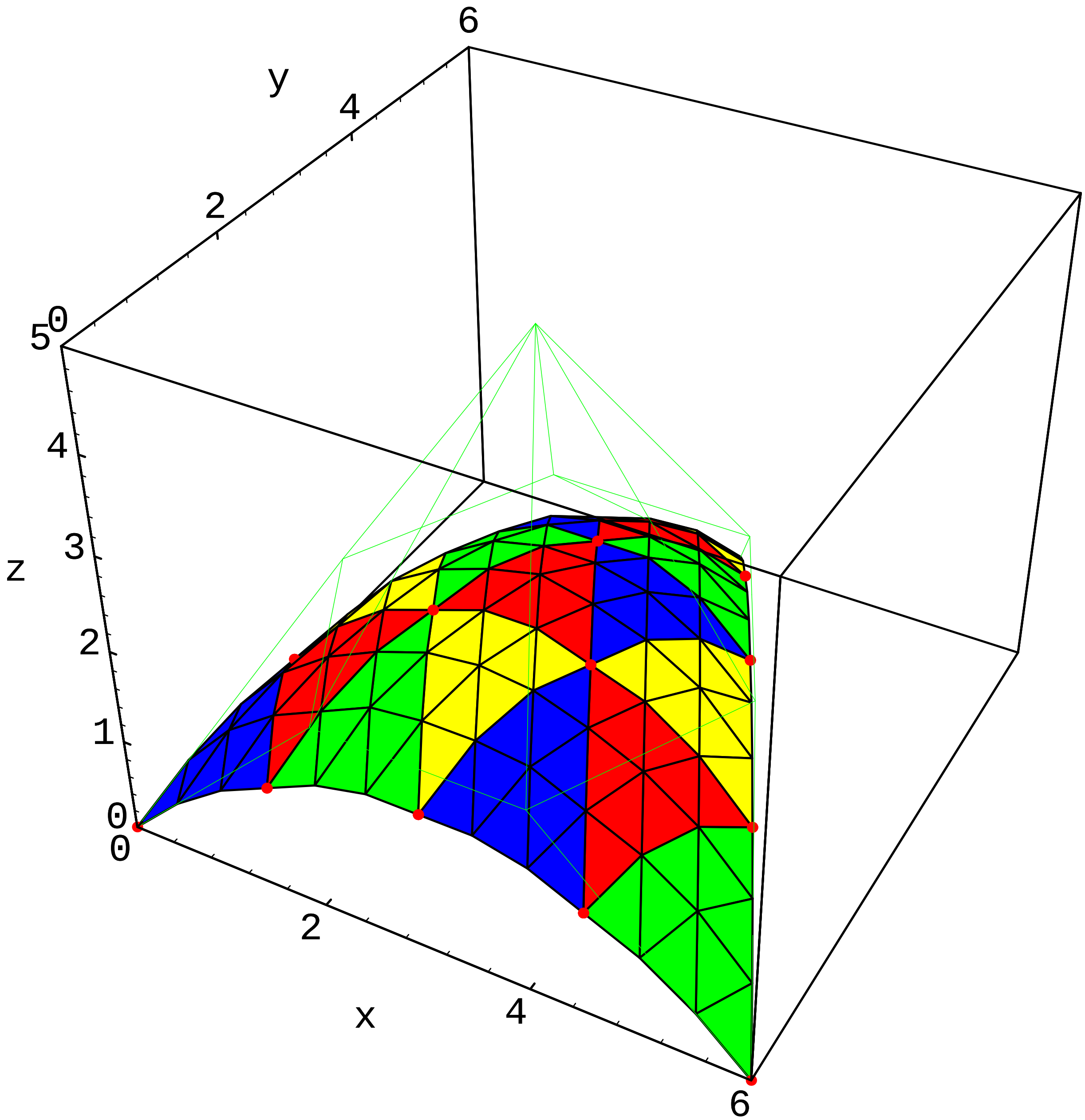


Figure 10.8: Subdivision, 2 iterations

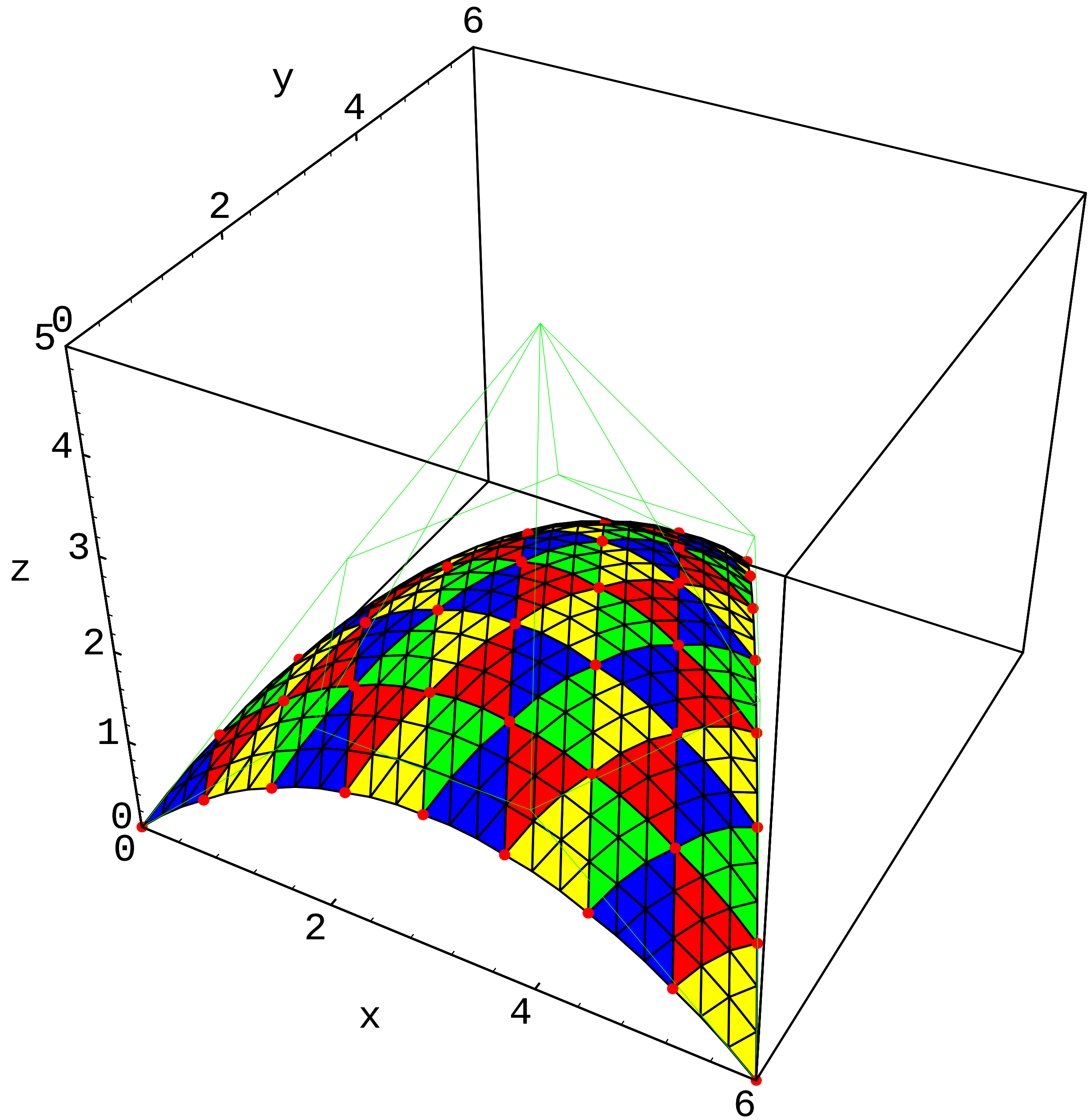


Figure 10.9: Subdivision, 3 iterations

Another pleasant application of the subdivision method is that it yields an efficient method for computing a control net $\mathcal{N}abc$ over a new reference triangle Δabc, from a control net $\mathcal{N}$ over an original reference triangle Δrst.

Let Δrst and Δabc be two reference triangles, and let $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$, and $(\lambda_3, \mu_3, \nu_3)$, be the barycentric coordinates of a, b, c, with respect to Δrts.

Given any arbitrary point d, if d has coordinates (λ, μ, ν) with respect to Δrst, and coordinates $(\lambda',\ \mu',\ \nu')$ with respect to Δabc, since

$$d = \lambda r + \mu s + \nu t = \lambda' a + \mu' b + \nu' c$$

and

$$\begin{aligned}
a = &\lambda_1 r + \mu_1 s + \nu_1 t, \\
b = &\lambda_2 r + \mu_2 s + \nu_2 t, \\
c = &\lambda_3 r + \mu_3 s + \nu_3 t,
\end{aligned}$$

we easily get

$$\begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{pmatrix} \begin{pmatrix} \lambda' \\ \mu' \\ \nu' \end{pmatrix}$$

and thus,

$$\begin{pmatrix} \lambda' \\ \mu' \\ \nu' \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{pmatrix}^{-1} \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix}$$

Thus, the coordinates (λ', μ', ν') of d with respect to Δabc can be computed from the coordinates (λ, μ, ν) of d with respect to Δrst, by inverting a matrix. In this case, this is easily done using determinants, by Cramer's formulae.

Now, given a reference triangle Δrst and a control net $\mathcal{N}$ over Δrst, we can compute the new control net $\mathcal{N}abc$ over the new reference triangle Δabc, using three subdivision steps as explained below.

In the first step, we compute the control net $\mathcal{N}ast$ over the reference triangle Δast, using `subdecas3ra`.

In the second step, we compute the control net $\mathcal{N}bat$ using `subdecas3sa`, and then the control net $\mathcal{N}abt$ over the triangle Δabt, using `transnetj`.

In the third step, we compute the control net $\mathcal{N}cab$ using `subdecas3ta`, and then the control net $\mathcal{N}abc$ over the triangle Δabc, using `transnetk`.

Note that in the second step, we need the coordinates of b with respect to the reference triangle Δast, and in the third step, we need the coordinates of c with respect to the reference triangle Δabt. This can be easily done by inverting a matrix of order 3, as explained earlier.

One should also observe that the above method is only correct if a does not belong to st, and b does not belong to at. In general, some adaptations are needed. We used the strategy explained below, and implemented in *Mathematica*.

Case 1: $a \notin st$.

Compute $\mathcal{N}ast$ using `subdecas3ra`.

Case 1a: $b \notin at$.

First, compute $\mathcal{N}bat$ using `subdecas3sa`, and then $\mathcal{N}abt$ using `transnetj`. Next, compute $\mathcal{N}cab$ using `subdecas3ta`, and then $\mathcal{N}abc$ using `transnetk`.

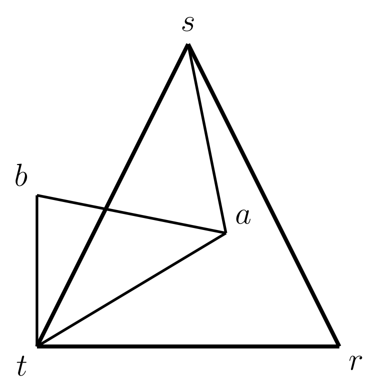


Figure 10.10: Case 1a: a not in st, b not in at

Case 1b: $b \in at$.

First, compute $\mathcal{N}tas$ from $\mathcal{N}ast$ using `transnetk` twice, then compute $\mathcal{N}bas$ using `subdecas3ra`, and then $\mathcal{N}abs$ using `transnetj`. Finally, compute $\mathcal{N}abc$ using `subdecas3ta`.

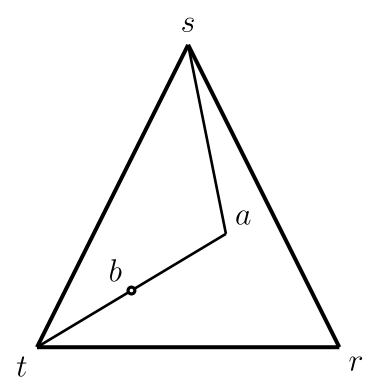


Figure 10.11: Case 1b: a not in st, $b \in at$

Case 2a: $s = a$ (and thus, $a \in st$).

In this case, $\Delta rst = \Delta rat$. First compute $\mathcal{N}art$ using `transnetj`, and then go back to case 1.

Case 2b: $a \in st$ and $s \neq a$.

Compute $\mathcal{N}ars$ using `subdecas3ta`, and then go back to case 1.

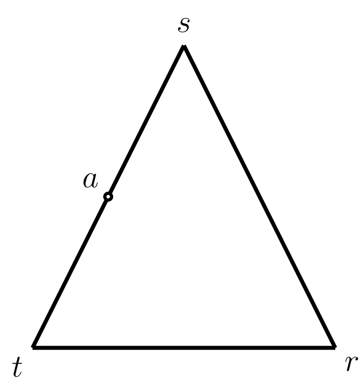


Figure 10.12: Case 2b: $a \in st$, $s \neq a$

As an example we can display a portion of a well known surface known as the "monkey saddle", defined by the equations

$$x = u,$$
$$y = v,$$
$$z = u^3 - 3uv^2.$$

Note that z is the real part of the complex number $(u+iv)^3$. It is easily shown that the monkey saddle is specified by the following triangular control net **monknet** over the standard reference triangle Δrst, where $r = (1, 0, 0)$, $s = (0, 1, 0)$, and $t = (0, 0, 1)$.

```
monknet = {{0, 0, 0}, {0, 1/3, 0}, {0, 2/3, 0}, {0, 1, 0},
          {1/3, 0, 0}, {1/3, 1/3, 0},  {1/3, 2/3, -1},
          {2/3, 0, 0}, {2/3, 1/3, 0},  {1, 0, 1}};
```

Using **newcnet3** twice to get some new nets **net1** and **net2**, and then subdividing both nets 3 times, we get the following picture.

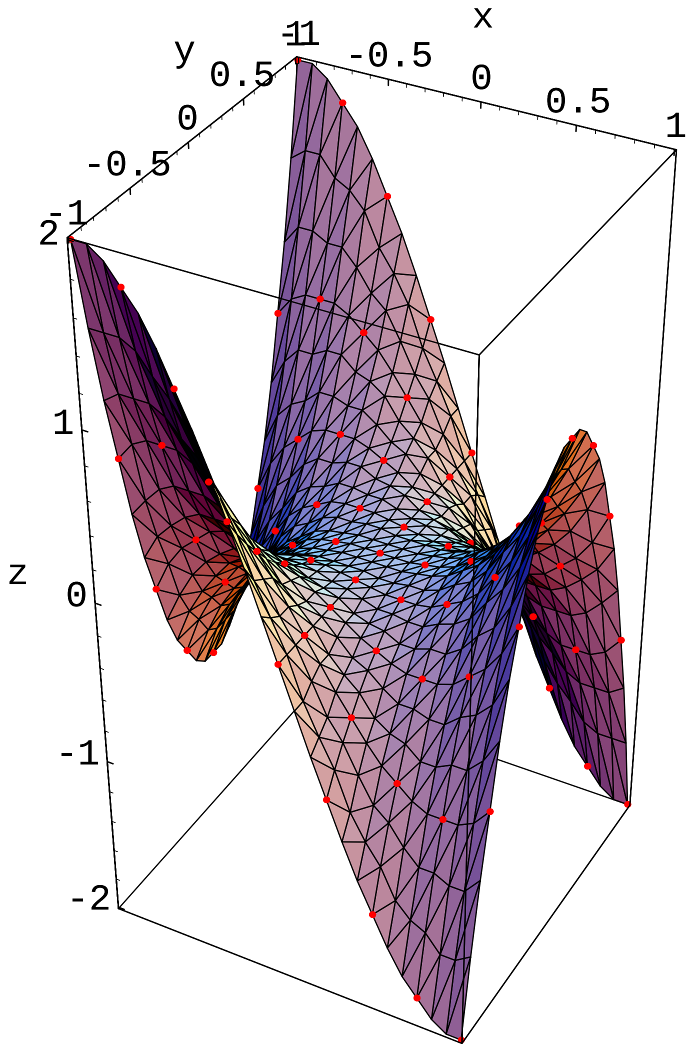


Figure 10.13: A monkey saddle, triangular subdivision

Another nice application of the subdivision algorithms, is an efficient method for computing the control points of a curve on a triangular surface patch, where the curve is the image of a line in the parameter plane, specified by two points a and b.

What we need is to compute the control points

$$d_i = f(\underbrace{a, \ldots, a}_{m-i}, \underbrace{b, \ldots, b}_{i}),$$

where m is the degree of the surface.

We could compute these polar values directly, but there is a much faster method. Indeed, assuming that the surface is defined by some net $\mathcal{N}$ over the reference triangle Δrts, if r does not belong to the line (a, b), we simply have to compute $\mathcal{N}rba$ using `newcnet3`, and the control points $(d_0, \ldots, d_m)$ are simply the bottom row of the net $\mathcal{N}rba$, assuming the usual representation of a triangular net as the list of rows

$$b_{i,\,0,\,m-i},\ b_{i,\,1,\,m-i-1},\ \ldots,\ b_{i,\,m-i,\,0}.$$

More precisely, we have the following cases.

Case 1: $r \notin ab$.

We compute $\mathcal{N}rba$ using `newcnet3`.

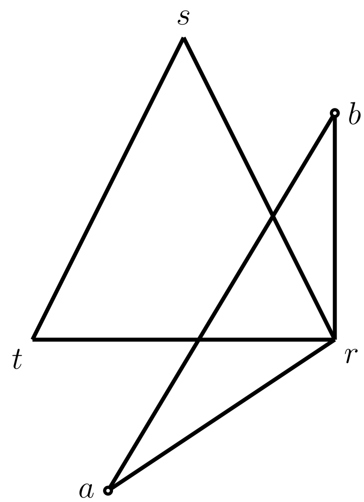


Figure 10.14: Case 1: r not in ab

Case 2a: $r \in ab$ and $a \in rt$.

We compute $\mathcal{N}sba$ using **`newcnet3`**.

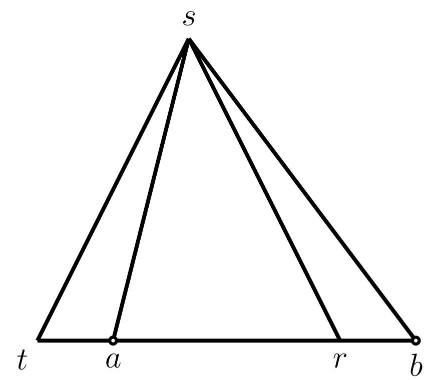


Figure 10.15: Case 2a: $r \in ab$ and $a \in rt$

Case 2b: $r \in ab$ and $a \notin rt$.

In this case, we must have $t \notin ab$, since $r \in ab$, and we compute $\mathcal{N}tba$ using `newcnet3`.

