Appendix A

Topological Preliminaries

A.1 Metric Spaces and Normed Vector Spaces

This Chapter provides a review of basic topological notions. For a comprehensive account, we highly recommend Munkres [37], Amstrong [3], Dixmier [11], Singer and Thorpe [45], Lang [29], or Schwartz [43]. Most spaces considered will have a topological structure given by a metric or a norm and we first review these notions. We begin with metric spaces.

Definition A.1 A *metric space* is a set, E, together with a function, $d\colon E \times E \to \mathbb{R}_+$, called a *metric or distance*, assigning a nonnegative real number, $d(x,\, y)$, to any two points, $x, y \in E$, and satisfying the following conditions for all $x, y, z \in E$:

(D1) $d(x,\, y) = d(y,\, x)$. (symmetry)

(D2) $d(x,\, y) \geq 0$ and $d(x,\, y) = 0$ iff $x = y$. (positivity)

(D3) $d(x,\, z) \leq d(x,\, y) + d(y,\, z)$. (triangular inequality)

Geometrically, condition (D3) expresses the fact that in a triangle with vertices x, y, z, the length of any side is bounded by the sum of the lengths of the other two sides. From (D3), we immediately get

$$|d(x,\, y) - d(y,\, z)| \leq d(x,\, z).$$

Let us give some examples of metric spaces. Recall that the *absolute value*, $|x|$, of a real number, $x \in \mathbb{R}$, is defined such that $|x| = x$ if $x \geq 0$, $|x| = -x$ if $x < 0$, and for a complex number, $x = a + ib$, by $|x| = \sqrt{a^2 + b^2}$.

Example A.1 Let $E = \mathbb{R}$ and $d(x,\, y) = |x - y|$, the absolute value of $x - y$. This is the so-called *natural metric* on $\mathbb{R}$.

Example A.2 Let $E = \mathbb{R}^n$ (or $E = \mathbb{C}^n$). We have the Euclidean metric,

$$d_2(x,\, y) = \left(|x_1 - y_1|^2 + \cdots + |x_n - y_n|^2\right)^{\frac{1}{2}},$$

the distance between the points $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$.

Example A.3 For every set, E, we can define the *discrete metric* defined such that $d(x,\, y) = 1$ iff $x \neq y$ and $d(x,\, x) = 0$.

Example A.4 For any $a, b \in \mathbb{R}$ such that $a < b$, we define the following sets:

1. $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$, (closed interval)
2. $]a, b[= \{x \in \mathbb{R} \mid a < x < b\}$, (open interval)
3. $[a, b[= \{x \in \mathbb{R} \mid a \leq x < b\}$, (interval closed on the left, open on the right)
4. $]a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$, (interval open on the left, closed on the right)

Let $E = [a, b]$, and $d(x,\, y) = |x - y|$. Then, $([a, b], d)$ is a metric space.

We will need to define the notion of proximity in order to define convergence of limits and continuity of functions. For this, we introduce some standard "small neighborhoods".

Definition A.2 Given a metric space, E, with metric, d, for every $a \in E$, for every $\rho \in \mathbb{R}$, with $\rho > 0$, the set

$$B(a, \rho) = \{x \in E \mid d(a,\, x) \leq \rho\}$$

is called the *closed ball of center a and radius ρ*, the set

$$B_0(a, \rho) = \{x \in E \mid d(a,\, x) < \rho\}$$

is called the *open ball of center a and radius ρ*, and the set

$$S(a,\, \rho) = \{x \in E \mid d(a,\, x) = \rho\}$$

is called the *sphere of center a and radius ρ*. It should be noted that ρ is finite (i.e. not $+\infty$). A subset, X, of a metric space, E, is *bounded* if there is a closed ball, $B(a, \rho)$, such that $X \subseteq B(a, \rho)$.

Clearly, $B(a, \rho) = B_0(a, \rho) \cup S(a,\, \rho)$.

In $E = \mathbb{R}$ with the distance $|x - y|$, an open ball of center a and radius ρ is the open interval $]a - \rho, a + \rho[$. In $E = \mathbb{R}^2$ with the Euclidean metric, an open ball of center a and radius ρ is the set of points inside the disk of center a and radius ρ, excluding the boundary points on the circle. In $E = \mathbb{R}^3$ with the Euclidean metric, an open ball of center a and radius ρ is the set of points inside the sphere of center a and radius ρ, excluding the boundary points on the sphere.

One should be aware that intuition can be misleading in forming a geometric image of a closed (or open) ball. For example, if d is the discrete metric, a closed ball of center a and radius $\rho < 1$ consists only of its center a, and a closed ball of center a and radius $\rho \geq 1$ consists of the entire space!

If $E = [a, b]$, and $d(x,\, y) = |x - y|$, as in example 4, an open ball, $B_0(a, \rho)$, with $\rho < b - a$, is in fact the interval, $[a, a + \rho[$, which is closed on the left.

We now consider a very important special case of metric spaces, normed vector spaces.

Definition A.3 Let E be a vector space over a field, K, where K is either the field, $\mathbb{R}$, of reals, or the field, $\mathbb{C}$, of complex numbers. A *norm on* E is a function, $\| \, \| \colon E \to \mathbb{R}_+$, assigning a nonnegative real number, $\|u\|$, to any vector, $u \in E$, and satisfying the following conditions for all $x, y, z \in E$:

(N1) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$. (positivity)

(N2) $\|\lambda x\| = |\lambda| \, \|x\|$. (scaling)

(N3) $\|x + y\| \leq \|x\| + \|y\|$. (convexity inequality)

A vector space, E, together with a norm, $\| \, \|$, is called a *normed vector space*.

From (N3), we easily get

$$|\|x\| - \|y\|| \leq \|x - y\|.$$

Given a normed vector space, E, if we define d such that

$$d(x,\, y) = \|x - y\|,$$

it is easily seen that d is a metric. Thus, every normed vector space is immediately a metric space. Note that the metric associated with a norm is *invariant under translation*, that is,

$$d(x + u,\, y + u) = d(x,\, y).$$

For this reason, we can restrict ourselves to open or closed balls of center 0.

Let us give some examples of normed vector spaces.

Example A.5 Let $E = \mathbb{R}$ and $\|x\| = |x|$, the absolute value of x. The associated metric is $|x - y|$, as in example 1.

Example A.6 Let $E = \mathbb{R}^n$ (or $E = \mathbb{C}^n$). There are three standard norms. For every $(x_1, \ldots, x_n) \in E$, we have the norm, $\|x\|_1$, defined such that,

$$\|x\|_1 = |x_1| + \cdots + |x_n|,$$

we have the Euclidean norm, $\|x\|_2$, defined such that,

$$\|x\|_2 = \left(|x_1|^2 + \cdots + |x_n|^2\right)^{\frac{1}{2}},$$

and the *sup*-norm, $\|x\|_\infty$, defined such that,

$$\|x\|_\infty = \max\{|x_i| \mid 1 \leq i \leq n\}.$$

Some work is required to show the convexity inequality for the Euclidean norm, but this can be found in any standard text. Note that the Euclidean distance is the distance associated with the Euclidean norm. The following proposition is easy to show:

Proposition A.1 *The following inequalities hold for all* $x \in \mathbb{R}^n$ *(or* $x \in \mathbb{C}^n$*):*

$$\begin{aligned} \|x\|_\infty &\leq \|x\|_1 \leq n\|x\|_\infty, \\ \|x\|_\infty &\leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty, \\ \|x\|_2 &\leq \|x\|_1 \leq \sqrt{n}\|x\|_2. \end{aligned}$$

In a normed vector space, we define a closed ball or an open ball of radius ρ as a closed ball or an open ball of center 0. We may use the notation $B(\rho)$ and $B_0(\rho)$.

We will now define the crucial notions of open sets and closed sets and of a topological space.

Definition A.4 Let E be a metric space with metric d. A subset, $U \subseteq E$, is an *open set* in E if either $U = \emptyset$ or, for every $a \in U$, there is some open ball, $B_0(a, \rho)$, such that, $B_0(a, \rho) \subseteq U$.[1] A subset, $F \subseteq E$, is a *closed set* in E if its complement, $E - F$, is open in E.

The set E itself is open, since for every $a \in E$, every open ball of center a is contained in E. In $E = \mathbb{R}^n$, given n intervals, $[a_i, b_i]$, with $a_i < b_i$, it is easy to show that the open n-cube,

$$\{(x_1, \ldots, x_n) \in E \mid a_i < x_i < b_i,\ 1 \leq i \leq n\},$$

is an open set. In fact, it is possible to find a metric for which such open n-cubes are open balls! Similarly, we can define the closed n-cube,

$$\{(x_1, \ldots, x_n) \in E \mid a_i \leq x_i \leq b_i,\ 1 \leq i \leq n\},$$

which is a closed set.

The open sets satisfy some important properties that lead to the definition of a topological space.

Proposition A.2 *Given a metric space,* E*, with metric,* d*, the family,* $\mathcal{O}$*, of open sets defined in Definition A.4 satisfies the following properties:*

(O1) For every finite family, $(U_i)_{1 \leq i \leq n}$*, of sets,* $U_i \in \mathcal{O}$*, we have* $U_1 \cap \cdots \cap U_n \in \mathcal{O}$*, i.e.,* $\mathcal{O}$ *is closed under finite intersections.*

(O2) For every arbitrary family, $(U_i)_{i \in I}$*, of sets,* $U_i \in \mathcal{O}$*, we have* $\bigcup_{i \in I} U_i \in \mathcal{O}$*, i.e.,* $\mathcal{O}$ *is closed under arbitrary unions.*

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[1] Recall that $\rho > 0$.

(O3) $\emptyset \in \mathcal{O}$ *and* $E \in \mathcal{O}$*, i.e.,* $\emptyset$ *and* E *belong to* $\mathcal{O}$*.*

Furthermore, for any two distinct points $a \neq b$ *in* E*, there exist two open sets,* U_a *and* U_b*, such that,* $a \in U_a$*,* $b \in U_b$*, and* $U_a \cap U_b = \emptyset$*.*

Proof. It is straightforward. For the last point, if we let $\rho = d(a, b)/3$ (in fact $\rho = d(a, b)/2$ works too), we can pick $U_a = B_0(a, \rho)$ and $U_b = B_0(b, \rho)$. By the triangle inequality, we must have $U_a \cap U_b = \emptyset$. $\square$

The above proposition leads to the very general concept of a topological space.

One should be careful that in general, the family of open sets is not closed under infinite intersections. For example, in $\mathbb{R}$ under the metric $|x - y|$, letting $U_n =]-1/n, +1/n[$, each U_n is open, but $\bigcap_n U_n = \{0\}$, which is not open.

A.2 Topological Spaces, Continuous Functions, Limits

Motivated by Proposition A.2, a topological space is defined in terms of a family of sets satisfing the properties of open sets stated in that proposition.

Definition A.5 Given a set, E, a *topology on* E *(or a topological structure on* E) is defined as a family, $\mathcal{O}$, of subsets of E called *open sets* and satisfying the following three properties:

(1) For every finite family, $(U_i)_{1 \leq i \leq n}$, of sets, $U_i \in \mathcal{O}$, we have $U_1 \cap \cdots \cap U_n \in \mathcal{O}$, *i.e.*, $\mathcal{O}$ is closed under finite intersections.

(2) For every arbitrary family, $(U_i)_{i \in I}$, of sets, $U_i \in \mathcal{O}$, we have $\bigcup_{i \in I} U_i \in \mathcal{O}$, *i.e.*, $\mathcal{O}$ is closed under arbitrary unions.

(3) $\emptyset \in \mathcal{O}$ and $E \in \mathcal{O}$, *i.e.*, $\emptyset$ and E belong to $\mathcal{O}$.

A set, E, together with a topology, $\mathcal{O}$, on E is called a *topological space*. Given a topological space, $(E, \mathcal{O})$, a subset, F, of E is a *closed set* if $F = E - U$ for some open set, $U \in \mathcal{O}$, *i.e.*, F is the complement of some open set.

