```
  {-20/17, -89/85, 128/85, 17/14}, {-98/87, -44/29, 146/87, 87/70},
  {-18/19, -2, 34/19, 19/14}, {-120/101, 0, 82/101, 101/70},
 {-120/101, -32/101, 82/101, 101/70}, {-120/101, -64/101, 254/303, 101/70},
  {-120/101, -100/101, 90/101, 101/70}, {-8/7, -48/35, 14/15, 3/2},
  {-1, 0, 2/5, 25/14}, {-1, -6/25, 2/5, 25/14}, {-1, -12/25, 2/5, 25/14},
  {-1, -18/25, 2/5, 25/14}, {-11/16, 0, 1/8, 16/7},
  {-11/16, -1/8, 1/8, 16/7}, {-11/16, -1/4, 1/8, 16/7}, {-1/3, 0, 0, 3},
  {-1/3, 0, 0, 3}, {0, 0, 0, 4}};
```

The result of subdividing twice over $[-1, 1] \times [-1, 1]$ and cutting off part of the surface to have a better understanding of its structure, is shown below:

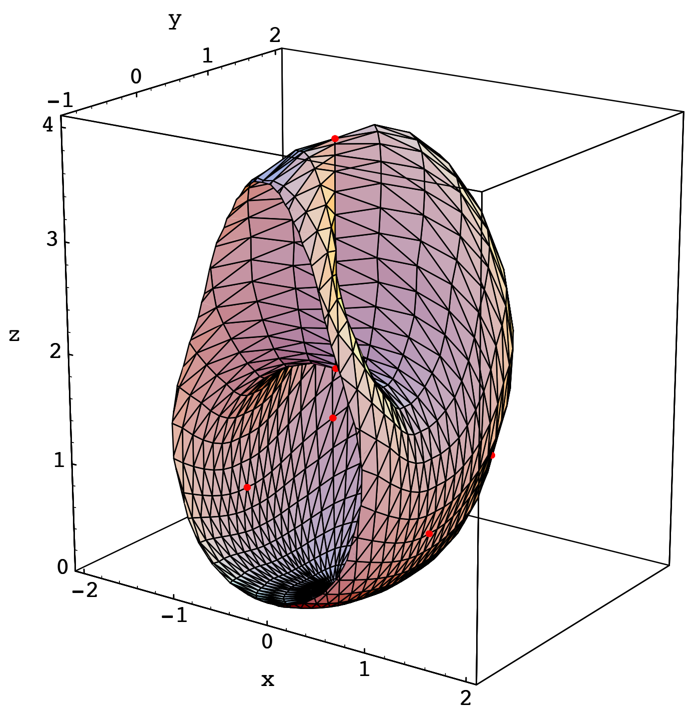

Figure 24.24: Another version of the cross-cap

Let us now consider a ruled surface, Plucker's conoïd.

Example 6: Plucker's conoïd.

This surface is defined by the implicit equation

$$z(x^2 + y^2) = 2xy.$$

Parametrically, we can use

$$\begin{aligned} x &= u, \\ y &= v, \\ z &= \frac{2uv}{u^2 + v^2}. \end{aligned}$$

Since z is discontinuous for $u = 0$, $v = 0$, some nasty things will happen, as we shall see. The problem is that the surface contains all points on the z-axis, those for which $x = y = 0$. We obtain the following triangular net of degree 2:

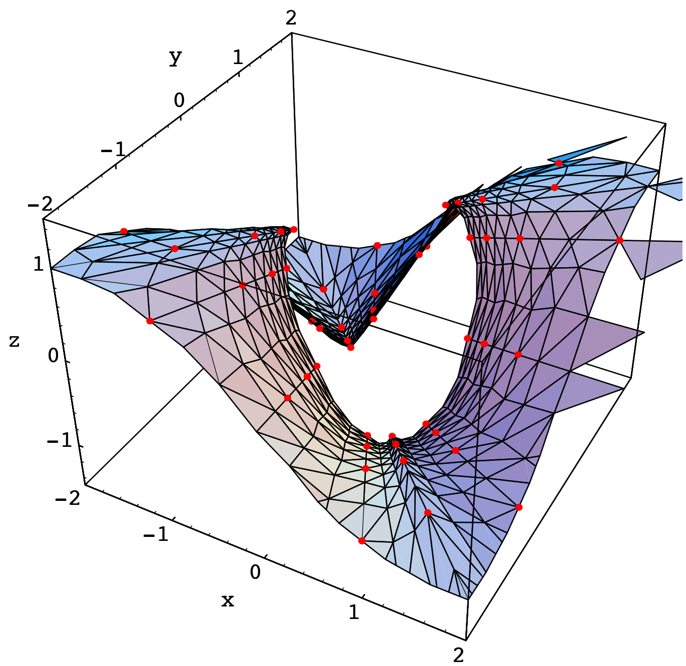

Figure 24.25: Plucker's conoïd, bad version

```
plucknet = {{0, 0, 0, 0}, {0, 1/2, 0, 0}, {0, 1, 0, 1},
        {1/2, 0, 0, 0}, {1/2, 1/2, 1, 0},
        {1, 0, 0, 1}};
```

Note that this net contains the zero vector. Thus, our program will attempt to resolve this base point. However, it will produce a net containing a point at infinity for one of its corners, and will yield a very poor triangulation. The result of subdividing 3 times after resolving the base point is shown in Figure 24.25.