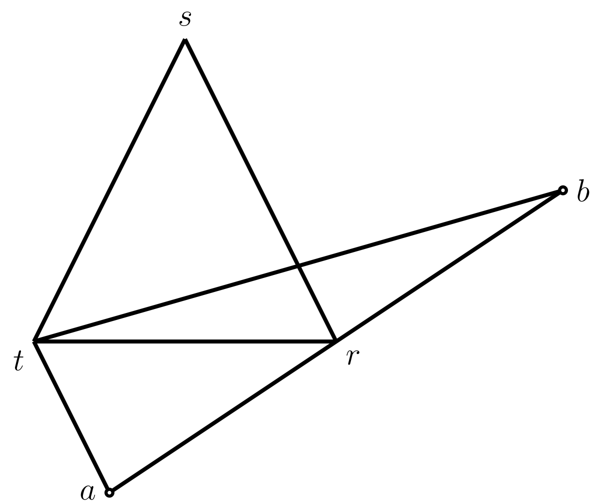


Figure 10.16: Case 2b: $r \in ab$ and a not in rt

Using the function *curvcpoly*, it is easy to render a triangular patch by rendering a number of u-curves and v-curves.

10.2 Subdivision Algorithms for Rectangular Patches

We now consider algorithms for approximating rectangular patches using recursive subdivision.

Given two affine frames $(\overline{r}_1, \overline{s}_1)$ and $(\overline{r}_2, \overline{s}_2)$ for the affine line $\mathbb{A}$, given a rectangular control net

$$\mathcal{N} = (b_{i,\,j})_{(i,j)\in\square_{p,q}},$$

recall that in terms of the polar form

$$f\colon (\mathbb{A})^p \times (\mathbb{A})^q \to \mathcal{E}$$

of the bipolynomial surface $F\colon \mathbb{A}\times\mathbb{A} \to \mathcal{E}$ of degree $\langle p, q\rangle$ defined by $\mathcal{N}$, for every $(i,j) \in \square_{p,q}$, we have

$$b_{i,\,j} = f(\underbrace{\overline{r}_1, \ldots, \overline{r}_1}_{p-i}, \underbrace{\overline{s}_1, \ldots, \overline{s}_1}_{i}; \underbrace{\overline{r}_2, \ldots, \overline{r}_2}_{q-j}, \underbrace{\overline{s}_2, \ldots, \overline{s}_2}_{j}).$$

Unlike subdividing triangular patches, subdividing rectangular patches is quite simple.

Indeed, it is possible to subdivide a rectangular control net $\mathcal{N}$ in two ways. The first way is to compute the two nets $\mathcal{N}[r_1, u; *]$ and $\mathcal{N}[u, s_1; *]$, where

$$\mathcal{N}[r_1, u; *]_{i,\,j} = f(\underbrace{\overline{r}_1, \ldots, \overline{r}_1}_{p-i}, \underbrace{\overline{u}, \ldots, \overline{u}}_{i}; \underbrace{\overline{r}_2, \ldots, \overline{r}_2}_{q-j}, \underbrace{\overline{s}_2, \ldots, \overline{s}_2}_{j}),$$

with $0 \leq i \leq p$, and $0 \leq j \leq q$, and

$$\mathcal{N}[u, s_1; *]_{i,\,j} = f(\underbrace{\overline{u}, \ldots, \overline{u}}_{p-i}, \underbrace{\overline{s}_1, \ldots, \overline{s}_1}_{i}; \underbrace{\overline{r}_2, \ldots, \overline{r}_2}_{q-j}, \underbrace{\overline{s}_2, \ldots, \overline{s}_2}_{j}),$$

with $0 \leq i \leq p$, and $0 \leq j \leq q$.

This can be achieved in $q+1$ calls to the version of the de Casteljau algorithm performing subdivision (in the case of curves).

This algorithm has been implemented in *Mathematica* as the function `urecdecas`.

The second way is to compute the two nets $\mathcal{N}[*; r_2, v]$ and $\mathcal{N}[*; v, s_2]$, where

$$\mathcal{N}[*; r_2, v]_{i,\,j} = f(\underbrace{\overline{r}_1, \ldots, \overline{r}_1}_{p-i}, \underbrace{\overline{s}_1, \ldots, \overline{s}_1}_{i}; \underbrace{\overline{r}_2, \ldots, \overline{r}_2}_{q-j}, \underbrace{\overline{v}, \ldots, \overline{v}}_{j}),$$

with $0 \leq i \leq p$, and $0 \leq j \leq q$, and

$$\mathcal{N}[*; v, s_2]_{i,\,j} = f(\underbrace{\overline{r}_1, \ldots, \overline{r}_1}_{p-i}, \underbrace{\overline{s}_1, \ldots, \overline{s}_1}_{i}; \underbrace{\overline{v}, \ldots, \overline{v}}_{q-j}, \underbrace{\overline{s}_2, \ldots, \overline{s}_2}_{j}),$$

with $0 \leq i \leq p$, and $0 \leq j \leq q$.

This can be achieved in $p+1$ calls to the version of the de Casteljau algorithm performing subdivision (in the case of curves).

This algorithm has been implemented in *Mathematica* as the function `vrecdecas`.

Then, given an input net $\mathcal{N}$ over $[r_1, s_1] \times [r_2, s_2]$, for any $\overline{u}, \overline{v} \in \mathbb{A}$, we can subdivide the net $\mathcal{N}$ into four subnets $\mathcal{N}[r_1, u; r_2, v], \mathcal{N}[u, s_1; r_2, v], \mathcal{N}[r_1, u; v, s_2], \mathcal{N}[u, s_1; v, s_2]$, by first subdviding $\mathcal{N}$ into $\mathcal{N}[*; r_2, v]$ and $\mathcal{N}[*; v, s_2]$, using the function `vrecdecas`, and then by splitting each of these two nets using `urecdecas`.

The four nets have the common corner $F(\overline{u}, \overline{v})$.

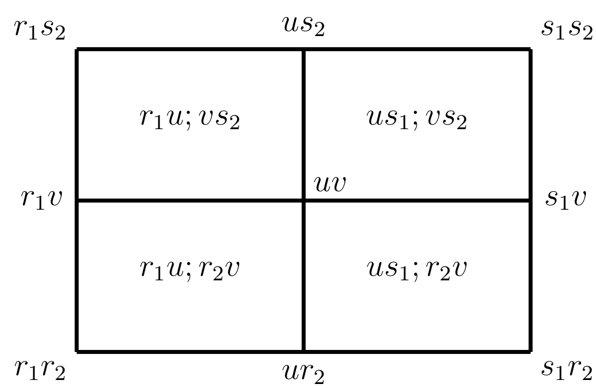


Figure 10.17: Subdividing a rectangular patch

In order to implement these algorithms, we represent a rectangular control net

$$\mathcal{N} = (b_{i,\,j})_{(i,j)\in\square_{p,q}}$$

as the list of $p+1$ rows

$$b_{i,\,0},\ b_{i,\,1},\ \ldots,\ b_{i,\,q},$$

where $0 \leq i \leq p$.

This has the advantage that we can view $\mathcal{N}$ as a rectangular array *net*, with $net[i, j] = b_{i,\,j}$.

The function `makerecnet` converts an input net into such a two dimensional array.

The subdivision algorithm is implemented in *Mathematica* by the function `recdecas`, which uses the functions `vrecdecas` and `urecdecas`.

In turn, these functions use the function `subdecas`, which performs the subdivision of a control polygon.

It turns out that an auxiliary function `rectrans` converting a matrix given as a list of columns into a linear list of rows, is needed.

As in the case of triangular patches, using the function `recdecas`, starting from a list consisting of a single control net `net`, we can repeatedly subdivide the nets in a list of nets, in order to obtain an approximation of the surface patch specified by the control net `net`.

The function `recsubdiv4` shown below performs `n` recursive steps of subdivision, starting with an input control net `net`.

The function `recitersub4` takes a list of nets and subdivides each net in this list into four subnets.

The function `recsubdiv4` returns a list of rectangular nets. In order to render the surface patch, it is necessary to link the nodes in each net. This is easily done, and is left as an exercise.

The functions `urecdecas` and `vrecdecas` can also be used to compute the control net $\mathcal{N}[a, b; c, d]$ over new affine bases $[a, b]$ and $[c, d]$, from a control net $\mathcal{N}$ over some affine bases $[r_1, s_1]$ and $[r_2, s_2]$.

If $d \neq r_2$ and $b \neq r_1$, we first compute $\mathcal{N}[r_1, s_1; r_2, d]$ using `vrecdecas`, then $\mathcal{N}[r_1, b; r_2, d]$ using `urecdecas`, and then $\mathcal{N}[r_1, b; c, d]$ using `vrecdecas`, and finally $\mathcal{N}[a, b; c, d]$ using `urecdecas`. It is easy to care of the cases where $d = r_2$ or $b = r_1$.

Let us go back to the example of the monkey saddle, to illustrate the use of the functions `recsubdiv4` and `recnewnet`.

It is easily shown that the monkey saddle is specified by the following rectangular control net of degree $(3, 2)$ `sqmonknet1`, over $[0, 1] \times [0, 1]$:

```
sqmonknet1 =  {{0, 0, 0}, {0, 1/2, 0},
   {0, 1, 0}, {1/3, 0, 0},
   {1/3, 1/2, 0}, {1/3, 1, -1},
   {2/3, 0, 0}, {2/3, 1/2, 0},
   {2/3, 1, -2}, {1, 0, 1},
   {1, 1/2, 1}, {1, 1, -2}}
```

Using **`recnewnet`**, we can compute a rectangular net `sqmonknet` over $[-1, 1] \times [-1, 1]$:

```
sqmonknet =   {{-1, -1, 2}, {-1, 0, -4},
   {-1, 1, 2}, {-1/3, -1, 2},
   {-1/3, 0, 0}, {-1/3, 1, 2},
   {1/3, -1, -2}, {1/3, 0, 0},
   {1/3, 1, -2}, {1, -1, -2},
   {1, 0, 4}, {1, 1, -2}}
```

Finally, we show the output of the subdivision algorithm **recsubdiv4**, for $n = 1, 2, 3$. The advantage of rectangular nets is that we get the patch over $[-1, 1] \times [-1, 1]$ directly, as opposed to the union of two triangular patches.

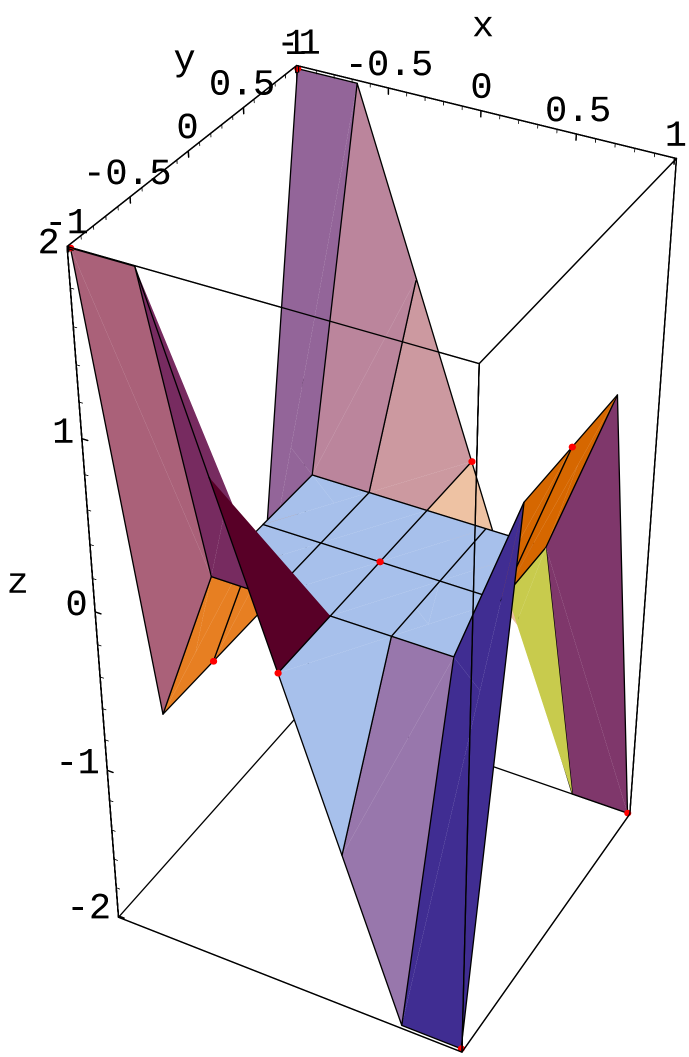


Figure 10.18: A monkey saddle, rectangular subdivision, 1 iteration

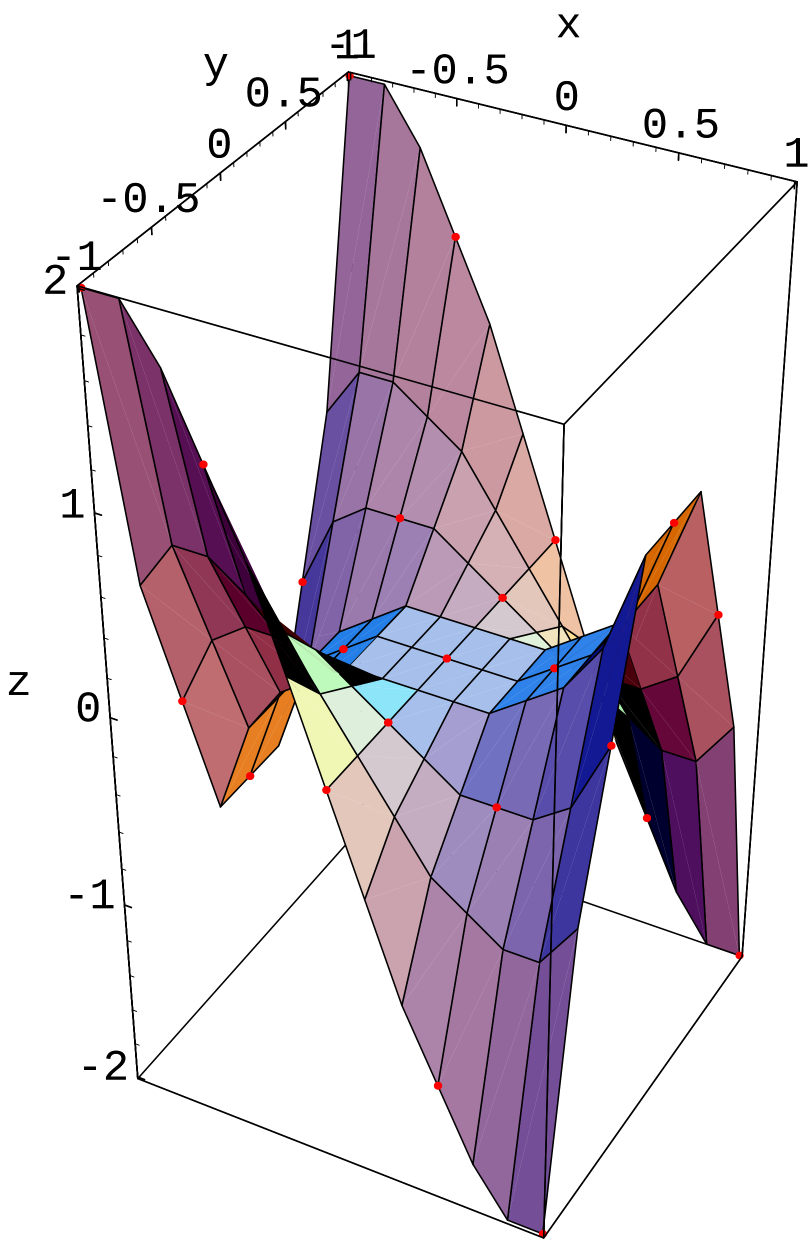


Figure 10.19: A monkey saddle, rectangular subdivision, 2 iterations

The final picture (corresponding to 3 iterations) is basically as good as the triangulation shown earlier, and is obtained faster.

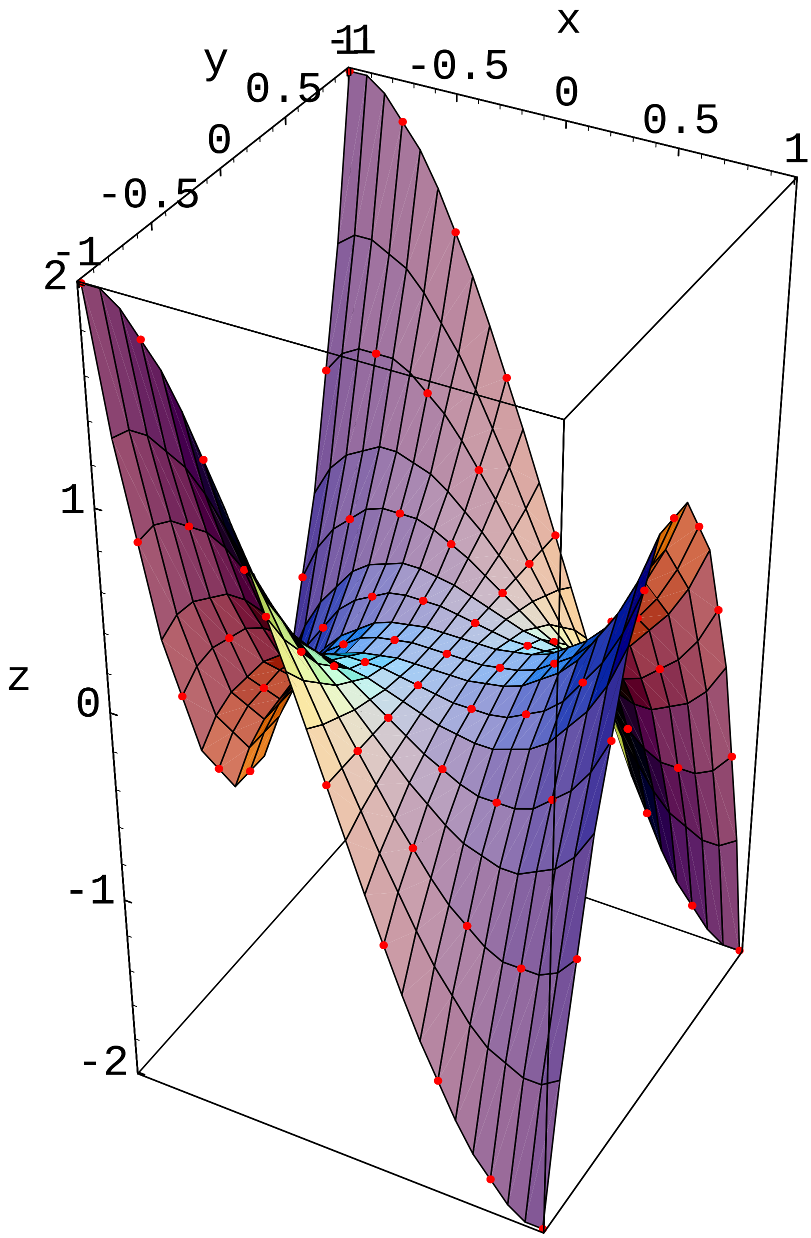


Figure 10.20: A monkey saddle, rectangular subdivision, 3 iterations

Actually, it is possible to convert a triangular net of degree m into a rectangular net of degree (m, m), and conversely to convert a rectangular net of degree (p, q) into a triangular net of degree $p + q$, but we will postpone this until we deal with rational surfaces.

