Chapter 6

The Classification Theorem for Compact Surfaces

6.1 Cell Complexes

It is remarkable that the compact (two-dimensional) polyhedra can be characterized up to homeomorphism. This situation is exceptional, as such a result is known to be essentially impossible for compact m-manifolds for $m \geq 4$, and still open for compact 3-manifolds (although some progress has been made recently with the proof of the Poincaré conjecture).

One of the reasons why there is a classification theorem for surfaces is that surfaces can be triangulated. In fact, it is possible to characterize the compact (two-dimensional) polyhedra in terms of a simple extension of the notion of a complex, called *cell complex* by Ahlfors and Sario. What happens is that it is possible to define an equivalence relation on cell complexes and it can be shown that every cell complex is equivalent to some specific normal form. Furthermore, every cell complex has a geometric realization which is a surface, and equivalent cell complexes have homeomorphic geometric realizations. Also, every cell complex is equivalent to a triangulated 2-complex. Finally, we can show that the geometric realizations of distinct normal forms are not homeomorphic.

The classification theorem for compact surfaces is presented (in slightly different ways)

Figure 6.1: Lars Ahlfors, 1907-1996

Figure 6.2: Karl Seifert, 1907-1996 (left) and William Thurston, 1946- (right)

in Massey [32] Amstrong [3], and Kinsey [26]. In the above references, the presentation is sometimes quite informal. The classification theorem is also presented in Ahlfors and Sario [1], and there, the presentation is formal and not always easy to follow. We tried to strike a middle ground in the degree of formality. It should be noted that the combinatorial part of the proof (Section 6.2) is heavily inspired by the proof given in Seifert and Threlfall [44]. One should also take a look at Chapter 1 of Thurston [47], especially Problem 1.3.12. Thurston's book is also highly recommended as a wonderful and insighful introduction to the topology and geometry of three-dimensional manifolds, but that's another story.

The first step is to define cell complexes. The intuitive idea is to generalize a little bit the notion of a triangulation and consider objects made of oriented faces, each face having some boundary. A boundary is a cyclically ordered list of oriented edges. We can think of each face as a circular closed disk and of the edges in a boundary as circular arcs on the boundaries of these disks. A cell complex represents the surface obtained by identifying identical boundary edges.

Technically, in order to deal with the notion of orientation, given any set, X, it is convenient to introduce the set, $X^{-1} = \{x^{-1} \mid x \in X\}$, of formal inverses of elements in X, where it is assumed that $X \cap X^{-1} = \emptyset$. We will say that the elements of $X \cup X^{-1}$ are *oriented*. It is also convenient to assume that $(x^{-1})^{-1} = x$, for every $x \in X$. It turns out that cell complexes can be defined using only faces and boundaries and that the notion of a vertex can be defined from the way edges occur in boundaries. This way of dealing with vertices is a bit counterintuitive, but we haven't found a better way to present cell complexes. We now give precise definitions.

Definition 6.1 A *cell complex*, K, consists of a triple, $K = (F, E, B)$, where F is a finite nonempty set of *faces*, E is a finite set of *edges*, and $B\colon (F \cup F^{-1}) \to (E \cup E^{-1})^*$ is the *boundary function*, which assigns to each oriented face, $A \in F \cup F^{-1}$, a cyclically ordered, sequence $a_1 \ldots a_n$, of oriented edges in $E \cup E^{-1}$, the *boundary of* A, in such a way that $B(A^{-1}) = a_n^{-1} \ldots a_1^{-1}$ (the reversal of the sequence $a_1^{-1} \ldots a_n^{-1}$). For all $A_1, A_2 \in F$, if $A_1 \neq A_2$, then $B(A_1) \neq B(A_2)$ (distinct faces have distinct boundaries). By a cyclically ordered sequence, we mean that we do not distinguish between the sequence $a_1 \ldots a_n$ and any sequence obtained from it by a cyclic permutation. In particular, the successor of a_n is a_1. Furthermore, the following conditions must hold:

(1) Every oriented edge, $a \in E \cup E^{-1}$, occurs either once or twice as an element of a boundary. In particular, this means that if a occurs twice in some boundary, then it does not occur in any other boundary.

(2) K is connected. This means that K is not the union of two disjoint systems satisfying condition (1).

It is possible that $F = \{A\}$ and $E = \emptyset$, in which case $B(A) = B(A^{-1}) = \epsilon$, the empty sequence.

For short, we will often say face and edge, rather than oriented face or oriented edge. As we said earlier, the notion of a vertex is defined in terms of faces and boundaries. The intuition is that a vertex is adjacent to pairs of incoming and outgoing edges. Using inverses of edges, we can define a vertex as the sequence of incoming edges into that vertex. When the vertex is not a boundary vertex, these edges form a cyclic sequence and when the vertex is a border vertex, such a sequence has two endpoints with no successors. The definition of a vertex given in Ahlfors and Sario [1] (see 39C) does not stipulate explicitly some of the conditions that a vertex should satisfy so we give a more detailed definition of a vertex.

Definition 6.2 Given a cell complex, $K = (F, E, B)$, for any edge, $a \in E \cup E^{-1}$, a *successor* of a is an edge b such that b is the successor of a in some boundary $B(A)$ (the string ab occurs in some boundary). If a occurs in two places in the set of boundaries, it has a *a pair of successors* (possibly identical) and otherwise, it has a *single successor*. A one-element sequence, $\alpha = (a)$, is an *inner vertex* iff aa^{-1} occurs in a single boundary (in this case, a does not appear in any other boundary); the cyclically ordered set, $\alpha = (a, b)$ $(a \neq b)$, is an *inner vertex* iff either $a = b^{-1}$ and if there is a face whose boundary is aa or $a \neq b^{-1}$ and ab^{-1} occurs twice in the set of boundaries; a cyclically ordered set, $\alpha = (a_1, \ldots, a_n)$, with $n \geq 3$, is an *inner vertex* if every a_i occurs in two places in the set of boundaries, if the successors of each a_i occur in α and if a_i has a_{i-1}^{-1} and a_{i+1}^{-1} as pair of successors, see Figure 6.3 (note that a_1 has a_n^{-1} and a_2^{-1} as pair of successors, and a_n has a_{n-1}^{-1} and a_1^{-1} as pair of successors). A *border vertex* is a cyclically ordered set, $\alpha = (a_1, \ldots, a_n)$, with $n \geq 2$ such that the above condition holds for all i, with $2 \leq i \leq n-1$, while a_1 and a_n occur once in the set of boundaries, a_1 has a_2^{-1} as only successor and a_n has a_{n-1}^{-1} as only successor. We consider that $(a_1, \ldots, a_n)$ and $(a_n, \ldots, a_1)$ represent the same vertex. An edge, $a \in E \cup E^{-1}$, is a *border edge* if it occurs once in a single boundary and, otherwise, an *inner edge*.

For example, if K has a single face with boundary $aba^{-1}b^{-1}$, then K has a single inner vertex, (a^{-1}, b, a, b^{-1}), as illustrated in Figure 6.4 (a). The corresponding surface is the torus. If K has a single face with boundary $aabb$, then K has a single inner vertex, (a^{-1}, a, b^{-1}, b), as illustrated in Figure 6.4 (b) and the corresponding surface is the Klein bottle. If K has a single face with boundary $abab$, then K has two inner vertices (b^{-1}, a) and (a^{-1}, b), as illustrated in Figure 6.4 (c). The corresponding surface is the projective plane.

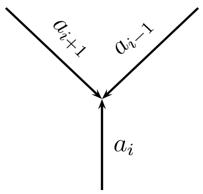

Figure 6.3: An inner vertex ($n \geq 3$)

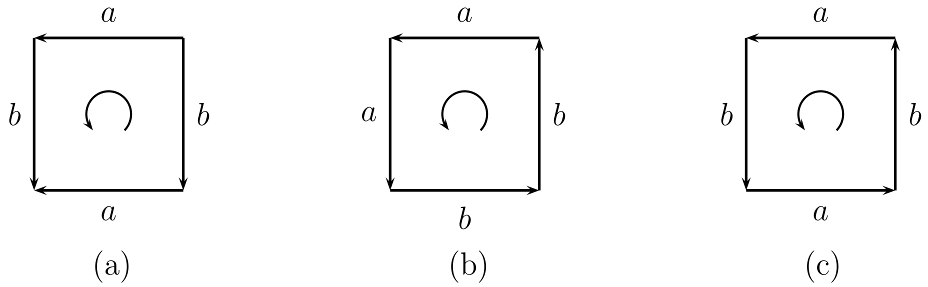

Figure 6.4: (a) A torus (boundary $aba^{-1}b^{-1}$). (b) A Klein bottle (boundary $aabb$). (c) A projective plane (boundary $abab$).