It is possible to overcome the problem by switching to polar coordinates. Letting $u = \rho\cos\theta$ and $v = \rho\sin\theta$, we get

$$z\rho^2 = 2\rho^2 \sin\theta\cos\theta,$$

which implies either $\rho = 0$ or $z = \sin 2\theta$. If we forget about the troublesome z-axis, we get

$$\begin{aligned}
x &= \rho\cos\theta,\\
y &= \rho\sin\theta,\\
z &= \sin 2\theta.
\end{aligned}$$

Switching to $\tan(\theta/2)$, we get

$$\begin{aligned}
x &= \frac{(1-u^2)(1+u^2)v}{(u^2+1)^2},\\
y &= \frac{2u(1+u^2)v}{(u^2+1)^2},\\
z &= \frac{4u(1-u^2)}{(u^2+1)^2},
\end{aligned}$$

We obtain the following triangular net of degree 5:

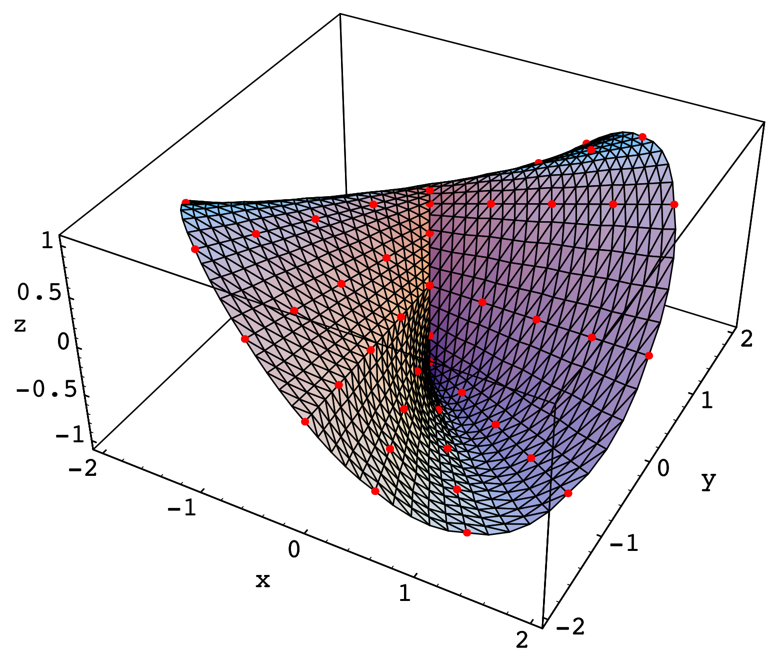

Figure 24.26: Plucker's conoïd

```
plucknet2 = {{0, 0, 0, 1}, {1/5, 0, 0, 1}, {2/5, 0, 0, 1}, {3/5, 0, 0, 1},
   {4/5, 0, 0, 1}, {1, 0, 0, 1}, {0, 0, 4/5, 1}, {1/5, 1/10, 4/5, 1},
   {2/5, 1/5, 4/5, 1}, {3/5, 3/10, 4/5, 1}, {4/5, 2/5, 4/5, 1},
   {0, 0, 4/3, 6/5}, {1/6, 1/6, 4/3, 6/5}, {1/3, 1/3, 4/3, 6/5},
   {1/2, 1/2, 4/3, 6/5}, {0, 0, 5/4, 8/5}, {1/8, 1/4, 5/4, 8/5},
   {1/4, 1/2, 5/4, 8/5}, {0, 0, 2/3, 12/5},
   {0, 1/3, 2/3, 12/5}, {0, 0, 0, 4}};
```

The result of subdividing 3 times over $[-1, 1] \times [-2, 2]$ is shown in Figure 24.26.

Another pretty ruled surface is the right conoïd.

Example 7: The right conoïd.

The right conoïd is defined in polar coordinates by

$$\begin{aligned} x &= \rho \cos\theta, \\ y &= \rho \sin\theta, \\ z &= 2\sin\theta. \end{aligned}$$

Switching to $\tan(\theta/4)$, we get

$$x = \frac{(u^4 - 6u^2 + 1)v}{(u^2+1)^2},$$

$$y = \frac{4uv(1-u^2)}{(u^2+1)^2},$$
$$z = \frac{8u(1-u^2)}{(u^2+1)^2},$$

We obtain the following triangular net of degree 5:

```
rconet =   {{0, 0, 0, 1}, {1/5, 0, 0, 1}, {2/5, 0, 0, 1}, {3/5, 0, 0, 1},
   {4/5, 0, 0, 1}, {1, 0, 0, 1}, {0, 0, 8/5, 1}, {1/5, 1/5, 8/5, 1},
   {2/5, 2/5, 8/5, 1}, {3/5, 3/5, 8/5, 1}, {4/5, 4/5, 8/5, 1},
   {0, 0, 8/3, 6/5}, {0, 1/3, 8/3, 6/5}, {0, 2/3, 8/3, 6/5},
   {0, 1, 8/3, 6/5}, {0, 0, 5/2, 8/5}, {-1/4, 1/4, 5/2, 8/5},
   {-1/2, 1/2, 5/2, 8/5}, {0, 0, 4/3, 12/5}, {-1/3, 0, 4/3, 12/5},
   {0, 0, 0, 4}};
```