Chapter 11

Polynomial Spline Surfaces

11.1 Joining Polynomial Surfaces

We now attempt to generalize the idea of splines to polynomial surfaces. As we shall see, this is far more subtle than it is for curves.

In the case of a curve, the parameter space is the affine line $\mathbb{A}$, and the only reasonable choice is to divide the affine line into intervals, and to view the curve as the result of joining curve segments defined over these intervals.

However, in the case of a surface, the parameter space is the affine *plane* $\mathcal{P}$, and even if we just want to subdivide the plane into convex regions, there is a tremendous variety of ways of doing so.

Thus, we will restrict our attention to subdivisions of the plane into convex polygons, where the edges are line segments.

In fact, we will basically only consider subdivisions made of rectangles or of (equilateral) triangles.

First, we will find necessary and sufficient conditions on polar forms for two surface patches to meet with C^n continuity.

We will restrict our attention to total degree polynomial surfaces. This is not a real restriction, since it is always possible to convert a rectangular net to a triangular net (see section **??**).

We need to review the definition of a *directional derivative*.

Definition 11.1.1 Let E and F be two normed affine spaces, say $E = \mathbb{A}^m$ and $F = \mathbb{A}^n$, let Ω be a nonempty open subset of E, and let $f\colon \Omega \to F$ be any function. For any $a \in \Omega$, for any $\overrightarrow{u} \neq \overrightarrow{0}$ in $\overrightarrow{E}$, the *directional derivative of f at a w.r.t. the vector $\overrightarrow{u}$*, denoted as $\mathrm{D}_u f(a)$, is the limit (if it exists)

$$\lim_{t \to 0,\, t \in U} \frac{f(a + t\overrightarrow{u}) - f(a)}{t},$$

where $U = \{t \in \mathbb{R} \mid a + t\overrightarrow{u} \in \Omega,\, t \neq 0\}$.

Since the map $t \mapsto a + t\overrightarrow{u}$ is continuous, and since $\Omega - \{a\}$ is open, the inverse image U of $\Omega - \{a\}$ under the above map is open, and the definition of the limit in definition 11.1.1 makes sense.

The directional derivative is sometimes called the Gâteaux derivative.

Let A and B be two adjacent convex polygons in the plane, and let (r, s) be the line segment along which they are adjacent (where $r, s \in \mathcal{P}$ are distinct vertices of A and B).

Given two polynomial surface F and G of degree m, for any point $a \in \mathcal{P}$, we say that *F and G agree to kth order at a*, iff

$$\mathrm{D}_{u_1} \ldots \mathrm{D}_{u_i} F(a) = \mathrm{D}_{u_1} \ldots \mathrm{D}_{u_i} G(a),$$

for all $\overrightarrow{u_1}, \ldots, \overrightarrow{u_i} \in \mathbb{R}^2$, where $0 \leq i \leq k$.

Definition 11.1.2 Let A and B be two adjacent convex polygons in the plane, and let (r, s) be the line segment along which they are adjacent (where $r, s \in \mathcal{P}$ are distinct vertices of A and B). Given two polynomial surfaces F_A and F_B of degree m, *F_A and F_B join with C^k continuity along (r, s)*, iff F_A and F_B agree to kth order for all $a \in (r, s)$.

Recall that lemma **??** tells us that for any $a \in (r, s)$, F_A and F_B agree to kth order at a iff their polar forms $f_A \colon \mathcal{P}^m \to \mathcal{E}$ and $f_B \colon \mathcal{P}^m \to \mathcal{E}$ agree on all multisets of points that contain at least $m - k$ copies of a, that is, iff

$$f_A(u_1, \ldots, u_k, \underbrace{a, \ldots, a}_{m-k}) = f_B(u_1, \ldots, u_k, \underbrace{a, \ldots, a}_{m-k}),$$

for all $u_1, \ldots, u_k \in \mathcal{P}$.

Using this fact, we can prove the following crucial lemma.

Lemma 11.1.3 *Let A and B be two adjacent convex polygons in the plane, and let (r, s) be the line segment along which they are adjacent (where $r, s \in \mathcal{P}$ are distinct vertices of A and B). Given two polynomial surface F_A and F_B of degree m, F_A and F_B join with C^k continuity along (r, s) iff their polar forms $f_A\colon \mathcal{P}^m \to \mathcal{E}$ and $f_B\colon \mathcal{P}^m \to \mathcal{E}$ agree on all multisets of points that contain at least $m - k$ points on the line (r, s), that is, iff*

$$\begin{aligned} & f_A(u_1, \ldots, u_k, a_{k+1}, \ldots, a_m) \\ & \qquad\qquad = f_B(u_1, \ldots, u_k, a_{k+1}, \ldots, a_m), \end{aligned}$$

for all $u_1, \ldots, u_k \in \mathcal{P}$, and all $a_{k+1}, \ldots, a_m \in (r, s)$.

As a consequence of lemma 11.1.3, we obtain the necessary and sufficient conditions on control nets for F_A and F_B for having C^n continuity along (r, s).

Let $A = \Delta prs$ and $B = \Delta qrs$ be two reference triangles in the plane, sharing the edge (r, s).

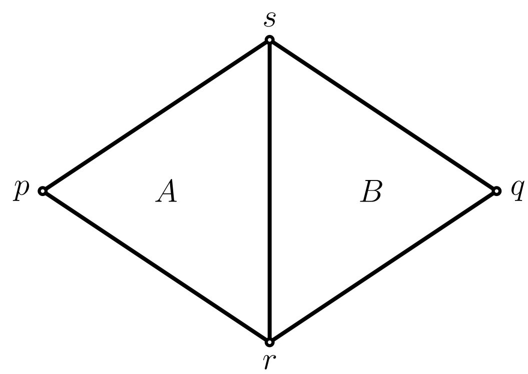


Figure 11.1: Two adjacent reference triangles

Then, lemma 11.1.3 tells us that F_A and F_B join with C^n continuity along (r, s) iff

$$f_A(p^i q^j r^k s^l) = f_B(p^i q^j r^k s^l),$$

for all i, j, k, l such that $i+j+k+l = m$, and $k+l \geq m-n$ $(0 \leq n \leq m)$.

For $n = 0$, we just have

$$f_A(r^k s^{m-k}) = f_B(r^k s^{m-k}),$$

with $0 \leq k \leq m$, which means that the control points of the boundary curves along (r, s) must agree. This is natural, the two surfaces join along this curve!

Let us now see what the continuity conditions mean for $m = 3$ and $n = 1, 2, 3$.

For C^1 continuity, the following 10 polar values must agree:

$$\begin{aligned}
f_A(r, r, r) &= f_B(r, r, r), \\
f_A(r, r, s) &= f_B(r, r, s), \\
f_A(r, s, s) &= f_B(r, s, s), \\
f_A(s, s, s) &= f_B(s, s, s), \\
f_A(p, r, r) &= f_B(p, r, r), \\
f_A(p, r, s) &= f_B(p, r, s), \\
f_A(p, s, s) &= f_B(p, s, s), \\
f_A(q, s, s) &= f_B(q, s, s), \\
f_A(q, r, s) &= f_B(q, r, s), \\
f_A(q, r, r) &= f_B(q, r, r).
\end{aligned}$$

Denoting these common polar values as $f_{A,B}(\cdot, \cdot, \cdot)$, note that these polar values naturally form the vertices of three diamonds,

$$\begin{gathered}(f_{A,B}(p,r,r),\ f_{A,B}(r,r,r),\ f_{A,B}(q,r,r),\ f_{A,B}(s,r,r)),\\ (f_{A,B}(p,r,s),\ f_{A,B}(r,r,s),\ f_{A,B}(q,r,s),\ f_{A,B}(s,r,s)),\\ (f_{A,B}(p,s,s),\ f_{A,B}(r,s,s),\ f_{A,B}(q,s,s),\ f_{A,B}(s,s,s)),\end{gathered}$$

images of the diamond (p, r, q, s). In particular, the vertices of each of these diamonds must be coplanar, but this is not enough to ensure C^1 continuity. The above conditions are depicted in the following diagram:

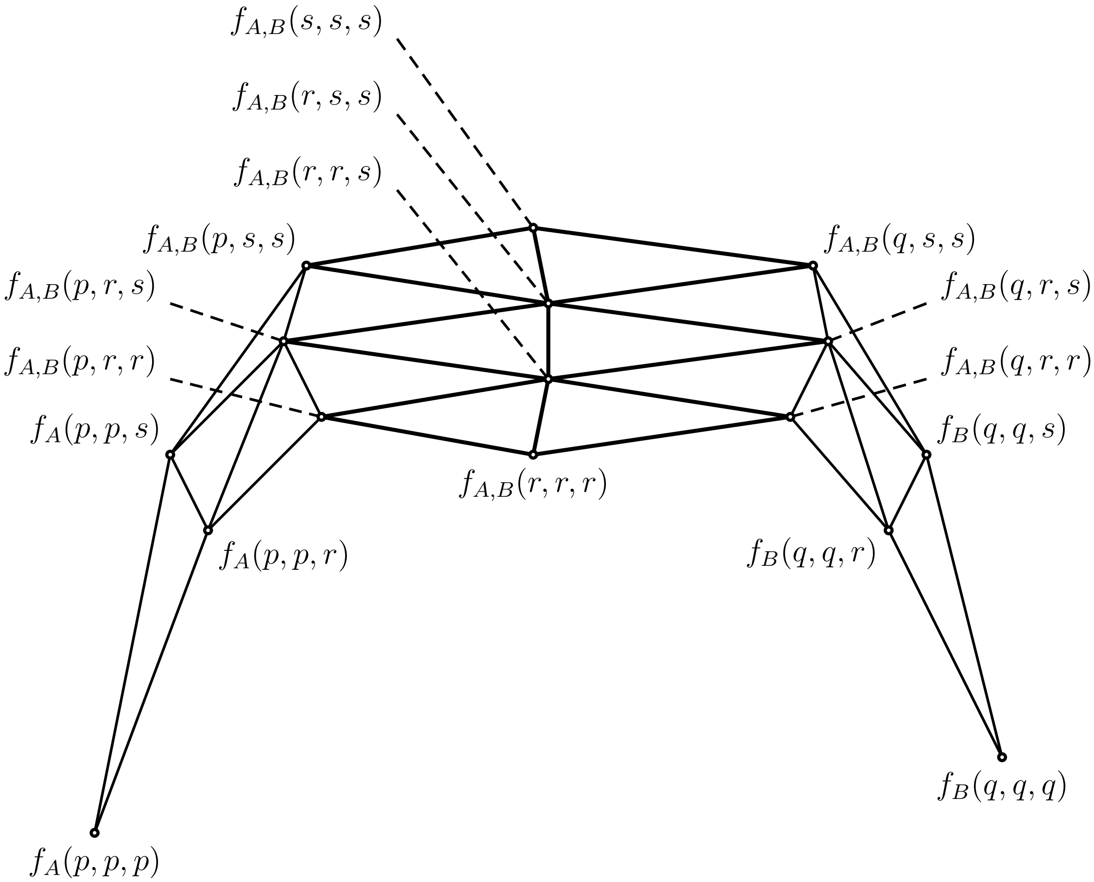


Figure 11.2: Control nets of cubic surfaces joining with C^1 continuity

We can view this diagram as three pairs of overlaping de Casteljau diagrams each with one shell.

Let us now consider C^2 continuity, i.e., $n = 2$. In addition to the 10 constraints necessary for C^1 continuity, we have 6 additional equations among polar values:

$$
\begin{aligned}
f_A(p,p,r) &= f_B(p,p,r),\\
f_A(p,p,s) &= f_B(p,p,s),\\
f_A(p,q,r) &= f_B(p,q,r),\\
f_A(p,q,s) &= f_B(p,q,s),\\
f_A(q,q,r) &= f_B(q,q,r),\\
f_A(q,q,s) &= f_B(q,q,s).
\end{aligned}
$$

Again, denoting these common polar values as $f_{A,B}(\cdot,\cdot,\cdot)$, note that these polar values naturally form the vertices of four diamonds, images of the diamond (p,r,q,s). For example, the left two diamonds are

$$
\begin{aligned}
&(f_{A,B}(p,p,r),\ f_{A,B}(r,p,r),\ f_{A,B}(q,p,r),\ f_{A,B}(s,p,r)),\\
&(f_{A,B}(p,p,s),\ f_{A,B}(r,p,s),\ f_{A,B}(q,p,s),\ f_{A,B}(s,p,s)).
\end{aligned}
$$

In particular, the vertices of each of these diamonds must be coplanar, but this is not enough to ensure C^2 continuity.

Note that the polar values $f_A(p,q,r) = f_B(p,q,r)$ and $f_A(p,q,s) = f_B(p,q,s)$ are not control points of the original nets. The above conditions are depicted in the following diagram:

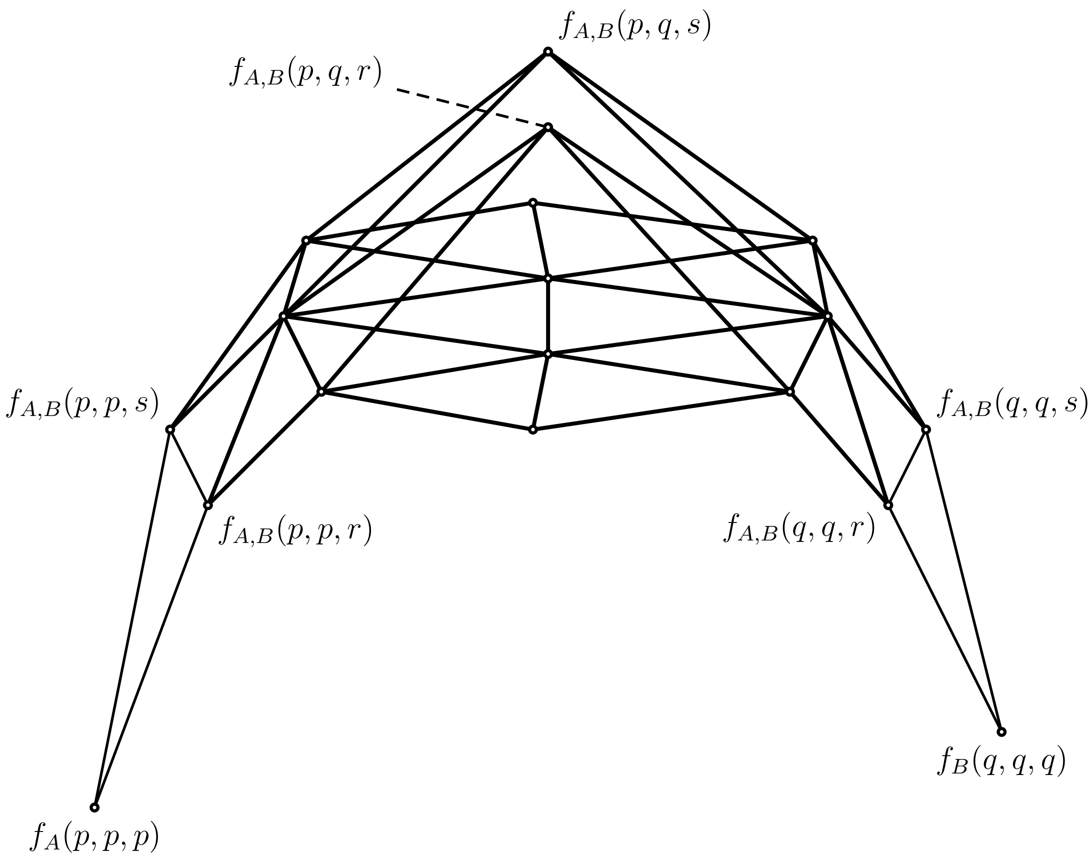


Figure 11.3: Control nets of cubic surfaces joining with C^2 continuity

We can view this diagram as two pairs of overlaping de Casteljau diagrams each with two shells.

Finally, in the case of C^3 continuity, i.e., $n = 3$, all the control points agree, which means that $f_A = f_B$.

In general, C^n continuity is ensured by the overlaping of $m - n + 1$ pairs of de Casteljau diagrams, each with n shells.

We now investigate the realizability of the continuity conditions in the two cases where the parameter plane is subdivided into rectangles, or triangles. We assume that the parameter plane has its natural euclidean structure.

11.2 Spline Surfaces with Triangular Patches

We study what happens with the continuity conditions between surface patches, if the parameter plane is divided into equilateral triangles.

In the case of spline curves, recall that it was possible to achieve C^{m-1} continuity with curve segments of degree m. Also, spline curves have *local flexibility*, which means that changing some control points in a small area does not affect the entire spline curve.

In the case of surfaces, the situation is not as pleasant. For simplicity, we will consider surface patches of degree m joining with the *same* degree of continuity n for all common edges.

First, we will prove that if $2m \leq 3n + 1$, then it is generally impossible to construct a spline surface.

More precisely, given any 4 adjacent patches as shown in the Figure below, if f_C and f_D are known, then f_A and f_B are completely determined.

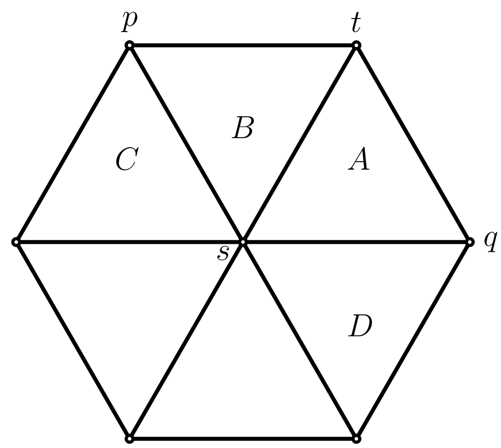


Figure 11.4: Constraints on triangular patches

The proof is more complicated than it might appear. The difficulty is that even though A and D join with C^n continuity along (s, q), A and B join with C^n continuity along (s, t), and B and C join with C^n continuity along (s, p), there is no reference triangle containing all of these three edges!

Lemma 11.2.1 *Surface splines consisting of triangular patches of degree $m \geq 1$ joining with C^n continuity cannot be locally flexible if $2m \leq 3n + 1$. This means that given any four adjacent patches D, A, B, C as in the previous figure, if f_D and f_C are known, then f_A and f_B are completely determined. Furthermore, when $2m = 3n + 2$, there is at most one free control point for every two internal adjacent patches.*

Proof. The idea is to show that the two control nets of polar values $f_A(s^i t^j q^l)$ and $f_B(s^i t^j p^k)$ are completely determined, where $i + j + l = m$ in the first case, and $i + j + k = m$ in the second case.