It is possible that an open set is also a closed set. For example, $\emptyset$ and E are both open and closed. When a topological space contains a proper nonempty subset, U, which is both open and closed, the space E is said to be *disconnected*. Connected spaces will be studied in Section A.3.

A topological space, $(E, \mathcal{O})$, is said to satisfy the *Hausdorff separation axiom (or* T_2*-separation axiom)* if for any two distinct points, $a \neq b$ in E, there exist two open sets, U_a and U_b, such that, $a \in U_a$, $b \in U_b$, and $U_a \cap U_b = \emptyset$. When the T_2-separation axiom is satisfied, we also say that $(E, \mathcal{O})$ is a *Hausdorff space*.

As shown by Proposition A.2, any metric space is a topological Hausdorff space, the family of open sets being in fact the family of arbitrary unions of open balls. Similarly, any normed vector space is a topological Hausdorff space, the family of open sets being the family of arbitrary unions of open balls. The topology, $\mathcal{O}$, consisting of all subsets of E is called the *discrete topology*.

Remark: Most (if not all) spaces used in analysis are Hausdorff spaces. Intuitively, the Hausdorff separation axiom says that there are enough "small" open sets. Without this axiom, some counter-intuitive behaviors may arise. For example, a sequence may have more than one limit point (or a compact set may not be closed). Nevertheless, non-Hausdorff topological spaces arise naturally in algebraic geometry. But even there, some substitute for separation is used.

One of the reasons why topological spaces are important is that the definition of a topology only involves a certain family, $\mathcal{O}$, of sets, and not **how** such family is generated from a metric or a norm. For example, different metrics or different norms can define the same family of open sets. Many topological properties only depend on the family $\mathcal{O}$ and not on the specific metric or norm. But the fact that a topology is definable from a metric or a norm is important, because it usually implies nice properties of a space. All our examples will be spaces whose topology is defined by a metric or a norm.

By taking complements, we can state properties of the closed sets dual to those of Definition A.5. Thus, $\emptyset$ and E are closed sets and the closed sets are closed under finite unions and arbitrary intersections. It is also worth noting that the Hausdorff separation axiom implies that for every $a \in E$, the set $\{a\}$ is closed. Indeed, if $x \in E - \{a\}$, then $x \neq a$, and so there exist open sets, U_a and U_x, such that $a \in U_a$, $x \in U_x$, and $U_a \cap U_x = \emptyset$. Thus, for every $x \in E - \{a\}$, there is an open set, U_x, containing x and contained in $E - \{a\}$, showing by (O3) that $E - \{a\}$ is open and thus, that the set $\{a\}$ is closed.

Given a topological space, $(E, \mathcal{O})$, given any subset, A, of E, since $E \in \mathcal{O}$ and E is a closed set, the family, $\mathcal{C}_A = \{F \mid A \subseteq F,\ F \text{ a closed set}\}$, of closed sets containing A is nonempty, and since any arbitrary intersection of closed sets is a closed set, the intersection, $\bigcap \mathcal{C}_A$, of the sets in the family $\mathcal{C}_A$ is the smallest closed set containing A. By a similar reasoning, the union of all the open subsets contained in A is the largest open set contained in A.

Definition A.6 Given a topological space, $(E, \mathcal{O})$, given any subset, A, of E, the smallest closed set containing A is denoted by $\overline{A}$ and is called the *closure or adherence of* A. A subset, A, of E is *dense in* E if $\overline{A} = E$. The largest open set contained in A is denoted by $\overset{\circ}{A}$ and is called the *interior of* A. The set, $\mathrm{Fr}\, A = \overline{A} \cap \overline{E - A}$, is called the *boundary (or frontier) of* A. We also denote the boundary of A by ∂A.

Remark: The notation $\overline{A}$ for the closure of a subset, A, of E is somewhat unfortunate, since $\overline{A}$ is often used to denote the set complement of A in E. Still, we prefer it to more cumbersome notations such as $\mathrm{clo}(A)$ and we denote the complement of A in E by $E - A$.

By definition, it is clear that a subset, A, of E is closed iff $A = \overline{A}$. The set $\mathbb{Q}$ of rationals is dense in $\mathbb{R}$. It is easily shown that $\overline{A} = \overset{\circ}{A} \cup \partial A$ and $\overset{\circ}{A} \cap \partial A = \emptyset$. Another useful characterization of $\overline{A}$ is given by the following proposition:

Proposition A.3 *Given a topological space, $(E, \mathcal{O})$, given any subset, A, of E, the closure, $\overline{A}$, of A is the set of all points, $x \in E$, such that for every open set, U, containing x, then $U \cap A \neq \emptyset$.*

Proof. If $A = \emptyset$, since $\emptyset$ is closed, the proposition holds trivially. Thus, assume that $A \neq \emptyset$. First, assume that $x \in \overline{A}$. Let U be any open set such that $x \in U$. If $U \cap A = \emptyset$, since U is open, then $E - U$ is a closed set containing A and since $\overline{A}$ is the intersection of all closed sets containing A, we must have $x \in E - U$, which is impossible. Conversely, assume that $x \in E$ is a point such that for every open set, U, containing x, then $U \cap A \neq \emptyset$. Let F be any closed subset containing A. If $x \notin F$, since F is closed, then $U = E - F$ is an open set such that $x \in U$, and $U \cap A = \emptyset$, a contradiction. Thus, we have $x \in F$ for every closed set containing A, that is, $x \in \overline{A}$. $\square$

Often, it is necessary to consider a subset, A, of a topological space E, and to view the subset A as a topological space. The following proposition shows how to define a topology on a subset:

Proposition A.4 *Given a topological space, $(E, \mathcal{O})$, given any subset, A, of E, let*

$$\mathcal{U} = \{U \cap A \mid U \in \mathcal{O}\}$$

be the family of all subsets of A obtained as the intersection of any open set in $\mathcal{O}$ with A. The following properties hold:

(1) The space $(A, \mathcal{U})$ is a topological space.

(2) If E is a metric space with metric d, then the restriction, $d_A \colon A \times A \to \mathbb{R}_+$, of the metric, d, to A defines a metric space. Furthermore, the topology induced by the metric, d_A, agrees with the topology defined by $\mathcal{U}$, as above.

Proof. Left as an exercise. $\square$

Proposition A.4 suggests the following definition:

Definition A.7 Given a topological space, $(E, \mathcal{O})$, given any subset, A, of E, the *subspace topology on A induced by $\mathcal{O}$* is the family, $\mathcal{U}$, of open sets defined such that

$$\mathcal{U} = \{U \cap A \mid U \in \mathcal{O}\}$$

is the family of all subsets of A obtained as the intersection of any open set in $\mathcal{O}$ with A. We say that $(A, \mathcal{U})$ has the *subspace topology*. If (E, d) is a metric space, the restriction, $d_A \colon A \times A \to \mathbb{R}_+$, of the metric, d, to A is called the *subspace metric*.

For example, if $E = \mathbb{R}^n$ and d is the Euclidean metric, we obtain the subspace topology on the closed n-cube,

$$\{(x_1, \ldots, x_n) \in E \mid a_i \leq x_i \leq b_i,\ 1 \leq i \leq n\}.$$

One should realize that every open set, $U \in \mathcal{O}$, which is entirely contained in A is also in the family, $\mathcal{U}$, but $\mathcal{U}$ may contain open sets that are not in $\mathcal{O}$. For example, if $E = \mathbb{R}$ with $|x - y|$, and $A = [a, b]$, then sets of the form $[a, c[$, with $a < c < b$ belong to $\mathcal{U}$, but they are not open sets for $\mathbb{R}$ under $|x - y|$. However, there is agreement in the following situation.

Proposition A.5 *Given a topological space, $(E, \mathcal{O})$, given any subset, A, of E, if $\mathcal{U}$ is the subspace topology, then the following properties hold.*

(1) If A is an open set, $A \in \mathcal{O}$, then every open set, $U \in \mathcal{U}$, is an open set, $U \in \mathcal{O}$.

(2) If A is a closed set in E, then every closed set w.r.t. the subspace topology is a closed set w.r.t. $\mathcal{O}$.

Proof. Left as an exercise. □

The concept of product topology is also useful. We have the following proposition:

Proposition A.6 *Given n topological spaces, $(E_i, \mathcal{O}_i)$, let $\mathcal{B}$ be the family of subsets of $E_1 \times \cdots \times E_n$ defined as follows:*

$$\mathcal{B} = \{U_1 \times \cdots \times U_n \mid U_i \in \mathcal{O}_i,\ 1 \leq i \leq n\},$$

and let $\mathcal{P}$ be the family consisting of arbitrary unions of sets in $\mathcal{B}$, including $\emptyset$. Then, $\mathcal{P}$ is a topology on $E_1 \times \cdots \times E_n$.

Proof. Left as an exercise. □

Definition A.8 Given n topological spaces, $(E_i, \mathcal{O}_i)$, the *product topology on $E_1 \times \cdots \times E_n$* is the family, $\mathcal{P}$, of subsets of $E_1 \times \cdots \times E_n$ defined as follows: if

$$\mathcal{B} = \{U_1 \times \cdots \times U_n \mid U_i \in \mathcal{O}_i,\ 1 \leq i \leq n\},$$

then $\mathcal{P}$ is the family consisting of arbitrary unions of sets in $\mathcal{B}$, including $\emptyset$.

If each $(E_i, \| \, \|_i)$ is a normed vector space, there are three natural norms that can be defined on $E_1 \times \cdots \times E_n$:

$$\begin{aligned}
\|(x_1, \ldots, x_n)\|_1 &= \|x_1\|_1 + \cdots + \|x_n\|_n, \\
\|(x_1, \ldots, x_n)\|_2 &= \Big(\|x_1\|_1^2 + \cdots + \|x_n\|_n^2\Big)^{\frac{1}{2}}, \\
\|(x_1, \ldots, x_n)\|_\infty &= \max\{\|x_1\|_1, \ldots, \|x_n\|_n\}.
\end{aligned}$$

It is easy to show that they all define the same topology, which is the product topology. One can also verify that when $E_i = \mathbb{R}$, with the standard topology induced by $|x - y|$, the topology product on $\mathbb{R}^n$ is the standard topology induced by the Euclidean norm.

Definition A.9 Two metrics, d_1 and d_2, on a space, E, are *equivalent* if they induce the same topology, $\mathcal{O}$, on E (i.e., they define the same family, $\mathcal{O}$, of open sets). Similarly, two norms, $\| \, \|_1$ and $\| \, \|_2$, on a space, E, are *equivalent* if they induce the same topology, $\mathcal{O}$, on E.

Remark: Given a topological space, $(E, \mathcal{O})$, it is often useful, as in Proposition A.6, to define the topology $\mathcal{O}$ in terms of a subfamily, $\mathcal{B}$, of subsets of E. We say that a family, $\mathcal{B}$, of subsets of E is a *basis for the topology* $\mathcal{O}$ if $\mathcal{B}$ is a subset of $\mathcal{O}$ and if every open set, U, in $\mathcal{O}$ can be obtained as some union (possibly infinite) of sets in $\mathcal{B}$ (agreeing that the empty union is the empty set). It is immediately verified that if a family, $\mathcal{B} = (U_i)_{i\in I}$, is a basis for the topology of $(E, \mathcal{O})$, then $E = \bigcup_{i\in I} U_i$ and the intersection of any two sets, $U_i, U_j \in \mathcal{B}$, is the union of some sets in the family $\mathcal{B}$ (again, agreeing that the empty union is the empty set). Conversely, a family, $\mathcal{B}$, with these properties is the basis of the topology obtained by forming arbitrary unions of sets in $\mathcal{B}$.