The result of subdividing 3 times over $[-1,1] \times [-2,2]$ is shown below:

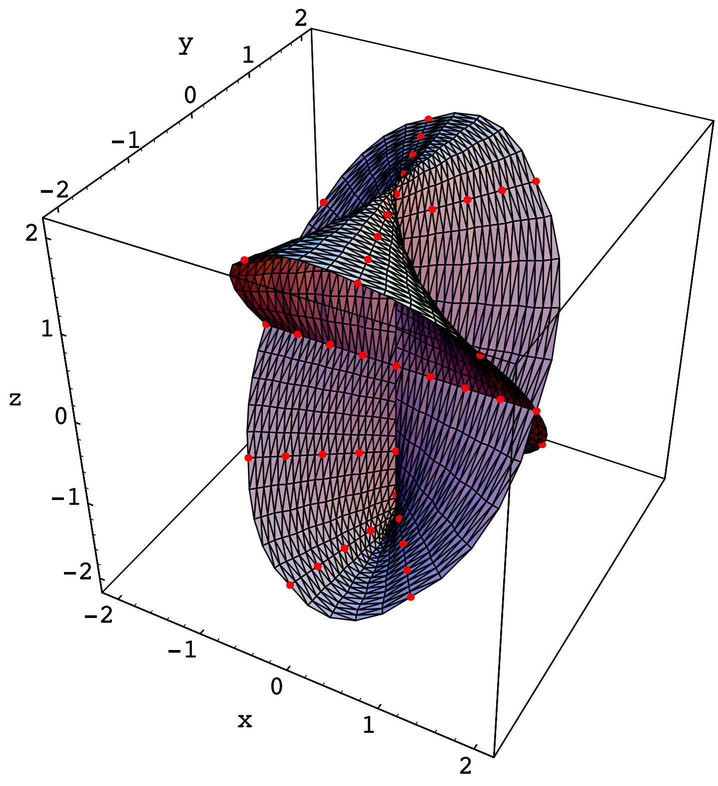

Figure 24.27: A right conoïd

We now consider several ways of viewing the Klein bottle in $\mathbb{A}^3$. Recall that as a topological space, the Klein bottle is the quotient of a rectangle whose vertices appear in the order A, B, C, D in a closed path from A, by identifying AB and DC, and AD and CB. Note that AD and CB have opposite orientations, which causes a twist. Equivalently, the Klein bottle is obtained from a torus, by identifying pairs of points symmetric with respect to the origin. To see that this yields a Klein bottle, consider a half torus obtained by slicing the torus into two half bent cylinders, and observe that the two boundary circles will be glued in the appropriate manner (with a twist). It is shown in Do Carmo [51] that the following map from $\mathbb{A}^2$ to $\mathbb{A}^4$

$$\begin{aligned}
x &= (a + r\cos\varphi)\cos\theta,\\
y &= (a + r\cos\varphi)\sin\theta,\\
z &= r\sin\varphi\cos(\theta/2),\\
t &= r\sin\varphi\sin(\theta/2),
\end{aligned}$$

induces an embedding of the Klein bottle in $\mathbb{A}^4$. Again, we can consider the four projections in $\mathbb{A}^3$. It turns out that the surfaces obtained by dropping z or t are identical and constitute a "crossed torus", and similarly the surfaces obtained by dropping x and y are identical, but stranger than the crossed torus.

First, we express all the trigonometric functions in terms of $u = \tan(\theta/4)$ and $v = \tan(\varphi/4)$, getting

$$\begin{aligned}
x &= \frac{(u^4 - 6u^2 + 1)((a + r)v^4 + 2(a - 3r)v^2 + a + r)}{(u^2 + 1)^2(v^2 + 1)^2},\\
y &= \frac{4u(1 - u^2)((a + r)v^4 + 2(a - 3r)v^2 + a + r)}{(u^2 + 1)^2(v^2 + 1)^2},\\
z &= \frac{4rv(1 - u^4)(1 - v^2)}{(u^2 + 1)^2(v^2 + 1)^2},\\
t &= \frac{8ruv(1 + u^2)(1 - v^2)}{(u^2 + 1)^2(v^2 + 1)^2}.
\end{aligned}$$

We now consider views of the surfaces obtained by dropping x, y, and z (the same surface is obtained by dropping z or t).

Example 8: Klein bottle 1.

