We now turn to less esoteric surfaces, the nondegenerate quadrics.

Example 13: The ellipsoid.

An ellipsoid is defined by the implicit equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

A nice parametric representation is given by:

$$\begin{aligned} x &= a\cos\theta\sin\varphi, \\ y &= b\sin\theta\sin\varphi, \\ z &= c\cos\varphi. \end{aligned}$$

We have already considered some rational specifications of the ellipsoid in section 23.1. In this example, we get the entire ellipsoid by switching to $u = \tan(\theta/2)$ and $v = \tan(\varphi/4)$. We get

$$\begin{aligned} x &= \frac{4av(1-u^2)(1-v^2)}{(u^2+1)(v^2+1)^2}, \\ y &= \frac{8buv(1-v^2)}{(u^2+1)(v^2+1)^2}, \\ z &= \frac{c(u^2+1)(v^4-6v^2+1)}{(u^2+1)(v^2+1)^2}. \end{aligned}$$

The following control net is obtained for $a = 4$, $b = 3$, and $c = 2$:

```
ellnet6 =   {{0, 0, 2, 1}, {8/3, 0, 2, 1}, {80/17, 0, 18/17, 17/15},
   {36/7, 0, -2/7, 7/5}, {4, 0, -10/7, 28/15}, {2, 0, -2, 8/3},
   {0, 0, -2, 4}, {0, 0, 2, 1}, {8/3, 4/5, 2, 1},
   {80/17, 24/17, 18/17, 17/15}, {36/7, 10/7, -2/7, 7/5},
   {4, 6/7, -10/7, 28/15}, {2, 0, -2, 8/3}, {0, 0, 2, 16/15},
   {9/4, 3/2, 2, 16/15}, {216/55, 144/55, 54/55, 11/9},
   {100/23, 60/23, -10/23, 23/15}, {7/2, 3/2, -3/2, 32/15}, {0, 0, 2, 6/5},
   {14/9, 2, 2, 6/5}, {8/3, 24/7, 6/7, 7/5}, {28/9, 10/3, -2/3, 9/5},
   {0, 0, 2, 7/5}, {16/21, 16/7, 2, 7/5}, {32/25, 96/25, 18/25, 5/3},
   {0, 0, 2, 5/3}, {0, 12/5, 2, 5/3}, {0, 0, 2, 2}};
```

The entire surface is obtained by subdividing over $[-1, 1] \times [-1, 1]$. The result of subdividing twice is shown in Figure 24.34.

Example 14: The elliptic paraboloid.

An elliptic paraboloid is defined by the implicit equation

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

A nice parametric representation is given by:

$$\begin{aligned} x &= av\cos\theta, \\ y &= bv\sin\theta, \\ z &= v^2. \end{aligned}$$

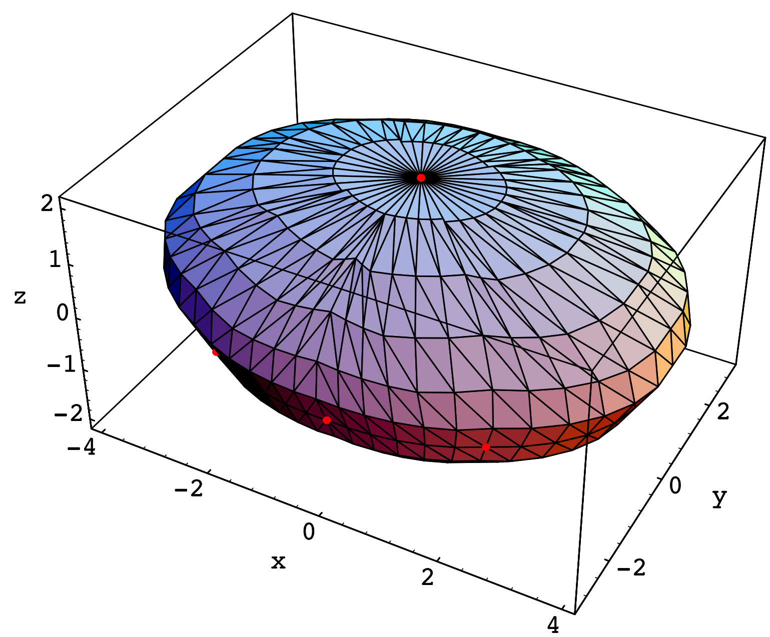

Figure 24.34: An ellipsoid

Switching to $u = \tan(\theta/2)$, we get

$$\begin{aligned} x &= \frac{av(1-u^2)}{u^2+1}, \\ y &= \frac{2buv}{u^2+1}, \\ z &= \frac{v^2(u^2+1)}{u^2+1}. \end{aligned}$$