A *subbasis for* $\mathcal{O}$ is a family, $\mathcal{S}$, of subsets of E such that the family, $\mathcal{B}$, of all finite intersections of sets in $\mathcal{S}$ (including E itself, in case of the empty intersection) is a basis of $\mathcal{O}$.

We now consider the fundamental property of continuity.

Definition A.10 Let $(E, \mathcal{O}_E)$ and $(F, \mathcal{O}_F)$ be topological spaces and let $f \colon E \to F$ be a function. For every $a \in E$, we say that f *is continuous at* a if for every open set, $V \in \mathcal{O}_F$, containing $f(a)$, there is some open set, $U \in \mathcal{O}_E$, containing a, such that $f(U) \subseteq V$. We say that f *is continuous* if it is continuous at every $a \in E$.

Define a *neighborhood of* $a \in E$ as any subset, N of E, containing some open set, $O \in \mathcal{O}$, such that $a \in O$. Now, if f is continuous at a and N is any neighborhood of $f(a)$, then there is some open set, $V \subseteq N$, containing $f(a)$ and since f is continuous at a, there is some open set, U, containing a, such that $f(U) \subseteq V$. Since $V \subseteq N$, the open set, U, is a subset of $f^{-1}(N)$ containing a and $f^{-1}(N)$ is a neighborhood of a. Conversely, if $f^{-1}(N)$ is a neighborhood of a whenever N is any neighborhood of $f(a)$, it is immediate that f is continuous at a. Thus, we can restate Definition A.10 as follows:

The function, f, is continuous at $a \in E$ iff for every neighborhood, N, of $f(a) \in F$, then $f^{-1}(N)$ is a neighborhood of a.

It is also easy to check that f is continuous on E iff $f^{-1}(V)$ is an open set in $\mathcal{O}_E$ for every open set, $V \in \mathcal{O}_F$.

If E and F are metric spaces defined by metrics d_1 and d_2, we can show easily that f is continuous at a iff

for every $\epsilon > 0$, there is some $\eta > 0$, such that, for every $x \in E$,

$$\text{if } d_1(a, x) \leq \eta, \text{ then } d_2(f(a), f(x)) \leq \epsilon.$$

Similarly, if E and F are normed vector spaces defined by norms $\| \|_1$ and $\| \|_2$, we can show easily that f is continuous at a iff

for every $\epsilon > 0$, there is some $\eta > 0$, such that, for every $x \in E$,

$$\text{if } \|x - a\|_1 \leq \eta, \text{ then } \|f(x) - f(a)\|_2 \leq \epsilon.$$

It is worth noting that continuity is a topological notion, in the sense that equivalent metrics (or equivalent norms) define exactly the same notion of continuity.

If $(E, \mathcal{O}_E)$ and $(F, \mathcal{O}_F)$ are topological spaces and $f\colon E \to F$ is a function, for every nonempty subset, $A \subseteq E$, of E, we say that f *is continuous on* A if the restriction of f to A is continuous with respect to $(A, \mathcal{U})$ and $(F, \mathcal{O}_F)$, where $\mathcal{U}$ is the subspace topology induced by $\mathcal{O}_E$ on A.

Given a product, $E_1 \times \cdots \times E_n$, of topological spaces, as usual, we let $\pi_i\colon E_1 \times \cdots \times E_n \to E_i$ be the projection function such that, $\pi_i(x_1, \ldots, x_n) = x_i$. It is immediately verified that each π_i is continuous.

Given a topological space, $(E, \mathcal{O})$, we say that a point, $a \in E$, is *isolated* if $\{a\}$ is an open set in $\mathcal{O}$. Then, if $(E, \mathcal{O}_E)$ and $(F, \mathcal{O}_F)$ are topological spaces, any function, $f\colon E \to F$, is continuous at every isolated point, $a \in E$. In the discrete topology, every point is isolated. In a nontrivial normed vector space, $(E, \| \|)$, (with $E \neq \{0\}$), no point is isolated. To show this, we show that every open ball, $B_0(u, \rho,)$, contains some vectors different from u. Indeed, since E is nontrivial, there is some $v \in E$ such that $v \neq 0$, and thus $\lambda = \|v\| > 0$ (by (N1)). Let

$$w = u + \frac{\rho}{\lambda + 1} v.$$

Since $v \neq 0$ and $\rho > 0$, we have $w \neq u$. Then,

$$\|w - u\| = \left\| \frac{\rho}{\lambda + 1} v \right\| = \frac{\rho\lambda}{\lambda + 1} < \rho,$$

which shows that $\|w - u\| < \rho$, for $w \neq u$.

The following proposition is easily shown:

Proposition A.7 *Given topological spaces, $(E, \mathcal{O}_E)$, $(F, \mathcal{O}_F)$, and $(G, \mathcal{O}_G)$, and two functions, $f\colon E \to F$ and $g\colon F \to G$, if f is continuous at $a \in E$ and g is continuous at $f(a) \in F$, then $g \circ f\colon E \to G$ is continuous at $a \in E$. Given n topological spaces, $(F_i, \mathcal{O}_i)$, for every function, $f\colon E \to F_1 \times \cdots \times F_n$, then f is continuous at $a \in E$ iff every $f_i\colon E \to F_i$ is continuous at a, where $f_i = \pi_i \circ f$.*

One can also show that in a metric space, (E, d), the norm $d\colon E \times E \to \mathbb{R}$ is continuous, where $E \times E$ has the product topology and that for a normed vector space, $(E, \| \|)$, the norm $\| \|\colon E \to \mathbb{R}$ is continuous.

Given a function, $f\colon E_1 \times \cdots \times E_n \to F$, we can fix $n-1$ of the arguments, say $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$, and view f as a function of the remaining argument,

$$x_i \mapsto f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n),$$

where $x_i \in E_i$. If f is continuous, it is clear that each f_i is continuous.

One should be careful that the converse is false! For example, consider the function $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined such that,

$$f(x,y) = \frac{xy}{x^2+y^2} \quad \text{if } (x,y) \neq (0,0), \quad \text{and} \quad f(0,0) = 0.$$

The function f is continuous on $\mathbb{R} \times \mathbb{R} - \{(0,0)\}$, but on the line $y = mx$, with $m \neq 0$, we have $f(x,y) = \frac{m}{1+m^2} \neq 0$, and thus, on this line, $f(x,y)$ does not approach 0 when (x,y) approaches $(0,0)$.

The following proposition is useful for showing that real-valued functions are continuous.

Proposition A.8 *If E is a topological space and $(\mathbb{R}, |x-y|)$ denotes the reals under the standard topology, for any two functions, $f\colon E \to \mathbb{R}$ and $g\colon E \to \mathbb{R}$, for any $a \in E$, for any $\lambda \in \mathbb{R}$, if f and g are continuous at a, then $f+g$, λf, $f \cdot g$, are continuous at a, and f/g is continuous at a if $g(a) \neq 0$.*

Proof. Left as an exercise.

Using Proposition A.8, we can show easily that every real polynomial function is continuous.

The notion of isomorphism of topological spaces is defined as follows:

Definition A.11 Let $(E, \mathcal{O}_E)$ and $(F, \mathcal{O}_F)$ be topological spaces and let $f\colon E \to F$ be a function. We say that f *is a homeomorphism between E and F* if f is bijective and both $f\colon E \to F$ and $f^{-1}\colon F \to E$ are continuous.

One should be careful that a bijective continuous function $f\colon E \to F$ is not necessarily an homeomorphism. For example, if $E = \mathbb{R}$ with the discrete topology and $F = \mathbb{R}$ with the standard topology, the identity is not a homeomorphism. Another interesting example involving a parametric curve is given below. Let $L\colon \mathbb{R} \to \mathbb{R}^2$ be the function, defined such that,

$$L_1(t) = \frac{t(1+t^2)}{1+t^4},$$
$$L_2(t) = \frac{t(1-t^2)}{1+t^4}.$$

If we think of $(x(t), y(t)) = (L_1(t), L_2(t))$ as a geometric point in $\mathbb{R}^2$, the set of points $(x(t), y(t))$ obtained by letting t vary in $\mathbb{R}$ from $-\infty$ to $+\infty$ defines a curve having the shape of a “figure eight”, with self-intersection at the origin, called the “lemniscate of Bernoulli”. The map L is continuous and, in fact bijective, but its inverse, L^{-1}, is not continuous. Indeed, when we approach the origin on the branch of the curve in the upper left quadrant (i.e., points such that, $x \leq 0$, $y \geq 0$), then t goes to $-\infty$, and when we approach the origin on the branch of the curve in the lower right quadrant (i.e., points such that, $x \geq 0$, $y \leq 0$), then t goes to $+\infty$.

We also review the concept of limit of a sequence. Given any set, E, a *sequence* is any function, $x\colon \mathbb{N} \to E$, usually denoted by $(x_n)_{n\in\mathbb{N}}$, or $(x_n)_{n\geq 0}$, or even as (x_n).

Definition A.12 Given a topological space, $(E, \mathcal{O})$, we say that *a sequence, $(x_n)_{n\in\mathbb{N}}$, converges to some $a \in E$* if for every open set, U, containing a, there is some $n_0 \geq 0$, such that, $x_n \in U$, for all $n \geq n_0$. We also say that *a is a limit of $(x_n)_{n\in\mathbb{N}}$*.

When E is a metric space with metric d, it is easy to show that this is equivalent to the fact that,

for every $\epsilon > 0$, there is some $n_0 \geq 0$, such that, $d(x_n, a) \leq \epsilon$, for all $n \geq n_0$.

When E is a normed vector space with norm $\| \, \|$, it is easy to show that this is equivalent to the fact that,

for every $\epsilon > 0$, there is some $n_0 \geq 0$, such that, $\|x_n - a\| \leq \epsilon$, for all $n \geq n_0$.

The following proposition shows the importance of the Hausdorff separation axiom.

Proposition A.9 *Given a topological space, $(E, \mathcal{O})$, if the Hausdorff separation axiom holds, then every sequence has at most one limit.*

Proof. Left as an exercise.

It is worth noting that the notion of limit is topological, in the sense that a sequence converge to a limit, b, iff it converges to the same limit b in any equivalent metric (and similarly for equivalent norms).

We still need one more concept of limit for functions.

Definition A.13 Let $(E, \mathcal{O}_E)$ and $(F, \mathcal{O}_F)$ be topological spaces, let A be some nonempty subset of E, and let $f\colon A \to F$ be a function. For any $a \in \overline{A}$ and any $b \in F$, we say that *$f(x)$ approaches b as x approaches a with values in A* if for every open set, $V \in \mathcal{O}_F$, containing b, there is some open set, $U \in \mathcal{O}_E$, containing a, such that, $f(U \cap A) \subseteq V$. This is denoted by

$$\lim_{x\to a, x\in A} f(x) = b.$$

First, note that by Proposition A.3, since $a \in \overline{A}$, for every open set, U, containing a, we have $U \cap A \neq \emptyset$, and the definition is nontrivial. Also, even if $a \in A$, the value, $f(a)$, of f at a plays no role in this definition. When E and F are metric space with metrics d_1 and d_2, it can be shown easily that the definition can be stated as follows:

for every $\epsilon > 0$, there is some $\eta > 0$, such that, for every $x \in A$,

$$\text{if } d_1(x, a) \leq \eta, \text{ then } d_2(f(x), b) \leq \epsilon.$$

When E and F are normed vector spaces with norms $\| \, \|_1$ and $\| \, \|_2$, it can be shown easily that the definition can be stated as follows:

for every $\epsilon > 0$, there is some $\eta > 0$, such that, for every $x \in A$,

$$\text{if } \|x - a\|_1 \leq \eta, \text{ then } \|f(x) - b\|_2 \leq \epsilon.$$

We have the following result relating continuity at a point and the previous notion:

Proposition A.10 *Let $(E, \mathcal{O}_E)$ and $(F, \mathcal{O}_F)$ be two topological spaces and let $f\colon E \to F$ be a function. For any $a \in E$, the function f is continuous at a iff $f(x)$ approaches $f(a)$ when x approaches a (with values in E).*

Proof. Left as a trivial exercise.