If K has a single face with boundary aa^{-1}, then K has a single inner vertex, (a), as illustrated in Figure 6.5 (a) and the corresponding surface is the sphere. If K has a single face with boundary aa, then K has a single inner vertex, (a^{-1}, a), as illustrated in Figure 6.5 (b) and the corresponding surface is again the projective plane.

If K has a single face with boundary aah, then K has no inner vertex and one boundary vertex, (h, a^{-1}, a, h^{-1}), see Figure 6.6 (a). The corresponding surface is the Möbius strip. If K has a single face with boundary $aachc^{-1}$, then K has one inner vertex (a^{-1}, a, c^{-1}), and one boundary vertex, (h, c, h^{-1}), see Figure 6.6 (b). The corresponding surface is again the

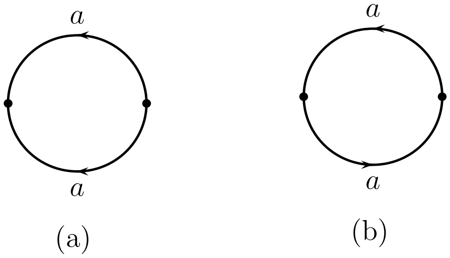

Figure 6.5: (a) A sphere (boundary aa^{-1}). (b) A projective plane (boundary aa).

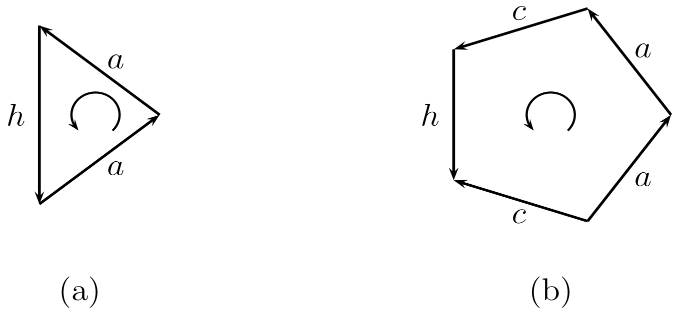

Figure 6.6: (a) A Möbius strip (boundary aah). (b) Another Möbius strip (boundary $aachc^{-1}$).

Möbius strip.

Given any edge, $a \in E \cup E^{-1}$, we can determine a unique vertex, α, as follows: The neighbors of a in the vertex α are the inverses of its successor(s). Repeat this step in both directions until either the cycle closes or we hit sides with only one successor. The vertex α in question is the list of the incoming edges into it. For this reason, we say that a *leads to* α. Note that when a vertex, $\alpha = (a)$, contains a single edge, a, there must be a unique occurrence of the form aa^{-1} in some boundary. Also, if ab with $a \neq b$ occurs only once (in a single boundary), then (a, b^{-1}) is a border vertex.

Vertices can also characterized in another way which will be useful later on. Intuitively, two edges a and b are equivalent iff they have the same terminal vertex.

We define a relation, λ, on edges as follows: $a\lambda b$ iff b^{-1} is the successor of a in some boundary. Note that this relation is symmetric. Indeed, if ab^{-1} appears in the boundary of some face A, then ba^{-1} appears in the boundary of A^{-1}. Let Λ be the reflexive and transitive closure of λ. Since λ is symmetric, Λ is an equivalence relation. We leave as an easy exercise to prove that the equivalence class of an edge, a, is the vertex, α, that a leads to. Thus, vertices induce a partition of $E \cup E^{-1}$. We say that an edge, a, is an edge from a vertex α to a vertex β if $a^{-1} \in \alpha$ and $a \in \beta$. Then, by a familiar reasoning, we can show that the fact that K is connected implies that there is a path between any two vertices.

Figure 6.7 shows a cell complex with border. The cell complex has three faces with boundaries abc, bed^{-1}, and adf^{-1}. It has one inner vertex $b^{-1}ad^{-1}$ and three border vertices edf, $c^{-1}be^{-1}$, and $ca^{-1}f^{-1}$.

If we fold the above cell complex by identifying the two edges labeled d, we get a tetrahedron with one face omitted, the face opposite the inner vertex, the endpoint of edge a.

There is a natural way to view a triangulated complex as a cell complex and it is not hard to see that the following conditions allow us to view a cell complex as a triangulated complex:

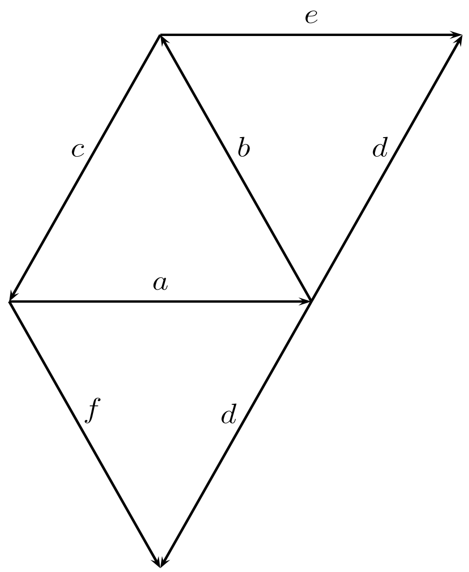

Figure 6.7: A cell complex with border

(C1) If a, b are distinct edges leading to the same vertex, then a^{-1} and b^{-1} lead to distinct vertices.

(C2) The boundary of every face is a triple, abc.

(C3) Different faces have different boundaries.

It is easy to see that a and a^{-1} cannot lead to the same vertex and that in a face, abc, the edges a, b, c are distinct.

6.2 Normal Form for Cell Complexes

We now introduce a notion of elementary subdivision of cell complexes which is crucial in obtaining the classification theorem.

Definition 6.3 Given any two cells complexes, K and K', we say that K' *is an elementary subdivision of* K if K' is obtained from K by one of the following two operations:

(P1) Any two edges, a and a^{-1}, in K are replaced by bc and $c^{-1}b^{-1}$ in all boundaries, where b, c are distinct edges of K' not in K.

(P2) Any face, A, in K with boundary, $a_1 \ldots a_p a_{p+1} \ldots a_n$, is replaced by two faces, A' and A'', in K', with boundaries, $a_1 \ldots a_p d$ and $d^{-1} a_{p+1} \ldots a_n$, where d is an edge in K' not in K. Of course, the corresponding replacement is applied to A^{-1}.

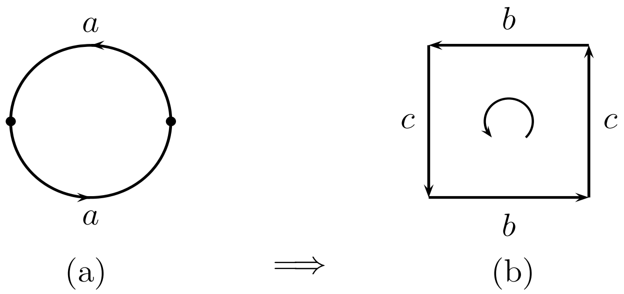

Figure 6.8: Example of elementary subdivision (P1)

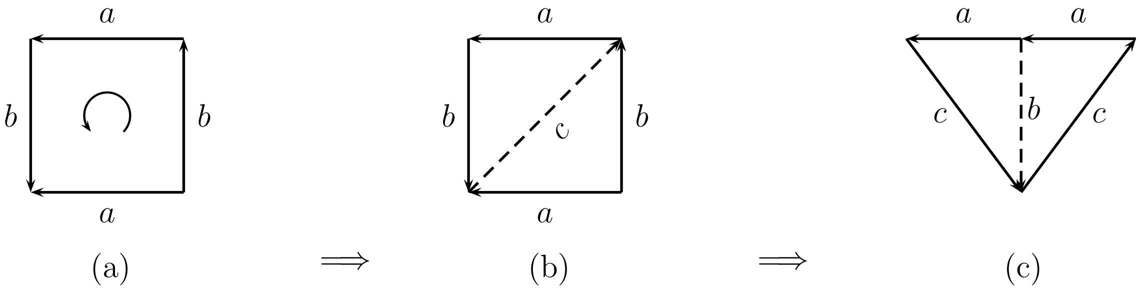

Figure 6.9: Example of elementary subdivision (P2) and its inverse

We say that a cell complex, K', is a *refinement* of a cell complex, K, if K and K' are related in the reflexive and transitive closure of the elementary subdivision relation and we say that K and K' are *equivalent* if they are related in the least equivalence relation containing the elementary subdivision relation.