The following control net is obtained for $a = 1$, and $b = 1/2$:

```
parabnet3 = {{0, 0, 0, 1}, {1/4, 0, 0, 1},
   {1/2, 0, 1/6, 1}, {3/4, 0, 1/2, 1},
   {1, 0, 1, 1}, {0, 0, 0, 1}, {1/4, 1/12, 0, 1}, {1/2, 1/6, 1/6, 1},
   {3/4, 1/4, 1/2, 1}, {0, 0, 0, 7/6}, {1/7, 1/7, 0, 7/6},
   {2/7, 2/7, 2/7, 7/6}, {0, 0, 0, 3/2}, {0, 1/6, 0, 3/2}, {0, 0, 0, 2}};
```

Subdividing 3 times over over $[-1, 1] \times [-2, 2]$, we obtain the following:

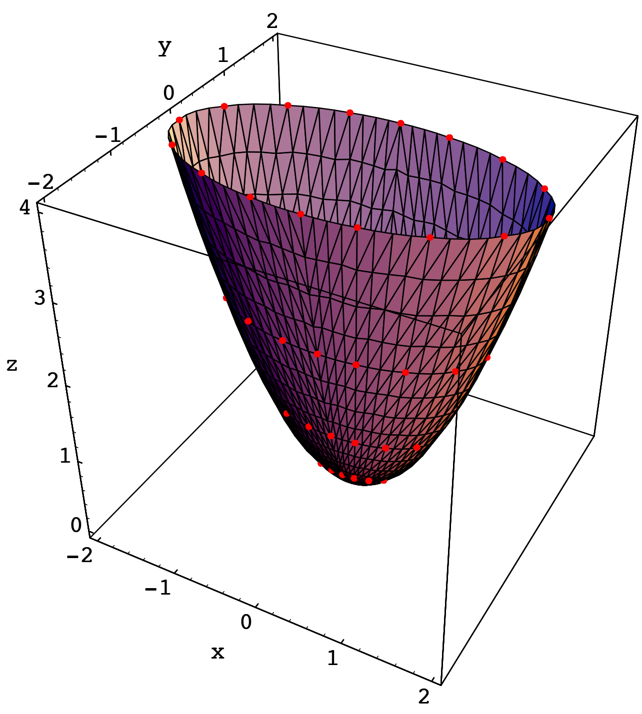

Figure 24.35: An elliptic paraboloid

Example 15: The hyperbolic paraboloid.

An hyperbolic paraboloid is defined by the implicit equation

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

A nice parametric representation is given by:

$$
\begin{aligned}
x &= av\cos\theta,\\
y &= bv\sin\theta,\\
z &= v^2\cos 2\theta.
\end{aligned}
$$

Switching to $u = \tan(\theta/4)$, we get

$$
\begin{aligned}
x &= \frac{av(1-u^2)(1+u^2)}{(u^2+1)^2},\\
y &= \frac{2buv(1+u^2)}{(u^2+1)^2},\\
z &= \frac{v^2(u^4-6u^2+1)}{(u^2+1)^2}.
\end{aligned}
$$

The following control net is obtained for $a = 1$, $b = 1$:

```
phypnet2 = {{0, 0, 0, 1}, {1/6, 0, 0, 1},
   {1/3, 0, 1/15, 1}, {1/2, 0, 1/5, 1},
   {2/3, 0, 2/5, 1}, {5/6, 0, 2/3, 1}, {1, 0, 1, 1}, {0, 0, 0, 1},
   {1/6, 1/15, 0, 1}, {1/3, 2/15, 1/15, 1}, {1/2, 1/5, 1/5, 1},
   {2/3, 4/15, 2/5, 1}, {5/6, 1/3, 2/3, 1}, {0, 0, 0, 17/15},
   {5/34, 2/17, 0, 17/15}, {5/17, 4/17, 0, 17/15}, {15/34, 6/17, 0, 17/15},
   {10/17, 8/17, 0, 17/15}, {0, 0, 0, 7/5}, {5/42, 1/6, 0, 7/5},
   {5/21, 1/3, -2/21, 7/5}, {5/14, 1/2, -2/7, 7/5}, {0, 0, 0, 28/15},
   {1/14, 3/14, 0, 28/15}, {1/7, 3/7, -1/7, 28/15}, {0, 0, 0, 8/3},
   {0, 1/4, 0, 8/3}, {0, 0, 0, 4}};
```

Subdividing twice over $[-1,1]\times[-2,2]$, we obtain the following:

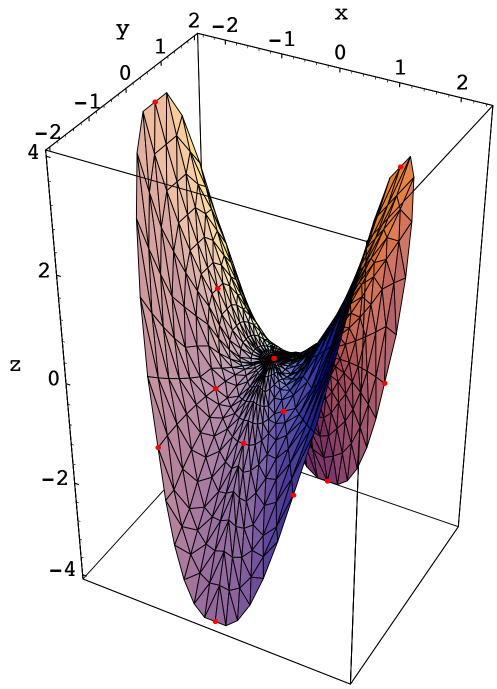

Figure 24.36: An hyperbolic paraboloid

Example 16: The hyperboloid of one sheet.

An hyperboloid of one sheet is defined by the implicit equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} + 1.$$

A nice parametric representation is given by:

$$\begin{aligned} x &= a\cos\theta\cosh\varphi, \\ y &= b\sin\theta\cosh\varphi, \\ z &= c\sinh\varphi. \end{aligned}$$

Switching to $u = \tan(\theta/4)$ and $v = \tanh(\varphi/2)$, we get

$$\begin{aligned} x &= \frac{a(u^4 - 6u^2 + 1)(1 + v^2)}{(u^2 + 1)^2(1 - v^2)}, \\ y &= \frac{4buv(1 - u^2)(1 + v^2)}{(u^2 + 1)^2(1 - v^2)}, \\ z &= \frac{2cv(u^2 + 1)^2}{(u^2 + 1)^2(1 - v^2)}. \end{aligned}$$

The following control net is obtained for $a = 2$, $b = 1$, and $c = 2$:

```
hypnet = {{2, 0, 0, 1}, {2, 0, 2/3, 1},
   {16/7, 0, 10/7, 14/15}, {3, 0, 5/2, 4/5},
```

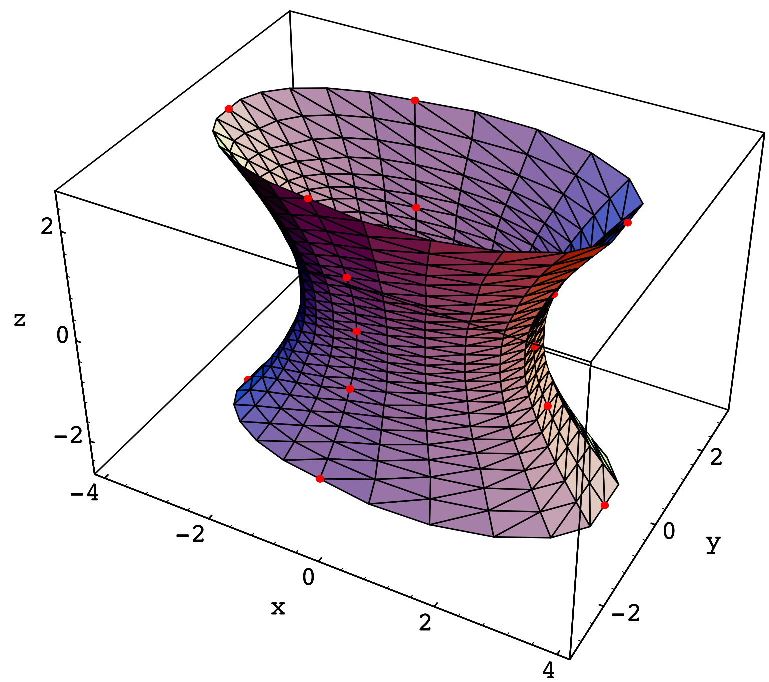

Figure 24.37: An hyperboloid of one sheet

```
{14/3, 0, 40/9, 3/5}, {10, 0, 10, 1/3}, {4, 0, 4, 0}, {2, 2/3, 0, 1},
{2, 2/3, 2/3, 1}, {16/7, 11/14, 10/7, 14/15}, {3, 13/12, 5/2, 4/5},
{14/3, 16/9, 40/9, 3/5}, {10, 4, 10, 1/3}, {18/17, 20/17, 0, 17/15},
{18/17, 20/17, 12/17, 17/15}, {54/47, 66/47, 72/47, 47/45},
{18/13, 2, 36/13, 13/15}, {2, 32/9, 16/3, 3/5}, {-2/7, 9/7, 0, 7/5},
{-2/7, 9/7, 16/21, 7/5}, {-10/19, 29/19, 32/19, 19/15},
{-6/5, 11/5, 16/5, 1}, {-10/7, 1, 0, 28/15}, {-10/7, 1, 6/7, 28/15},
{-2, 7/6, 2, 8/5}, {-2, 1/2, 0, 8/3}, {-2, 1/2, 1, 8/3}, {-2, 0, 0, 4}};
```

Subdividing twice over $[-1, 1] \times [-1/2, 1/2]$, we obtain the surface shown in Figure 24.37.