This surface is obtained by dropping x, which yields

$$\begin{aligned}
x &= \frac{4u(1 - u^2)((a + r)v^4 + 2(a - 3r)v^2 + a + r)}{(u^2 + 1)^2(v^2 + 1)^2},\\
y &= \frac{4rv(1 - u^4)(1 - v^2)}{(u^2 + 1)^2(v^2 + 1)^2},\\
z &= \frac{8ruv(1 + u^2)(1 - v^2)}{(u^2 + 1)^2(v^2 + 1)^2}.
\end{aligned}$$

This surface gives quite a bit of work to our polarizing program, but the program does come back with a net of degree 8 shown below:

```
kleinetx =   {{0, 0, 0, 1}, {0, rr/2, 0, 1}, {0, (14*rr)/15, 0, 15/14},
   {0, (20*rr)/17, 0, 17/14}, {0, (120*rr)/101, 0, 101/70},
   {0, rr, 0, 25/14}, {0, (11*rr)/16, 0, 16/7}, {0, rr/3, 0, 3},
   {0, 0, 0, 4}, {(4*aa + 4*rr)/8, 0, 0, 1},
   {(4*aa + 4*rr)/8, rr/2, rr/7, 1},
   {(14*((8*aa - 24*rr)/168 + (4*aa + 4*rr)/8))/15, (14*rr)/15, (4*rr)/15,
    15/14}, {(14*((8*aa - 24*rr)/56 + (4*aa + 4*rr)/8))/17, (20*rr)/17,
    (28*rr)/85, 17/14}, {(70*((8*aa - 24*rr)/28 + (9*(4*aa + 4*rr))/70))/
     101, (120*rr)/101, (32*rr)/101, 101/70},
  {(14*((5*(8*aa - 24*rr))/84 + (4*aa + 4*rr)/7))/25, rr, (6*rr)/25, 25/14},
```

```
{(7*((5*(8*aa - 24*rr))/56 + (5*(4*aa + 4*rr))/28))/16, (11*rr)/16, rr/8,
 16/7}, {((8*aa - 24*rr)/8 + (4*aa + 4*rr)/4)/3, rr/3, 0, 3},
{(7*(4*aa + 4*rr))/30, 0, 0, 15/14},
{(7*(4*aa + 4*rr))/30, (7*rr)/15, (4*rr)/15, 15/14},
{(105*((8*aa - 24*rr)/84 + (4*aa + 4*rr)/4))/121, (105*rr)/121,
 (60*rr)/121, 121/105}, {(35*((8*aa - 24*rr)/28 + (4*aa + 4*rr)/4))/46,
 (25*rr)/23, (14*rr)/23, 46/35},
{(210*((8*aa - 24*rr)/14 + (9*(4*aa + 4*rr))/35))/331, (360*rr)/331,
 (192*rr)/331, 331/210}, {(42*
    ((5*(8*aa - 24*rr))/42 + (2*(4*aa + 4*rr))/7))/83, (75*rr)/83,
 (36*rr)/83, 83/42}, {(7*((5*(8*aa - 24*rr))/28 + (5*(4*aa + 4*rr))/14))/
  18, (11*rr)/18, (2*rr)/9, 18/7},
{(14*((-4*aa - 4*rr)/56 + (3*(4*aa + 4*rr))/8))/17, 0, 0, 17/14},
{(14*((-4*aa - 4*rr)/56 + (3*(4*aa + 4*rr))/8))/17, (7*rr)/17,
 (32*rr)/85, 17/14}, {(35*((8*aa - 24*rr)/56 + (-4*aa - 4*rr)/56 +
      (3*(4*aa + 4*rr))/8 + (-8*aa + 24*rr)/560))/46, (35*rr)/46,
 (16*rr)/23, 46/35}, {(35*((3*(8*aa - 24*rr))/56 + (-4*aa - 4*rr)/56 +
      (3*(4*aa + 4*rr))/8 + (3*(-8*aa + 24*rr))/560))/53, (50*rr)/53,
 (89*rr)/106, 53/35}, {(70*((3*(8*aa - 24*rr))/28 +
      (3*(-4*aa - 4*rr))/140 + (27*(4*aa + 4*rr))/70 +
      (3*(-8*aa + 24*rr))/280))/129, (40*rr)/43, (100*rr)/129, 129/70},
{(14*((5*(8*aa - 24*rr))/28 + (-4*aa - 4*rr)/28 + (3*(4*aa + 4*rr))/7 +
      (-8*aa + 24*rr)/56))/33, (25*rr)/33, (6*rr)/11, 33/14},
{(70*((-4*aa - 4*rr)/14 + (4*aa + 4*rr)/2))/101, 0, 0, 101/70},
{(70*((-4*aa - 4*rr)/14 + (4*aa + 4*rr)/2))/101, (34*rr)/101,
 (48*rr)/101, 101/70}, {(210*
    ((8*aa - 24*rr)/42 + (-4*aa - 4*rr)/14 + (4*aa + 4*rr)/2 +
      (-8*aa + 24*rr)/140))/331, (204*rr)/331, (288*rr)/331, 331/210},
{(70*((8*aa - 24*rr)/14 + (-4*aa - 4*rr)/14 + (4*aa + 4*rr)/2 +
      (3*(-8*aa + 24*rr))/140))/129, (98*rr)/129, (44*rr)/43, 129/70},
{(10*((8*aa - 24*rr)/7 + (3*(-4*aa - 4*rr))/35 +
      (18*(4*aa + 4*rr))/35 + (3*(-8*aa + 24*rr))/70))/23, (120*rr)/161,
 (144*rr)/161, 23/10}, {(14*
    ((5*(-4*aa - 4*rr))/28 + (5*(4*aa + 4*rr))/8))/25, 0, 0, 25/14},
{(14*((5*(-4*aa - 4*rr))/28 + (5*(4*aa + 4*rr))/8))/25, (6*rr)/25,
 (14*rr)/25, 25/14}, {(42*((5*(8*aa - 24*rr))/168 +
      (5*(-4*aa - 4*rr))/28 + (5*(4*aa + 4*rr))/8 + (-8*aa + 24*rr)/56))/
  83, (36*rr)/83, (84*rr)/83, 83/42},
{(14*((5*(8*aa - 24*rr))/56 + (5*(-4*aa - 4*rr))/28 +
      (5*(4*aa + 4*rr))/8 + (3*(-8*aa + 24*rr))/56))/33, (6*rr)/11,
 (38*rr)/33, 33/14}, {(7*((5*(-4*aa - 4*rr))/14 + (3*(4*aa + 4*rr))/4))/
  16, 0, 0, 16/7}, {(7*((5*(-4*aa - 4*rr))/14 + (3*(4*aa + 4*rr))/4))/16,
 rr/8, (5*rr)/8, 16/7}, {(7*
   ((8*aa - 24*rr)/28 + (5*(-4*aa - 4*rr))/14 + (3*(4*aa + 4*rr))/4 +
      (-8*aa + 24*rr)/28))/18, (2*rr)/9, (10*rr)/9, 18/7},
{((5*(-4*aa - 4*rr))/8 + (7*(4*aa + 4*rr))/8)/3, 0, 0, 3},
{((5*(-4*aa - 4*rr))/8 + (7*(4*aa + 4*rr))/8)/3, 0, (2*rr)/3, 3},
{0, 0, 0, 4}};
```