Operation (P1) is Seifert and Threlfall's *cutting of dimension* 1 and operation (P2) is Seifert and Threlfall's *cutting of dimension* 2, see Seifert and Threlfall [44], Chapter VI, Section 37.

For example, if we apply (P1) twice to the cell complex for the projective plane shown in Figure 6.8 (a), we get the cell complex shown in Figure 6.8 (b).

As another example, we can apply (P2) to the cell complex, K, consisting of a single face with boundary $aba^{-1}b$ to obtain a cell complex with two faces with boundaries, abc and $c^{-1}a^{-1}b$. Then, we can glue these two faces along the edge labeled b using $(\mathrm{P2})^{-1}$ and we get a cell complex with boundary $aacc$, that is, a Klein bottle. This sequence of operations is shown in Figure 6.9.

As we will see shortly, every cell complex is equivalent to some special cell complex in normal form. First, we show that a topological space, $|K|$, can be associated with a cell complex, K, that this space is the same for all cell complexes equivalent to K and that it is a surface.

Given a cell complex, K, we associate with K a topological space, $|K|$, as follows. Let us first assume that no face has the empty sequence as a boundary. Then, we assign to each face, A, a circular disk, and if the boundary of A is $a_1 \ldots a_m$, we divide the boundary of the disk into m oriented arcs. These arcs, in clockwise order, are named $a_1 \ldots a_m$, while the opposite arcs are named $a_1^{-1} \ldots a_m^{-1}$. We then form the quotient space obtained by identifying arcs having the same name in the various disks (this requires using homeomorphisms between arcs named identically, *etc.*).

We leave as an exercise to prove that equivalent cell complexes are mapped to homeomorphic spaces, and that if K represents a triangulated complex, then $|K|$ is homeomorphic to K_g.

When K has a single face A with the null boundary, by (P2), K is equivalent to the cell complex with two faces, A', A'', where A' has boundary d and A'' has boundary d^{-1}. In this case, $|K|$ must be homeomorphic to a sphere.

In order to show that the space, $|K|$, associated with a cell complex is a surface, we prove that every cell complex can be refined to a triangulated 2-complex.

Proposition 6.1 *Every cell complex, K, can be refined to a triangulated 2-complex.*

Proof. Details are given in Ahlfors and Sario [1] and we only indicate the main steps. The idea is to subdivide the cell complex by adding new edges. Informally, it is helpful to view the process as adding new vertices and new edges, but since vertices are not primitive objects, this must be done via the refinement operations (P1) and (P2). The first step is to split every edge a into two edges b and c where $b \neq c$, using (P1), introducing new border vertices (b, c^{-1}). The effect is that a and a^{-1} lead to distinct vertices for every (new) edge a. Then, for every boundary $B = a_1 \ldots a_n$, we have $n \geq 2$ and, intuitively, we create a "central vertex", $\beta = (d_1, \ldots, d_n)$, and we join this vertex β to every vertex including the newly created vertices (except β itself). This is done as follows: first, using (P2), split the boundary $B = a_1 \ldots a_n$ into $a_1 d$ and $d^{-1} a_2 \ldots a_n$, and then using (P1), split d into $d_1 d_n^{-1}$, getting boundaries $d_n^{-1} a_1 d_1$ and $d_1^{-1} a_2 \ldots a_n d_n$. Applying (P2) to the boundary $d_1^{-1} a_2 \ldots a_n d_n$, we get the boundaries $d_1^{-1} a_2 d_2$, $d_2^{-1} a_3 d_3, \ldots$, $d_{n-1}^{-1} a_n d_n$, and $\beta = (d_1, \ldots, d_n)$ is indeed an inner vertex. At the end of this step, it is easy to verify that (C2) and (C3) are satisfied, but (C1) may not. Finally, we split each new triangular boundary, $a_1 a_2 a_3$, into four subtriangles, by joining the middles of its three sides. This is done by getting $b_1 c_1 b_2 c_2 b_3 c_3$, using (P1), and then $c_1 b_2 d_3$, $c_2 b_3 d_1$, $c_3 b_1 d_2$, and $d_1^{-1} d_2^{-1} d_3^{-1}$, using (P2). The resulting cell complex also satisfies (C1) and, in fact, what we have done is to provide a triangulation. □

Next, we need to define cell complexes in normal form. First, we need to define what we mean by orientability of a cell complex, and to explain how we compute its Euler-Poincaré characteristic.

Definition 6.4 Given a cell complex, $K = (F, E, B)$, an *orientation of* K is the choice of one of the two oriented faces, A, A^{-1}, for every face, $A \in F$. An orientation is *coherent* if for every edge, a, if a occurs twice in the boundaries, then a occurs in the boundary of a face, A_1, and in the boundary of a face, A_2^{-1}, where $A_1 \neq A_2$. In other words, if a occurs twice in the boundaries, then it appears once as a and once as a^{-1}. A cell complex, K, is *orientable* if is has some coherent orientation. A *contour* of a cell complex is a cyclically ordered sequence, $(a_1, \ldots, a_n)$, of edges such that a_i and a_{i+1}^{-1} lead to the same vertex and the a_i belong to a single boundary.

It is easily seen that equivalence of cell complexes preserves orientability. In counting contours, we do not distinguish between $(a_1, \ldots, a_n)$ and $(a_n^{-1}, \ldots, a_1^{-1})$. It is easily verified that (P1) and (P2) do not change the number of contours.

Given a cell complex, $K = (F, E, B)$, the number of vertices is denoted as n_0, the number n_1 of edges is the number of elements in E, and the number n_2 of faces is the number of elements in F. The Euler-Poincaré characteristic of K is $n_0 - n_1 + n_2$. It is easily seen that (P1) increases n_1 by 1, creates one more vertex, and leaves n_2 unchanged. Also, (P2) increases n_1 and n_2 by 1 and leaves n_0 unchanged. Thus, equivalence preserves the Euler-Poincaré characteristic. However, we need a small adjustment in the case where K has a single face A with the null boundary. In this case, we agree that K has the "null vertex", ϵ. We now define the normal forms of cell complexes. As we shall see, these normal forms have a single face and a single inner vertex.

Definition 6.5 A *cell complex in normal form, or canonical cell complex* is a cell complex, $K = (F, E, B)$, where $F = \{A\}$ is a singleton set, and either

(I) $E = \{a_1, \ldots, a_p, b_1, \ldots, b_p, c_1, \ldots, c_q, h_1, \ldots, h_q\}$ and

$$B(A) = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_p b_p a_p^{-1} b_p^{-1} c_1 h_1 c_1^{-1} \cdots c_q h_q c_q^{-1},$$

where $p \geq 0$, $q \geq 0$, or

(II) $E = \{a_1, \ldots, a_p, c_1, \ldots, c_q, h_1, \ldots, h_q\}$ and

$$B(A) = a_1 a_1 \cdots a_p a_p c_1 h_1 c_1^{-1} \cdots c_q h_q c_q^{-1},$$

where $p \geq 1$, $q \geq 0$.

Some examples of normal forms of surfaces without boundaries are shown in Figure 6.10.