Both the hyperbolic paraboloid and the hyperboloid of one sheet are ruled surfaces. The last nondegenerate quadric is the hyperboloid of two sheets.

Example 17: The hyperboloid of two sheets.

An hyperboloid of two sheets is defined by the implicit equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} - 1.$$

A nice parametric representation is given by:

$$\begin{aligned} x &= a \cos\theta \sinh\varphi, \\ y &= b \sin\theta \sinh\varphi, \\ z &= c \cosh\varphi. \end{aligned}$$

Switching to $u = \tan(\theta/4)$ and $v = \tanh(\varphi/2)$, we get

$$x = \frac{2av(u^4 - 6u^2 + 1)}{(u^2+1)^2(1-v^2)},$$
$$y = \frac{8buv(1-u^2)}{(u^2+1)^2(1-v^2)},$$
$$z = \frac{c(v^2+1)(u^2+1)^2}{(u^2+1)^2(1-v^2)}.$$

The following control net is obtained for $a = 2$, $b = 1$, and $c = 2$:

```
hypnet3 = {{0, 0, 2, 1}, {2/3, 0, 2, 1},
   {10/7, 0, 16/7, 14/15}, {5/2, 0, 3, 4/5},
   {40/9, 0, 14/3, 3/5}, {10, 0, 10, 1/3}, {4, 0, 4, 0}, {0, 0, 2, 1},
   {2/3, 4/15, 2, 1}, {10/7, 4/7, 16/7, 14/15}, {5/2, 1, 3, 4/5},
   {40/9, 16/9, 14/3, 3/5}, {10, 4, 10, 1/3}, {0, 0, 2, 17/15},
   {4/17, 8/17, 2, 17/15}, {24/47, 48/47, 110/47, 47/45},
   {12/13, 24/13, 42/13, 13/15}, {16/9, 32/9, 50/9, 3/5}, {0, 0, 2, 7/5},
  {-8/21, 10/21, 2, 7/5}, {-16/19, 20/19, 46/19, 19/15}, {-8/5, 2, 18/5, 1},
   {0, 0, 2, 28/15}, {-6/7, 2/7, 2, 28/15}, {-2, 2/3, 8/3, 8/5},
   {0, 0, 2, 8/3}, {-1, 0, 2, 8/3}, {0, 0, 2, 4}};
```

Only the upper cup is obtained for $-1 \leq v \leq 1$. The lower cup is obtained by symmetry. Subdividing twice over $[-1, 1] \times [0, 4/5]$, and using symmetry, we obtain the following:

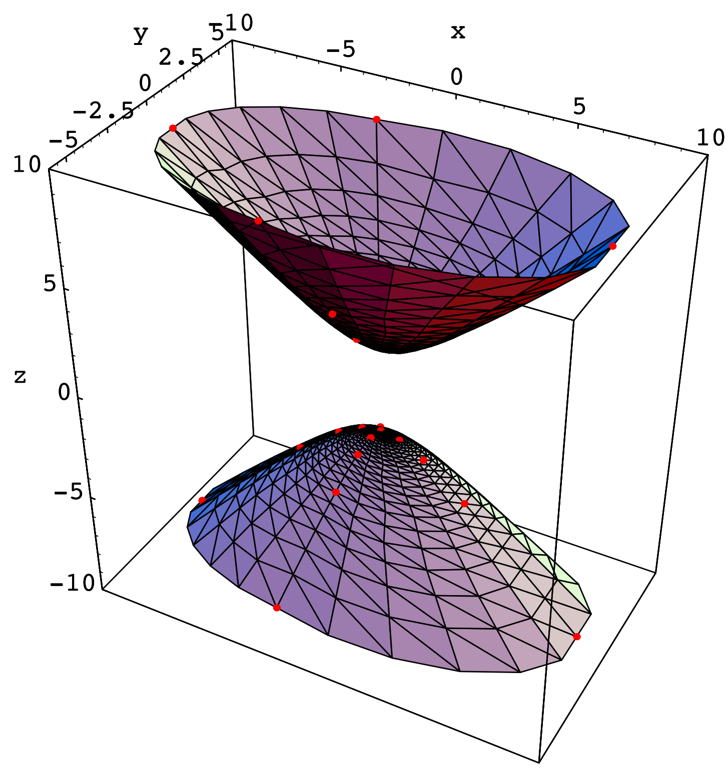

Figure 24.38: An hyperboloid of two sheets

We now return to the torus, to show that it is obtained over $[-1,1]\times[-1,1]$ using a control net of degree 8.