Setting $rr = 1$ and $aa = 2$, we get the following net:

```
kleinx = {{0, 0, 0, 1}, {0, 1/2, 0, 1},
   {0, 14/15, 0, 15/14}, {0, 20/17, 0, 17/14},
   {0, 120/101, 0, 101/70}, {0, 1, 0, 25/14}, {0, 11/16, 0, 16/7},
   {0, 1/3, 0, 3}, {0, 0, 0, 4}, {3/2, 0, 0, 1}, {3/2, 1/2, 1/7, 1},
   {61/45, 14/15, 4/15, 15/14}, {19/17, 20/17, 28/85, 17/14},
   {88/101, 120/101, 32/101, 101/70}, {52/75, 1, 6/25, 25/14},
   {5/8, 11/16, 1/8, 16/7}, {2/3, 1/3, 0, 3}, {14/5, 0, 0, 15/14},
   {14/5, 7/15, 4/15, 15/14}, {305/121, 105/121, 60/121, 121/105},
   {95/46, 25/23, 14/23, 46/35}, {528/331, 360/331, 192/331, 331/210},
   {104/83, 75/83, 36/83, 83/42}, {10/9, 11/18, 2/9, 18/7},
   {60/17, 0, 0, 17/14}, {60/17, 7/17, 32/85, 17/14},
   {291/92, 35/46, 16/23, 46/35}, {273/106, 50/53, 89/106, 53/35},
   {84/43, 40/43, 100/129, 129/70}, {16/11, 25/33, 6/11, 33/14},
   {360/101, 0, 0, 101/70}, {360/101, 34/101, 48/101, 101/70},
   {1052/331, 204/331, 288/331, 331/210}, {332/129, 98/129, 44/43, 129/70},
   {304/161, 120/161, 144/161, 23/10}, {3, 0, 0, 25/14},
   {3, 6/25, 14/25, 25/14}, {221/83, 36/83, 84/83, 83/42},
  {71/33, 6/11, 38/33, 33/14}, {33/16, 0, 0, 16/7}, {33/16, 1/8, 5/8, 16/7},
   {11/6, 2/9, 10/9, 18/7}, {1, 0, 0, 3}, {1, 0, 2/3, 3}, {0, 0, 0, 4}};
```

It turns out that the entire surface is obtained by subdividing over $[-1, 1] \times [-1, 1]$. Subdividing twice and cutting off one end, we obtain the picture shown in Figure 24.28.

Clearly, this surface has chambers too. We now consider the surface obtained by dropping y. It is obtained from the previous one by a rotation by $\pi/4$ about the x axis.

Example 9: Klein bottle 2.