Since D and A join with C^n continuity along (s, q), B and C join with C^n continuity along (s, p), and A and B join with C^n continuity along (s, t),

$f_A(s^i t^j p^k q^l)$ is determined for all $j + k \leq n$, $f_B(s^i t^j p^k q^l)$ is determined for all $j + l \leq n$, and

$$f_A(s^i t^j p^k q^l) = f_B(s^i t^j p^k q^l)$$

for all $k + l \leq n$.

These conditions do not seem to be sufficient to show that f_A and f_B are completely determined, but we haven't yet taken advantage of the symmetries of the situation.

Indeed, note that (p, q) and (s, t) have the same middle point, so that $p + q = s + t$.

We first reformulate the C^n-continuity conditions between A and B, using the identity $p + q = s + t$.

Recall that these conditions are

$$f_A(s^i t^j p^k q^l) = f_B(s^i t^j p^k q^l)$$

for all $k + l \leq n$.

Replacing p by $s+t-q$ on the left-hand side and q by $s+t-p$ on the right-hand side, we get

$$\sum_{i_1+i_2+i_3=k} (-1)^{i_3} \frac{k!}{i_1!i_2!i_3!}\, f_A(s^{i+i_1}t^{j+i_2}q^{l+i_3}) = \sum_{j_1+j_2+j_3=l} (-1)^{j_3} \frac{l!}{j_1!j_2!j_3!}\, f_B(s^{i+j_1}t^{j+j_2}p^{k+j_3}),$$

where $k+l \leq n$, and $i+j+k+l=m$.

This is an equation relating some affine combination of polar values from a triangular net of $\frac{(k+1)(k+2)}{2}$ polar values associated with A and some affine combination of polar values from a triangular net of $\frac{(l+1)(l+2)}{2}$ polar values associated with B.

A similar rewriting of the C^n-continuity equations between A and D and between C and B shows that the polar values $f_A(s^{m-j-l}t^j q^l)$ are known for $0 \leq l \leq m-j$ and $0 \leq j \leq n$, and that the polar values $f_B(s^{m-j-k}t^j p^k)$ are known for $0 \leq k \leq m-j$ and $0 \leq j \leq n$.

On Figure 11.5, the polar values of the form $f_A(s^{m-j-l}t^j q^l)$ are located in the trapezoid (s, q, x, y) and the polar values of the form $f_B(s^{m-j-k}t^j p^k)$ are located in the trapezoid (s, p, z, y).

If $n = 2h$ and $m = 3h$, the polar values associated with A and B that are not already determined are contained in the diamond (t, u, v, w), and there are $(m-n)^2 = h^2$ such polar values, since $f_A(s^i t^j) = f_B(s^i t^j)$ along (s, t) (where $i + j = m$).

If $n = 2h+1$ and $m = 3h+2$, the polar values associated with A and B that are not already determined are also contained in the diamond (t, u, v, w), and there are $(m-n)^2 = (h+1)^2$ such polar values.

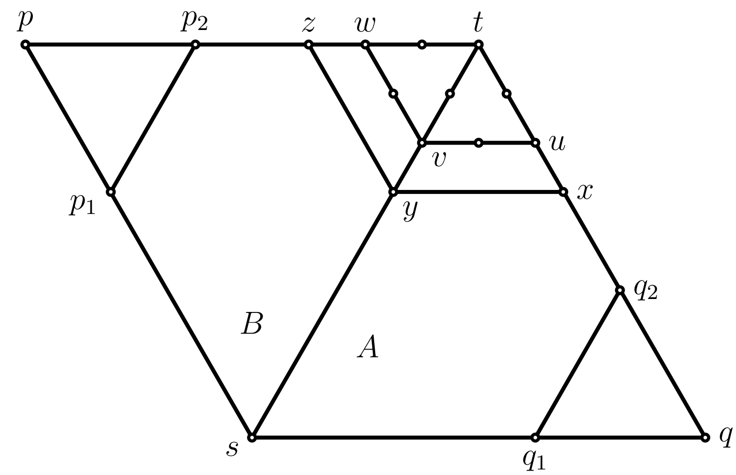


Figure 11.5: Determining polar values in A and B

In either case, the polar values in the diamond (t, u, v, w) can be determined inductively from right to left and from bottom up.

Knowing that we must have $2m \geq 3n + 2$ to have local flexibility, and thus, to find any reasonable scheme to constuct triangular spline surfaces, the problem remains to actually find a method for contructing spline surfaces when $2m = 3n + 2$.

Such a method using convolutions is described by Ramshaw [**?**], but it is not practical.

Instead of presenting this method, we attempt to understand better what are the constraints on triangular patches when $n = 2N$ and $m = 3N + 1$. The key is to look at "derived surfaces".

Given a polynomial surface $F\colon \mathcal{P} \to \mathcal{E}$ of degree m, for any vector $\overrightarrow{u} \in \mathbb{R}^2$, the map $\mathrm{D}_u F\colon \mathcal{P} \to \overrightarrow{\mathcal{E}}$ defined by the directional derivative of F in the fixed direction $\overrightarrow{u}$, is a polynomial surface of degree $m-1$, called a *derived surface* of F.

Given two triangular surfaces $F\colon \mathcal{P} \to \mathcal{E}$ and $G\colon \mathcal{P} \to \mathcal{E}$, the following lemmas show that if F and G join with C^n continuity along a line L and if $\overrightarrow{u}$ is parallel to L, then $\mathrm{D}_u F$ and $\mathrm{D}_u G$ also join with C^n continuity along L.

Lemma 11.2.2 *Given two triangular surfaces $F\colon \mathcal{P} \to \mathcal{E}$ and $G\colon \mathcal{P} \to \mathcal{E}$, if F and G meet with C^0 continuity along a line L, and if $\overrightarrow{u} \in \mathbb{R}^2$ is parallel to L, then $\mathrm{D}_u F$ and $\mathrm{D}_u G$ also meet with C^0 continuity along L.*

Lemma 11.2.3 *Given two triangular surfaces $F\colon \mathcal{P} \to \mathcal{E}$ and $G\colon \mathcal{P} \to \mathcal{E}$, if F and G meet with C^n continuity along a line L, and if $\overrightarrow{u}$ is parallel to L, then $\mathrm{D}_u F$ and $\mathrm{D}_u G$ also meet with C^n continuity along L.*

We can now derive necessary conditions on surfaces F and G of degree $3n + 1$ to join with C^{2n} continuity.

Consider three vectors $\overrightarrow{\alpha}$, $\overrightarrow{\beta}$, $\overrightarrow{\gamma}$, parallel to the three directions of the edges of triangles in the triangular grid, and such that

$$\overrightarrow{\alpha} + \overrightarrow{\beta} + \overrightarrow{\gamma} = \overrightarrow{0}.$$

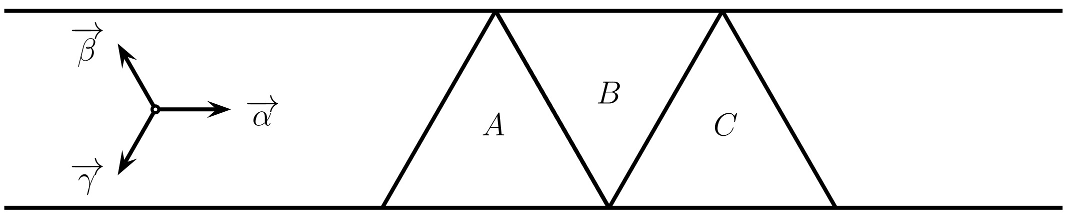


Figure 11.6: A stripe in the parameter plane for triangular patches

Lemma 11.2.4 *Given a spline surface $F\colon \mathcal{P} \to \mathcal{E}$ of degree $3n+1$ having C^{2n} continuity, for any three vectors $\overrightarrow{\alpha}$, $\overrightarrow{\beta}$, $\overrightarrow{\gamma}$, parallel to the three directions of the edges of triangles in the triangular grid, and such that $\overrightarrow{\alpha} + \overrightarrow{\beta} + \overrightarrow{\gamma} = \overrightarrow{0}$, for every triangle A, the derived surface $D_{\alpha}^{n+1} D_{\beta}^{n+1} F_A$ is the same in any stripe in the direction $\overrightarrow{\gamma}$, the derived surface $D_{\beta}^{n+1} D_{\gamma}^{n+1} F_A$ is the same in any stripe in the direction $\overrightarrow{\alpha}$, and the derived surface $D_{\alpha}^{n+1} D_{\gamma}^{n+1} F_A$ is the same in any stripe in the direction $\overrightarrow{\beta}$.*

From lemma 11.2.4, in order to find spline surfaces of degree $3n+1$ with C^{2n} continuity, it is natural to attempt to satisfy the conditions

$$\mathrm{D}_{\alpha}^{n+1}\mathrm{D}_{\beta}^{n+1}F_A = \mathrm{D}_{\beta}^{n+1}\mathrm{D}_{\gamma}^{n+1}F_A = \mathrm{D}_{\alpha}^{n+1}\mathrm{D}_{\gamma}^{n+1}F_A = \overrightarrow{0},$$

for all triangles A.

Each derived surface patch has degree $n-1$, and thus, setting it to zero corresponds to $\frac{(n+1)n}{2}$ conditions.

If we can show that for $\overrightarrow{\alpha}$, $\overrightarrow{\beta}$, $\overrightarrow{\gamma}$, these conditions are independent, we have a total of $\frac{3(n+1)n}{2}$ conditions.

A surface of degree $3n + 1$ is determined by $\frac{(3n+3)(3n+2)}{2}$ control points. Subtracting the $\frac{3(n+1)n}{2}$ conditions, we see that each patch F_A is specified by $3(n+1)^2$ control points.

We can show that these conditions are indeed independent using tensors.

In summary, we were led to consider surface splines of degree $3n + 1$ with C^{2n} continuity, satisfying the independent conditions

$$\mathrm{D}_{\alpha}^{n+1}\mathrm{D}_{\beta}^{n+1}F_A = \mathrm{D}_{\beta}^{n+1}\mathrm{D}_{\gamma}^{n+1}F_A = \mathrm{D}_{\alpha}^{n+1}\mathrm{D}_{\gamma}^{n+1}F_A = \overrightarrow{0}.$$

Each patch is then defined by $3(n+1)^2$ control points. Such spline surfaces do exist, and their existence can be shown using convolutions.

Unfortunately, to the best of our knowledge, no nice scheme involving de Boor control points is known for such triangular spline surfaces.

This is one of the outstanding open problems for spline surfaces, as discussed very lucidly by Ramshaw [**?**].

Next we will see that we have better luck with rectangular spline surfaces.

11.3 Spline Surfaces with Rectangular Patches

We now study what happens with the continuity conditions between surface patches, if the parameter plane is divided into rectangles.

For simplicity, we will consider surface patches of degree m joining with the *same* degree of continuity n for all common edges.

First, we will prove that if $m \leq 2n+1$, then it is generally impossible to construct a spline surface.

More precisely, given any 4 adjacent patches as shown in the figure below, if f_B and f_D are known, then f_A is completely determined.

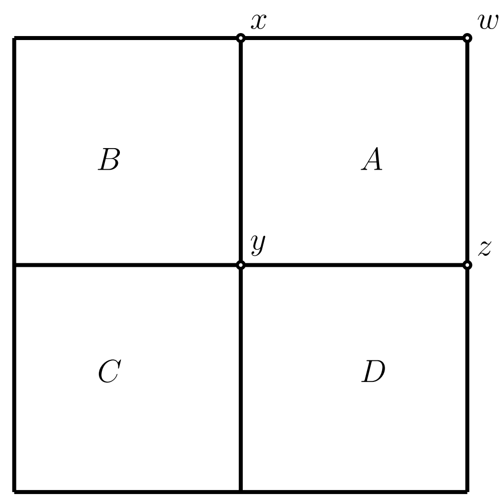


Figure 11.7: Constraints on rectangular patches

As opposed to the triangular case, the proof is fairly simple.

Lemma 11.3.1 *Surface splines consisting of rectangular patches of degree $m \geq 1$ joining with C^n continuity cannot be locally flexible if $m \leq 2n+1$. This means that given any three adjacent patches A, B, D as in the previous figure, if f_B and f_D are known, then f_A is completely determined. Furthermore, when $m = 2n+2$, there is at most one free control point for every two internal adjacent patches.*

Thus, in order to have rectangular spline surfaces with C^n continuity, we must have $m \geq 2n+2$.

We shall consider the case of rectangular spline surfaces of degree $2n$ meeting with C^{n-1} continuity.

One can prove using convolutions (see Ramshaw [**?**]) that such spline surfaces exist, but the construction is not practical.

Instead, as in the case of triangular spline surfaces, we will look for necessary conditions in terms of derived surfaces. This time, we will be successful in finding a nice class of spline surfaces specifiable in terms of de Boor control points.

Lemma 11.3.2 *Given two (triangular) surfaces $F: \mathcal{P} \to \mathcal{E}$ and $G: \mathcal{P} \to \mathcal{E}$ of degree $2n$, if F and G meet with C^{n-1} continuity along a line L, and if $\overrightarrow{u}$ is parallel to L, then $D_u^{n+1} F = D_u^{n+1} G$.*

We can now derive necessary conditions on surfaces F and G of degree $2n$ to join with C^{n-1} continuity.

Lemma 11.3.3 *Given a spline surface $F: \mathcal{P} \to \mathcal{E}$ of degree $2n$ having C^{n-1} continuity, for any horizontal vector $\overrightarrow{\alpha}$, and any vertical vector $\overrightarrow{\beta}$, for every rectangle A, the derived surface $D_\alpha^{n+1} F_A$ is the same in any stripe in the direction $\overrightarrow{\alpha}$, and the derived surface $D_\beta^{n+1} F_A$ is the same in any stripe in the direction $\overrightarrow{\beta}$.*

In view of lemma 11.3.3, it makes sense to look for rectangular spline surfaces of degree $2n$ with continuity C^{n-1} satisfying the constraints

$$\mathrm{D}_{\alpha}^{n+1} F_A = \mathrm{D}_{\beta}^{n+1} F_A = \overrightarrow{0}$$

for all rectangles A.

Since $\mathrm{D}_{\alpha}^{n+1} F_A$ has degree $n-1$, setting it to zero corresponds to $\frac{(n+1)n}{2}$ constraints, and thus, we have a total of $(n+1)n$ constraints.

A surface of degree $2n$ is specified by $\frac{(2n+2)(2n+1)}{2}$ control points, and subtracting the $(n+1)n$ constraints, we find that each rectangular patch is determined by $(n+1)^2$ control points.

However, note that a surface of degree $2n$ such that

$$\mathrm{D}_{\alpha}^{n+1} F_A = \mathrm{D}_{\beta}^{n+1} F_A = \overrightarrow{0}$$

is equivalent to a bipolynomial surface of bidegree $\langle n, n\rangle$.

Thus, in the present case of rectangular spline surfaces, we discover that bipolynomial spline surfaces of bidegree $\langle n, n\rangle$ are an answer to our quest.

Furthermore, since each rectangle is the product of two intervals, we can easily adapt what we have done for spline curves to bipolynomial spline surfaces. In fact, we can do this for bipolynomial spline surfaces of bidegree $\langle p, q \rangle$.

Given a knot sequences $(\overline{s}_i)$ along the u-direction, and a knot sequences $(\overline{t}_j)$ along the v-direction, we have de Boor control points of the form

$$x_{i,j} = f(\overline{s}_{i+1}, \ldots, \overline{s}_{i+p}; \overline{t}_{j+1}, \ldots, \overline{t}_{j+q}).$$

The patches of the spline surface have domain rectangles of the form

$$R_{k,l} = [\overline{s}_k, \overline{s}_{k+1}] \times [\overline{t}_l, \overline{t}_{l+1}],$$

where $\overline{s}_k < \overline{s}_{k+1}$ and $\overline{t}_l < \overline{t}_{l+1}$.

The patch defined on the rectangle $R_{k,l}$ has the $(p+1)(q+1)$ de Boor control points $x_{i,j}$, where $k - p \leq i \leq k$ and $l - q \leq i \leq l$.

Two patches adjacent in the u-direction meet with C^{p-r} continuity, where r is the multiplicity of the knot $\overline{s}_i$ that divides them, and two patches adjacent in the v-direction meet with C^{q-r} continuity, where r is the multiplicity of the knot $\overline{t}_j$ that divides them.

The progressive version of the de Casteljau algorithm can be generalized quite easily. Since the study of bipolynomial spline surfaces of bidegree $\langle p, q\rangle$ basically reduces to the study of spline curves, we will not elaborate any further, and leave this topic as an interesting project.

In summary, contrary to the case of triangular spline surfaces, in the case of rectangular spline surfaces, we were able to generalize the treatment of spline curves in terms of knot sequences and de Boor control points to bipolynomial spline surfaces.

The challenge of finding such a scheme for triangular spline surfaces remains open.

11.4 Subdivision Surfaces

A Quick History of Subdivision Surfaces

The idea of defining a curve or a surface via a limit process involving subdivision goes back to Chaikin, who (in 1974) defined a simple subdivision scheme applying to curves defined by a closed control polygon [**?**].

Soon after that, Riesenfeld [**?**] realized that
Chaikin's scheme was simply the de Boor subdivision method for quadratic uniform B-splines, i.e., the process of recursively inserting a knot at the midpoint of every interval in a cyclic knot sequence.

In 1978, two subdivision schemes for surfaces were proposed by Doo and Sabin [**?**, **?**, **?**], and by Catmull and Clark [**?**].

The main difference between the two schemes is the following.

After one round of subdivision the Doo-Sabin scheme produces a mesh whose vertices all have the same degree 4, and most faces are rectangular, except for faces arising from original vertices of degree not equal to four and from nonrectangular faces.

After one round of subdivision, the number of nonrectangular faces remains constant, and it turns out that these faces shrink and tend to a limit which is their common centroid.

The centroid of each nonrectangular face is referred to as an *extraordinary point*.

Furthermore, large regions of the mesh define biquadratic B-splines.

The limit surface is C^1-continuous except at extraordinary points.

On the other hand, after one round of subdivision, the Catmull-Clark scheme produces rectangular faces, and most vertices have degree 4, except for vertices arising from original nonrectangular faces and from vertices of degree not equal to four, also referred to as extraordinary points.

The limit surface is C^2-continuous except at extraordinary points. Large regions of the mesh define bicubic B-splines.

Several years later, Charles Loop in his Master's thesis (1987) introduced a subdivision scheme based on a mesh consisting strictly of triangular faces [**?**].

In Loop's scheme, every triangular face is refined into four subtriangles. Most vertices have degree six, except for original vertices whose degree is not equal to six, referred to as extraordinary points.

Large regions of the mesh define triangular splines based on hexagons consisting of 24 small triangles each of degree four (each edge of such an hexagon consists of two edges of a small triangle). The limit surface is C^2-continuous except at extraordinary points.

Doo-Sabin's Scheme

During every round of the subdivision process, new vertices and new faces are created as follows.