Example 18: The elliptic torus.

We saw in section 24.3 that an elliptic torus is defined by:

$$\begin{aligned} x &= (a - b\,\sin\varphi)\cos\theta,\\ y &= (a - b\,\sin\varphi)\sin\theta,\\ z &= c\,\cos\varphi. \end{aligned}$$

In order to obtain the entire torus over $[-1,1]\times[-1,1]$, we switch to $u=\tan(\theta/4)$ and $v=\tan(\varphi/4)$. We get

$$\begin{aligned} x &= \frac{(a(1+v^2)^2 - 4bv(1-v^2))(u^4-6u^2+1)}{(1+u^2)^2(1+v^2)^2},\\ y &= \frac{4u(1-u^2)(a(1+v^2)^2 - 4bv(1-v^2))}{(1+u^2)^2(1+v^2)^2},\\ z &= \frac{c(v^4-6v^2+1)(1+u^2)^2}{(1+u^2)^2(1+v^2)^2}. \end{aligned}$$

The following control net is obtained for a$=1/2$, $b=1$, $c=1$. Such a torus is self intersecting.

```
tornet3=   {{1/2, 0, 1, 1}, {0, 0, 1, 1}, {-13/30, 0, 11/15, 15/14},
   {-23/34, 0, 5/17, 17/14}, {-139/202, 0, -19/101, 101/70},
   {-1/2, 0, -3/5, 25/14}, {-3/16, 0, -7/8, 16/7}, {1/6, 0, -1, 3},
   {1/2, 0, -1, 4}, {1/2, 1/4, 1, 1}, {0, -1/28, 1, 1},
   {-13/30, -5/18, 11/15, 15/14}, {-23/34, -67/170, 5/17, 17/14},
   {-139/202, -36/101, -19/101, 101/70}, {-1/2, -14/75, -3/5, 25/14},
   {-3/16, 1/16, -7/8, 16/7}, {1/6, 1/3, -1, 3}, {11/30, 7/15, 1, 15/14},
   {1/30, -1/15, 1, 15/14}, {-63/242, -125/242, 87/121, 121/105},
   {-41/92, -67/92, 6/23, 46/35}, {-333/662, -216/331, -77/331, 331/210},
   {-75/166, -28/83, -53/83, 83/42}, {-1/3, 1/9, -8/9, 18/7},
   {5/34, 10/17, 1, 17/14}, {3/34, -6/85, 1, 17/14},
   {1/46, -5/8, 16/23, 46/35}, {-4/53, -189/212, 11/53, 53/35},
   {-55/258, -106/129, -13/43, 129/70}, {-25/66, -16/33, -23/33, 33/14},
   {-19/202, 60/101, 1, 101/70}, {29/202, -4/101, 1, 101/70},
   {211/662, -190/331, 219/331, 331/210}, {27/86, -110/129, 17/129, 129/70},
   {33/322, -136/161, -9/23, 23/10}, {-3/10, 1/2, 1, 25/14},
   {9/50, 1/50, 1, 25/14}, {91/166, -65/166, 51/83, 83/42},
   {41/66, -43/66, 1/33, 33/14}, {-7/16, 11/32, 1, 16/7},
   {3/16, 3/32, 1, 16/7}, {2/3, -5/36, 5/9, 18/7}, {-1/2, 1/6, 1, 3},
   {1/6, 1/6, 1, 3}, {-1/2, 0, 1, 4}};
```

Subdividing twice over $[0,1]\times[-1,1]$, we obtain half of the torus displayed in Figure 24.39, showing clearly the self intersection.

We now show consider another pretty surface, the Whitney umbrella.