Another important proposition relating the notion of convergence of a sequence to continuity is stated without proof.

Proposition A.11 *Let $(E, \mathcal{O}_E)$ and $(F, \mathcal{O}_F)$ be two topological spaces and let $f\colon E \to F$ be a function.*

(1) If f is continuous, then for every sequence, $(x_n)_{n \in \mathbb{N}}$, in E, if (x_n) converges to a, then $(f(x_n))$ converges to $f(a)$.

(2) If E is a metric space and $(f(x_n))$ converges to $f(a)$ whenever (x_n) converges to a, for every sequence, $(x_n)_{n \in \mathbb{N}}$, in E, then f is continuous.

Remark: A special case of Definition A.13 shows up in the following case: $E = \mathbb{R}$, and F is some arbitrary topological space. Let A be some nonempty subset of $\mathbb{R}$, and let $f\colon A \to F$ be some function. For any $a \in A$, we say that f *is continuous on the right at a* if

$$\lim_{x \to a, x \in A \cap [a, +\infty[} f(x) = f(a).$$

We can define continuity on the left at a in a similar fashion.

We now turn to connectivity properties of topological spaces.

A.3 Connected Sets

Connectivity properties of topological spaces play a very important role in understanding the topology of surfaces. This section gathers the facts needed to have a good understanding of the classification theorem for compact (bordered) surfaces. The main references are Ahlfors and Sario [1] and Massey [32, 33]. For general backgroud on topology, geometry, and algebraic topology, we also highly recommend Bredon [7] and Fulton [18].

Definition A.14 A topological space, $(E, \mathcal{O})$, is *connected* if the only subsets of E that are both open and closed are the empty set and E itself. Equivalently, $(E, \mathcal{O})$ is connected if E cannot be written as the union, $E = U \cup V$, of two disjoint nonempty open sets, U, V, if E cannot be written as the union, $E = U \cup V$, of two disjoint nonempty closed sets. A subset, $S \subseteq E$, is *connected* if it is connected in the subspace topology on S induced by $(E, \mathcal{O})$. A connected open set is called a *region* and a closed set is a *closed region* if its interior is a connected (open) set.

Intuitively, if a space is not connected, it is possible to define a continuous function which is constant on disjoint "connected components" and which takes possibly distinct values on disjoint components. This can be stated in terms of the concept of a locally constant function. Given two topological spaces, X, Y, a function, $f\colon X \to Y$, is *locally constant* if for every $x \in X$, there is an open set, $U \subseteq X$, such that $x \in X$ and f is constant on U.

We claim that a locally constant function is continuous. In fact, we will prove that $f^{-1}(V)$ is open for every subset, $V \subseteq Y$ (not just for an open set V). It is enough to show that $f^{-1}(y)$ is open for every $y \in Y$, since for every subset $V \subseteq Y$,

$$f^{-1}(V) = \bigcup_{y \in V} f^{-1}(y),$$

and open sets are closed under arbitrary unions. However, either $f^{-1}(y) = \emptyset$ if $y \in Y - f(X)$ or f is constant on $U = f^{-1}(y)$ if $y \in f(X)$ (with value y), and since f is locally constant, for every $x \in U$, there is some open set, $W \subseteq X$, such that $x \in W$ and f is constant on W, which implies that $f(w) = y$ for all $w \in W$ and thus, that $W \subseteq U$, showing that U is a union of open sets and thus, is open. The following proposition shows that a space is connected iff every locally constant function is constant:

Proposition A.12 *A topological space is connected iff every locally constant function is constant.*

Proof. First, assume that X is connected. Let $f\colon X \to Y$ be a locally constant function to some space Y and assume that f is not constant. Pick any $y \in f(Y)$. Since f is not constant, $U_1 = f^{-1}(y) \neq X$, and of course, $U_1 \neq \emptyset$. We proved just before Proposition A.12 that $f^{-1}(V)$ is open for every subset $V \subseteq Y$, and thus $U_1 = f^{-1}(y) = f^{-1}(\{y\})$ and

$U_2 = f^{-1}(Y - \{y\})$ are both open, nonempty, and clearly $X = U_1 \cup U_2$ and U_1 and U_2 are disjoint. This contradicts the fact that X is connected and f must be constant.

Assume that every locally constant function, $f \colon X \to Y$, to a Hausdorff space, Y, is constant. If X is not connected, we can write $X = U_1 \cup U_2$, where both U_1, U_2 are open, disjoint, and nonempty. We can define the function, $f \colon X \to \mathbb{R}$, such that $f(x) = 1$ on U_1 and $f(x) = 0$ on U_2. Since U_1 and U_2 are open, the function f is locally constant, and yet not constant, a contradiction. □

The following standard proposition characterizing the connected subsets of $\mathbb{R}$ can be found in most topology texts (for example, Munkres [37], Schwartz [43]). For the sake of completeness, we give a proof.

Proposition A.13 *A subset of the real line, $\mathbb{R}$, is connected iff it is an interval, i.e., of the form $[a, b]$, $]\,a, b]$, where $a = -\infty$ is possible, $[a, b[$, where $b = +\infty$ is possible, or $]a, b[$, where $a = -\infty$ or $b = +\infty$ is possible.*

Proof. Assume that A is a connected nonempty subset of $\mathbb{R}$. The cases where $A = \emptyset$ or A consists of a single point are trivial. We show that whenever $a, b \in A$, $a < b$, then the entire interval $[a, b]$ is a subset of A. Indeed, if this was not the case, there would be some $c \in]a, b[$ such that $c \notin A$, and then we could write $A = (\,]-\infty, c[\, \cap A) \cup (\,]c + \infty[\, \cap A)$, where $]-\infty, c[\, \cap A$ and $]c + \infty[\, \cap A$ are nonempty and disjoint open subsets of A, contradicting the fact that A is connected. It follows easily that A must be an interval.

Conversely, we show that an interval, I, must be connected. Let A be any nonempty subset of I which is both open and closed in I. We show that $I = A$. Fix any $x \in A$ and consider the set, R_x, of all y such that $[x, y] \subseteq A$. If the set R_x is unbounded, then $R_x = [x, +\infty[$. Otherwise, if this set is bounded, let b be its least upper bound. We claim that b is the right boundary of the interval I. Because A is closed in I, unless I is open on the right and b is its right boundary, we must have $b \in A$. In the first case, $A \cap [x, b[\, = I \cap [x, b[\, = [x, b[$. In the second case, because A is also open in I, unless b is the right boundary of the interval I (closed on the right), there is some open set $]b - \eta, b + \eta[$ contained in A, which implies that $[x, b + \eta/2] \subseteq A$, contradicting the fact that b is the least upper bound of the set R_x. Thus, b must be the right boundary of the interval I (closed on the right). A similar argument applies to the set, L_y, of all x such that $[x, y] \subseteq A$ and either L_y is unbounded, or its greatest lower bound a is the left boundary of I (open or closed on the left). In all cases, we showed that $A = I$, and the interval must be connected. □

A characterization on the connected subsets of $\mathbb{R}^n$ is harder and requires the notion of arcwise connectedness. One of the most important properties of connected sets is that they are preserved by continuous maps.

Proposition A.14 *Given any continuous map, $f \colon E \to F$, if $A \subseteq E$ is connected, then $f(A)$ is connected.*

Proof. If $f(A)$ is not connected, then there exist some nonempty open sets, U, V, in F such that $f(A) \cap U$ and $f(A) \cap V$ are nonempty and disjoint, and

$$f(A) = (f(A) \cap U) \cup (f(A) \cap V).$$

Then, $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty and open since f is continuous and

$$A = (A \cap f^{-1}(U)) \cup (A \cap f^{-1}(V)),$$

with $A \cap f^{-1}(U)$ and $A \cap f^{-1}(V)$ nonempty, disjoint, and open in A, contradicting the fact that A is connected. $\square$

An important corollary of Proposition A.14 is that for every continuous function, $f \colon E \to \mathbb{R}$, where E is a connected space, then $f(E)$ is an interval. Indeed, this follows from Proposition A.13. Thus, if f takes the values a and b where $a < b$, then f takes all values $c \in [a, b]$. This is a very important property.

Even if a topological space is not connected, it turns out that it is the disjoint union of maximal connected subsets and these connected components are closed in E. In order to obtain this result, we need a few lemmas.

Lemma A.15 *Given a topological space, E, for any family, $(A_i)_{i \in I}$, of (nonempty) connected subsets of E, if $A_i \cap A_j \neq \emptyset$ for all $i, j \in I$, then the union, $A = \bigcup_{i \in I} A_i$, of the family, $(A_i)_{i \in I}$, is also connected.*

Proof. Assume that $\bigcup_{i \in I} A_i$ is not connected. Then, there exists two nonempty open subsets, U and V, of E such that $A \cap U$ and $A \cap V$ are disjoint and nonempty and such that

$$A = (A \cap U) \cup (A \cap V).$$

Now, for every $i \in I$, we can write

$$A_i = (A_i \cap U) \cup (A_i \cap V),$$

where $A_i \cap U$ and $A_i \cap V$ are disjoint, since $A_i \subseteq A$ and $A \cap U$ and $A \cap V$ are disjoint. Since A_i is connected, either $A_i \cap U = \emptyset$ or $A_i \cap V = \emptyset$. This implies that either $A_i \subseteq A \cap U$ or $A_i \subseteq A \cap V$. However, by assumption, $A_i \cap A_j \neq \emptyset$, for all $i, j \in I$, and thus, either both $A_i \subseteq A \cap U$ and $A_j \subseteq A \cap U$, or both $A_i \subseteq A \cap V$ and $A_j \subseteq A \cap V$, since $A \cap U$ and $A \cap V$ are disjoint. Thus, we conclude that either $A_i \subseteq A \cap U$ for all $i \in I$, or $A_i \subseteq A \cap V$ for all $i \in I$. But this proves that either

$$A = \bigcup_{i \in I} A_i \subseteq A \cap U,$$

or

$$A = \bigcup_{i \in I} A_i \subseteq A \cap V,$$

contradicting the fact that both $A \cap U$ and $A \cap V$ are disjoint and nonempty. Thus, A must be connected. $\square$

In particular, the above lemma applies when the connected sets in a family $(A_i)_{i \in I}$ have a point in common.

Lemma A.16 *If A is a connected subset of a topological space, E, then for every subset, B, such that $A \subseteq B \subseteq \overline{A}$, where $\overline{A}$ is the closure of A in E, the set B is connected.*

Proof. If B is not connected, then there are two nonempty open subsets, U, V, of E such that $B \cap U$ and $B \cap V$ are disjoint and nonempty, and

$$B = (B \cap U) \cup (B \cap V).$$

Since $A \subseteq B$, the above implies that

$$A = (A \cap U) \cup (A \cap V),$$

and since A is connected, either $A \cap U = \emptyset$, or $A \cap V = \emptyset$. Without loss of generality, assume that $A \cap V = \emptyset$, which implies that $A \subseteq A \cap U \subseteq B \cap U$. However, $B \cap U$ is closed in the subspace topology for B and since $B \subseteq \overline{A}$ and $\overline{A}$ is closed in E, the closure of A in B w.r.t. the subspace topology of B is clearly $B \cap \overline{A} = B$, which implies that $B \subseteq B \cap U$ (since the closure is the smallest closed set containing the given set). Thus, $B \cap V = \emptyset$, a contradiction. $\square$

In particular, Lemma A.16 shows that if A is a connected subset, then its closure, $\overline{A}$, is also connected. We are now ready to introduce the connected components of a space.

Definition A.15 Given a topological space, $(E, \mathcal{O})$, we say that two points, $a, b \in E$, are *connected* if there is some connected subset, A, of E such that $a \in A$ and $b \in A$.