Every vertex v of the current mesh yields a new vertex v_F called *image of v in F*, for every face F having v as a vertex.

Then, image vertices are connected to form three kinds of new faces: F-faces, E-faces, and V-faces.

An F-face is a smaller version of a face F, and it is obtained by connecting the image vertices of the boundary vertices of F in F. Note that if F is an n-sided face, so is the new F-face. This process is illustrated in Figure 11.8.

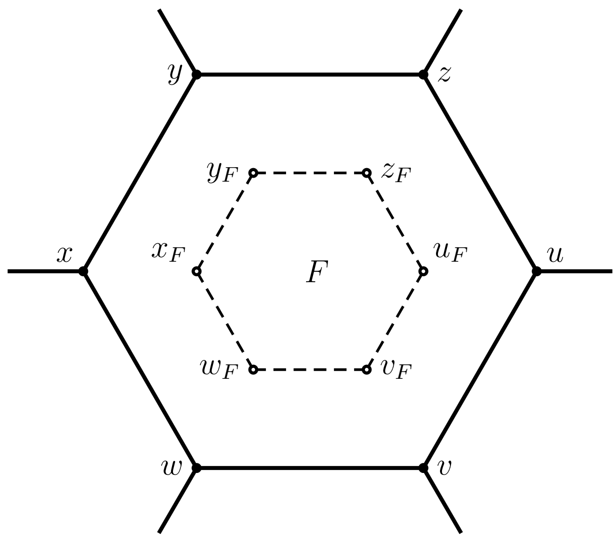


Figure 11.8: Vertices of a new F-face

A new E-face is created as follows.

For every edge E common to two faces F_1 and F_2, the four image vertices v_{F_1}, v_{F_2} of the end vertex v of E, and w_{F_1}, w_{F_2} of the other end vertex w of E are connected to form a rectangular face, as illustrated in Figure 11.9.

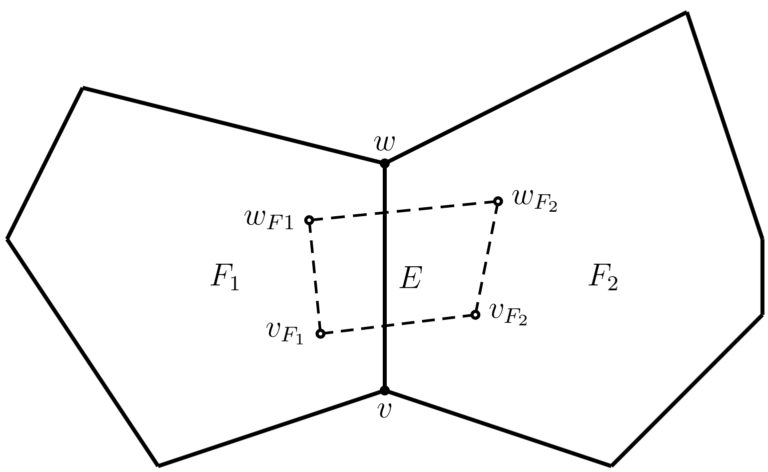


Figure 11.9: Vertices of a new E-face

A new V-face is obtained by connecting the image vertices v_F of a given vertex v in all the faces adjacent to v, provided that v has degree $n \geq 3$. If v has degree n, the new V-face is also n-sided. This process is illustrated in Figure 11.10.

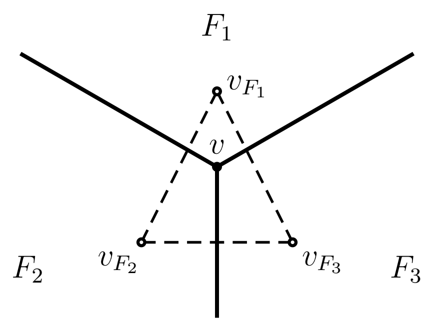


Figure 11.10: Vertices of a new V-face

Various rules are used to determine the image vertex v_F of a vertex v in some face F.

A simple scheme used by Doo is to compute the centroid c of the face F, and the image v_F of v in F as the midpoint of c and v (if F has n sides, the centroid of F is the barycenter of the weighted points $(v, 1/n)$, where the v's are the vertices of F).

Another rule is

$$v_i = \sum_{j=1}^{n} \alpha_{ij} w_j,$$

where the w_j are the vertices of the face F, and v_i is the image of w_i in F, with

$$\alpha_{ij} = \begin{cases} \frac{n+5}{4n} & \text{if } i = j, \\ \frac{3+2\cos(2\pi(i-j)/n)}{4n} & \text{if } i \neq j, \end{cases}$$

where $1 \leq i, j \leq n$ and $n \geq 3$ is the number of boundary edges of F.

Observe that after one round of subdivision, all vertices have degree four, and the number of nonrectangular faces remains constant.

It is also easy to check that these faces shrink and tend to a limit which is their common centroid.

However, it is not obvious that such subdivision schemes converge, and what kind of smoothness is obtained at extraordinary points.

These matters were investigated by Doo and Sabin [**?**] and by Peters and Reif [**?**].

This can be achieved by eigenvalue analysis, or better, using discrete Fourier transforms.

The Doo-Sabin method has been generalized to accomodate features such as creases, darts, or cusps, by Sederberg, Zheng, Sewell, and Sabin [**?**].

Such features are desirable in human modeling, for example, to model clothes or human skin.

Catmull-Clark's Scheme

Unlike the previous one, this method consists in subdividing every face into smaller rectangular faces obtained by connecting new face points, edge points, and vertex points.

Given a face F with vertices $v_1, \ldots, v_n$, the new face point v_F is computed as the centroid of the v_i, i.e.

$$v_F = \sum_{i=1}^{n} \frac{1}{n} v_i.$$

Given an edge E with endpoints v and w, if F_1 and F_2 are the two faces sharing E as a common edge, the new edge point v_E is the average of the four points v, w, v_{F_1}, v_{F_2}, where v_{F_1} and v_{F_2} are the centroids of F_1 and F_2, i.e.

$$v_E = \frac{v + w + v_{F_1} + v_{F_2}}{4}.$$

The computation of new vertex points is slightly more involved.

In fact, there are several different versions. The version presented in Catmull and Clark [**?**] is as follows.

Given a vertex v (an old one), if $\mathcal{F}$ denotes the average of the new face points of all (old) faces adjacent to v and $\mathcal{E}$ denotes the average of the midpoints of all (old) n edges incident with v, the new vertex point v' associated with v is

$$v' = \frac{1}{n}\mathcal{F} + \frac{2}{n}\mathcal{E} + \frac{n-3}{n}v.$$

New faces are then determined by connecting the new points as follows: each new face point v_F is connected by an edge to the new edge points v_E associated with the boundary edges E of the face F; each new vertex point v' is connected by an edge to the new edge points v_E associated with all the edges E incident with v.

Note that only rectangular faces are created.

Figure 11.11 shows this process. New face points are denoted as solid square points, new edges points are denoted as hollow round points, and new vertex points are denoted as hollow square points.

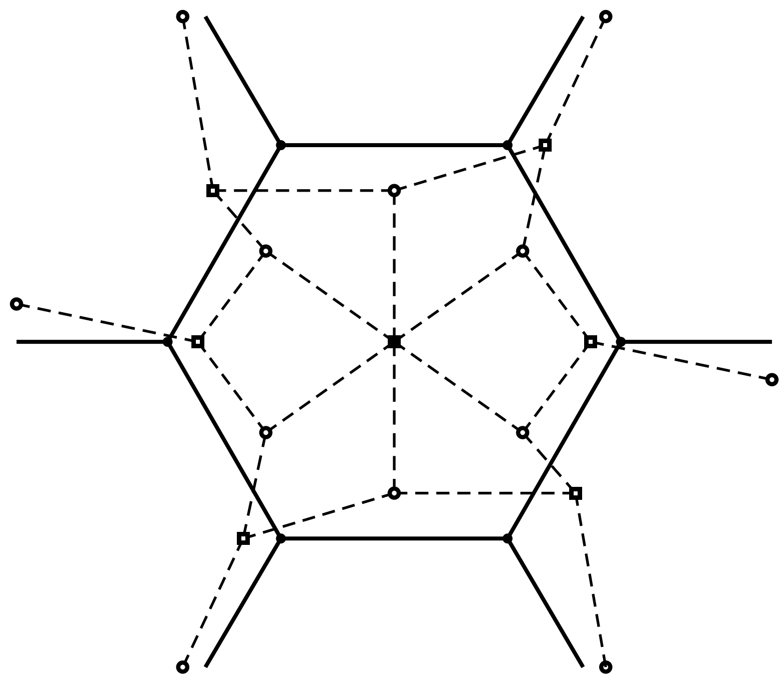


Figure 11.11: New face point, edge points, and vertex points

An older version of the rule for vertex points is

$$v' = \frac{1}{4}\mathcal{F} + \frac{1}{2}\mathcal{E} + \frac{1}{4}v,$$

but it was observed that the resulting surfaces could be too "pointy" (for example, starting from a tetrahedron).

Another version studied by Doo and Sabin is

$$v' = \frac{1}{n}\mathcal{F} + \frac{1}{n}\mathcal{E} + \frac{n-2}{n}v.$$

Doo and Sabin analyzed the tangent-plane continuity of this scheme using discrete Fourier transforms [**?**].

Observe that after one round of subdivision, all faces are rectangular, and the number of extraordinary points (vertices of degree different from four) remains constant.

The tangent-plane continuity of various versions of Catmull-Clark schemes are also investigated in Ball and Storry [**?**] (using discrete Fourier transforms), and C^1-continuity is investigated by Peters and Reif [**?**].

A more general study of the convergence of subdivision methods can be found in Zorin [**?**] (see also Zorin [**?**]).

It is also possible to accomodate boundary vertices and edges. DeRose, Kass, and Truong [**?**], have generalized the Catmull-Clark subdivision rules to accomodate sharp edges and creases.

Their work is inspired by previous work of Hoppe et al [**?**], in which the Loop scheme was extended to allow (infinitely) sharp creases, except that DeRose et al's method applies to Catmull-Clark surfaces.

The method of DeRose Kass, and Truong [**?**], also allows semi-sharp creases in addition to (infinitely) sharp creases.

This new scheme was used in modeling the character Geri in the short film *Geri's game*.

Loop's Scheme

Unlike the previous methods, Loop's method only applies to meshes whose faces are all triangles.

Loop's method consists in splitting each (triangular) face into four triangular faces, using rules to determine new edge points and new vertex points.

For every edge (rs), since exactly two triangles Δprs and Δqrs share the edge (rs), we compute the new edge point η_{rs} as the following convex combination:

$$\eta_{rs} = \frac{1}{8}\,p + \frac{3}{8}\,r + \frac{3}{8}\,s + \frac{1}{8}\,q,$$

as illustrated in Figure 11.12.

This corresponds to computing the affine combination of three points assigned respectively the weights $3/8, 3/8$, and $2/8$: the centroids of the two triangles Δprs and Δqrs, and the midpoint of the edge (rs).

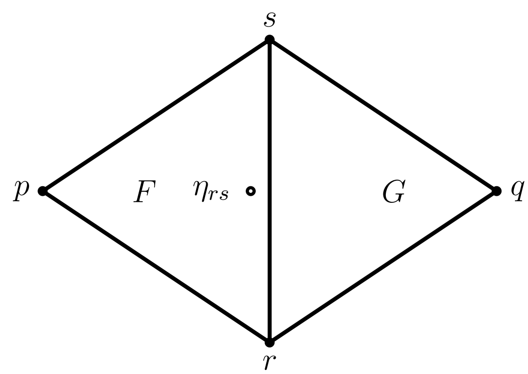


Figure 11.12: Loop's scheme for computing edge points

For any vertex v of degree n, if $p_0, \ldots, p_{n-1}$ are the other endpoints of all (old) edges incident with v, the new vertex point v' associated with v is

$$v' = (1 - \alpha_n) \left(\sum_{i=0}^{n-1} \frac{1}{n} p_i \right) + \alpha_n v,$$

where α_n is a coefficient dependent on n.

Loop's method is illustrated in Figure 11.13, where hollow round points denote new edge points, and hollow square points denote new vertex points.

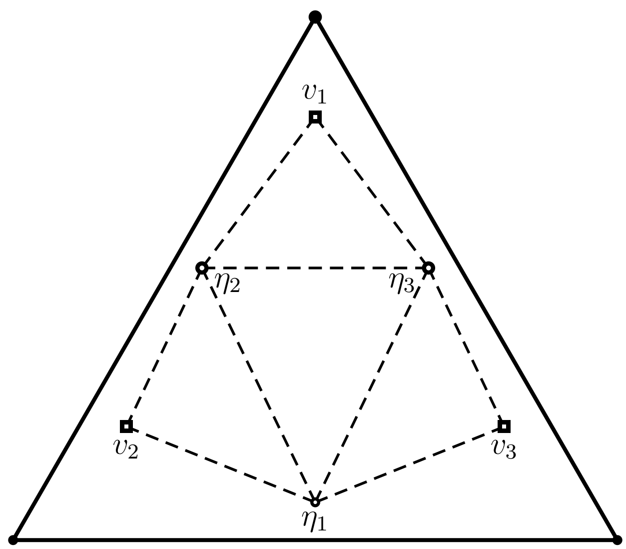


Figure 11.13: Loop's scheme for subdividing faces

Observe that after one round of subdivision, all vertices have degree six, except for vertices coming from orginal vertices of degree different from six, but such vertices are surrounded by ordinary vertices of degree six.

Vertices of degree different from six are called extraordinary points. Loop determined that the value $\alpha_n = 5/8$ produces good results [**?**], but in some cases, tangent plane continuity is lost at extraordinary points.

Large regions of the mesh define triangular splines based on hexagons consisting of small triangles each of degree four (each edge of such an hexagon consists of two edges of a small triangle).

Thus, ordinary points have a well defined limit that can be computed by subdividing the quartic triangular patches. The limit surface is C^2-continuous except at extraordinary points.

Loop's method was first formulated for surfaces without boundaries. Boundaries can be easily handled by treating the boundary curves a cubic B-splines, as in the Catmull-Clark scheme.

In his Master's thesis [**?**], Loop rigorously investigates the convergence and smoothness properties of his scheme.

He proves convergence of extraordinary points to a limit.

He also figures out in which interval α_n should belong, in order to insure convergence and better smoothness at extraordinary points.

Since the principles of Loop's analysis are seminal and yet quite simple, we will present its main lines.

Loop's Analysis of Convergence

As we already remarked, after one round of subdivision, extraordinary points are surrounded by ordinary points, which makes the analysis of convergence possible.

Since points are created during every iteration of the subdivision process, it is convenient to label points with the index of the subdivision round during which they are created.

Then, the rule for creating a new vertex point v^l associated with a vertex v^{l-1} can be written as

$$v^l = (1 - \alpha_n)q^{l-1} + \alpha_n v^{l-1},$$

where

$$q^{l-1} = \sum_{i=0}^{n-1} \frac{1}{n} p_i^{l-1}$$

is the centroid of the points $p_0^{l-1}, \ldots, p_{n-1}^{l-1}$, the other endpoints of all edges incident with v^{l-1}.

Loop proves that as l tends to ∞,

(1) Every extraordinary vertex v^l tends to the same limit as q^l;

(2) The ordinary vertices $p_0^l, \ldots, p_{n-1}^l$ surrounding v^l also tend to the same limit as q^l.

Since q^l is the centroid of ordinary points, this proves the convergence for extraordinary points. Keep in mind that the lower indices of the p_i^l are taken modulo n.

Proving that $\lim_{l\to\infty} v^l = \lim_{l\to\infty} q^l$ is fairly easy.

Using the fact that

$$p_i^l = \frac{1}{8}\,p_{i-1}^{l-1} + \frac{3}{8}\,p_i^{l-1} + \frac{3}{8}\,v^{l-1} + \frac{1}{8}\,p_{i+1}^{l-1}$$

and some calculations, it is easy to show that

$$v^l - q^l = \left(\alpha_n - \frac{3}{8}\right)(v^{l-1} - q^{l-1}).$$

By a trivial induction, we get

$$v^l - q^l = \left(\alpha_n - \frac{3}{8}\right)^l (v^0 - q^0).$$

Thus, if $-1 < \alpha_n - \frac{3}{8} < 1$, i,e,

$$-\frac{5}{8} < \alpha_n < \frac{11}{8},$$

we get convergence of v^l to q^l.

The value $\alpha_n = 5/8$ is certainly acceptable.

Proving (2) is a little more involved.

Loop makes a clever use of *discrete Fourier transforms*.

Discrete Fourier series deal with finite sequences $c \in \mathbb{C}^n$ of complex numbers.

It is convenient to view a finite sequence $c \in \mathbb{C}^n$ as a periodic sequence over $\mathbb{Z}$, by letting $c_k = c_h$ iff $k - h = 0 \bmod n$.

It is also more convenient to index n-tuples starting from 0 instead of 1, thus writing $c = (c_0, \ldots, c_{n-1})$.

Every sequence $c = (c_0, \ldots, c_{n-1}) \in \mathbb{C}^n$ of "Fourier coefficients" determines a periodic function $f_c\colon \mathbb{R} \to \mathbb{C}$ (of period 2π) known as *discrete Fourier series, or phase polynomial*, defined such that

$$f_c(\theta) = c_0 + c_1 e^{i\theta} + \cdots + c_{n-1} e^{i(n-1)\theta} = \sum_{k=0}^{n-1} c_k e^{ik\theta}.$$

Then, given any sequence $f = (f_0, \ldots, f_{n-1})$ of data points, it is desirable to find the "Fourier coefficients" $c = (c_0, \ldots, c_{n-1})$ of the discrete Fourier series f_c such that

$$f_c(2\pi k/n) = f_k,$$

for every k, $0 \leq k \leq n-1$.

The problem amounts to solving the linear system

$$F_n c = f,$$

where F_n is the symmetric $n \times n$-matrix (with complex coefficients)

$$F_n = \left(e^{i2\pi kl/n}\right)_{\substack{0 \leq k \leq n-1 \\ 0 \leq l \leq n-1}},$$

assuming that we index the entries in F_n over $[0, 1, \ldots, n-1] \times [0, 1, \ldots, n-1]$, the standard k-th row now being indexed by $k-1$ and the standard l-th column now being indexed by $l-1$.

The matrix F_n is called a *Fourier matrix*. Letting

$$\overline{F_n} = \left(e^{-i2\pi kl/n}\right)_{\substack{0\leq k\leq n-1\\ 0\leq l\leq n-1}}$$

be the conjugate of F_n, it is easily checked that

$$F_n\overline{F_n} = \overline{F_n}F_n = n\, I_n.$$

Thus, the Fourier matrix is invertible, and its inverse $F_n^{-1} = (1/n)\overline{F_n}$ is computed very cheaply.

The purpose of the *discrete Fourier transform* is to find the Fourier coefficients $c = (c_0, \ldots, c_{n-1})$ from the data points $f = (f_0, \ldots, f_{n-1})$.