This surface is defined by

$$\begin{aligned}
x &= \frac{(u^4 - 6u^2 + 1)((a + r)v^4 + 2(a - 3r)v^2 + a + r)}{(u^2 + 1)^2(v^2 + 1)^2},\\
y &= \frac{4rv(1 - u^4)(1 - v^2)}{(u^2 + 1)^2(v^2 + 1)^2},\\
z &= \frac{8ruv(1 + u^2)(1 - v^2)}{(u^2 + 1)^2(v^2 + 1)^2}.
\end{aligned}$$

For $rr = 2$ and $aa = 1$, the following net of degree 8 is obtained:

```
kleiny =    {{3, 0, 0, 1}, {3, 1/2, 0, 1}, {41/15, 14/15, 0, 15/14},
   {39/17, 20/17, 0, 17/14}, {183/101, 120/101, 0, 101/70},
   {7/5, 1, 0, 25/14}, {9/8, 11/16, 0, 16/7}, {1, 1/3, 0, 3}, {1, 0, 0, 4},
   {3, 0, 0, 1}, {3, 1/2, 1/7, 1}, {41/15, 14/15, 4/15, 15/14},
   {39/17, 20/17, 28/85, 17/14}, {183/101, 120/101, 32/101, 101/70},
   {7/5, 1, 6/25, 25/14}, {9/8, 11/16, 1/8, 16/7}, {1, 1/3, 0, 3},
   {11/5, 0, 0, 15/14}, {11/5, 7/15, 4/15, 15/14},
   {243/121, 105/121, 60/121, 121/105}, {39/23, 25/23, 14/23, 46/35},
   {441/331, 360/331, 192/331, 331/210}, {81/83, 75/83, 36/83, 83/42},
   {2/3, 11/18, 2/9, 18/7}, {15/17, 0, 0, 17/14},
```

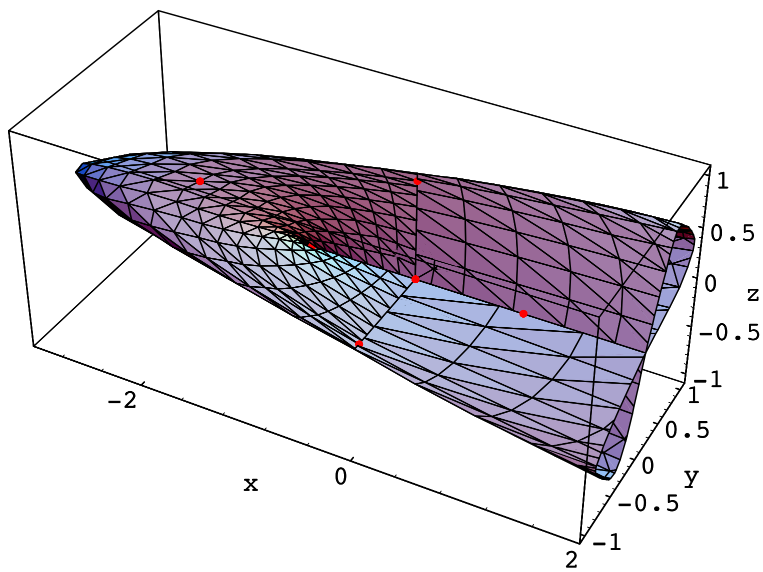

Figure 24.28: A Klein bottle, version 1

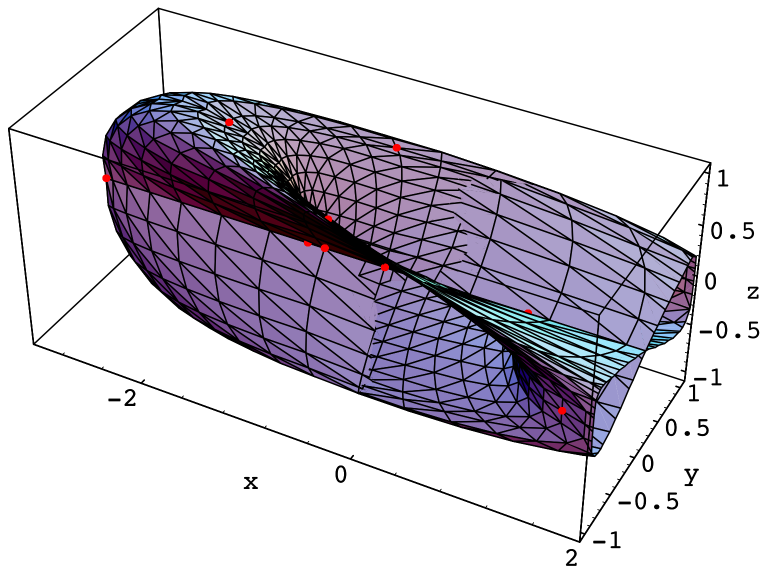

Figure 24.29: A Klein bottle, version 2

```
{15/17, 7/17, 32/85, 17/14}, {19/23, 35/46, 16/23, 46/35},
{39/53, 50/53, 89/106, 53/35}, {25/43, 40/43, 100/129, 129/70},
{1/3, 25/33, 6/11, 33/14}, {-57/101, 0, 0, 101/70},
{-57/101, 34/101, 48/101, 101/70}, {-151/331, 204/331, 288/331, 331/210},
{-37/129, 98/129, 44/43, 129/70}, {-29/161, 120/161, 144/161, 23/10},
{-9/5, 0, 0, 25/14}, {-9/5, 6/25, 14/25, 25/14},
{-127/83, 36/83, 84/83, 83/42}, {-37/33, 6/11, 38/33, 33/14},
{-21/8, 0, 0, 16/7}, {-21/8, 1/8, 5/8, 16/7}, {-20/9, 2/9, 10/9, 18/7},
{-3, 0, 0, 3}, {-3, 0, 2/3, 3}, {-3, 0, 0, 4}};
```

Again, the entire surface is obtained by subdividing over $[-1,1]\times[-1,1]$. Subdividing twice, and cutting off one end, we obtain the view of the surface shown in Figure 24.29.