It is immediately verified that the relation "a and b are connected in E" is an equivalence relation. Only transitivity is not obvious, but it follows immediately as a special case of Lemma A.15. Thus, the above equivalence relation defines a partition of E into nonempty disjoint *connected components*. The following proposition is easily proved using Lemma A.15 and Lemma A.16:

Proposition A.17 *Given any topological space, E, for any $a \in E$, the connected component containing a is the largest connected set containing a. The connected components of E are closed.*

The notion of a locally connected space is also useful.

Definition A.16 A topological space, $(E, \mathcal{O})$, is *locally connected* if for every $a \in E$, for every neighborhood, V, of a, there is a connected neighborhood, U, of a such that $U \subseteq V$.

As we shall see in a moment, it would be equivalent to require that E has a basis of connected open sets.

There are connected spaces that are not locally connected and there are locally connected spaces that are not connected. The two properties are independent.

Proposition A.18 *A topological space, E, is locally connected iff for every open subset, A, of E, the connected components of A are open.*

Proof. Assume that E is locally connected. Let A be any open subset of E and let C be one of the connected components of A. For any $a \in C \subseteq A$, there is some connected neigborhood, U, of a such that $U \subseteq A$ and since C is a connected component of A containing a, we must have $U \subseteq C$. This shows that for every $a \in C$, there is some open subset containing a contained in C, so C is open.

Conversely, assume that for every open subset, A, of E, the connected components of A are open. Then, for every $a \in E$ and every neighborhood, U, of a, since U contains some open set A containing a, the interior, $\mathring{U}$, of U is an open set containing a and its connected components are open. In particular, the connected component C containing a is a connected open set containing a and contained in U. $\square$

Proposition A.18 shows that in a locally connected space, the connected open sets form a basis for the topology. It is easily seen that $\mathbb{R}^n$ is locally connected. Another very important property of surfaces and more generally, manifolds, is to be arcwise connected. The intuition is that any two points can be joined by a continuous arc of curve. This is formalized as follows.

Definition A.17 Given a topological space, $(E, \mathcal{O})$, an *arc (or path)* is a continuous map, $\gamma\colon [a, b] \to E$, where $[a, b]$ is a closed interval of the real line, $\mathbb{R}$. The point $\gamma(a)$ is the *initial point* of the arc and the point $\gamma(b)$ is the *terminal point* of the arc. We say that γ *is an arc joining* $\gamma(a)$ *and* $\gamma(b)$. An arc is a *closed curve* if $\gamma(a) = \gamma(b)$. The set $\gamma([a, b])$ is the *trace* of the arc γ.

Typically, $a = 0$ and $b = 1$. In the sequel, this will be assumed.

One should not confuse an arc, $\gamma\colon [a, b] \to E$, with its trace. For example, γ could be constant, and thus, its trace reduced to a single point.

An arc is a *Jordan arc* if γ is a homeomorphism onto its trace. An arc, $\gamma\colon [a, b] \to E$, is a *Jordan curve* if $\gamma(a) = \gamma(b)$ and γ is injective on $[a, b[$. Since $[a, b]$ is connected, by Proposition A.14, the trace $\gamma([a, b])$ of an arc is a connected subset of E.

Given two arcs $\gamma\colon [0, 1] \to E$ and $\delta\colon [0, 1] \to E$ such that $\gamma(1) = \delta(0)$, we can form a new arc defined as follows:

Definition A.18 Given two arcs, $\gamma\colon [0, 1] \to E$ and $\delta\colon [0, 1] \to E$, such that $\gamma(1) = \delta(0)$, we can form their *composition (or product)*, $\gamma\delta$,, defined such that

$$\gamma\delta(t) = \begin{cases} \gamma(2t) & \text{if } 0 \leq t \leq 1/2; \\ \delta(2t - 1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

The *inverse*, γ^{-1}, *of the arc*, γ, is the arc defined such that $\gamma^{-1}(t) = \gamma(1-t)$, for all $t \in [0,1]$.

It is trivially verified that Definition A.18 yields continuous arcs.

Definition A.19 A topological space, E, is *arcwise connected* if for any two points, $a, b \in E$, there is an arc, $\gamma\colon [0,1] \to E$, joining a and b, i.e., such that $\gamma(0) = a$ and $\gamma(1) = b$. A topological space, E, is *locally arcwise connected* if for every $a \in E$, for every neighborhood, V, of a, there is an arcwise connected neighborhood, U, of a such that $U \subseteq V$.

The space $\mathbb{R}^n$ is locally arcwise connected, since for any open ball, any two points in this ball are joined by a line segment. Manifolds and surfaces are also locally arcwise connected. It is easy to verify that Proposition A.14 also applies to arcwise connectedness. The following theorem is crucial to the theory of manifolds and surfaces:

Theorem A.19 *If a topological space, E, is arcwise connected, then it is connected. If a topological space, E, is connected and locally arcwise connected, then E is arcwise connected.*

Proof. First, assume that E is arcwise connected. Pick any point, a, in E. Since E is arcwise connected, for every $b \in E$, there is a path, $\gamma_b\colon [0,1] \to E$, from a to b and so,

$$E = \bigcup_{b \in E} \gamma_b([0,1])$$

a union of connected subsets all containing a. By Lemma A.15, E is connected.

Now assume that E is connected and locally arcwise connected. For any point $a \in E$, let F_a be the set of all points, b, such that there is an arc, $\gamma_b\colon [0,1] \to E$, from a to b. Clearly, F_a contains a. We show that F_a is both open and closed. For any $b \in F_a$, since E is locally arcwise connected, there is an arcwise connected neighborhood U containing b (because E is a neighborhood of b). Thus, b can be joined to every point $c \in U$ by an arc, and since by the definition of F_a, there is an arc from a to b, the composition of these two arcs yields an arc from a to c, which shows that $c \in F_a$. But then $U \subseteq F_a$ and thus, F_a is open. Now assume that b is in the complement of F_a. As in the previous case, there is some arcwise connected neighborhood U containing b. Thus, every point $c \in U$ can be joined to b by an arc. If there was an arc joining a to c, we would get an arc from a to b, contradicting the fact that b is in the complement of F_a. Thus, every point $c \in U$ is in the complement of F_a, which shows that U is contained in the complement of F_a, and thus, that the the complement of F_a is open. Consequently, we have shown that F_a is both open and closed and since it is nonempty, we must have $E = F_a$, which shows that E is arcwise connected. $\square$

If E is locally arcwise connected, the above argument shows that the connected components of E are arcwise connected.

It is not true that a connected space is arcwise connected. For example, the space consisting of the graph of the function

$$f(x) = \sin(1/x),$$

where $x > 0$, together with the portion of the y-axis, for which $-1 \leq y \leq 1$, is connected, but not arcwise connected.

A trivial modification of the proof of Theorem A.19 shows that in a normed vector space, E, a connected open set is arcwise connected by polygonal lines (i.e., arcs consisting of line segments). This is because in every open ball, any two points are connected by a line segment. Furthermore, if E is finite dimensional, these polygonal lines can be forced to be parallel to basis vectors.

We now consider compactness.

A.4 Compact Sets

The property of compactness is very important in topology and analysis. We provide a quick review geared towards the study of surfaces and for details, we refer the reader to Munkres [37], Schwartz [43]. In this section, we will need to assume that the topological spaces are Hausdorff spaces. This is not a luxury, as many of the results are false otherwise.

There are various equivalent ways of defining compactness. For our purposes, the most convenient way involves the notion of open cover.

Definition A.20 Given a topological space, E, for any subset, A, of E, an *open cover,* $(U_i)_{i\in I}$*, of* A is a family of open subsets of E such that $A \subseteq \bigcup_{i\in I} U_i$. An *open subcover* of an open cover, $(U_i)_{i\in I}$, of A is any subfamily, $(U_j)_{j\in J}$, which is an open cover of A, with $J \subseteq I$. An open cover, $(U_i)_{i\in I}$, of A is *finite* if I is finite. The topological space, E, is *compact* if it is Hausdorff and for every open cover, $(U_i)_{i\in I}$, of E, there is a finite open subcover, $(U_j)_{j\in J}$, of E. Given any subset, A, of E, we say that A is *compact* if it is compact with respect to the subspace topology. We say that A is *relatively compact* if its closure $\overline{A}$ is compact.

It is immediately verified that a subset, A, of E is compact in the subspace topology relative to A iff for every open cover, $(U_i)_{i\in I}$, of A by open subsets of E, there is a finite open subcover, $(U_j)_{j\in J}$, of A. The property that every open cover contains a finite open subcover is often called the *Heine-Borel-Lebesgue* property. By considering complements, a Hausdorff space is compact iff for every family, $(F_i)_{i\in I}$, of closed sets, if $\bigcap_{i\in I} F_i = \emptyset$, then $\bigcap_{j\in J} F_j = \emptyset$ for some finite subset, J, of I.

Definition A.20 requires that a compact space be Hausdorff. There are books in which a compact space is not necessarily required to be Hausdorff. Following Schwartz, we prefer calling such a space *quasi-compact*.

Another equivalent and useful characterization can be given in terms of families having the finite intersection property. A family, $(F_i)_{i\in I}$, of sets has the *finite intersection property* if $\bigcap_{j\in J} F_j \neq \emptyset$ for every finite subset, J, of I. We have the following proposition:

Proposition A.20 *A topological Hausdorff space, E, is compact iff for every family, $(F_i)_{i\in I}$, of closed sets having the finite intersection property, then $\bigcap_{i\in I} F_i \neq \emptyset$.*

Proof. If E is compact and $(F_i)_{i\in I}$ is a family of closed sets having the finite intersection property, then $\bigcap_{i\in I} F_i$ cannot be empty, since otherwise we would have $\bigcap_{j\in J} F_j = \emptyset$ for some finite subset, J, of I, a contradiction. The converse is equally obvious. $\square$

Another useful consequence of compactness is as follows. For any family, $(F_i)_{i\in I}$, of closed sets such that $F_{i+1} \subseteq F_i$ for all $i \in I$, if $\bigcap_{i\in I} F_i = \emptyset$, then $F_i = \emptyset$ for some $i \in I$. Indeed, there must be some finite subset, J, of I such that $\bigcap_{j\in J} F_j = \emptyset$ and since $F_{i+1} \subseteq F_i$ for all $i \in I$, we must have $F_j = \emptyset$ for the smallest F_j in $(F_j)_{j\in J}$. Using this fact, we note that $\mathbb{R}$ is *not* compact. Indeed, the family of closed sets, $([n, +\infty[\,)_{n\geq 0}$, is decreasing and has an empty intersection.

Given a metric space, if we define a *bounded subset* to be a subset that can be enclosed in some closed ball (of finite radius), then any nonbounded subset of a metric space is not compact. However, a closed interval $[a, b]$ of the real line is compact.

Proposition A.21 *Every closed interval, $[a, b]$, of the real line is compact.*

Proof. We proceed by contradiction. Let $(U_i)_{i\in I}$ be any open cover of $[a, b]$ and assume that there is no finite open subcover. Let $c = (a + b)/2$. If both $[a, c]$ and $[c, b]$ had some finite open subcover, so would $[a, b]$, and thus, either $[a, c]$ does not have any finite subcover, or $[c, b]$ does not have any finite open subcover. Let $[a_1, b_1]$ be such a bad subinterval. The same argument applies and we split $[a_1, b_1]$ into two equal subintervals, one of which must be bad. Thus, having defined $[a_n, b_n]$ of length $(b - a)/2^n$ as an interval having no finite open subcover, splitting $[a_n, b_n]$ into two equal intervals, we know that at least one of the two has no finite open subcover and we denote such a bad interval by $[a_{n+1}, b_{n+1}]$. The sequence (a_n) is nondecreasing and bounded from above by b, and thus, by a fundamental property of the real line, it converges to its least upper bound, α. Similarly, the sequence (b_n) is nonincreasing and bounded from below by a and thus, it converges to its greatest lowest bound, β. Since $[a_n, b_n]$ has length $(b - a)/2^n$, we must have $\alpha = \beta$. However, the common limit $\alpha = \beta$ of the sequences (a_n) and (b_n) must belong to some open set, U_i, of the open cover and since U_i is open, it must contain some interval $[c, d]$ containing α. Then, because α is the common limit of the sequences (a_n) and (b_n), there is some N such that the intervals $[a_n, b_n]$ are all contained in the interval $[c, d]$ for all $n \geq N$, which contradicts the fact that none of the intervals $[a_n, b_n]$ has a finite open subcover. Thus, $[a, b]$ is indeed compact. $\square$

It is easy to adapt the argument of Proposition A.21 to show that in $\mathbb{R}^m$, every closed set, $[a_1, b_1] \times \cdots \times [a_m, b_m]$, is compact. At every stage, we need to divide into 2^m subpieces instead of 2.