The discrete Fourier transform is a linear map
$\widehat{\ }\colon \mathbb{C}^n \to \mathbb{C}^n$.

Now, the other major player in Fourier analysis is the convolution.

In the discrete case, it is natural to define the discrete convolution as a circular type of convolution rule.

The *discrete convolution* is a map $\star\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$, taking two sequences $c, d \in \mathbb{C}^n$, and forming the new sequence $c \star d$.

The Fourier transform and the convolution rule (discrete or not!) must be defined in such a way that they form a harmonious pair, which means that the transform of a convolution should be the product of the transforms, i.e.

$$\widehat{c \star d} = \widehat{c}\,\widehat{d},$$

where the multiplication on the right-hand side is just the inner product of $\widehat{c}$ and $\widehat{d}$ (vectors of length n).

Inspired by the continuous case, and following Strang [**?**], it is natural to define the *discrete Fourier transform* $\widehat{f}$ of a sequence $f = (f_0, \ldots, f_{n-1}) \in \mathbb{C}^n$ as

$$\widehat{f} = \overline{F_n} f,$$

or equivalently, as

$$\widehat{f}(k) = \sum_{j=0}^{n-1} f_j e^{-i2\pi jk/n}$$

for every k, $0 \leq k \leq n-1$.

We also define the *inverse discrete Fourier transform* (taking c back to f) as

$$\overline{\widehat{c}} = F_n\, c.$$

Since

$$\widehat{f}(k) = \sum_{j=0}^{n-1} f_j e^{-i2\pi jk/n},$$

in view of the formula $F_n \overline{F_n} = \overline{F_n} F_n = n\, I_n$, the Fourier coefficients $c = (c_0, \ldots, c_{n-1})$ are then given by the formulae

$$c_k = \frac{1}{n}\,\widehat{f}(k) = \frac{1}{n}\,\sum_{j=0}^{n-1} f_j e^{-i2\pi jk/n}.$$

Note the analogy with the continuous case, where the Fourier transform $\widehat{f}$ of the function f is given by

$$\widehat{f}(x) = \int_{-\infty}^{\infty} f(t) e^{-ixt} dt,$$

and the Fourier coefficients of the Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

are given by the formulae

$$c_k = \frac{1}{2\pi}\,\int_{-\pi}^{\pi} f(x) e^{-ikx} dx.$$

Remark. Others authors (including Strang in his older book [**?**]) define the discrete Fourier transform as $\widehat{f} = \frac{1}{n}\overline{F_n} f$.

The drawback of this choice is that the convolution rule has an extra factor of n.

Loop defines the discrete Fourier transform as $F_n f$, which causes problem with the convolution rule. We will come back to this point shortly!

The simplest definition of discrete convolution is, in our opinion, the definition in terms of *circulant matrices*.

We define the *circular shift matrix* S_n *(of order n)* as the matrix

$$S_n = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

consisting of cyclic permutations of its first column.

For any sequence $f = (f_0, \ldots, f_{n-1}) \in \mathbb{C}^n$, we define the *circulant matrix* $H(f)$ as

$$H(f) = \sum_{j=0}^{n-1} f_j S_n^j,$$

where $S_n^0 = I_n$, as usual.

For example, the circulant matrix associated with the sequence $f = (a, b, c, d)$ is

$$\begin{pmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{pmatrix}$$

We can now define the *convolution* $f \star g$ of two sequences $f = (f_0, \ldots, f_{n-1})$ and $g = (g_0, \ldots, g_{n-1})$ as

$$f \star g = H(f)\, g,$$

viewing f and g as column vectors.

Then, the miracle (which is not too hard to prove!) is that we have

$$H(f) F_n = F_n \widehat{f},$$

which means that the columns of the Fourier matrix F_n are the eigenvectors of the circulant matrix $H(f)$, and that the eigenvalue associated with the lth eigenvector is $(\widehat{f})_l$, the lth component of the Fourier transform $\widehat{f}$ of f (counting from 0).

After some calculations, we get

$$\widehat{f \star g} = \overline{F_n}(f \star g),$$

which can be rewritten

as the *(circular) convolution rule*

$$\widehat{f \star g} = \widehat{f}\,\widehat{g},$$

where the multiplication on the right-hand side is just the inner product of the vectors $\widehat{f}$ and $\widehat{g}$.

If the sequence $f = (f_0, \ldots, f_{n-1})$ is even, which means that $f_{-j} = f_j$ for all $j \in \mathbb{Z}$ (viewed as a periodic sequence), or equivalently, that $f_{n-j} = f_j$ for all j, $0 \leq j \leq n-1$, it is easily seen that the Fourier transform $\widehat{f}$ can be expressed as

$$\widehat{f}(k) = \sum_{j=0}^{n-1} f_j \cos\left(2\pi jk/n\right),$$

for every k, $0 \leq k \leq n-1$.

Similarly, the inverse Fourier transform (taking c back to f) is expressed as

$$\overline{\widehat{c}}(k) = \sum_{j=0}^{n-1} c_j \cos(2\pi jk/n),$$

for every k, $0 \leq k \leq n-1$.

Observe that it is the same as the (forward) discrete Fourier transform. This is what saves Loop's proof (see below)!

After this digression, we get back to Loop's Master's thesis [**?**].

However, we warn our readers that Loop defines the discrete Fourier transform as

$$\mathcal{F}(f) = F_n f,$$

(which is our inverse Fourier transform $\overline{\widehat{f}}$) and not as $\overline{F_n} f$, which is our Fourier transform $\widehat{f}$ (following Strang [**?**]).

Neverthless, even though Loop appears to be using an incorrect definition of the Fourier transform, what saves his argument is that for even sequences, his $\mathcal{F}(f)$ and our $\widehat{f}$ are identical, as observed earlier.

With these remarks in mind, we go back to Loop's proof that the ordinary vertices $p_0^l, \ldots, p_{n-1}^l$ surrounding v^l also tend to the same limit as q^l.

The trick is rewrite the equations

$$q^l = \sum_{i=0}^{n-1} \frac{1}{n} p_i^l$$

and

$$p_i^l = \frac{1}{8} p_{i-1}^{l-1} + \frac{3}{8} p_i^{l-1} + \frac{3}{8} v^{l-1} + \frac{1}{8} p_{i+1}^{l-1}$$

in terms of discrete convolutions.

To do so, define the sequences

$$M = \left(\frac{3}{8}, \frac{1}{8}, \underbrace{0, \ldots, 0}_{n-3}, \frac{1}{8}\right),$$

and

$$A = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right),$$

both of length n.

Note that these sequences are even!

We also define the sequence P^l as

$$P^l = (p_0^l, \ldots, p_{n-1}^l),$$

and treat q^l and v^l as constant sequences Q^l and V^l of length n.

Then, after some calculations, we get

$$P^l = \left(M - \frac{5}{8}\,A\right) \star P^{l-1} + Q^l.$$

Taking advantage of certain special properties of M and A, namely,

$$\sum_{j=0}^{n-1}\left(M-\frac{5}{8}\,A\right)_j=0,$$

we get

$$P^l=\left(M-\frac{5}{8}\,A\right)^{l\star}\star P^0+Q^l,$$

where $c^{n\star}$ stands for the n-fold convolution $\underbrace{c\star\cdots\star c}_{n}$.

At this stage, letting

$$R=M-\frac{5}{8}\,A,$$

all we have to prove is that $R^{l\star}$ tends to the null sequence as l goes to infinity.

Since both M and A are even sequences, applying the Fourier transform in its cosine form and the convolution rule, we have

$$\widehat{R^{l\star}} = (\widehat{R})^l,$$

and so, we just have to compute the discrete Fourier transform of R.

However, this is easy to do, and we get

$$(\widehat{R})_j = \begin{cases} 0 & \text{if } j = 0, \\ \frac{3}{8} + \frac{1}{4}\cos(2\pi j/n) & \text{if } j \neq 0. \end{cases}$$

Since the absolute value of the cosine is bounded by 1,

$$\frac{1}{8} \leq (\widehat{R})_j \leq \frac{5}{8}$$

for all j, $0 \leq j \leq n-1$, and thus

$$\lim_{l \to \infty} (\widehat{R})^l = 0_n,$$

which proves that

$$\lim_{l \to \infty} \widehat{R^{l\star}} = \lim_{l \to \infty} R^{l\star} = 0_n,$$

and consequently that

$$\lim_{l \to \infty} p_i^l = \lim_{l \to \infty} q^l.$$

Therefore, the faces surrounding extraordinary points converge to the same limit as the centroid of these faces.

Loop gives explicit formulae for the limit of extraordinary points.

He proves that q^l (and thus v^l) has the limit

$$(1-\beta_n)q^0 + \beta_n v^0, \quad \text{where} \quad \beta_n = \frac{3}{11-8\alpha_n}.$$

The bounds to insure convergence are the same as the bounds to insure convergence of v^l to q^l, namely

$$-\frac{5}{8} < \alpha_n < \frac{11}{8}.$$

In particular, $\alpha_n = 5/8$ yields $\beta_n = 1/2$. Loop also investigates the tangent plane continuity at these limit points.

He proves that tangent plane continuity is insured if α_n is chosen so that

$$-\frac{1}{4}\cos(2\pi/n) < \alpha_n < \frac{3}{4} + \frac{1}{4}\cos(2\pi/n).$$

For instance, for a vertex of degree three ($n = 3$), the values $\alpha_3 = 5/8$ is outside the correct range, as Loop first observed experimentally.

If α_n is chosen in the correct range, it is possible to find a formula for the tangent vector function at each extraordinary point.

Loop also discusses curvature continuity at extraordinary points, but his study is more tentative.

He proposes the following "optimal" value for α_n;

$$\alpha_n = \frac{3}{8} + \left(\frac{3}{8} + \frac{1}{4}\cos(2\pi/n)\right)^2.$$

Note that $\alpha_6 = 5/8$ is indeed this value for regular vertices (of degree $n = 6$).

In summary, Loop proves that his subdivision scheme is C^2-continuous, except at a finite number of extraordinary points. At extraordinary points, there is convergence, and there is a range of values from which α_n can be chosen to insure tangent plane continuity.

The implementation of the method is discussed, and it is nontrivial.

Stam [**?**] also implemented a method for computing points on Loop surfaces.

Loop's scheme was extended to accomodate sharp edges and creases on boundaries, see Hoppe et [**?**].

Chapter 12

Embedding an Affine Space in a Vector Space

12.1 Embedding an Affine Space as a Hyperplane in a Vector Space: the "Hat Construction"

Assume that we consider the real affine space E of dimension 3, and that we have some affine frame $(a_0, (\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_2}))$. With respect to this affine frame, every point $x \in E$ is represented by its coordinates (x_1, x_2, x_3), where

$$a = a_0 + x_1 \overrightarrow{v_1} + x_2 \overrightarrow{v_2} + x_3 \overrightarrow{v_3}.$$

A vector $\overrightarrow{u} \in \overrightarrow{E}$ is also represented by its coordinates (u_1, u_2, u_3) over the basis $(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_2})$.

One way to distinguish between points and vectors is to add a fourth coordinate, and to agree that points are represented by (row) vectors $(x_1, x_2, x_3, 1)$ whose fourth coordinate is 1, and that vectors are represented by (row) vectors $(v_1, v_2, v_3, 0)$ whose fourth coordinate is 0.

This "programming trick" works actually very well. Of course, we are opening the door for strange elements such as $(x_1, x_2, x_3, 5)$, where the fourth coordinate is neither 1 nor 0.

The question is, can we make sense of such elements, and of such a construction? The answer is “yes”. We will present a construction in which an affine space $(E, \overrightarrow{E})$ is embedded in a vector space $\widehat{E}$, in which $\overrightarrow{E}$ is embedded as a hyperplane passing through the origin, and E itself is embedded as an affine hyperplane, defined as $\omega^{-1}(1)$, for some linear form $\omega\colon \widehat{E} \to \mathbb{R}$.

The vector space $\widehat{E}$ has the universal property that for any vector space $\overrightarrow{F}$ and any affine map $f\colon E \to \overrightarrow{F}$, there is a unique linear map $\widehat{f}\colon \widehat{E} \to \overrightarrow{F}$ extending $f\colon E \to \overrightarrow{F}$.

Some Simple Geometric Transformations

Given an affine space $(E, \overrightarrow{E})$, every $\overrightarrow{u} \in \overrightarrow{E}$ induces a mapping $t_u\colon E \to E$, called a *translation*, and defined such that $t_u(a) = a + \overrightarrow{u}$, for every $a \in E$. Clearly, the set of translations is a vector space isomorphic to $\overrightarrow{E}$.

Given any point a and any scalar $\lambda \in \mathbb{R}$, we define the mapping $H_{a,\lambda}\colon E \to E$, called *dilatation (or central dilatation, or homothety) of center a and ratio λ*, and defined such that

$$H_{a,\lambda}(x) = a + \lambda\overrightarrow{ax},$$

for every $x \in E$.

$H_{a,\lambda}(a) = a$, and when $\lambda \neq 0$ and $x \neq a$, $H_{a,\lambda}(x)$ is on the line defined by a and x, and is obtained by "scaling" $\overrightarrow{ax}$ by λ. The effect is a uniform dilatation (or contraction, if $\lambda < 1$).

When $\lambda = 0$, $H_{a,0}(x) = a$ for all $x \in E$, and $H_{a,0}$ is the constant affine map sending every point to a.

If we assume $\lambda \neq 1$, note that $H_{a,\lambda}$ is never the identity, and since a is a fixed-point, $H_{a,\lambda}$ is never a translation.

We now consider the set $\widehat{E}$ of geometric transformations from E to E, consisting of the union of the (disjoint) sets of translations and dilatations of ratio $\lambda \neq 1$.

We would like to give this set the structure of a vector space, in such a way that both E and $\overrightarrow{E}$ can be naturally embedded into $\widehat{E}$. In fact, it will turn out that barycenters show up quite naturally too!

In order to "add" two dilatations H_{a_1,λ_1} and H_{a_2,λ_2}, it turns out that it is more convenient to consider dilatations of the form $H_{a,1-\lambda}$, where $\lambda \neq 0$. To see this, let us see the effect of such a dilatation on a point $x \in E$: we have

$$H_{a,1-\lambda}(x) = a + (1-\lambda)\overrightarrow{ax} = a + \overrightarrow{ax} - \lambda\overrightarrow{ax} = x + \lambda\overrightarrow{xa}.$$

For simplicity of notation, let us denote $H_{a,1-\lambda}$ as $\langle a, \lambda\rangle$. Then, we have

$$\langle a, \lambda\rangle(x) = x + \lambda\overrightarrow{xa}.$$

Lemma 12.1.1 *The set* $\widehat{E}$ *consisting of the disjoint union of the translations and the dilatations* $H_{a,1-\lambda} = \langle a, \lambda \rangle$, $\lambda \in \mathbb{R}, \lambda \neq 0$, *is a vector space under the following operations of addition and multiplication by a scalar:*

$$\langle a_1, \lambda_1 \rangle \,\widehat{+}\, \langle a_2, \lambda_2 \rangle = \lambda_1 \overrightarrow{a_2 a_1},$$

if $\lambda_1 + \lambda_2 = 0$*;*

$$\langle a_1, \lambda_1 \rangle \,\widehat{+}\, \langle a_2, \lambda_2 \rangle = \langle \frac{\lambda_1}{\lambda_1 + \lambda_2} a_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} a_2, \lambda_1 + \lambda_2 \rangle,$$

if $\lambda_1 + \lambda_2 \neq 0$*;*

$$\langle a, \lambda \rangle \,\widehat{+}\, \overrightarrow{u} = \langle a + \lambda^{-1} \overrightarrow{u}, \lambda \rangle;$$

$$\overrightarrow{u} \,\widehat{+}\, \overrightarrow{v} = \overrightarrow{u} + \overrightarrow{v};$$

$$\mu \cdot \langle a, \lambda \rangle = \langle a, \lambda\mu \rangle,$$

if $\mu \neq 0$, *and*

$$0 \cdot \langle a, \lambda \rangle = \overrightarrow{0},$$

$$\lambda \cdot \overrightarrow{u} = \lambda \overrightarrow{u}.$$

Furthermore, the map $\omega\colon \widehat{E} \to \mathbb{R}$ *defined such that*

$$\begin{aligned}\omega(\langle a, \lambda\rangle) &= \lambda,\\ \omega(\overrightarrow{u}) &= 0,\end{aligned}$$

is a linear form, $\omega^{-1}(0)$ *is a hyperplane isomorphic to* $\overrightarrow{E}$ *under the injective linear map* $i\colon \overrightarrow{E} \to \widehat{E}$ *such that* $i(\overrightarrow{u}) = t_u$ *(the translation associated with* $\overrightarrow{u}$*), and* $\omega^{-1}(1)$ *is an affine hyperplane isomorphic to* E *with direction* $i(\overrightarrow{E})$*, under the injective affine map* $j\colon E \to \widehat{E}$*, where* $j(a) = \langle a, 1\rangle$*, for every* $a \in E$*. Finally, for every* $a \in E$*, we have*

$$\widehat{E} = i(\overrightarrow{E}) \oplus \mathbb{R}j(a).$$

The following diagram illustrates the embedding of the affine space E into the vector space $\widehat{E}$, when E is an affine plane.

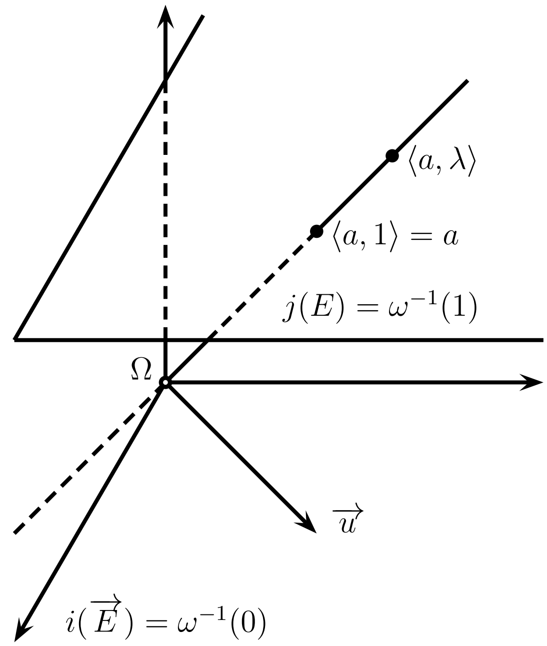


Figure 12.1: Embedding an affine space E into a vector space $\widehat{E}$

Note that $\widehat{E}$ is isomorphic to $\overrightarrow{E} \cup (E \times \mathbb{R}^*)$ (where $\mathbb{R}^* = \mathbb{R} - \{0\}$). Other authors, such as Ramshaw, use the notation E_* for $\widehat{E}$.