We now consider the surface obtained by dropping z.

Example 10: Klein bottle 3.

This surface is defined by

$$x = \frac{(u^4 - 6u^2 + 1)((a + r)v^4 + 2(a - 3r)v^2 + a + r)}{(u^2 + 1)^2(v^2 + 1)^2},$$

$$y = \frac{4u(1 - u^2)((a + r)v^4 + 2(a - 3r)v^2 + a + r)}{(u^2 + 1)^2(v^2 + 1)^2},$$

$$z = \frac{8ruv(1 + u^2)(1 - v^2)}{(u^2 + 1)^2(v^2 + 1)^2}.$$

For $rr = 2$ and $aa = 1$, the following net of degree 8 is obtained:

```
kleinz = {{3, 0, 0, 1}, {3, 0, 0, 1},
   {41/15, 0, 0, 15/14}, {39/17, 0, 0, 17/14},
   {183/101, 0, 0, 101/70}, {7/5, 0, 0, 25/14}, {9/8, 0, 0, 16/7},
   {1, 0, 0, 3}, {1, 0, 0, 4}, {3, 3/2, 0, 1}, {3, 3/2, 1/7, 1},
   {41/15, 61/45, 4/15, 15/14}, {39/17, 19/17, 28/85, 17/14},
   {183/101, 88/101, 32/101, 101/70}, {7/5, 52/75, 6/25, 25/14},
   {9/8, 5/8, 1/8, 16/7}, {1, 2/3, 0, 3}, {11/5, 14/5, 0, 15/14},
   {11/5, 14/5, 4/15, 15/14}, {243/121, 305/121, 60/121, 121/105},
   {39/23, 95/46, 14/23, 46/35}, {441/331, 528/331, 192/331, 331/210},
   {81/83, 104/83, 36/83, 83/42}, {2/3, 10/9, 2/9, 18/7},
   {15/17, 60/17, 0, 17/14}, {15/17, 60/17, 32/85, 17/14},
   {19/23, 291/92, 16/23, 46/35}, {39/53, 273/106, 89/106, 53/35},
   {25/43, 84/43, 100/129, 129/70}, {1/3, 16/11, 6/11, 33/14},
   {-57/101, 360/101, 0, 101/70}, {-57/101, 360/101, 48/101, 101/70},
  {-151/331, 1052/331, 288/331, 331/210}, {-37/129, 332/129, 44/43, 129/70},
   {-29/161, 304/161, 144/161, 23/10}, {-9/5, 3, 0, 25/14},
   {-9/5, 3, 14/25, 25/14}, {-127/83, 221/83, 84/83, 83/42},
   {-37/33, 71/33, 38/33, 33/14}, {-21/8, 33/16, 0, 16/7},
   {-21/8, 33/16, 5/8, 16/7}, {-20/9, 11/6, 10/9, 18/7}, {-3, 1, 0, 3},
   {-3, 1, 2/3, 3}, {-3, 0, 0, 4}};
```

Again, the entire surface is obtained by subdividing over $[-1,1] \times [-1,1]$. Subdividing twice, we obtain the picture shown in Figure 24.30.

This surface is a "crossed torus ". The surface obtained by dropping t is identical to the previous one.

We can actually compute a control net in $\mathbb{A}^4$ for the Klein bottle. This can be done by polarizing all five polynomials in the fractions below

$$
\begin{aligned}
x &= \frac{(u^4 - 6u^2 + 1)((a+r)v^4 + 2(a-3r)v^2 + a + r)}{(u^2+1)^2(v^2+1)^2}, \\
y &= \frac{4u(1-u^2)((a+r)v^4 + 2(a-3r)v^2 + a + r)}{(u^2+1)^2(v^2+1)^2}, \\
z &= \frac{4rv(1-u^4)(1-v^2)}{(u^2+1)^2(v^2+1)^2}, \\
t &= \frac{8ruv(1+u^2)(1-v^2)}{(u^2+1)^2(v^2+1)^2}.
\end{aligned}
$$

As in the case of the real projective plane, if we project the Klein bottle onto a hyperplane in $\mathbb{A}^4$, either from a center or parallel to a direction, we can see a "3D shadow" of the Klein bottle in $\mathbb{A}^3$. We leave such explorations as very challenging programming projects. Again interested readers should consult the book by Francis [65], where many beautiful pictures are shown, and shadows of four dimensional surfaces are discussed extensively.