Observe that canonical complexes of type (I) are orientable, whereas canonical complexes of type (II) are not. The sequences $c_i h_i c_i^{-1}$ yield q border vertices, (h_i, c_i, h_i^{-1}), and thus q contours (h_i), and in case (I), the single inner vertex,

$$(a_1^{-1}, b_1, a_1, b_1^{-1} \ldots, a_p^{-1}, b_p, a_p, b_p^{-1}, c_1^{-1}, \ldots, c_q^{-1}),$$

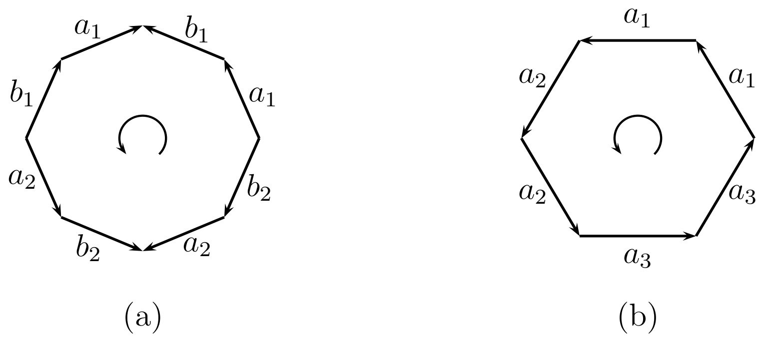

Figure 6.10: Examples of Normal Forms: (a) Type I; (b) Type II.

and in case (II), the single inner vertex,

$$(a_1^{-1}, a_1, \ldots, a_p^{-1}, a_p, c_1^{-1}, \ldots, c_q^{-1}).$$

Thus, in case (I), there are $q+1$ vertices, $2p+2q$ sides, and one face, and the Euler-Poincaré characteristic is $q+1-(2p+2q)+1 = 2-2p-q$, that is,

$$\chi(K) = 2 - 2p - q,$$

and in case (II), there are $q+1$ vertices, $p+2q$ sides, and one face, and the Euler-Poincaré characteristic is $q+1-(p+2q)+1 = 2-p-q$, that is,

$$\chi(K) = 2 - p - q.$$

Note that when $p = q = 0$, we do get $\chi(K) = 2$, which agrees with the fact that in this case, we assumed the existence of a null vertex and there is one face. This is the case of the sphere.

The above shows that distinct canonical complexes, K_1 and K_2, are inequivalent, since otherwise $|K_1|$ and $|K_2|$ would be homeomorphic, which would imply that K_1 and K_2 have the same number of contours, the same kind of orientability, and the same Euler-Poincaré characteristic.

It remains to prove that every cell complex is equivalent to a canonical cell complex, but first, it is helpful to give more intuition regarding the nature of the canonical complexes.

If a canonical cell complex has the border, $B(A) = a_1b_1a_1^{-1}b_1^{-1}$, we can think of the face A as a square whose opposite edges are oriented the same way, and labeled the same way, so that by identification of the opposite edges labeled a_1 and then of the edges labeled b_1, we get a surface homeomorphic to a torus. Figure 6.11 shows such a cell complex.

If we start with a sphere and glue a torus onto the surface of the sphere by removing some small disk from both the sphere and the torus and gluing along the boundaries of the

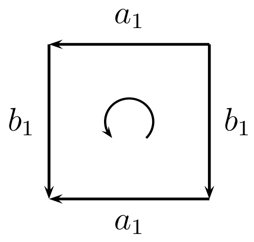

Figure 6.11: A cell complex corresponding to a torus

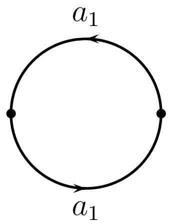

Figure 6.12: A cell complex corresponding to a projective plane

holes, it is as if we had added a handle to the sphere. For this reason, the string $a_1b_1a_1^{-1}b_1^{-1}$ is called a *handle*. A canonical cell complex with boundary $a_1b_1a_1^{-1}b_1^{-1}\cdots a_pb_pa_p^{-1}b_p^{-1}$ can be viewed as the result of attaching p handles to a sphere.

If a canonical cell complex has the border, $B(A) = a_1a_1$, we can think of the face A as a circular disk whose boundary is divided into two semi-circles both labeled a_1. The corresponding surface is obtained by identifying diametrically opposed points on the boundary and thus, it is homeomorphic to the projective plane. Figure 6.12 illustrates this situation.

There is a way of performing such an identification resulting in a surface with self-intersection called a *cross-cap*. As pointed out in Section 1.2, a nice description of the process of getting a cross-cap is given in Hilbert and Cohn-Vossen [22]. A string of the form aa is called a *cross-cap*. Generally, a canonical cell complex with boundary $a_1a_1\cdots a_pa_p$ can be viewed as the result of forming $p \geq 1$ cross-caps, starting from a circular disk with $p-1$ circular holes, and performing the cross-cap identifications on all p boundaries, including the original disk itself.

A string of the form $c_1h_1c_1^{-1}$ occurring in a border can be interpreted as a hole with boundary h_1. For instance, if the boundary of a canonical cell complex is $c_1h_1c_1^{-1}$, splitting the face A into the two faces A' and A'' with boundaries $c_1h_1c_1^{-1}d$ and d^{-1}, we can view the face A' as a disk with boundary d in which a small circular disk has been removed. Choosing any point on the boundary d of A', we can join this point to the boundary h_1 of the small circle by an edge c_1, and we get a path $c_1h_1c_1^{-1}d$. The path is a closed loop, and a string of the form $c_1h_1c_1^{-1}$ is called a *loop*. Figure 6.13 illustrates this situation.

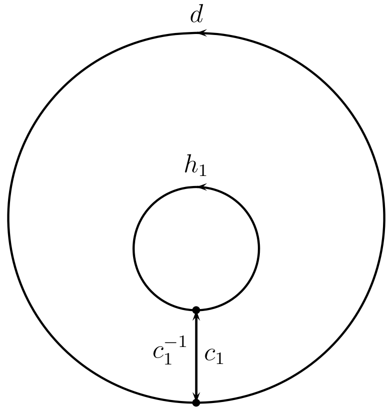

Figure 6.13: A disk with a hole

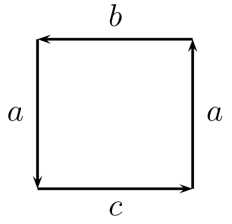

Figure 6.14: A cell complex corresponding to a Möbius strip

We now prove a combinatorial lemma which is the key to the classification of the compact surfaces. First, note that the inverse of the reduction step (P1), denoted by $(\mathrm{P1})^{-1}$, applies to a string of edges bc provided that $b \neq c$ and (b, c^{-1}) is a vertex. The result is that such a border vertex is eliminated. The inverse of the reduction step (P2), denoted by $(\mathrm{P2})^{-1}$, applies to two faces A_1 and A_2 such that $A_1 \neq A_2$, $A_1 \neq A_2^{-1}$, and $B(A_1)$ contains some edge d and $B(A_2)$ contains the edge d^{-1}. The result is that d (and d^{-1}) is eliminated. As a preview of the proof, we show that the following cell complex, obviously corresponding to a Möbius strip, is equivalent to the cell complex of type (II) with boundary $aachc^{-1}$. The boundary of the cell complex shown in Figure 6.14 is $abac$.

First using (P2), we split $abac$ into abd and $d^{-1}ac$. Since $abd = bda$ and the inverse face of $d^{-1}ac$ is $c^{-1}a^{-1}d = a^{-1}dc^{-1}$, by applying $(\mathrm{P2})^{-1}$, we get $bddc^{-1} = ddc^{-1}b$. We can now apply $(\mathrm{P1})^{-1}$, getting ddk. We are almost there, except that the complex with boundary ddk has no inner vertex. We can introduce one as follows. Split d into bc, getting $bcbck = cbckb$. Next, apply (P2), getting cba and $a^{-1}ckb$. Since $cba = bac$ and the inverse face of $a^{-1}ckb$ is $b^{-1}k^{-1}c^{-1}a = c^{-1}ab^{-1}k^{-1}$, by applying $(\mathrm{P2})^{-1}$ again, we get $baab^{-1}k^{-1} = aab^{-1}k^{-1}b$, which is of the form $aachc^{-1}$, with $c = b^{-1}$ and $h = k^{-1}$. Thus, the canonical cell complex with boundary $aachc^{-1}$ has the Möbius strip as its geometric realization. Intuitively, this corresponds to cutting out a small circular disk in a projective plane. This process is very nicely described in Hilbert and Cohn-Vossen [22].