Ramshaw calls the linear form $\omega\colon \widehat{E} \to \mathbb{R}$ a *weight (or flavor)*, and he says that an element $z \in \widehat{E}$ such that $\omega(z) = \lambda$ is *λ-heavy (or has flavor λ)* ([?]). The elements of $j(E)$ are 1-heavy and are called *points*, and the elements of $i(\overrightarrow{E})$ are 0-heavy and are called *vectors*. In general, the λ-heavy elements all belong to the hyperplane $\omega^{-1}(\lambda)$ parallel to $i(\overrightarrow{E})$.

Thus, intuitively, we can thing of $\widehat{E}$ as a stack of parallel hyperplanes, one for each λ, a little bit like an infinite stack of very thin pancakes! There are two privileged pancakes: one corresponding to E, for $\lambda = 1$, and one corresponding to $\overrightarrow{E}$, for $\lambda = 0$.

From now on, we will identify $j(E)$ and E, and $i(\overrightarrow{E})$ and $\overrightarrow{E}$. We will also write λa instead of $\langle a, \lambda\rangle$, which we will call a *weighted point*, and write $1a$ just as a. When we want to be more precise, we may also write $\langle a, 1\rangle$ as $\overline{a}$ (as Ramshaw does).

In particular, when we consider the homogenized version $\widehat{\mathbb{A}}$ of the affine space $\mathbb{A}$ associated with the base field $\mathbb{R}$ considered as an affine space, we write $\overline{\lambda}$ for $\langle \lambda, 1\rangle$, when viewing λ as a point in both $\mathbb{A}$ and $\widehat{\mathbb{A}}$, and simply λ, when viewing λ as a vector in $\mathbb{R}$ and in $\widehat{\mathbb{A}}$. The elements of $\widehat{\mathbb{A}}$ are called *Bézier sites*, by Ramshaw.

Then, in view of the fact that

$$\langle a + \overrightarrow{u}, 1\rangle = \langle a, 1\rangle \mathbin{\widehat{+}} \overrightarrow{u},$$

and since we are identifying $a + \overrightarrow{u}$ with $\langle a + \overrightarrow{u}, 1\rangle$ (under the injection j), in the simplified notation, the above reads as $a + \overrightarrow{u} = a \mathbin{\widehat{+}} \overrightarrow{u}$. Thus, we go one step further, and denote $a \mathbin{\widehat{+}} \overrightarrow{u}$ as $a + \overrightarrow{u}$.

From lemma 12.1.1, for every $a \in E$, every element of $\widehat{E}$ can be written uniquely as $\overrightarrow{u} \mathbin{\widehat{+}} \lambda a$. We also denote

$$\lambda a \mathbin{\widehat{+}} (-\mu) b$$

as

$$\lambda a \mathbin{\widehat{-}} \mu b.$$

Given any family $(a_i)_{i\in I}$ of points in E, and any family $(\lambda_i)_{i\in I}$ of scalars in $\mathbb{R}$, with finite support, it is easily shown by induction on the size of the support of $(\lambda_i)_{i\in I}$ that,

(1) If $\sum_{i\in I} \lambda_i = 0$, then

$$\sum_{i\in I} \langle a_i, \lambda_i \rangle = \sum_{i\in I} \lambda_i a_i,$$

where

$$\sum_{i\in I} \lambda_i a_i = \sum_{i\in I} \lambda_i \overrightarrow{ba_i}$$

for any $b \in E$, which, by lemma 5.2.1, is a vector independent of b, or

(2) If $\sum_{i\in I} \lambda_i \neq 0$, then

$$\sum_{i\in I} \langle a_i, \lambda_i \rangle = \langle \sum_{i\in I} \frac{\lambda_i}{\sum_{i\in I} \lambda_i} a_i, \sum_{i\in I} \lambda_i \rangle.$$

Thus, we see how barycenters reenter the scene quite naturally, and that in $\widehat{E}$, we can make sense of $\sum_{i\in I} \langle a_i, \lambda_i \rangle$, regardless of the value of $\sum_{i\in I} \lambda_i$.

When $\sum_{i\in I}\lambda_i = 1$, the element $\sum_{i\in I}\langle a_i, \lambda_i\rangle$ belongs to the hyperplane $\omega^{-1}(1)$, and thus, it is a point. When $\sum_{i\in I}\lambda_i = 0$, the linear combination of points $\sum_{i\in I}\lambda_i a_i$ is a vector, and when $I = \{1, \ldots, n\}$, we allow ourselves to write

$$\lambda_1 a_1 \,\widehat{+}\, \cdots \,\widehat{+}\, \lambda_n a_n,$$

where some of the occurrences of $\widehat{+}$ can be replaced by $\widehat{-}$, as

$$\lambda_1 a_1 + \cdots + \lambda_n a_n,$$

where the occurrences of $\widehat{-}$ (if any) are replaced by $-$.

In fact, we have the following slightly more general property, which is left as an exercise.

Lemma 12.1.2 *Given any affine space $(E, \overrightarrow{E})$, for any family $(a_i)_{i\in I}$ of points in E, for any family $(\lambda_i)_{i\in I}$ of scalars in $\mathbb{R}$, with finite support, and any family $(\overrightarrow{v_j})_{j\in J}$ of vectors in $\overrightarrow{E}$ also with finite support, and with $I \cap J = \emptyset$, the following properties hold:*

(1) If $\sum_{i\in I} \lambda_i = 0$, then

$$\sum_{i\in I} \langle a_i, \lambda_i \rangle \,\widehat{+}\, \sum_{j\in J} \overrightarrow{v_j} = \sum_{i\in I} \lambda_i a_i + \sum_{j\in J} \overrightarrow{v_j},$$

where

$$\sum_{i\in I} \lambda_i a_i = \sum_{i\in I} \lambda_i \overrightarrow{ba_i}$$

for any $b \in E$, which, by lemma 5.2.1, is a vector independent of b, or

(2) If $\sum_{i\in I} \lambda_i \neq 0$, then

$$\sum_{i\in I} \langle a_i, \lambda_i \rangle \,\widehat{+}\, \sum_{j\in J} \overrightarrow{v_j}$$

$$= \langle \sum_{i\in I} \frac{\lambda_i}{\sum_{i\in I} \lambda_i} a_i + \sum_{j\in J} \frac{\overrightarrow{v_j}}{\sum_{i\in I} \lambda_i}, \sum_{i\in I} \lambda_i \rangle.$$

The above formulae show that we have some kind of extended barycentric calculus.

Operations on weighted points and vectors were introduced by H. Grassmann, in his book published in 1844! This calculus will be helpful in dealing with rational curves.

There is also a nice relationship between affine frames in $(E, \overrightarrow{E})$ and bases of $\widehat{E}$, stated in the following lemma.

Lemma 12.1.3 *Given any affine space $(E, \overrightarrow{E})$, for any affine frame $(a_0, (\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m}))$ for E, the family $(\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m}, a_0)$ is a basis for $\widehat{E}$, and for any affine frame $(a_0, \ldots, a_m)$ for E, the family $(a_0, \ldots, a_m)$ is a basis for $\widehat{E}$.*

Furthermore, given any element $\langle x, \lambda \rangle \in \widehat{E}$, if

$$x = a_0 + x_1\overrightarrow{a_0a_1} + \cdots + x_m\overrightarrow{a_0a_m}$$

over the affine frame $(a_0, (\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m}))$ in E, then the coordinates of $\langle x, \lambda \rangle$ over the basis

$$(\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m}, a_0)$$

in $\widehat{E}$, are

$$(\lambda x_1, \ldots, \lambda x_m, \lambda).$$

For any vector $\overrightarrow{v} \in \overrightarrow{E}$, *if*

$$\overrightarrow{v} = v_1\overrightarrow{a_0a_1} + \cdots + v_m\overrightarrow{a_0a_m}$$

over the basis

$$(\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m})$$

in $\overrightarrow{E}$, *then over the basis*

$$(\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m}, a_0)$$

in $\widehat{E}$, *the coordinates of* $\overrightarrow{v}$ *are*

$$(v_1, \ldots, v_m, 0).$$

For any element $\langle a, \lambda\rangle$, where $\lambda \neq 0$, if the barycentric coordinates of a w.r.t. the affine basis $(a_0, \ldots, a_m)$ in E are $(\lambda_0, \ldots, \lambda_m)$ with $\lambda_0 + \cdots + \lambda_m = 1$, then the coordinates of $\langle a, \lambda\rangle$ w.r.t. the basis $(a_0, \ldots, a_m)$ in $\widehat{E}$ are

$$(\lambda\lambda_0, \ldots, \lambda\lambda_m).$$

If a vector $\overrightarrow{v} \in \overrightarrow{E}$ is expressed as

$$\begin{aligned}\overrightarrow{v} = v_1\overrightarrow{a_0a_1} + \cdots + v_m\overrightarrow{a_0a_m} \\ &= -(v_1 + \cdots + v_m)a_0 + v_1a_1 + \cdots + v_ma_m,\end{aligned}$$

with respect to the affine basis $(a_0, \ldots, a_m)$ in E, then its coordinates w.r.t. the basis $(a_0, \ldots, a_m)$ in $\widehat{E}$ are

$$(-(v_1 + \cdots + v_m), v_1, \ldots, v_m).$$

The following diagram shows the basis $(\overrightarrow{a_0a_1}, \overrightarrow{a_0a_2}, a_0)$ corresponding to the affine frame (a_0, a_1, a_2) in E.

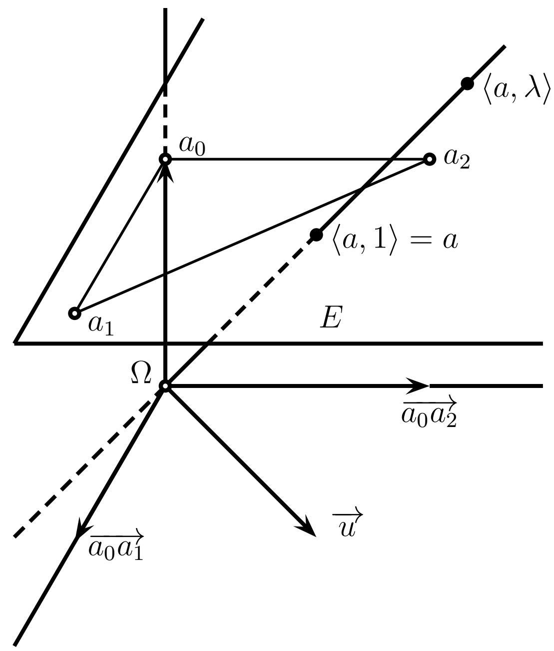


Figure 12.2: The basis $(\overrightarrow{a_0a_1}, \overrightarrow{a_0a_2}, a_0)$ in $\widehat{E}$

If $(x_1, \ldots, x_m)$ are the coordinates of x w.r.t. to the affine frame $(a_0, (\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m}))$ in E, then, $(x_1, \ldots, x_m, 1)$ are the coordinates of x in $\widehat{E}$, i.e., the last coordinate is 1, and if $\overrightarrow{u}$ has coordinates $(u_1, \ldots, u_m)$ with respect to the basis $(\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m})$ in $\overrightarrow{E}$, then $\overrightarrow{u}$ has coordinates $(u_1, \ldots, u_m, 0)$ in $\widehat{E}$, i.e., the last coordinate is 0.

The following diagram shows the affine frame (a_0, a_1, a_2) in E viewed as a basis in $\widehat{E}$.

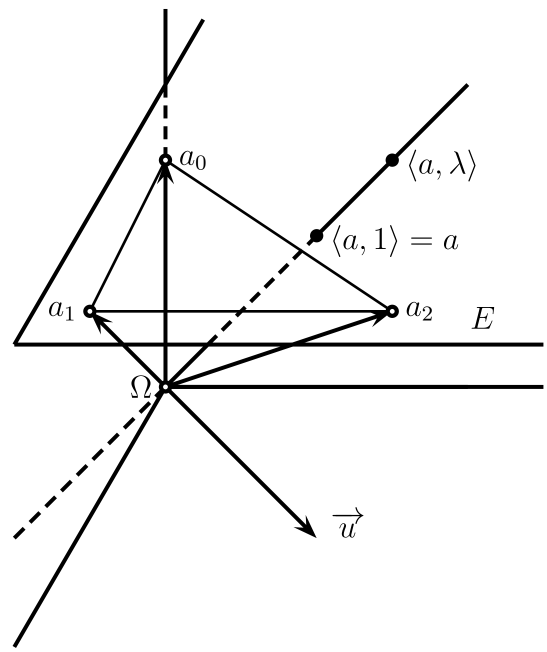


Figure 12.3: The basis (a_0, a_1, a_2) in $\widehat{E}$

Now that we have defined $\widehat{E}$ and investigated the relationship between affine frames in E and bases in $\widehat{E}$, we can give one more construction of a vector space $\mathcal{F}$ from E and $\overrightarrow{E}$, that will allow us to “visualize” in a much more intuitive fashion the structure of $\widehat{E}$ and of its operations $\widehat{+}$ and $\cdot$.

Definition 12.1.4 Given any affine space $(E, \overrightarrow{E})$, we define the vector space $\mathcal{F}$ as the direct sum $\overrightarrow{E} \oplus \mathbb{R}$, where $\mathbb{R}$ denotes the field $\mathbb{R}$ considered as a vector space (over itself). Denoting the unit vector in $\mathbb{R}$ as $\overrightarrow{1}$, since $\mathcal{F} = \overrightarrow{E} \oplus \mathbb{R}$, every vector $\overrightarrow{v} \in \mathcal{F}$ can be written as $\overrightarrow{v} = \overrightarrow{u} + \lambda\overrightarrow{1}$, for some unique $\overrightarrow{u} \in \overrightarrow{E}$, and some unique $\lambda \in \mathbb{R}$. Then, for any choice of an origin Ω_1 in E, we define the map $\widehat{\Omega}\colon \widehat{E} \to \mathcal{F}$, as follows:

$$\widehat{\Omega}(\theta) = \begin{cases} \lambda(\overrightarrow{1} + \overrightarrow{\Omega_1 a}) & \text{if } \theta = \langle a, \lambda\rangle,\ a \in E,\ \lambda \neq 0; \\ \overrightarrow{u} & \text{if } \theta = \overrightarrow{u},\ \overrightarrow{u} \in \overrightarrow{E}. \end{cases}$$

The idea is that, once again, viewing $\mathcal{F}$ as an affine space under its canonical structure, E is embedded in $\mathcal{F}$ as the hyperplane $H = \overrightarrow{1} + \overrightarrow{E}$, with direction $\overrightarrow{E}$, the hyperplane $\overrightarrow{E}$ in $\mathcal{F}$.

Then, every point $a \in E$ is in bijection with the point $A = \overrightarrow{1} + \overrightarrow{\Omega_1 a}$, in the hyperplane H. Denoting the origin $\overrightarrow{0}$ of the canonical affine space $\mathcal{F}$ as Ω, the map $\widehat{\Omega}$ maps a point $\langle a, \lambda \rangle \in E$ to a point in $\mathcal{F}$, as follows: $\widehat{\Omega}(\langle a, \lambda \rangle)$ is the point on the line passing through both the origin Ω of $\mathcal{F}$ and the point $A = \overrightarrow{1} + \overrightarrow{\Omega_1 a}$ in the hyperplane $H = \overrightarrow{1} + \overrightarrow{E}$, such that

$$\widehat{\Omega}(\langle a, \lambda \rangle) = \lambda \overrightarrow{\Omega A} = \lambda(\overrightarrow{1} + \overrightarrow{\Omega_1 a}).$$

The following lemma shows that $\widehat{\Omega}$ is an isomorphism of vector spaces.

Lemma 12.1.5 *Given any affine space $(E, \overrightarrow{E})$, for any choice Ω_1 of an origin in E, the map $\widehat{\Omega}\colon \widehat{E} \to \mathcal{F}$ is a linear isomorphism between $\widehat{E}$ and the vector space $\mathcal{F}$ of definition 12.1.4. The inverse of $\widehat{\Omega}$ is given by*

$$\widehat{\Omega}^{-1}(\overrightarrow{u} + \lambda\overrightarrow{1}) = \begin{cases} \langle \Omega_1 + \lambda^{-1}\overrightarrow{u}, \lambda\rangle) & \text{if } \lambda \neq 0; \\ \overrightarrow{u} & \text{if } \lambda = 0. \end{cases}$$

The following diagram illustrates the embedding of the affine space E into the vector space $\mathcal{F}$, when E is an affine plane.

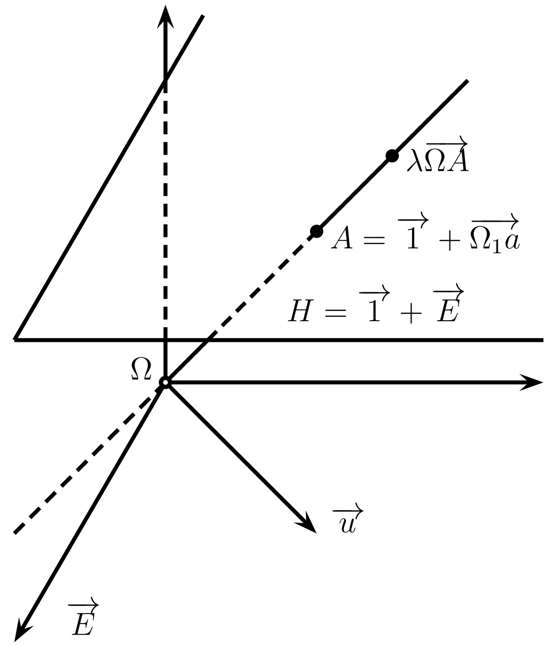


Figure 12.4: Embedding an affine space E into a vector space $\mathcal{F}$

We now consider the universal property of $\widehat{E}$. Other authors, such as Ramshaw, use the notation f_* for $\widehat{f}$. First, we define rigorously the notion of homogenization of an affine space.

Definition 12.1.6 Given any affine space $(E, \overrightarrow{E})$, an *homogenization (or linearization) of* $(E, \overrightarrow{E})$, is a triple $\langle \mathcal{E}, j, \omega \rangle$, where $\mathcal{E}$ is a vector space, $j: E \to \mathcal{E}$ is an injective affine map with associated injective linear map $i: \overrightarrow{E} \to \mathcal{E}$, $\omega: \mathcal{E} \to \mathbb{R}$ is a linear form, such that $\omega^{-1}(0) = i(\overrightarrow{E})$, $\omega^{-1}(1) = j(E)$, and for every vector space $\overrightarrow{F}$ and every affine map $f: E \to \overrightarrow{F}$, there is a unique linear map $\widehat{f}: \mathcal{E} \to \overrightarrow{F}$ extending f, i.e. $f = \widehat{f} \circ j$, as in the following diagram:

$$
\begin{array}{ccc}
E & \xrightarrow{j} & \mathcal{E} \\
& {\scriptstyle f} \searrow & \downarrow \widehat{f} \\
& & \overrightarrow{F}
\end{array}
$$

Thus, $j(E) = \omega^{-1}(1)$ is an affine hyperplane with direction $i(\overrightarrow{E}) = \omega^{-1}(0)$.