The following two propositions give very important properties of the compact sets, and they only hold for Hausdorff spaces:

Proposition A.22 *Given a topological Hausdorff space, E, for every compact subset, A, and every point, b, not in A, there exist disjoint open sets, U and V, such that $A \subseteq U$ and $b \in V$. As a consequence, every compact subset is closed.*

Proof. Since E is Hausdorff, for every $a \in A$, there are some disjoint open sets, U_a and V_b, containing a and b respectively. Thus, the family, $(U_a)_{a\in A}$, forms an open cover of A. Since A is compact there is a finite open subcover, $(U_j)_{j\in J}$, of A, where $J \subseteq A$, and then $\bigcup_{j\in J} U_j$ is an open set containing A disjoint from the open set $\bigcap_{j\in J} V_j$ containing b. This shows that every point, b, in the complement of A belongs to some open set in this complement and thus, that the complement is open, i.e., that A is closed. $\square$

Actually, the proof of Proposition A.22 can be used to show the following useful property:

Proposition A.23 *Given a topological Hausdorff space, E, for every pair of compact disjoint subsets, A and B, there exist disjoint open sets, U and V, such that $A \subseteq U$ and $B \subseteq V$.*

Proof. We repeat the argument of Proposition A.22 with B playing the role of b and use Proposition A.22 to find disjoint open sets, U_a, containing $a \in A$ and, V_a, containing B. $\square$

The following proposition shows that in a compact topological space, every closed set is compact:

Proposition A.24 *Given a compact topological space, E, every closed set is compact.*

Proof. Since A is closed, $E - A$ is open and from any open cover, $(U_i)_{i\in I}$, of A, we can form an open cover of E by adding $E - A$ to $(U_i)_{i\in I}$ and, since E is compact, a finite subcover, $(U_j)_{j\in J} \cup \{E - A\}$, of E can be extracted such that $(U_j)_{j\in J}$ is a finite subcover of A. $\square$

Remark: Proposition A.24 also holds for quasi-compact spaces, i.e., the Hausdorff separation property is not needed.

Putting Proposition A.23 and Proposition A.24 together, we note that if X is compact, then for every pair of disjoint closed, sets A and B, there exist disjoint open sets, U and V, such that $A \subseteq U$ and $B \subseteq V$. We say that X is a *normal* space.

Proposition A.25 *Given a compact topological space, E, for every $a \in E$, for every neighborhood, V, of a, there exists a compact neighborhood, U, of a such that $U \subseteq V$.*

Proof. Since V is a neighborhood of a, there is some open subset, O, of V containing a. Then the complement, $K = E - O$, of O is closed and since E is compact, by Proposition A.24, K is compact. Now, if we consider the family of all closed sets of the form, $K \cap F$, where F is any closed neighborhood of a, since $a \notin K$, this family has an empty intersection and thus, there is a finite number of closed neighborhood, $F_1, \ldots, F_n$, of a, such that $K \cap F_1 \cap \cdots \cap F_n = \emptyset$. Then, $U = F_1 \cap \cdots \cap F_n$ is a compact neigborhood of a contained in $O \subseteq V$. $\square$

It can be shown that in a normed vector space of finite dimension, a subset is compact iff it is closed and bounded. For $\mathbb{R}^n$, this is easy.

In a normed vector space of infinite dimension, there are closed and bounded sets that are not compact!

More could be said about compactness in metric spaces but we will only need the notion of Lebesgue number, which will be discussed a little later. Another crucial property of compactness is that it is preserved under continuity.

Proposition A.26 *Let E be a topological space and let F be a topological Hausdorff space. For every compact subset, A, of E, for every continuous map, $f\colon E \to F$, the subspace $f(A)$ is compact.*

Proof. Let $(U_i)_{i \in I}$ be an open cover of $f(A)$. We claim that $(f^{-1}(U_i))_{i \in I}$ is an open cover of A, which is easily checked. Since A is compact, there is a finite open subcover, $(f^{-1}(U_j))_{j \in J}$, of A, and thus, $(U_j)_{j \in J}$ is an open subcover of $f(A)$. $\square$

As a corollary of Proposition A.26, if E is compact, F is Hausdorff, and $f\colon E \to F$ is continuous and bijective, then f is a homeomorphism. Indeed, it is enough to show that f^{-1} is continuous, which is equivalent to showing that f maps closed sets to closed sets. However, closed sets are compact and Proposition A.26 shows that compact sets are mapped to compact sets, which, by Proposition A.22, are closed.

It can also be shown that if E is a compact nonempty space and $f\colon E \to \mathbb{R}$ is a continuous function, then there are points $a, b \in E$ such that $f(a)$ is the minimum of $f(E)$ and $f(b)$ is the maximum of $f(E)$. Indeed, $f(E)$ is a compact subset of $\mathbb{R}$ and thus, a closed and bounded set which contains its greatest lower bound and its least upper bound.

Another useful notion is that of local compactness. Indeed, manifolds and surfaces are locally compact.

Definition A.21 A topological space, E, is *locally compact* if it is Hausdorff and for every $a \in E$, there is some compact neighborhood, K, of a.

From Proposition A.25, every compact space is locally compact but the converse is false. It can be shown that a normed vector space of finite dimension is locally compact.

Proposition A.27 *Given a locally compact topological space, E, for every $a \in E$, for every neighborhood, N, of a, there exists a compact neighborhood, U, of a, such that $U \subseteq N$.*

Proof. For any $a \in E$, there is some compact neighborhood, V, of a. By Proposition A.25, every neigborhood of a relative to V contains some compact neighborhood U of a relative to V. But every neighborhood of a relative to V is a neighborhood of a relative to E and every neighborhood N of a in E yields a neighborhood, $V \cap N$, of a in V and thus, for every neighborhood, N, of a, there exists a compact neighborhood, U, of a such that $U \subseteq N$. $\square$

It is much harder to deal with noncompact surfaces (or manifolds) than it is to deal with compact surfaces (or manifolds). However, surfaces (and manifolds) are locally compact and it turns out that there are various ways of embedding a locally compact Hausdorff space into a compact Hausdorff space. The most economical construction consists in adding just one point. This construction, known as the *Alexandroff compactification*, is technically useful, and we now describe it and sketch the proof that it achieves its goal.

To help the reader's intuition, let us consider the case of the plane, $\mathbb{R}^2$. If we view the plane, $\mathbb{R}^2$, as embedded in 3-space, $\mathbb{R}^3$, say as the xOy plane of equation $z = 0$, we can consider the sphere, Σ, of radius 1 centered on the z-axis at the point $(0, 0, 1)$ and tangent to the xOy plane at the origin (sphere of equation $x^2 + y^2 + (z - 1)^2 = 1$). If N denotes the north pole on the sphere, i.e., the point of coordinates $(0, 0, 2)$, then any line, D, passing through the north pole and not tangent to the sphere (i.e., not parallel to the xOy plane) intersects the xOy plane in a unique point, M, and the sphere in a unique point, P, other than the north pole, N. This, way, we obtain a bijection between the xOy plane and the punctured sphere Σ, i.e., the sphere with the north pole N deleted. This bijection is called a *stereographic projection*. The Alexandroff compactification of the plane consists in putting the north pole back on the sphere, which amounts to adding a single point at infinity ∞ to the plane. Intuitively, as we travel away from the origin O towards infinity (in any direction!), we tend towards an ideal point at infinity ∞. Imagine that we "bend" the plane so that it gets wrapped around the sphere, according to stereographic projection. A simpler example consists in taking a line and getting a circle as its compactification. The Alexandroff compactification is a generalization of these simple constructions.

Definition A.22 Let $(E, \mathcal{O})$ be a locally compact space. Let ω be any point not in E, and let $E_\omega = E \cup \{\omega\}$. Define the family, $\mathcal{O}_\omega$, as follows:

$$\mathcal{O}_\omega = \mathcal{O} \cup \{(E - K) \cup \{\omega\} \mid K \text{ compact in } E\}.$$

The pair, $(E_\omega, \mathcal{O}_\omega)$, is called the *Alexandroff compactification (or one point compactification) of* $(E, \mathcal{O})$.

The following theorem shows that $(E_\omega, \mathcal{O}_\omega)$ is indeed a topological space, and that it is compact.

Theorem A.28 *Let E be a locally compact topological space. The Alexandroff compactification, E_ω, of E is a compact space such that E is a subspace of E_ω and if E is not compact, then $\overline{E} = E_\omega$.*

Proof. The verification that $\mathcal{O}_\omega$ is a family of open sets is not difficult but a bit tedious. Details can be found in Munkres [37] or Schwartz [43]. Let us show that E_ω is compact. For every open cover, $(U_i)_{i\in I}$, of E_ω, since ω must be covered, there is some U_{i_0} of the form

$$U_{i_0} = (E - K_0) \cup \{\omega\}$$

where K_0 is compact in E. Consider the family, $(V_i)_{i\in I}$, defined as follows:

$$\begin{aligned} V_i &= U_i \quad \text{if} \quad U_i \in \mathcal{O}, \\ V_i &= E - K \quad \text{if} \quad U_i = (E - K) \cup \{\omega\}, \end{aligned}$$

where K is compact in E. Then, because each K is compact and thus closed in E (since E is Hausdorff), $E - K$ is open, and every V_i is an open subset of E. Furthermore, the family, $(V_i)_{i\in(I-\{i_0\})}$, is an open cover of K_0. Since K_0 is compact, there is a finite open subcover, $(V_j)_{j\in J}$, of K_0, and thus, $(U_j)_{j\in J\cup\{i_0\}}$ is a finite open cover of E_ω.

Let us show that E_ω is Hausdorff. Given any two points, $a, b \in E_\omega$, if both $a, b \in E$, since E is Hausdorff and every open set in $\mathcal{O}$ is an open set in $\mathcal{O}_\omega$, there exist disjoint open sets, U, V (in $\mathcal{O}$), such that $a \in U$ and $b \in V$. If $b = \omega$, since E is locally compact, there is some compact set, K, containing an open set, U, containing a and then, U and $V = (E-K)\cup\{\omega\}$ are disjoint open sets (in $\mathcal{O}_\omega$) such that $a \in U$ and $b \in V$.

The space E is a subspace of E_ω because for every open set, U, in $\mathcal{O}_\omega$, either $U \in \mathcal{O}$ and $E \cap U = U$ is open in E, or $U = (E - K) \cup \{\omega\}$, where K is compact in E, and thus, $U \cap E = E - K$, which is open in E, since K is compact in E and thus, closed (since E is Hausdorff). Finally, if E is not compact, for every compact subset, K, of E, $E - K$ is nonempty and thus, for every open set, $U = (E-K)\cup\{\omega\}$, containing ω, we have $U \cap E \neq \emptyset$, which shows that $\omega \in \overline{E}$ and thus, that $\overline{E} = E_\omega$. $\square$

Finally, in studying surfaces and manifolds, an important property is the existence of a countable basis for the topology. Indeed, this property guarantees the existence of triangulations of surfaces, a crucial property.