Lemma 6.2 *Every cell complex, K, is equivalent to some canonical cell complex.*

Proof. All the steps are given in Ahlfors and Sario [1] and, in a slightly different and more informal manner, in Seifert and Threlfall [44] and Massey [32]. We will only give the keys steps, referring the reader to the above sources for details.

The proof proceeds by steps that bring the original cell complex closer to normal form.

Step 1. Elimination of strings aa^{-1} in boundaries.

Given a boundary of the form $aa^{-1}X$, where X denotes some string of edges (possibly empty), we can use (P2) to replace $aa^{-1}X$ by the two boundaries ad and $d^{-1}a^{-1}X$, where d is new. But then, using (P1), we can contract ad to a new edge c (and $d^{-1}a^{-1}$ to c^{-1}). But now, using $(\mathrm{P2})^{-1}$, we can eliminate c. The net result is the elimination of aa^{-1}.

Step 2. Vertex Reduction.

If $p = 0, q = 0$, there is only the empty vertex and there is nothing to do. Otherwise, the purpose of this step is to obtain a cell complex with a single inner vertex and where border vertices correspond to loops. First, we perform step 1 until all occurrences of the form aa^{-1} have been eliminated.

Consider an inner vertex, $\alpha = (b_1, \ldots, b_m)$. If b_i^{-1} also belongs to α for all i, $1 \leq i \leq m$, and there is another inner vertex, β, since all vertices are connected, there is some inner vertex, $\delta \neq \alpha$, directly connected to α, which means that either some b_i or b_i^{-1} belongs to δ. But since the vertices form a partition of $E \cup E^{-1}$, $\alpha = \delta$, a contradiction.

Thus, if $\alpha = (b_1, \ldots, b_m)$ is not the only inner vertex, we can assume by relabeling that b_1^{-1} does not belong to α. Also, we must have $m \geq 2$, since otherwise there would be a string $b_1b_1^{-1}$ in some boundary, contrary to the fact that we performed step 1 all the way. Thus, there is a string $b_1b_2^{-1}$ in some boundary. We claim that we can eliminate b_2. Indeed, since α is an inner vertex, b_2 must occur twice in the set of boundaries, and thus, since b_2^{-1} is a successor of b_1, there are boundaries of the form $b_1b_2^{-1}X_1$ and b_2X_2, and using (P2), we can split $b_1b_2^{-1}X_1$ into $b_1b_2^{-1}c$ and $c^{-1}X_1$, where c is new. Since b_2 differs from b_1, b_1^{-1}, c, c^{-1}, we can eliminate b_2 by $(\mathrm{P2})^{-1}$ applied to $b_2X_2 = X_2b_2$ and $b_1b_2^{-1}c = b_2^{-1}cb_1$, getting $X_2cb_1 = cb_1X_2$. This has the effect of shrinking α. Indeed, the existence of the boundary cb_1X_2 implies that c and b_1^{-1} lead to the same vertex and the existence of the boundary $b_1b_2^{-1}c$ implies that c^{-1} and b_2^{-1} lead to the same vertex, and if b_2^{-1} does not belong to α, then b_2 is dropped, or if b_2^{-1} belongs to α, then c^{-1} is added to α, but both b_2 and b_2^{-1} are dropped.

This process can be repeated until $\alpha = (b_1)$, at which stage, b_1, is eliminated using step 1. Thus, it is possible to eliminate all inner vertices except one. In the event that there was no inner vertex, we can always create one using (P1) and (P2) as in the proof of Proposition 6.1. Thus, from now on, we will assume that there is a single inner vertex.

We now show that border vertices can be reduced to the form (h, c, h^{-1}). The previous argument shows that we can assume that there is a single inner vertex α. A border vertex is of the form, $\beta = (h, b_1, \ldots, b_m, k)$, where h, k are border edges, and the b_i are inner edges. We claim that there is some border vertex, $\beta = (h, b_1, \ldots, b_m, k)$, where some b_i^{-1} belongs to the inner vertex, α. Indeed, since K is connected, every border vertex is connected to α, and thus, there is a least one border vertex, $\beta = (h, b_1, \ldots, b_m, k)$, directly connected to α by some edge. Observe that h^{-1} and b_1^{-1} lead to the same vertex and, similarly, b_m^{-1} and k^{-1} lead to the same vertex. Thus, if no b_i^{-1} belongs to α, either h^{-1} or k^{-1} belongs to α, which would imply that either b_1^{-1} or b_m^{-1} is in α. Thus, such an edge from β to α must be one of the b_i^{-1}. Then by the reasoning used in the case of an inner vertex, we can eliminate all b_j except b_i, and the resulting vertex is of the form (h, b_i, k). If $h \neq k^{-1}$, we can also eliminate b_i since h^{-1} does not belong to (h, b_i, k), and the vertex (h, k) can be eliminated using $(\mathrm{P1})^{-1}$.

One can verify that reducing a border vertex to the form (h, c, h^{-1}) does not undo the reductions already performed and thus, at the end of step 2, we either obtain a cell complex with a null inner node and loop vertices, or a single inner vertex and loop vertices.

Step 3. Reduction to a single face and introduction of cross-caps.

We may still have several faces. We claim that if there are at least two faces, then for every face, A, there is some face, B, such that $B \neq A$, $B \neq A^{-1}$, and there is some edge, a, both in the boundary of A and in the boundary of B. If this was not the case, there would be some face, A, such that for every face, B, such that $B \neq A$ and $B \neq A^{-1}$, every edge, a, in the boundary of B does not belong to the boundary of A. Then, every inner edge, a, occurring in the boundary of A must have both of its occurrences in the boundary of A and, of course, every border edge in the boundary of A occurs once in the boundary of A alone. But then, the cell complex consisting of the face A alone and the edges occurring in its boundary would form a proper subsystem of K, contradicting the fact that K is connected.

Thus, if there are at least two faces, from the above claim and using $(\mathrm{P2})^{-1}$, we can reduce the number of faces down to one. It easy to check that no new vertices are introduced and loops are unaffected.

Next, if some boundary contains two occurrences of the same edge, a, i.e., it is of the form, $aXaY$, where X, Y denote strings of edges, with $X, Y \neq \epsilon$, we show how to make the two occurrences of a adjacent. Symbolically, we show that the following pseudo-rewrite rule is admissible:

$$aXaY \simeq bbY^{-1}X, \quad \text{or} \quad aaXY \simeq bYbX^{-1}.$$

Indeed, $aXaY$ can be split into aXb and $b^{-1}aY$, and since we also have the boundary

$$(b^{-1}aY)^{-1} = Y^{-1}a^{-1}b = a^{-1}bY^{-1},$$

together with $aXb = Xba$, we can apply $(\mathrm{P2})^{-1}$ to Xba and $a^{-1}bY^{-1}$, obtaining $XbbY^{-1} = bbY^{-1}X$, as claimed. Thus, we can introduce cross-caps.

Using the formal rule $aXaY \simeq bbY^{-1}X$ again does not alter the previous loops and cross-caps. By repeating step 3, we convert boundaries of the form $aXaY$ to boundaries with cross-caps.

Step 4. Introduction of handles.

The purpose of this step is to convert boundaries of the form $aUbVa^{-1}Xb^{-1}Y$ to boundaries $cdc^{-1}d^{-1}YXVU$ containing handles. First, we prove the pseudo-rewrite rule

$$aUVa^{-1}X \simeq bVUb^{-1}X.$$

First, we split $aUVa^{-1}X$ into $aUc = Uca$ and $c^{-1}Va^{-1}X = a^{-1}Xc^{-1}V$, and then we apply $(\mathrm{P2})^{-1}$ to Uca and $a^{-1}Xc^{-1}V$, getting $UcXc^{-1}V = c^{-1}VUcX$. Letting $b = c^{-1}$, the rule follows.

Now we apply the rule to $aUbVa^{-1}Xb^{-1}Y$, and we get

$$\begin{aligned} aUbVa^{-1}Xb^{-1}Y &\simeq a_1bVUa_1^{-1}Xb^{-1}Y \\ &\simeq a_1b_1a_1^{-1}XVUb_1^{-1}Y = a_1^{-1}XVUb_1^{-1}Ya_1b_1 \\ &\simeq a_2^{-1}b_1^{-1}YXVUa_2b_1 = a_2b_1a_2^{-1}b_1^{-1}YXVU. \end{aligned}$$

Iteration of this step preserves existing loops, cross-caps and handles.