Lemma 12.1.7 *Given any affine space* $(E, \overrightarrow{E})$ *and any vector space* $\overrightarrow{F}$*, for any affine map* $f \colon E \to \overrightarrow{F}$*, there is a unique linear map* $\widehat{f} \colon \widehat{E} \to \overrightarrow{F}$ *extending* f*, such that*

$$\widehat{f}(\overrightarrow{u} \,\widehat{+}\, \lambda a) = \lambda f(a) + \overrightarrow{f}(\overrightarrow{u}),$$

for all $a \in E$*, all* $\overrightarrow{u} \in \overrightarrow{E}$*, and all* $\lambda \in \mathbb{R}$*, where* $\overrightarrow{f}$ *is the linear map associated with* f*. In particular, when* $\lambda \neq 0$*, we have*

$$\widehat{f}(\overrightarrow{u} \,\widehat{+}\, \lambda a) = \lambda f(a + \lambda^{-1}\overrightarrow{u}).$$

Lemma 12.1.7 shows that $\langle \widehat{E}, j, \omega \rangle$, is an homogenization of $(E, \overrightarrow{E})$. As a corollary, we obtain the following lemma.

Lemma 12.1.8 *Given two affine spaces E and F and an affine map $f\colon E \to F$, there is a unique linear map $\widehat{f}\colon \widehat{E} \to \widehat{F}$ extending f, as in the diagram below,*

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} \\ \widehat{E} & \xrightarrow[\widehat{f}]{} & \widehat{F} \end{array}$$

such that

$$\widehat{f}(\overrightarrow{u} \mathbin{\widehat{+}} \lambda a) = \overrightarrow{f}(\overrightarrow{u}) \mathbin{\widehat{+}} \lambda f(a),$$

for all $a \in E$, all $\overrightarrow{u} \in \overrightarrow{E}$, and all $\lambda \in \mathbb{R}$, where $\overrightarrow{f}$ is the linear map associated with f. In particular, when $\lambda \neq 0$, we have

$$\widehat{f}(\overrightarrow{u} \mathbin{\widehat{+}} \lambda a) = \lambda f(a + \lambda^{-1}\overrightarrow{u}).$$

From a practical point of view, lemma 12.1.8 shows us how to homogenize an affine map to turn it into a linear map between the two homogenized spaces.

Assume that E and F are of finite dimension, and that $(a_0, (\overrightarrow{u_1}, \ldots, \overrightarrow{u_n}))$ is an affine basis of E, with origin a_0, and $(b_0, (\overrightarrow{v_1}, \ldots, \overrightarrow{v_m}))$ is an affine basis of F, with origin b_0.

Then, with respect to the two bases $(\overrightarrow{u_1}, \ldots, \overrightarrow{u_n}, a_0)$ in $\widehat{E}$ and $(\overrightarrow{v_1}, \ldots, \overrightarrow{v_m}, b_0)$ in $\widehat{F}$, a linear map $h\colon \widehat{E} \to \widehat{F}$ is given by an $(m+1) \times (n+1)$ matrice A.

If this linear map h is equal to the homogenized version $\widehat{f}$ of an affine map f, since

$$\widehat{f}(\overrightarrow{u} \mathbin{\widehat{+}} \lambda a) = \overrightarrow{f}(\overrightarrow{u}) \mathbin{\widehat{+}} \lambda f(a),$$

since over the basis $(\overrightarrow{u_1}, \ldots, \overrightarrow{u_n}, a_0)$ in $\widehat{E}$, points are represented by vectors whose last coordinate is 1, and vectors are represented by vectors whose last coordinate is 0, the last row of the matrix $A = M(\widehat{f})$ with respect to the given bases is

$$(0, 0, \ldots, 0, 1),$$

with m occurrences of 0, the last column contains the coordinates

$$(\mu_1, \ldots, \mu_m, 1)$$

of $f(a_0)$ with respect to the basis $(\overrightarrow{v_1}, \ldots, \overrightarrow{v_m}, b_0)$,

the submatrix of A obtained by deleting the last row and the last column is the matrix of the linear map $\overrightarrow{f}$ with respect to the bases $(\overrightarrow{u_1}, \ldots, \overrightarrow{u_n})$ and $(\overrightarrow{v_1}, \ldots, \overrightarrow{v_m})$, and since

$$f(a_0 + \overrightarrow{u}) = \widehat{f}(\overrightarrow{u} \mathbin{\widehat{+}} a_0),$$

given any $x \in E$ and $y \in F$, with coordinates
$(x_1, \ldots, x_n, 1)$ and $(y_1, \ldots, y_m, 1)$,
for $X = (x_1, \ldots, x_n, 1)^\top$ and $Y = (y_1, \ldots, y_m, 1)^\top$, we have

$$y = f(x) \quad \text{iff} \quad Y = AX.$$

For example, consider the following affine map $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ defined as follows:

$$\begin{aligned} y_1 &= ax_1 + bx_2 + \mu_1, \\ y_2 &= cx_1 + dx_2 + \mu_2. \end{aligned}$$

The matrix of $\widehat{f}$ is

$$\begin{pmatrix} a & b & \mu_1 \\ c & d & \mu_2 \\ 0 & 0 & 1 \end{pmatrix}$$

and we have

$$\begin{pmatrix} y_1 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & \mu_1 \\ c & d & \mu_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}$$

In $\widehat{E}$, we have

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} a & b & \mu_1 \\ c & d & \mu_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

which means that the homogeneous map $\widehat{f}$ is is obtained from f by "adding the variable of homogeneity x_3":

$$\begin{aligned} y_1 &= ax_1 + bx_2 + \mu_1 x_3, \\ y_2 &= cx_1 + dx_2 + \mu_2 x_3, \\ y_3 &= x_3. \end{aligned}$$

We now show how to homogenize multiaffine maps.

Lemma 12.1.9 *Given any affine space E and any vector space $\overrightarrow{F}$, for any m-affine map $f\colon E^m \to \overrightarrow{F}$, there is a unique m-linear map $\widehat{f}\colon (\widehat{E})^m \to \overrightarrow{F}$ extending f, such that, if*

$$f(a_1 + \overrightarrow{v_1}, \ldots, a_m + \overrightarrow{v_m}) = f(a_1, \ldots, a_m) + \sum_{\substack{S\subseteq\{1,\ldots,m\},\, k=\mathrm{card}(S)\\ S=\{i_1,\ldots,i_k\},\, k\geq 1}} f_S(\overrightarrow{v_{i_1}}, \ldots, \overrightarrow{v_{i_k}}),$$

for all $a_1 \ldots, a_m \in E$, and all $\overrightarrow{v_1}, \ldots, \overrightarrow{v_m} \in \overrightarrow{E}$, where the f_S are uniquely determined multilinear maps (by lemma **??***), then*

$$\begin{aligned}\widehat{f}(\overrightarrow{v_1} \,\widehat{+}\, \lambda_1 a_1, \ldots, \overrightarrow{v_m} \,\widehat{+}\, \lambda_m a_m) &= \lambda_1 \cdots \lambda_m f(a_1, \ldots, a_m) + \\ &\sum_{\substack{S\subseteq\{1,\ldots,m\},\, k=\mathrm{card}(S)\\ S=\{i_1,\ldots,i_k\},\, k\geq 1}} \Big(\prod_{\substack{j\in\{1,\ldots,m\}\\ j\notin S}} \lambda_j\Big)\, f_S(\overrightarrow{v_{i_1}}, \ldots, \overrightarrow{v_{i_k}}),\end{aligned}$$

for all $a_1 \ldots, a_m \in E$, all $\overrightarrow{v_1}, \ldots, \overrightarrow{v_m} \in \overrightarrow{E}$, and all $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$. Furthermore, for $\lambda_i \neq 0$, $1 \leq i \leq m$, we have

$$\widehat{f}(\overrightarrow{v_1} \,\widehat{+}\, \lambda_1 a_1, \ldots, \overrightarrow{v_m} \,\widehat{+}\, \lambda_m a_m) = \lambda_1 \cdots \lambda_m f(a_1 + \lambda_1^{-1}\overrightarrow{v_1}, \ldots, a_m + \lambda_m^{-1}\overrightarrow{v_m}).$$

12.2 Differentiating Affine Polynomial Functions Using Their Homogenized Polar Forms, Osculating Flats

Let $\delta = \overrightarrow{1}$, the unit (vector) in $\mathbb{R}$. When dealing with derivatives, it is also more convenient to denote the vector $\overrightarrow{ab}$ as $b - a$.

For any $\overline{a} \in \mathbb{A}$, the derivative $\mathrm{D}F(\overline{a})$ is the limit,

$$\lim_{t \to 0,\, t \neq 0} \frac{F(\overline{a} + t\delta) - F(\overline{a})}{t},$$

if it exists. However, since $\widehat{F}$ agrees with F on $\mathbb{A}$, we have

$$F(\overline{a} + t\delta) - F(\overline{a}) = \widehat{F}(\overline{a} + t\delta) - \widehat{F}(\overline{a}),$$

and thus, we need to see what is the limit of

$$\frac{\widehat{F}(\overline{a} + t\delta) - \widehat{F}(\overline{a})}{t},$$

when $t \to 0$, $t \neq 0$, with $t \in \mathbb{R}$.

Recall that since $F\colon \mathbb{A} \to \mathcal{E}$, where $\mathcal{E}$ is an affine space, the derivative $\mathrm{D}F(\overline{a})$ of F at $\overline{a}$ is a *vector* in $\overrightarrow{\mathcal{E}}$, and not a point in $\mathcal{E}$. However, the structure of $\widehat{\mathcal{E}}$ takes care of this, since $\widehat{F}(\overline{a}+t\delta)-\widehat{F}(\overline{a})$ is indeed a vector (remember our convention that $-$ is an abbreviation for $\widehat{-}$).

Since

$$\widehat{F}(\overline{a}+t\delta) = \widehat{f}(\underbrace{\overline{a}+t\delta, \ldots, \overline{a}+t\delta}_{m}),$$

where $\widehat{f}$ is the homogenized version of the polar form f of F, and $\widehat{F}$ is the homogenized version of F, since

$$\widehat{F}(\overline{a}+t\delta)-\widehat{F}(\overline{a}) = \widehat{f}(\underbrace{\overline{a}+t\delta, \ldots, \overline{a}+t\delta}_{m}) - \widehat{f}(\underbrace{\overline{a}, \ldots, \overline{a}}_{m}),$$

by multilinearity and symmetry, we have

$$\widehat{F}(\overline{a}+t\delta)-\widehat{F}(\overline{a}) =$$

$$m\,t\,\widehat{f}(\underbrace{\overline{a}, \ldots, \overline{a}}_{m-1}, \delta) + \sum_{k=2}^{k=m} \binom{m}{k}\, t^k\, \widehat{f}(\underbrace{\overline{a}, \ldots, \overline{a}}_{m-k}, \underbrace{\delta, \ldots, \delta}_{k}),$$

and thus,

$$\lim_{t\to 0,\, t\neq 0} \frac{\widehat{F}(\overline{a}+t\delta)-\widehat{F}(\overline{a})}{t} = m\widehat{f}(\underbrace{\overline{a}, \ldots, \overline{a}}_{m-1}, \delta).$$

However, since $\widehat{F}$ extends F on $\mathbb{A}$, we have $\mathrm{D}F(\overline{a}) = \mathrm{D}\widehat{F}(\overline{a})$, and thus, we showed that

$$\mathrm{D}F(\overline{a}) = m\widehat{f}(\underbrace{\overline{a}, \ldots, \overline{a}}_{m-1}, \delta).$$

This shows that the derivative of F at $\overline{a} \in \mathbb{A}$ can be computed by evaluating the homogenized version $\widehat{f}$ of the polar form f of F, by replacing just one occurrence of $\overline{a}$ in $\widehat{f}(\overline{a}, \ldots, \overline{a})$ by δ.

More generally, we have the following useful lemma.

Lemma 12.2.1 *Given an affine polynomial function $F\colon \mathbb{A} \to \mathcal{E}$ of polar degree m, where $\mathcal{E}$ is a normed affine space, the k-th derivative $\mathrm{D}^k F(\overline{a})$ can be computed from the homogenized polar form $\widehat{f}$ of F as follows, where $1 \leq k \leq m$:*

$$\mathrm{D}^k F(\overline{a}) = m(m-1)\cdots(m-k+1)\,\widehat{f}(\underbrace{\overline{a}, \ldots, \overline{a}}_{m-k}, \underbrace{\delta, \ldots, \delta}_{k}).$$

Since coefficients of the form $m(m-1)\cdots(m-k+1)$ occur a lot when taking derivatives, following Knuth, it is useful to introduce the *falling power* notation. We define the *falling power* $m^{\underline{k}}$, as

$$m^{\underline{k}} = m(m-1)\cdots(m-k+1),$$

for $0 \leq k \leq m$, with $m^{\underline{0}} = 1$, and with the convention that $m^{\underline{k}} = 0$ when $k > m$.

Using the falling power notation, the previous lemma reads as

$$\mathrm{D}^k F(\overline{a}) = m^{\underline{k}}\,\widehat{f}(\underbrace{\overline{a}, \ldots, \overline{a}}_{m-k}, \underbrace{\delta, \ldots, \delta}_{k}).$$

We also get the following explicit formula in terms of control points.

Lemma 12.2.2 *Given an affine polynomial function $F\colon \mathbb{A} \to \mathcal{E}$ of polar degree m, where $\mathcal{E}$ is a normed affine space, for any $\overline{r}, \overline{s} \in \mathbb{A}$, with $r \neq s$, the k-th derivative $\mathrm{D}^k F(\overline{r})$ can be computed from the polar form f of F as follows, where $1 \leq k \leq m$:*

$$\mathrm{D}^k F(\overline{r}) = \frac{m^{\underline{k}}}{(s-r)^k} \sum_{i=0}^{i=k} \binom{k}{i} (-1)^{k-i} f(\underbrace{\overline{r}, \ldots, \overline{r}}_{m-i}, \underbrace{\overline{s}, \ldots, \overline{s}}_{i}).$$

If F is specified by the sequence of $m+1$ control points $b_i = f(\overline{r}^{\,m-i} \overline{s}^{\,i})$, $0 \leq i \leq m$, the above lemma shows that the k-th derivative $\mathrm{D}^k F(\overline{r})$ of F at $\overline{r}$, depends only on the $k+1$ control points $b_0, \ldots, b_k$ In terms of the control points $b_0, \ldots, b_k$, the formula of lemma 7.4.1 reads as follows:

$$\mathrm{D}^k F(\overline{r}) = \frac{m^{\underline{k}}}{(s-r)^k} \sum_{i=0}^{i=k} \binom{k}{i} (-1)^{k-i} b_i.$$

In particular, if $b_0 \neq b_1$, then $\mathrm{D}F(\overline{r})$ is the velocity vector of F at b_0, and it is given by

$$\mathrm{D}F(\overline{r}) = \frac{m}{s-r}\,\overrightarrow{b_0b_1} = \frac{m}{s-r}\,(b_1 - b_0),$$

the last expression making sense in $\widehat{\mathcal{E}}$.

In terms of the de Casteljau diagram

$$\mathrm{D}F(\overline{t}) = \frac{m}{s-r}(b_{1,m-1} - b_{0,m-1}).$$

Similarly, the acceleration vector $\mathrm{D}^2F(\overline{r})$ is given by

$$\mathrm{D}^2F(\overline{r}) = \frac{m(m-1)}{(s-r)^2}\,(\overrightarrow{b_0b_2} - 2\overrightarrow{b_0b_1}) = \frac{m(m-1)}{(s-r)^2}\,(b_2 - 2b_1 + b_0),$$

the last expression making sense in $\widehat{\mathcal{E}}$.

Later on when we deal with surfaces, it will be necessary to generalize the above results to directional derivatives. However, we have basically done all the work already.

Let us assume that E and $\mathcal{E}$ are normed affine spaces, and consider a map $F\colon E \to \mathcal{E}$. Recall from definition 11.1.1, that if A is any open subset of E, for any $a \in A$, for any $\overrightarrow{u} \neq \overrightarrow{0}$ in $\overrightarrow{E}$, the *directional derivative of F at a w.r.t. the vector $\overrightarrow{u}$*, denoted as $\mathrm{D}_u F(a)$, is the limit, if it exists,

$$\lim_{t \to 0, t \in U, t \neq 0} \frac{F(a + t\overrightarrow{u}) - F(a)}{t},$$

where $U = \{t \in \mathbb{R} \mid a + t\overrightarrow{u} \in A\}$.

If $F\colon E \to \mathcal{E}$ is a polynomial function of degree m, with polar form the symmetric multiaffine map $f\colon E^m \to \mathcal{E}$, then

$$F(a + t\overrightarrow{u}) - F(a) = \widehat{F}(a + t\overrightarrow{u}) - \widehat{F}(a),$$

where $\widehat{F}$ is the homogenized version of F, that is, the polynomial map $\widehat{F}\colon \widehat{E} \to \widehat{\mathcal{E}}$ associated with the homogenized version $f\colon (\widehat{E})^m \to \widehat{\mathcal{E}}$ of the polar form $f\colon E^m \to \mathcal{E}$ of $F\colon E \to \mathcal{E}$.

Thus, $\mathrm{D}_u F(a)$ exists iff the limit

$$\lim_{t\to 0,\, t\neq 0} \frac{\widehat{F}(a + t\overrightarrow{u}) - \widehat{F}(a)}{t}$$

exists, and in this case, this limit is $\mathrm{D}_u F(a) = \mathrm{D}_u \widehat{F}(a)$. We get

$$\mathrm{D}_u F(a) = m\widehat{f}(\underbrace{a, \ldots, a}_{m-1}, \overrightarrow{u}).$$

By a simple, induction, we can prove the following lemma.

Lemma 12.2.3 *Given an affine polynomial function $F\colon E \to \mathcal{E}$ of polar degree m, where E and $\mathcal{E}$ are normed affine spaces, for any k nonzero vectors $\overrightarrow{u_1}$, …, $\overrightarrow{u_k} \in \overrightarrow{E}$, where $1 \leq k \leq m$, the k-th directional derivative $\mathrm{D}_{u_1} \ldots \mathrm{D}_{u_k} F(a)$ can be computed from the homogenized polar form $\widehat{f}$ of F as follows:*

$$\mathrm{D}_{u_1} \ldots \mathrm{D}_{u_k} F(a) = m^{\underline{k}}\, \widehat{f}(\underbrace{a, \ldots, a}_{m-k}, \overrightarrow{u_1}, \ldots, \overrightarrow{u_k}).$$

If E has finite dimension,

$$\mathrm{D}^k F(a)(\overrightarrow{u_1}, \ldots, \overrightarrow{u_k}) = m^{\underline{k}}\, \widehat{f}(\underbrace{a, \ldots, a}_{m-k}, \overrightarrow{u_1}, \ldots, \overrightarrow{u_k}).$$