Example 19: The Whitney umbrella

The Whitney umbrella is given by

$$\begin{aligned} x &= 2uv,\\ y &= u,\\ z &= v^2. \end{aligned}$$

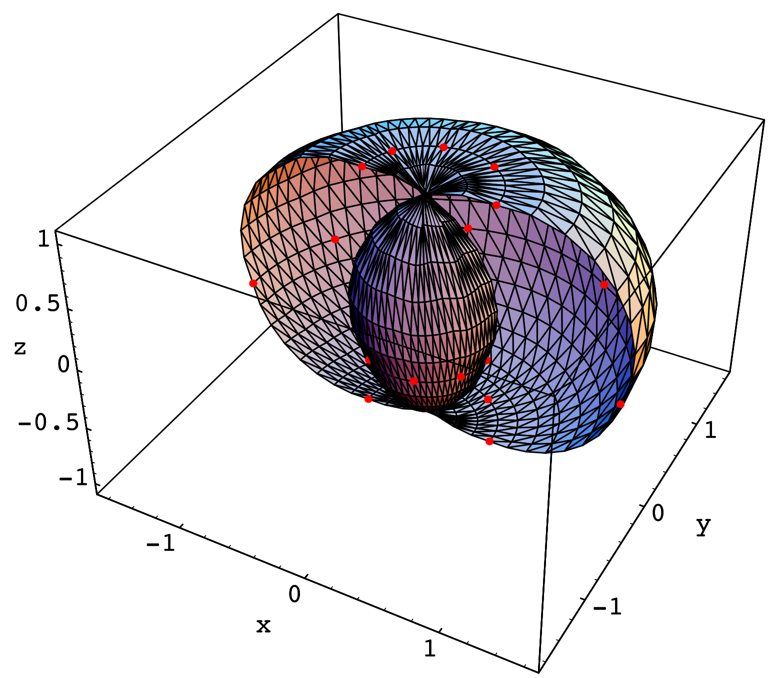

Figure 24.39: A spindle torus

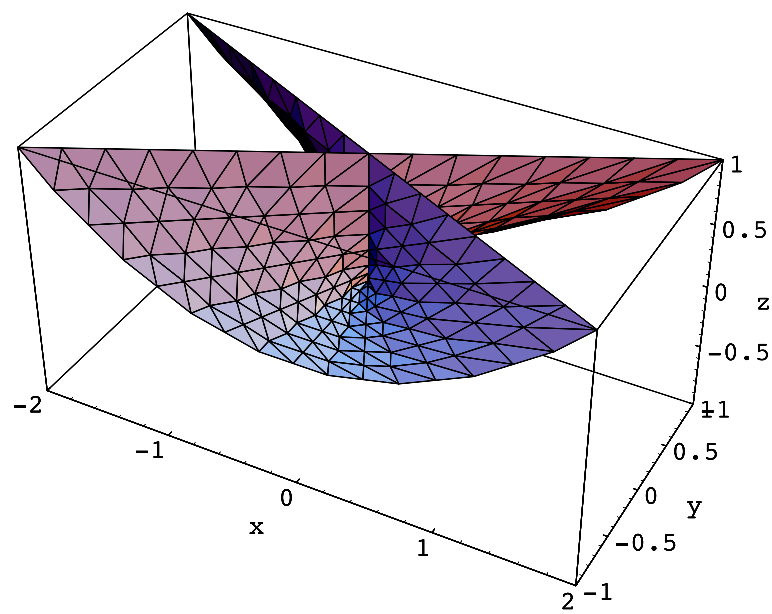

Figure 24.40: The Whitney Umbrella

The following control net is obtained:

```
whitnet = {{0, 0, 0, 1}, {0, 0, 0, 1}, {0, 0, 1, 1},
       {0, 1/2, 0, 1}, {1, 1/2, 0, 1},
       {0, 1, 0, 1}};
```

Subdividing 3 times over $[-1,1]\times[-1,1]$, we obtain the surafce shown in Figure 24.40.

We now consider an example related to Riemann surfaces.

Example 20: The real part of the Riemann surface of $Y^2 = X$.

If $X = x + iy$ and $Y = z + iw$ are complex numbers, the surface in $\mathbb{A}^4$ defined by the set of points

$$\{(x, y, z, w) \in \mathbb{A}^4 \mid (z + iw)^2 = x + iy\}$$

is the *Riemann surface* associated with the equation $Y^2 = X$. We can't visualize this surface directly, but we can visualize its real part, the surface which is the set of points

$$\{(x, y, z) \in \mathbb{A}^4 \mid (z + iw)^2 = x + iy\}.$$

Letting $x + iy = \rho(\cos\theta + i\sin\theta)$ and $z + iw = v(\cos\varphi + i\sin\varphi)$, we get $v^2 = \rho$ and $\varphi = \theta$ or $\varphi = \theta + \pi$, and thus, the real part of the Riemann surface is defined by:

$$\begin{aligned}
x &= v^2\cos\theta,\\
y &= v^2\sin\theta,\\
z &= v\cos(\theta/2).
\end{aligned}$$