Definition A.23 A topological space is called *second-countable* if there is a countable basis for its topology, i.e., if there is a countable family, $(U_i)_{i\geq 0}$, of open sets such that every open set of E is a union of open sets U_i.

It is easily seen that $\mathbb{R}^n$ is second-countable and more generally, that every normed vector space of finite dimension is second-countable. It can also be shown that if E is a locally compact space that has a countable basis, then E_ω also has a countable basis (and in fact, is metrizable). We have the following properties.

Proposition A.29 *Given a second-countable topological space, E, every open cover, $(U_i)_{i\in I}$, of E contains some countable subcover.*

Proof. Let $(O_n)_{n\geq 0}$ be a countable basis for the topology. Then, all sets O_n contained in some U_i can be arranged into a countable subsequence, $(\Omega_m)_{m\geq 0}$, of $(O_n)_{n\geq 0}$ and for every Ω_m, there is some U_{i_m} such that $\Omega_m \subseteq U_{i_m}$. Furthermore, every U_i is some union of sets Ω_j, and thus, every $a \in E$ belongs to some Ω_j, which shows that $(\Omega_m)_{m\geq 0}$ is a countable open subcover of $(U_i)_{i\in I}$. $\square$

As an immediate corollary of Proposition A.29, a locally connected second-countable space has countably many connected components.

In second-countable Hausdorff spaces, compactness can be characterized in terms of accumulation points (this is also true of metric spaces).

Definition A.24 Given a topological Hausdorff space, E, given any sequence, (x_n), of points in E, a point, $l \in E$, is an *accumulation point (or cluster point)* of the sequence (x_n) if every open set, U, containing l contains x_n for infinitely many n.

Clearly, if l is a limit of the sequence, (x_n), then it is an accumulation point, since every open set, U, containing a contains all x_n except for finitely many n.

Proposition A.30 *A second-countable topological Hausdorff space, E, is compact iff every sequence, (x_n), has some accumulation point.*

Proof. Assume that every sequence, (x_n), has some accumulation point. Let $(U_i)_{i\in I}$ be some open cover of E. By Proposition A.29, there is a countable open subcover, $(O_n)_{n\geq 0}$, for E. Now, if E is not covered by any finite subcover of $(O_n)_{n\geq 0}$, we can define a sequence, (x_m), by induction as follows:

Let x_0 be arbitrary and for every $m \geq 1$, let x_m be some point in E not in $O_1 \cup \cdots \cup O_m$, which exists, since $O_1 \cup \cdots \cup O_m$ is not an open cover of E. We claim that the sequence, (x_m), does not have any accumulation point. Indeed, for every $l \in E$, since $(O_n)_{n\geq 0}$ is an open cover of E, there is some O_m such that $l \in O_m$, and by construction, every x_n with $n \geq m+1$ does not belong to O_m, which means that $x_n \in O_m$ for only finitely many n and l is not an accumulation point.

Conversely, assume that E is compact, and let (x_n) be any sequence. If $l \in E$ is not an accumulation point of the sequence, then there is some open set, U_l, such that $l \in U_l$ and $x_n \in U_l$ for only finitely many n. Thus, if (x_n) does not have any accumulation point, the family, $(U_l)_{l\in E}$, is an open cover of E and since E is compact, it has some finite open subcover, $(U_l)_{l\in J}$, where J is a finite subset of E. But every U_l with $l \in J$ is such that $x_n \in U_l$ for only finitely many n, and since J is finite, $x_n \in \bigcup_{l\in J} U_l$ for only finitely many n, which contradicts the fact that $(U_l)_{l\in J}$ is an open cover of E, and thus contains all the x_n. Thus, (x_n) has some accumulation point. $\square$

Remark: It should be noted that the proof that if E is compact, then every sequence has some accumulation point, holds for any arbitrary compact space (the proof does not use a countable basis for the topology). The converse also holds for metric spaces. We will prove this converse since it is a major property of metric spaces. It is also convenient to have such a characterization of compactness when dealing with fractal geometry.

Given a metric space in which every sequence has some accumulation point, we first prove the existence of a *Lebesgue number*.

Lemma A.31 *Given a metric space, E, if every sequence, (x_n), has an accumulation point, for every open cover, $(U_i)_{i\in I}$, of E, there is some $\delta > 0$ (a Lebesgue number for $(U_i)_{i\in I}$) such that, for every open ball, $B_0(a,\epsilon)$, of diameter $\epsilon \leq \delta$, there is some open subset, U_i, such that $B_0(a,\epsilon) \subseteq U_i$.*

Proof. If there was no δ with the above property, then, for every natural number, n, there would be some open ball, $B_0(a_n, 1/n)$, which is not contained in any open set, U_i, of the open cover, $(U_i)_{i\in I}$. However, the sequence, (a_n), has some accumulation point, a, and since $(U_i)_{i\in I}$ is an open cover of E, there is some U_i such that $a \in U_i$. Since U_i is open, there is some open ball of center a and radius ϵ contained in U_i. Now, since a is an accumulation point of the sequence, (a_n), every open set containing a contain a_n for infinitely many n and thus, there is some n large enough so that

$$1/n \leq \epsilon/2 \quad \text{and} \quad a_n \in B_0(a, \epsilon/2),$$

which implies that

$$B_0(a_n, 1/n) \subseteq B_0(a,\epsilon) \subseteq U_i,$$

a contradiction. □

By a previous remark, since the proof of Proposition A.30 implies that in a compact topological space, every sequence has some accumulation point, by Lemma A.31, in a compact metric space, every open cover has a Lebesgue number. This fact can be used to prove another important property of compact metric spaces, the uniform continuity theorem.

Definition A.25 Given two metric spaces, (E, d_E) and (F, d_F), a function, $f\colon E \to F$, is *uniformly continuous* if for every $\epsilon > 0$, there is some $\eta > 0$, such that, for all $a, b \in E$,

$$\text{if} \quad d_E(a,b) \leq \eta \quad \text{then} \quad d_F(f(a), f(b)) \leq \epsilon.$$

The *uniform continuity theorem* can be stated as follows:

Theorem A.32 *Given two metric spaces, (E, d_E) and (F, d_F), if E is compact and $f\colon E \to F$ is a continuous function, then it is uniformly continuous.*

Proof. Consider any $\epsilon > 0$ and let $(B_0(y, \epsilon/2))_{y\in F}$ be the open cover of F consisting of open balls of radius $\epsilon/2$. Since f is continuous, the family,

$$(f^{-1}(B_0(y, \epsilon/2)))_{y\in F},$$

is an open cover of E. Since, E is compact, by Lemma A.31, there is a Lebesgue number, δ, such that for every open ball, $B_0(a, \eta)$, of diameter $\eta \leq \delta$, then $B_0(a, \eta) \subseteq f^{-1}(B_0(y, \epsilon/2))$, for some $y \in F$. In particular, for any $a, b \in E$ such that $d_E(a, b) \leq \eta = \delta/2$, we have $a, b \in B_0(a, \delta)$ and thus, $a, b \in f^{-1}(B_0(y, \epsilon/2))$, which implies that $f(a), f(b) \in B_0(y, \epsilon/2)$. But then, $d_F(f(a), f(b)) \leq \epsilon$, as desired. $\square$

We now prove another lemma needed to obtain the characterization of compactness in metric spaces in terms of accumulation points.

Lemma A.33 *Given a metric space, E, if every sequence, (x_n), has an accumulation point, then for every $\epsilon > 0$, there is a finite open cover, $B_0(a_0, \epsilon) \cup \cdots \cup B_0(a_n, \epsilon)$, of E by open balls of radius ϵ.*

Proof. Let a_0 be any point in E. If $B_0(a_0, \epsilon) = E$, then the lemma is proved. Otherwise, assume that a sequence, $(a_0, a_1, \ldots, a_n)$, has been defined, such that $B_0(a_0, \epsilon) \cup \cdots \cup B_0(a_n, \epsilon)$ does not cover E. Then, there is some a_{n+1} not in $B_0(a_0, \epsilon) \cup \cdots \cup B_0(a_n, \epsilon)$ and either

$$B_0(a_0, \epsilon) \cup \cdots \cup B_0(a_{n+1}, \epsilon) = E,$$

in which case the lemma is proved, or we obtain a sequence, $(a_0, a_1, \ldots, a_{n+1})$, such that $B_0(a_0, \epsilon) \cup \cdots \cup B_0(a_{n+1}, \epsilon)$ does not cover E. If this process goes on forever, we obtain an infinite sequence, (a_n), such that $d(a_m, a_n) > \epsilon$ for all $m \neq n$. Since every sequence in E has some accumulation point, the sequence, (a_n), has some accumulation point, a. Then, for infinitely many n, we must have $d(a_n, a) \leq \epsilon/3$ and thus, for at least two distinct natural numbers, p, q, we must have $d(a_p, a) \leq \epsilon/3$ and $d(a_q, a) \leq \epsilon/3$, which implies $d(a_p, a_q) \leq 2\epsilon/3$, contradicting the fact that $d(a_m, a_n) > \epsilon$ for all $m \neq n$. Thus, there must be some n such that

$$B_0(a_0, \epsilon) \cup \cdots \cup B_0(a_n, \epsilon) = E.$$

$\square$

A metric space satisfying the condition of Lemma A.33 is sometimes called *precompact* (or *totally bounded*). We now obtain the *Weierstrass-Bolzano* property.

Theorem A.34 *A metric space, E, is compact iff every sequence, (x_n), has an accumulation point.*

Proof. We already observed that the proof of Proposition A.30 shows that for any compact space (not necessarily metric), every sequence, (x_n), has an accumulation point. Conversely, let E be a metric space, and assume that every sequence, (x_n), has an accumulation point. Given any open cover, $(U_i)_{i\in I}$, for E, we must find a finite open subcover of E. By Lemma

A.31, there is some $\delta > 0$ (a Lebesgue number for $(U_i)_{i\in I}$) such that, for every open ball, $B_0(a,\epsilon)$, of diameter $\epsilon \leq \delta$, there is some open subset, U_j, such that $B_0(a,\epsilon) \subseteq U_j$. By Lemma A.33, for every $\delta > 0$, there is a finite open cover, $B_0(a_0,\delta) \cup \cdots \cup B_0(a_n,\delta)$, of E by open balls of radius δ. But from the previous statement, every open ball, $B_0(a_i,\delta)$, is contained in some open set, U_{j_i}, and thus, $\{U_{j_1},\ldots,U_{j_n}\}$ is an open cover of E. $\square$

Another very useful characterization of compact metric spaces is obtained in terms of Cauchy sequences. Such a characterization is quite useful in fractal geometry (and elsewhere). First, recall the definition of a Cauchy sequence and of a complete metric space.

Definition A.26 Given a metric space, (E,d), a sequence, $(x_n)_{n\in\mathbb{N}}$, in E is a *Cauchy sequence* if the following condition holds:

for every $\epsilon > 0$, there is some $p \geq 0$, such that, for all $m, n \geq p$, then $d(x_m,\, x_n) \leq \epsilon$.

If every Cauchy sequence in (E,d) converges we say that (E,d) is a *complete metric space*.

First, let us show the following easy proposition:

Proposition A.35 *Given a metric space, E, if a Cauchy sequence, (x_n), has some accumulation point, a, then a is the limit of the sequence, (x_n).*

Proof. Since (x_n) is a Cauchy sequence, for every $\epsilon > 0$, there is some $p \geq 0$, such that, for all $m, n \geq p$, then $d(x_m,\, x_n) \leq \epsilon/2$. Since a is an accumulation point for (x_n), for infinitely many n, we have $d(x_n,a) \leq \epsilon/2$, and thus, for at least some $n \geq p$, we have $d(x_n,a) \leq \epsilon/2$. Then, for all $m \geq p$,

$$d(x_m,a) \leq d(x_m,x_n) + d(x_n,a) \leq \epsilon,$$

which shows that a is the limit of the sequence (x_n). $\square$

Recall that a metric space is *precompact* (or *totally bounded*) if for every $\epsilon > 0$, there is a finite open cover, $B_0(a_0,\epsilon) \cup \cdots \cup B_0(a_n,\epsilon)$, of E by open balls of radius ϵ. We can now prove the following theorem.