Step 5. Transformation of handles into cross-caps.

At this point, one of the obstacle to the canonical form is that we may still have a mixture of handles and cross-caps. We now show that a handle and a cross-cap is equivalent to three cross-caps. For this, we apply the pseudo-rewrite rule $aaXY \simeq bYbX^{-1}$. We have

$$\begin{aligned} aaXbcb^{-1}c^{-1}Y &\simeq a_1b^{-1}c^{-1}Ya_1c^{-1}b^{-1}X^{-1} = b^{-1}c^{-1}Ya_1c^{-1}b^{-1}X^{-1}a_1 \\ &\simeq b_1^{-1}b_1^{-1}a_1^{-1}Xc^{-1}Ya_1c^{-1} = c^{-1}Ya_1c^{-1}b_1^{-1}b_1^{-1}a_1^{-1}X \\ &\simeq c_1^{-1}c_1^{-1}X^{-1}a_1b_1b_1Ya_1 = a_1b_1b_1Ya_1c_1^{-1}c_1^{-1}X^{-1} \\ &\simeq a_2a_2Xc_1c_1b_1b_1Y. \end{aligned}$$

At this stage, we can prove that all boundaries consist of loops, cross-caps, or handles. The details can be found in Ahlfors and Sario [1].

Step 6. Grouping loops togehter.

Finally, we have to group the loops together. This can be done using the pseudo-rewrite rule

$$aUVa^{-1}X \simeq bVUb^{-1}X.$$

Indeed, we can write

$$chc^{-1}Xdkd^{-1}Y = c^{-1}Xdkd^{-1}Ych \simeq c_1^{-1}dkd^{-1}YXc_1h = c_1hc_1^{-1}dkd^{-1}YX,$$

showing that any two loops can be brought next to each other, without altering other successions.

When all this is done, we have obtained a canonical form and the proof is complete. □

Remarks: For the benefit of the reader, we compare the proof given here (due to Ahlfors and Sario [1]) to other proofs, in particular, the proof given in Seifert and Threlfall [44] (Chapter VI). The first point is that Ahlfors and Sario's proof applies to surfaces with boundaries whereas Seifert and Threlfall first give a proof for surfaces without boundaries in Section 38 and then they show how to modify this proof to deal with boundaries in Section 40. As we said earlier, Ahlfors and Sario use the same elementary transformations as Seifert and Threlfall. Unlike Seifert and Threlfall, who begin by gluing cells sharing a common edge using $(P2)^{-1}$ to obtain a complex with a single cell (their Step 1), Ahlfors and Sario reduce the complex to a single cell at the beginning of their Step 3. Step 1 of Ahlfors and Sario (eliminating strings aa^{-1}) is identical to Step 2 of Seifert and Threlfall. Step 2 of Ahlfors and Sario (reduction to a single vertex) is similar to Step 3 of Seifert and Threlfall, except that Ahlfors and Sario deal with border vertices. Step 3 of Ahlfors and Sario (cross-cap introduction) is analogous to Step 4 of Seifert and Threlfall and also preserves loops. Step 4 of Ahlfors and Sario (handle introduction) is analogous to Step 5 of Seifert and Threlfall and also preserves loops. Step 5 of Ahlfors and Sario (transformation of handles into cross-caps) is analogous to Step 6 of Seifert and Threlfall. Finally, Step 6 of Ahlfors and Sario consists in grouping loops together. This step is achieved by Seifert and Threlfall in Section 40.

The proof given by Fréchet and Fan [17] is identical to Seifert and Threlfall's proof [44]. Massey [32] gives a similar proof except that he does not use the transformation rule (P1) in eliminating pairs of the form aa^{-1}. In this respect, Massey's proof is closer to Brahana's proof [6] (1921). The cutting rule (P2) seems to have been first introduced by Brahana although Brahana states that the *method of cutting* was first presented by Veblen in a Seminar given in 1915. Every proof using the method of cutting uses the proof steps first presented in Brahana [6], but Brahana only deals with surfaces without boundaries.

Readers familiar with formal grammars or rewrite rules may be intrigued by the use of the "rewrite rules"

$$aXaY \simeq bbY^{-1}X$$

or

$$aUVa^{-1}X \simeq bVUb^{-1}X.$$

These rules are context-sensitive, since X and Y stand for parts of boundaries, but they also apply to objects not traditionally found in formal language theory or rewrite rule theory. Indeed, the objects being rewritten are cell complexes, which can be viewed as certain kinds of graphs. Furthermore, since boundaries are invariant under cyclic permutations, these rewrite rules apply modulo cyclic permutations, something that I have never encountered in the rewrite rule literature. Thus, it appears that a formal treatment of such rewrite rules has not been given yet, which poses an interesting challenge to researchers in the field of rewrite rule theory. For example, are such rewrite systems confluent, can normal forms be easily found?

We have already observed that identification of the edges in the boundary, $aba^{-1}b^{-1}$, yields a torus. We have also noted that identification of the two edges in the boundary, aa, yields the projective plane. Lemma 6.2 implies that the cell complex consisting of a single face, A, and the boundary, $abab^{-1}$, is equivalent to the canonical cell complex, $ccbb$. This follows immediately from the pseudo-rewrite rule $aXaY \simeq bbY^{-1}X$. However, it is easily seen that identification of edges in the boundary $abab^{-1}$ yields the Klein bottle. The lemma also showed that the cell complex with boundary, $aabbcc$, is equivalent to the cell complex with boundary, $aabcb^{-1}c^{-1}$. Thus, intuitively, it seems that the corresponding space is a simple combination of a projective plane and a torus, or of three projective planes.

We will see shortly that there is an operation on surfaces (the connected sum) which allows us to interpret the canonical cell complexes as combinations of elementary surfaces, the sphere, the torus, and the projective plane.

6.3 Proof of the Classification Theorem

Having the key Lemma 6.2 at hand, we can finally prove the fundamental theorem of the classification of triangulated compact surfaces and compact bordered surfaces.

Theorem 6.3 *Two (two-dimensional) compact polyhedra or compact bordered polyhedra (triangulated compact surfaces or compact bordered surfaces) are homeomorphic iff they agree in character of orientability, number of contours, and Euler-Poincaré characteristic.*

Proof. If $M_1 = (K_1)_g$ and $M_2 = (K_2)_g$ are homeomorphic, we know that M_1 is orientable iff M_2 is orientable, and the restriction of the homeomorphism between M_1 and M_2 to the boundaries, ∂M_1 and ∂M_2, is a homeomorphism, which implies that ∂M_1 and ∂M_2 have the same number of arcwise components, that is, the same number of contours. Also, we have stated that homeomorphic spaces have isomorphic homology groups and, by Theorem 5.8, they have the same Euler-Poincaré characteristic. Conversely, by Lemma 6.2, since any cell complex is equivalent to a canonical cell complex, the triangulated 2-complexes K_1 and K_2, viewed as cell complexes, are equivalent to canonical cell complexes C_1 and C_2. However, we know that equivalence preserves orientability, the number of contours, and the Euler-Poincaré characteristic, which implies that C_1 and C_2 are identical. But then, $M_1 = (K_1)_g$ and $M_2 = (K_2)_g$ are both homeomorphic to $|C_1| = |C_2|$. $\square$

This completes the combinatorial part of the proof of the classification theorem. In order to finally get a version of Theorem 6.3 for compact surfaces or compact bordered surfaces (not necessarily triangulated), it is necessary to prove that every surface and every bordered surface can be triangulated. As we said in Section 1.1, this is indeed true, but the proof is far from trivial. Radó's proof (going back) to 1925 is presented in Alhfors and Sario [1]. Simpler and shorter proofs were given later by Doyle and Moran [14] (1968) and Thomassen

[46] (1992). We will present Carsten Thomassen's proof which we consider to be the most easily accessible in Appendix D.

It is interesting to note that 3-manifolds can be triangulated (E. Moise, 1952) but that Markov showed that deciding whether two triangulated 4-manifolds are homeomorphic is undecidable (1958). For the record, we state the following theorem putting all the pieces of the puzzle together.