Switching to $u = \tan(\theta/4)$, we get

$$x = \frac{(u^4 - 6u^2 + 1)v^2}{(u^2+1)^2},$$
$$y = \frac{4u(1-u^2)v^2}{(u^2+1)^2},$$
$$z = \frac{(1-u^2)(1+u^2)v}{(u^2+1)^2}.$$

The following triangular control net is obtained:

```
riemnet =   {{0, 0, 0, 1}, {0, 0, 1/6, 1},
   {1/15, 0, 1/3, 1}, {1/5, 0, 1/2, 1},
   {2/5, 0, 2/3, 1}, {2/3, 0, 5/6, 1}, {1, 0, 1, 1}, {0, 0, 0, 1},
   {0, 0, 1/6, 1}, {1/15, 1/15, 1/3, 1}, {1/5, 1/5, 1/2, 1},
   {2/5, 2/5, 2/3, 1}, {2/3, 2/3, 5/6, 1}, {0, 0, 0, 17/15},
   {0, 0, 5/34, 17/15}, {0, 2/17, 5/17, 17/15}, {0, 6/17, 15/34, 17/15},
   {0, 12/17, 10/17, 17/15}, {0, 0, 0, 7/5}, {0, 0, 5/42, 7/5},
   {-2/21, 2/21, 5/21, 7/5}, {-2/7, 2/7, 5/14, 7/5}, {0, 0, 0, 28/15},
   {0, 0, 1/14, 28/15}, {-1/7, 0, 1/7, 28/15}, {0, 0, 0, 8/3},
   {0, 0, 0, 8/3}, {0, 0, 0, 4}};
```

The surface is self-intersecting for $\theta = \pi$, which corresponds to $x < 0$ and $y = 0$. Indeed, in this case, $X = x$ is a negative real number, and it does not have any real square root. Otherwise, X has two square roots Y and $-Y$ with real parts. This is why the surface is symmetric with respect to the xOy plane. Subdividing twice over $[-1, 1] \times [-2, 2]$, we obtain the following:

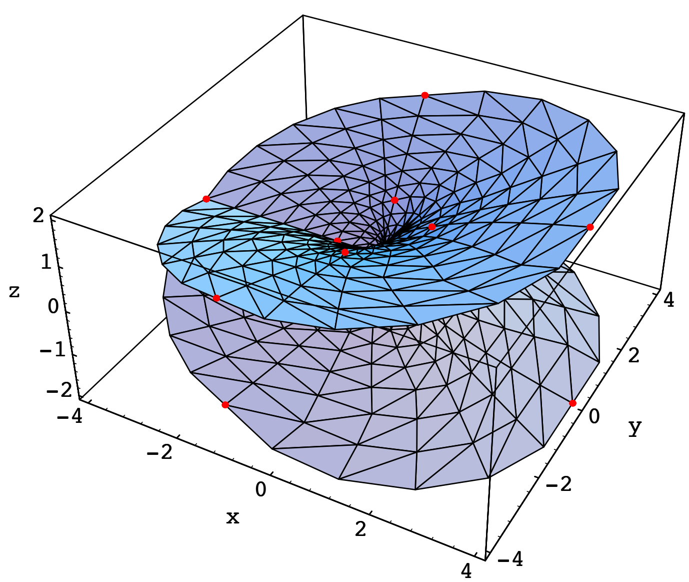

Figure 24.41: Real part of the Riemann surface $Y^2 = X$

We conclude with a classic, the Möbius strip.

Example 21: The Möbius strip.

As we saw in section 23.1, the Moëbius strip can be defined parametrically as

$$\begin{aligned} x &= (2 + v \sin\frac{\theta}{2}) \cos\theta \\ y &= (2 + v \sin\frac{\theta}{2}) \sin\theta \\ z &= v \cos\frac{\theta}{2}. \end{aligned}$$

Switching to $u = \tan\frac{\theta}{4}$, we get

$$\begin{aligned} x &= \frac{2[1 - 5u^2 - 5u^4 + u^6 + (u - 6u^3 + u^5)v]}{1 + 3u^2 + 3u^4 + u^6}, \\ v &= \frac{8[u - u^5 + (u^2 - u^4)v]}{1 + 3u^2 + 3u^4 + u^6}, \\ z &= \frac{(1 + u^2 - u^4 - u^6)v}{1 + 3u^2 + 3u^4 + u^6}. \end{aligned}$$