Theorem A.36 *A metric space, E, is compact iff it is precompact and complete.*

Proof. Let E be compact. For every $\epsilon > 0$, the family of all open balls of radius ϵ is an open cover for E and since E is compact, there is a finite subcover, $B_0(a_0,\epsilon) \cup \cdots \cup B_0(a_n,\epsilon)$, of E by open balls of radius ϵ. Thus, E is precompact. Since E is compact, by Theorem A.34, every sequence, (x_n), has some accumulation point. Thus, every Cauchy sequence, (x_n), has some accumulation point, a, and, by Proposition A.35, a is the limit of (x_n). Thus, E is complete.

Now, assume that E is precompact and complete. We prove that every sequence, (x_n), has an accumulation point. By the other direction of Theorem A.34, this shows that E

is compact. Given any sequence, (x_n), we construct a Cauchy subsequence, (y_n), of (x_n) as follows: Since E is precompact, letting $\epsilon = 1$, there exists a finite cover, $\mathcal{U}_1$, of E by open balls of radius 1. Thus, some open ball, B_o^1, in the cover, $\mathcal{U}_1$, contains infinitely many elements from the sequence (x_n). Let y_0 be any element of (x_n) in B_o^1. By induction, assume that a sequence of open balls, $(B_o^i)_{1\leq i\leq m}$, has been defined, such that every ball, B_o^i, has radius $\frac{1}{2^i}$, contains infinitely many elements from the sequence (x_n) and contains some y_i from (x_n) such that

$$d(y_i, y_{i+1}) \leq \frac{1}{2^i},$$

for all i, $0 \leq i \leq m-1$. Then, letting $\epsilon = \frac{1}{2^{m+1}}$, because E is precompact, there is some finite cover, $\mathcal{U}_{m+1}$, of E by open balls of radius ϵ and thus, of the open ball B_o^m. Thus, some open ball, B_o^{m+1}, in the cover, $\mathcal{U}_{m+1}$, contains infinitely many elements from the sequence, (x_n), and we let y_{m+1} be any element of (x_n) in B_o^{m+1}. Thus, we have defined by induction a sequence, (y_n), which is a subsequence of, (x_n), and such that

$$d(y_i, y_{i+1}) \leq \frac{1}{2^i},$$

for all i. However, for all $m, n \geq 1$, we have

$$d(y_m, y_n) \leq d(y_m, y_{m+1}) + \cdots + d(y_{n-1}, y_n) \leq \sum_{i=m}^{n} \frac{1}{2^i} \leq \frac{1}{2^{m-1}},$$

and thus, (y_n) is a Cauchy sequence. Since E is complete, the sequence, (y_n), has a limit, and since it is a subsequence of (x_n), the sequence, (x_n), has some accumulation point. $\square$

If (E, d) is a nonempty complete metric space, every map, $f\colon E \to E$, for which there is some k such that $0 \leq k < 1$ and

$$d(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in E$, has the very important property that it has a unique fixed point, that is, there is a unique, $a \in E$, such that $f(a) = a$. A map as above is called a *contracting mapping*. Furthermore, the fixed point of a contracting mapping can be computed as the limit of a fast converging sequence.

The fixed point property of contracting mappings is used to show some important theorems of analysis, such as the implicit function theorem and the existence of solutions to certain differential equations. It can also be used to show the existence of fractal sets defined in terms of iterated function systems, a topic that we intend to discuss later on. Since the proof is quite simple, we prove the fixed point property of contracting mappings. First, observe that a contracting mapping is (uniformly) continuous.

Proposition A.37 *If (E, d) is a nonempty complete metric space, every contracting mapping, $f\colon E \to E$, has a unique fixed point. Furthermore, for every $x_0 \in E$, defining the sequence, (x_n), such that $x_{n+1} = f(x_n)$, the sequence, (x_n), converges to the unique fixed point of f.*

Proof. First, we prove that f has at most one fixed point. Indeed, if $f(a) = a$ and $f(b) = b$, since

$$d(a,b) = d(f(a), f(b)) \leq kd(a,b)$$

and $0 \leq k < 1$, we must have $d(a,b) = 0$, that is, $a = b$.

Next, we prove that (x_n) is a Cauchy sequence. Observe that

$$\begin{aligned} d(x_2, x_1) &\leq kd(x_1, x_0), \\ d(x_3, x_2) &\leq kd(x_2, x_1) \leq k^2 d(x_1, x_0), \\ \cdots & \quad \cdots \\ d(x_{n+1}, x_n) &\leq kd(x_n, x_{n-1}) \leq \cdots \leq k^n d(x_1, x_0). \end{aligned}$$

Thus, we have

$$\begin{aligned} d(x_{n+p}, x_n) &\leq d(x_{n+p}, x_{n+p-1}) + d(x_{n+p-1}, x_{n+p-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq (k^{p-1} + k^{p-2} + \cdots + k + 1)k^n d(x_1, x_0) \\ &\leq \frac{k^n}{1-k}\, d(x_1, x_0). \end{aligned}$$

We conclude that $d(x_{n+p}, x_n)$ converges to 0 when n goes to infinity, which shows that (x_n) is a Cauchy sequence. Since E is complete, the sequence (x_n) has a limit, a. Since f is continuous, the sequence $(f(x_n))$ converges to $f(a)$. But $x_{n+1} = f(x_n)$ converges to a and so $f(a) = a$, the unique fixed point of f. $\square$

Note that no matter how the starting point x_0 of the sequence (x_n) is chosen, (x_n) converges to the unique fixed point of f. Also, the convergence is fast, since

$$d(x_n, a) \leq \frac{k^n}{1-k}\, d(x_1, x_0).$$

The Hausdorff distance between compact subsets of a metric space provides a very nice illustration of some of the theorems on complete and compact metric spaces just presented. It can also be used to define certain kinds of fractal sets and thus, we indulge into a short digression on the Hausdorff distance.

Definition A.27 Given a metric space, (X, d), for any subset, $A \subseteq X$, for any, $\epsilon \geq 0$, define the *ϵ-hull* of A as the set

$$V_\epsilon(A) = \{x \in X,\ \exists a \in A \mid d(a,x) \leq \epsilon\}.$$

Given any two nonempty bounded subsets, A, B of X, define $D(A,B)$, *the Hausdorff distance between A and B*, by

$$D(A,B) = \inf\{\epsilon \geq 0 \mid A \subseteq V_\epsilon(B) \text{ and } B \subseteq V_\epsilon(A)\}.$$

Note that since we are considering nonempty bounded subsets, $D(A, B)$ is well defined (i.e., not infinite). However, D is not necessarily a distance function. It is a distance function if we restrict our attention to nonempty compact subsets of X (actually, it is also a metric on closed and bounded subsets). We let $\mathcal{K}(X)$ denote the set of all nonempty compact subsets of X. The remarkable fact is that D is a distance on $\mathcal{K}(X)$ and that if X is complete or compact, then so it $\mathcal{K}(X)$. The following theorem is taken from Edgar [15].

Theorem A.38 *If (X, d) is a metric space, then the Hausdorff distance, D, on the set, $\mathcal{K}(X)$, of nonempty compact subsets of X is a distance. If (X, d) is complete, then $(\mathcal{K}(X), D)$ is complete and if (X, d) is compact, then $(\mathcal{K}(X), D)$ is compact.*

Proof. Since (nonempty) compact sets are bounded, $D(A, B)$ is well defined. Clearly, D is symmetric. Assume that $D(A, B) = 0$. Then, for every $\epsilon > 0$, $A \subseteq V_\epsilon(B)$, which means that for every $a \in A$, there is some $b \in B$ such that $d(a, b) \leq \epsilon$, and thus, that $A \subseteq \overline{B}$. Since B is closed, $\overline{B} = B$, and we have $A \subseteq B$. Similarly, $B \subseteq A$, and thus, $A = B$. Clearly, if $A = B$, we have $D(A, B) = 0$. It remains to prove the triangle inequality. If $B \subseteq V_{\epsilon_1}(A)$ and $C \subseteq V_{\epsilon_2}(B)$, then

$$V_{\epsilon_2}(B) \subseteq V_{\epsilon_2}(V_{\epsilon_1}(A)),$$

and since

$$V_{\epsilon_2}(V_{\epsilon_1}(A)) \subseteq V_{\epsilon_1+\epsilon_2}(A),$$

we get

$$C \subseteq V_{\epsilon_2}(B) \subseteq V_{\epsilon_1+\epsilon_2}(A).$$

Similarly, we can prove that

$$A \subseteq V_{\epsilon_1+\epsilon_2}(C),$$

and thus, the triangle inequality follows.

Next, we need to prove that if (X, d) is complete, then $(\mathcal{K}(X), D)$ is also complete. First, we show that if (A_n) is a sequence of nonempty compact sets converging to a nonempty compact set A in the Hausdorff metric, then

$$A = \{x \in X \mid \text{there is a sequence, } (x_n), \text{ with } x_n \in A_n \text{ converging to } x\}.$$

Indeed, if (x_n) is a sequence with $x_n \in A_n$ converging to x and (A_n) converges to A then, for every $\epsilon > 0$, there is some x_n such that $d(x_n, x) \leq \epsilon/2$ and there is some $a_n \in A$ such that $d(a_n, x_n) \leq \epsilon/2$ and thus, $d(a_n, x) \leq \epsilon$, which shows that $x \in \overline{A}$. Since A is compact, it is closed, and $x \in A$. Conversely, since (A_n) converges to A, for every $x \in A$, for every $n \geq 1$, there is some $x_n \in A_n$ such that $d(x_n, x) \leq 1/n$ and the sequence (x_n) converges to x.

Now, let (A_n) be a Cauchy sequence in $\mathcal{K}(X)$. It can be proven that (A_n) converges to the set

$$A = \{x \in X \mid \text{there is a sequence, } (x_n), \text{ with } x_n \in A_n \text{ converging to } x\},$$

and that A is nonempty and compact. To prove that A is compact, one proves that it is totally bounded and complete. Details are given in Edgar [15].

Finally, we need to prove that if (X, d) is compact, then $(\mathcal{K}(X), D)$ is compact. Since we already know that $(\mathcal{K}(X), D)$ is complete if (X, d) is, it is enough to prove that $(\mathcal{K}(X), D)$ is totally bounded if (X, d) is, which is fairly easy. $\square$

In view of Theorem A.38 and Theorem A.37, it is possible to define some nonempty compact subsets of X in terms of fixed points of contracting maps. We will see later on how this can be done in terms of iterated function systems, yielding a large class of fractals.

Finally, returning to second-countable spaces, we give another characterization of accumulation points.

Proposition A.39 *Given a second-countable topological Hausdorff space, E, a point, l, is an accumulation point of the sequence, (x_n), iff l is the limit of some subsequence, (x_{n_k}), of (x_n).*

Proof. Clearly, if l is the limit of some subsequence (x_{n_k}) of (x_n), it is an accumulation point of (x_n).

Conversely, let $(U_k)_{k \geq 0}$ be the sequence of open sets containing l, where each U_k belongs to a countable basis of E, and let $V_k = U_1 \cap \cdots \cap U_k$. For every $k \geq 1$, we can find some $n_k > n_{k-1}$ such that $x_{n_k} \in V_k$, since l is an accumulation point of (x_n). Now, since every open set containing l contains some U_{k_0} and since $x_{n_k} \in U_{k_0}$ for all $k \geq 0$, the sequence (x_{n_k}) has limit l. $\square$

Remark: Proposition A.39 also holds for metric spaces.

As promised, we show how certain fractals can be defined by iterated function systems, using Theorem A.38 and Theorem A.37.