Theorem 6.4 *Two compact surfaces or compact bordered surfaces are homeomorphic iff they agree in character of orientability, number of contours, and Euler-Poincaré characteristic.*

We now explain somewhat informally what is the connected sum operation and how it can be used to interpret the canonical cell complexes. We will also indicate how the canonical cell complexes can be used to determine the fundamental groups of the compact surfaces and compact bordered surfaces.

Definition 6.6 Given two surfaces, S_1 and S_2, their *connected sum*, $S_1 \sharp S_2$, is the surface obtained by choosing two small regions, D_1 and D_2, on S_1 and S_2, both homeomorphic to some disk in the plane, and letting h be a homeomorphism between the boundary circles C_1 and C_2 of D_1 and D_2, by forming the quotient space of $(S_1 - \overset{\circ}{D}_1) \cup (S_2 - \overset{\circ}{D}_2)$, by the equivalence relation defined by the relation $\{(a, h(a)) \mid a \in C_1\}$.

Intuitively, $S_1 \sharp S_2$ is formed by cutting out some small circular hole in each surface, and gluing the two surfaces along the boundaries of these holes. It can be shown that $S_1 \sharp S_2$ is a surface and that it does not depend on the choice of D_1, D_2, and h. Also, if S_2 is a sphere, then $S_1 \sharp S_2$ is homeomorphic to S_1. It can also be shown that the Euler-Poincaré characteristic of $S_1 \sharp S_2$ is given by the formula

$$\chi(S_1 \sharp S_2) = \chi(S_1) + \chi(S_2) - 2.$$

Then, we can give an interpretation of the geometric realization of a canonical cell complex. It turns out to be the connected sum of some elementary surfaces. Ignoring borders for the time being, assume that we have two canonical cell complexes S_1 and S_2 represented by circular disks with borders

$$B_1 = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_{p_1} b_{p_1} a_{p_1}^{-1} b_{p_1}^{-1}$$

and

$$B_2 = c_1 d_1 c_1^{-1} d_1^{-1} \cdots c_{p_2} d_{p_2} c_{p_2}^{-1} d_{p_2}^{-1}.$$

Cutting a small hole with boundary h_1 in S_1 amounts to forming the new boundary

$$B_1' = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_{p_1} b_{p_1} a_{p_1}^{-1} b_{p_1}^{-1} h_1,$$

and similarly, cutting a small hole with boundary h_2 in S_2 amounts to forming the new boundary

$$B_2' = c_1 d_1 c_1^{-1} d_1^{-1} \cdots c_{p_2} d_{p_2} c_{p_2}^{-1} d_{p_2}^{-1} h_2^{-1}.$$

If we now glue S_1 and S_2 along h_1 and h_2, we get a figure looking like two convex polygons glued together along one edge, and by deformation, we get a circular disk with boundary

$$B = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_{p_1} b_{p_1} a_{p_1}^{-1} b_{p_1}^{-1} c_1 d_1 c_1^{-1} d_1^{-1} \cdots c_{p_2} d_{p_2} c_{p_2}^{-1} d_{p_2}^{-1}.$$

A similar reasoning applies to cell complexes of type (II).

As a consequence, the geometric realization of a cell complex of type (I) is either a sphere, or the connected sum of $p \geq 1$ tori, and the geometric realization of a cell complex of type (II) is the connected sum of $p \geq 1$ projective planes. Furthermore, the equivalence of the cell complexes consisting of a single face, A, and the boundaries, $abab^{-1}$ and $aabb$, shows that the connected sum of two projective planes is homeomorphic to the Klein bottle. Also, the equivalence of the cell complexes with boundaries $aabbcc$ and $aabcb^{-1}c^{-1}$ shows that the connected sum of a projective plane and a torus is equivalent to the connected sum of three projective planes. Thus, we obtain another form of the classification theorem for compact surfaces.

Theorem 6.5 *Every orientable compact surface is homeomorphic either to a sphere or to a connected sum of tori. Every nonorientable compact surface is homeomorphic either to a projective plane, or a Klein bottle, or the connected sum of a projective plane or a Klein bottle with some tori.*

If bordered compact surfaces are considered, a similar theorem holds, but holes have to be made in the various spaces forming the connected sum. For more details, the reader is referred to Massey [32], in which it is also shown how to build models of bordered surfaces by gluing strips to a circular disk.

6.4 Other Combinatorial Proofs

Most proofs of the classification theorem reproduce Brahana's proof or slight modifications of his proof. The main modification has to do with the introduction of a special rule to split an edge into two edges (and, conversely, to merge two incident edges into a single edge), a transformation not used by Brahana, who eliminates occurrences of the pattern aa^{-1} during the reduction to a single vertex. Such a splitting (or merging) is used by Seifert and Threlfall [44] (*subdivision or gluing of dimension* 1, Chapter 6, page 138) in addition to the rule for splitting a polygon or merging two polygons along an edge (*subdivision or gluing of dimension* 2, page 138). Brahana's reduction algorithm uses two phases. During the first phase (Reduction 1, page 147), the surface is transformed to a representation with a single vertex. During the second phase (Reduction 2, page 147-151), handles and cross-caps are normalized and in the case of a non-orientable surface, handles are converted to pairs of cross-caps. An early textbook presentation of the classification theorem appears in de Kerékjártó [25] (who also considers non-compact surfaces). Other early textbook presentations appear in Levi [30] and Reidemeister [39].

Seifert and Threlfall decompose phase 1 into three steps, where the second step performs *side cancellations* (removal of pairs aa^{-1}) using edge splitting, polygon gluing and edge merging. Phase 2 consists of three steps identical to those used by Brahana. Furthermore, Seifert and Threlfall also extend the reduction procedure to (compact) surfaces with boundaries [44] (Section 40).

Proofs modeled after Seifert and Threlfall's proof are also given in Fréchet and Fan [17], Massey [32], Henle [21], Kinsey [26], Bloch [5] and Fulton [18] (although Massey does not use edge splitting-merging rules). A proof involving *surgery* is given in Amstrong [3] and Andrews [4] (1988).

Other proofs have been given by Burgess [8] (1985), Thomassen [46] (1992) and Francis and Weeks (Conway's ZIP proof) [16] (1999). Those three proofs adopt a notion of normal form based on the notion of connected sum, as in Theorem 6.5, except that Burgess replaces disjoint discs with either a Möbius strip or a punctured torus. Conway's ZIP proof is quite intuitive and Francis and Weeks provide many amusing illustrations. Thomassen's proof is the most elementary and even yields a formula for the Euler characteristic. On the other hands, since these proofs rely on normal forms different from the one used in Section 6.2, they do not give quite as much information.

6.5 Application of the Main Theorem: Determining the Fundamental Groups of Compact Surfaces

We now explain briefly how the canonical forms can be used to determine the fundamental groups of the compact (bordered) surfaces. This is done in two steps. The first step consists in defining a group structure on certain closed paths in a cell complex. The second step consists in showing that this group is isomorphic to the fundamental group of $|K|$.

Given a cell complex, $K = (F, E, B)$, recall that a vertex, α, is an equivalence class of edges, under the equivalence relation, Λ, induced by the relation, λ, defined such that, $a\lambda b$ iff b^{-1} is the successor of a in some boundary. Every inner vertex, $\alpha = (b_1, \ldots, b_m)$, can be cyclically ordered such that b_i has b_{i-1}^{-1} and b_{i+1}^{-1} as successors and, for a border vertex, $\alpha = (b_1, \ldots, b_m)$, the same is true for $2 \leq i \leq m-1$, but b_1 only has b_2^{-1} as successor and b_m only has b_{m-1}^{-1} as successor. An edge from α to β is any edge $a \in \beta$ such that $a^{-1} \in \alpha$. For every edge, a, we will call the vertex that a defines the *target* of a and the vertex that a^{-1} defines the *source* of a. Clearly, a is an edge between its source and its target. We now define certain paths in a cell complex, and a notion of deformation of paths.