The following triangular control net is obtained:

```
mobnet =  {{2, 0, 0, 1}, {2, 0, 1/7, 1}, {2, 0, 2/7, 1}, {2, 0, 3/7, 1},
   {2, 0, 4/7, 1}, {2, 0, 5/7, 1}, {2, 0, 6/7, 1}, {2, 0, 1, 1},
   {2, 8/7, 0, 1}, {43/21, 8/7, 1/7, 1}, {44/21, 8/7, 2/7, 1},
   {15/7, 8/7, 3/7, 1}, {46/21, 8/7, 4/7, 1}, {47/21, 8/7, 5/7, 1},
   {16/7, 8/7, 6/7, 1}, {4/3, 2, 0, 8/7}, {17/12, 31/15, 2/15, 8/7},
  {3/2, 32/15, 4/15, 8/7}, {19/12, 11/5, 2/5, 8/7}, {5/3, 34/15, 8/15, 8/7},
   {7/4, 7/3, 2/3, 8/7}, {2/5, 12/5, 0, 10/7}, {11/25, 64/25, 3/25, 10/7},
   {12/25, 68/25, 6/25, 10/7}, {13/25, 72/25, 9/25, 10/7},
   {14/25, 76/25, 12/25, 10/7}, {-10/17, 40/17, 0, 68/35},
   {-2/3, 130/51, 5/51, 68/35}, {-38/51, 140/51, 10/51, 68/35},
   {-14/17, 50/17, 5/17, 68/35}, {-22/15, 28/15, 0, 20/7},
   {-5/3, 2, 1/15, 20/7}, {-28/15, 32/15, 2/15, 20/7}, {-2, 1, 0, 32/7},
   {-9/4, 1, 0, 32/7}, {-2, 0, 0, 8}};
```

Subdividing twice over $[-1, 1] \times [-1, 1]$, we obtain the following:

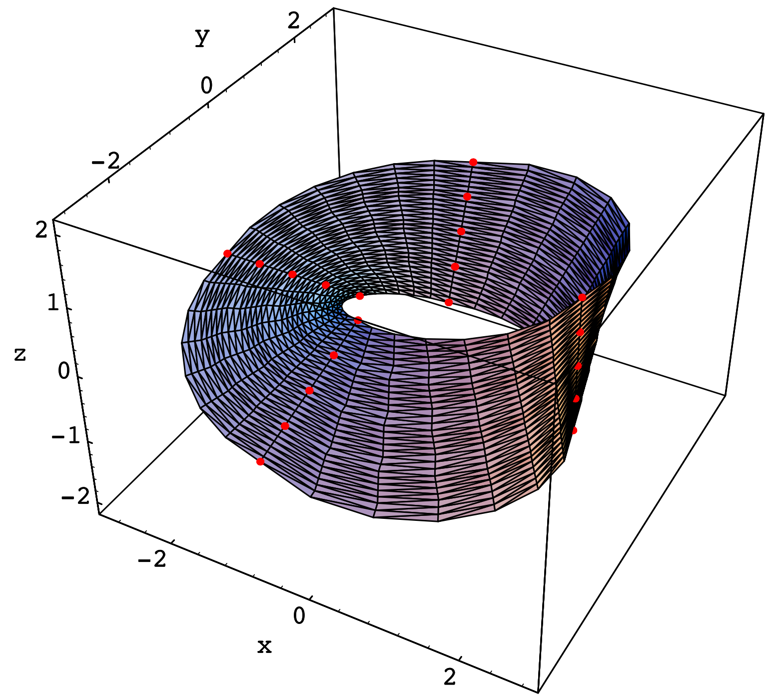

Figure 24.42: The Möbius strip

This concludes the tour in the surface gallery. I hope that it was enjoyable and instructive. We will now take a very quick look at rational spline surfaces.

24.8 Rational Spline Surfaces

In this section, we consider very briefly the problem of designing spline surfaces consisting of rational patches. As in the case of curves, we focus on parametric continuity. Since we view a rational surface as the projection under $\Pi\colon \widehat{\mathcal{E}} \to \widetilde{\mathcal{E}}$ of a polynomial surface $F\colon \mathbb{A}^2 \to \widehat{\mathcal{E}}$, we can choose to enforce C^n continuity before or after projection. As in the case of curves, it is technically easier to enforce C^n-continuity before projection, and since it is easily shown that the projected surface also has at least C^n continuity, this is the approach that we advocate. In the case of rectangular spline surfaces, this means that we will be dealing with de Boor control points $\langle a, w\rangle$ with weights, and thus, that we have some extra freedom (choosing these weights). Otherwise, the algorithms used for polynomial surfaces remain unchanged. Such spline surfaces (based on knot sequences that are not necessarily uniform) are often referred to as NURBS (for non-uniform rational B-splines). These ideas are briefly developed in Farin [57], and in Piegl and Tiller [139]. In the case of triangular patches, the situation is no better than in the polynomial case, except that the extra freedom provided by the weight may turn out to be helpful. Unfortunately, we are unaware of a practical method based on the Boor control points.