Definition 6.7 Given a cell complex, $K = (F, E, B)$, a *polygon in* K is any nonempty string, $a_1 \ldots a_m$, of edges such that a_i and a_{i+1}^{-1} lead to the same vertex or, equivalently, such that the target of a_i is equal to the source of a_{i+1}. The source of the path, $a_1 \ldots a_m$, is the source of a_1 (i.e., the vertex that a_1^{-1} leads to), and the target of the path, $a_1 \ldots a_m$, is the target of a_m (i.e., the vertex that a_m leads to). The polygon is *closed* if its source and target coincide. The product of two paths, $a_1 \ldots a_m$ and $b_1 \ldots b_n$, is defined if the target of a_m is equal to the source of b_1 and is the path $a_1 \ldots a_m b_1 \ldots b_n$. Given two paths, $p_1 = a_1 \ldots a_m$ and $p_2 = b_1 \ldots b_n$, with the same source and the same target, we say that p_2 *is an immediate deformation of* p_1 if p_2 is obtained from p_1 by either deleting some subsequence of the form aa^{-1}, or deleting some subsequence X which is the boundary of some face. The smallest equivalence relation containing the immediate deformation relation is called *path-homotopy*.

It is easily verified that path-homotopy is compatible with the composition of paths. Then, for any vertex, α_0, the set of equivalence classes of path-homotopic polygons forms a group, $\pi(K, \alpha_0)$. It is also easy to see that any two groups, $\pi(K, \alpha_0)$ and $\pi(K, \alpha_1)$, are isomorphic, and that if K_1 and K_2 are equivalent cell complexes, then $\pi(K_1, \alpha_0)$ and $\pi(K_2, \alpha_0)$ are isomorphic. Thus, the group, $\pi(K, \alpha_0)$, only depends on the equivalence class of the cell complex, K. Furthermore, it can be proved that the group, $\pi(K, \alpha_0)$, is isomorphic to the fundamental group, $\pi(|K|, (\alpha_0)_g)$, associated with the geometric realization, $|K|$, of K (this is proved in Ahlfors and Sario [1]). It is then possible to determine what these groups are, by considering the canonical cell complexes.

Let us first assume that there are no borders, which corresponds to $q = 0$. In this case, there is only one (inner) vertex, and all polygons are closed. For an orientable cell complex (of type (I)), the fundamental group is the group presented by the generators $\{a_1, b_1, \ldots, a_p, b_p\}$, and satisfying the single equation

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_p b_p a_p^{-1} b_p^{-1} = 1.$$

When $p = 0$, it is the trivial group reduced to 1. For a nonorientable cell complex (of type (II)), the fundamental group is the group presented by the generators $\{a_1, \ldots, a_p\}$, and satisfying the single equation

$$a_1 a_1 \cdots a_p a_p = 1.$$

In the presence of borders, which corresponds to $q \geq 1$, it is easy to see that the closed polygons are products of a_i, b_i, and the $d_i = c_i h_i c_i^{-1}$. For cell complexes of type (I), these generators satisfy the single equation

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_p b_p a_p^{-1} b_p^{-1} d_1 \cdots d_q = 1,$$

and for cell complexes of type (II), these generators satisfy the single equation

$$a_1 a_1 \cdots a_p a_p d_1 \cdots d_q = 1.$$

Using these equations, d_q can be expressed in terms of the other generators, and we get a free group. In the orientable case, we get a free group with $2q + p - 1$ generators, and in the nonorientable case, we get a free group with $p + q - 1$ generators.

The above result shows that there are only two kinds of complexes having a trivial group, namely, for orientable complexes for which $p = q = 0$, or $p = 0$ and $q = 1$. The corresponding (bordered) surfaces are a *sphere* and a *closed disk* (a bordered surface). We can also figure out for which other surfaces the fundamental group is abelian. This happens in the orientable case when $p = 1$ and $q = 0$, a *torus*, or $p = 0$ and $q = 2$, an *annulus*, and in the nonorientable case when $p = 1$ and $q = 0$, a *projective plane*, or $p = 1$ and $q = 1$, a *Möbius strip*.

It is also possible to use the above results to determine the homology groups, $H_1(K)$, of the (bordered) surfaces, since it can be shown that $H_1(K) = \pi(K, a)/[\pi(K, a), \pi(K, a)]$, where $[\pi(K, a), \pi(K, a)]$ is the *commutator subgroup of* $\pi(K, a)$ (see Ahlfors and Sario [1]). Recall that for any group, G, the commutator subgroup is the subgroup of G generated by all elements of the form $aba^{-1}b^{-1}$ (the *commutators*). It is a normal subgroup of G, since for any $h \in G$ and any $d \in [G, G]$, we have $hdh^{-1} = (hdh^{-1}d^{-1})d$, which is also in G. Then, $G/[G, G]$ is abelian and $[G, G]$ is the smallest subgroup of G for which $G/[G, G]$ is abelian.

Applying the above to the fundamental groups of the surfaces, in the orientable case, we see that the commutators cause a lot of cancellation, and we get the equation

$$d_1 + \cdots + d_q = 0,$$

whereas in the nonorientable case, we get the equation

$$2a_1 + \cdots + 2a_p + d_1 + \cdots + d_q = 0.$$

If $q > 0$, we can express d_q in terms of the other generators and, in the orientable case, we get a free abelian group with $2p + q - 1$ generators, whereas in the nonorientable case, a free abelian group with $p + q - 1$ generators. When $q = 0$, in the orientable case, we get a free abelian group with $2p$ generators, and in the nonorientable case, since we have the equation

$$2(a_1 + \cdots + a_p) = 0,$$

there is an element of order 2, and we get the direct sum of a free abelian group of order $p - 1$ with $\mathbb{Z}/2\mathbb{Z}$.

The number p is called the *genus* of the surface. Intuitively, it counts the number of holes in the surface, which is certainly the case in the orientable case, but in the nonorientable case, it is considered that the projective plane has one hole and the Klein bottle has two holes. Of course, the genus of a surface is the number of copies of tori occurring in the canonical connected sum of the surface when orientable (which, when $p = 0$, yields the sphere), or the number of copies of projective planes occurring in the canonical connected sum of the surface when nonorientable. In terms of the Euler-Poincaré characteristic, for an orientable surface, the genus g is given by the formula

$$g = (2 - \chi - q)/2,$$

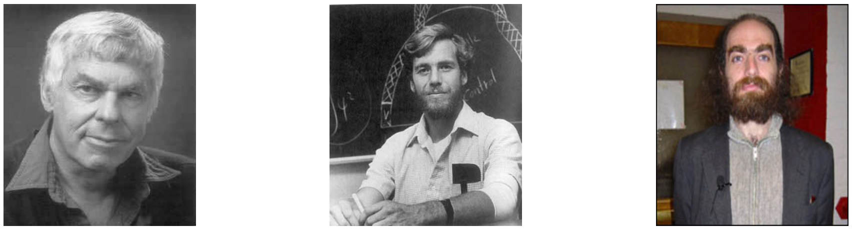

Figure 6.15: Stephen Smale, 1930- (left), Michael Freedman 1951- (middle) and Grigori Perelman, 1966- (right)

and for a nonorientable surface, the genus g is given by the formula

$$g = 2 - \chi - q,$$

where q is the number of contours.

It is rather curious that bordered surfaces, orientable or not, have free groups as fundamental groups (free abelian groups for the homology groups $H_1(K)$). It is also shown in Massey [32] that every bordered surface, orientable or not, can be embedded in $\mathbb{R}^3$. This is not the case for nonorientable surfaces (with an empty border).

Finally, we conclude with a few words about the *Poincaré conjecture*. We observed that the only surface which is simply connected (with a trivial fundamental group) is the sphere. Poincaré conjectured in the early 1900's that the same thing holds for compact simply-connected 3-manifolds, that is, any compact simply-connected 3-manifold is homeomorphic to the 3-sphere S^3.

Remarkably, this famous problem was finally settled by Grigori Perelman in 2006 using a tool from differential geometry known as *Ricci flow*. Now, there is at least some hope to have a classification theory of compact 3-manifolds (recall that 3-manifolds can be triangulated, a result of E. Moise, 1952, see Massey [32]). The generalization of the Poincaré conjecture was shown to be true by Stephen Smale for $m > 4$ in 1960, and true for $m = 4$ by Michael Freedman in 1982. Smale, Freedman and Perelman all received the Fields Medal for their ground-breaking work but Perelman declined this prestigious award!