We now consider other ways of obtaining the Klein bottle. Going back to the map $\mathcal{H}$ from $\mathbb{A}^3$ to $\mathbb{A}^4$ defined as

$$(x, y, z) \mapsto (xy, yz, xz, x^2 - y^2),$$

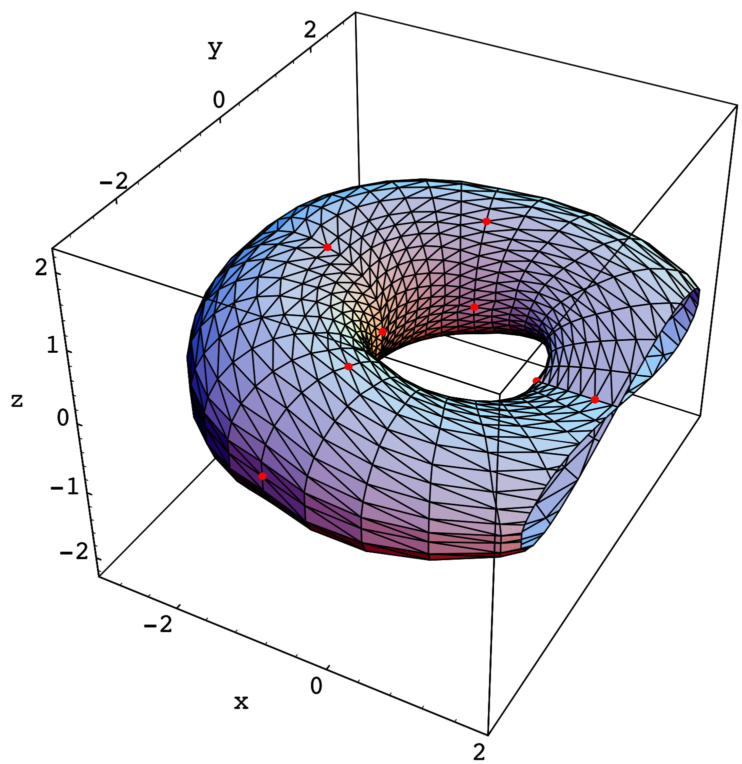

Figure 24.30: A Klein bottle, version 3

we can show that when this map is restricted to the torus T^2 defined by the implicit equation

$$\left(\frac{x^2}{b^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-\frac{a^2}{b^2}-1\right)^2=\frac{4a^2}{b^2}\left(1-\frac{z^2}{c^2}\right),$$

we have $\mathcal{H}(x,y,z)=\mathcal{H}(x',y',z')$ iff $(x',y',z')=(x,y,z)$ or $(x',y',z')=(-x,-y,-z)$. In other words, the inverse image of every point in $\mathcal{H}(T^2)$ consists of two antipodal points. Thus, the map $\mathcal{H}$ induces an injective map from the Klein bottle onto $\mathcal{H}(T^2)$, which is obviously continuous, and since the Klein bottle is compact, a it is a homeomorphism. Thus, the map $\mathcal{H}$ allows us to realize concretely the Klein bottle in $\mathbb{A}^4$, using any parameterization of the torus. For example, we can use the parameterization

$$\begin{aligned}x&=(a-b\sin\varphi)\cos\theta,\\ y&=(a-b\sin\varphi)\sin\theta,\\ z&=c\cos\varphi,\end{aligned}$$

and to avoid an ennoying factor 1/2, use the map $\mathcal{H}'$

$$(x,y,z)\mapsto(2xy,yz,xz,x^2-y^2),$$

which is clearly equivalent to $\mathcal{H}$. We get

$$\begin{aligned}x&=(a-b\sin\varphi)^2\sin 2\theta,\\ y&=c(a-b\sin\varphi)\cos\varphi\sin\theta,\\ z&=c(a-b\sin\varphi)\cos\varphi\cos\theta,\\ t&=(a-b\sin\varphi)^2\cos 2\theta.\end{aligned}$$

Switching to $u=\tan(\theta/2)$ and $v=\tan(\varphi/2)$, we get

$$\begin{aligned}x&=\frac{4u(1-u^2)(a(1+v^2)-2bv)^2}{(u^2+1)^2(v^2+1)^2},\\ y&=\frac{2cu(1+u^2)(1-v^2)(a(1+v^2)-2bv)}{(u^2+1)^2(v^2+1)^2},\\ z&=\frac{c(1-u^2)(1+u^2)(1-v^2)(a(1+v^2)-2bv)}{(u^2+1)^2(v^2+1)^2},\\ t&=\frac{(u^4-6u^2+1)(a(1+v^2)-2bv)^2}{(u^2+1)^2(v^2+1)^2}.\end{aligned}$$

Let us look at the projection of $\mathcal{H}'(T^2)$ obtained by dropping z.

Example 11: Klein bottle 4.

This surface reminicent of the surface of example 10, is defined by

$$\begin{aligned}x&=\frac{4u(1-u^2)(a(1+v^2)-2bv)^2}{(u^2+1)^2(v^2+1)^2},\\ y&=\frac{2cu(1+u^2)(1-v^2)(a(1+v^2)-2bv)}{(u^2+1)^2(v^2+1)^2},\\ z&=\frac{(u^4-6u^2+1)(a(1+v^2)-2bv)^2}{(u^2+1)^2(v^2+1)^2}.\end{aligned}$$

Actually, when $-1\le u\le 1$ and $-1\le v\le 1$, we obtain half of the surface, but the other half is obtained by the symmetry $y\mapsto -y$. It is a crossed-torus. For $a=8$, $b=2$, and $c=10$, the following net of degree 8 is obtained (actually, the coordinates x and y are swapped, but this is unimportant):

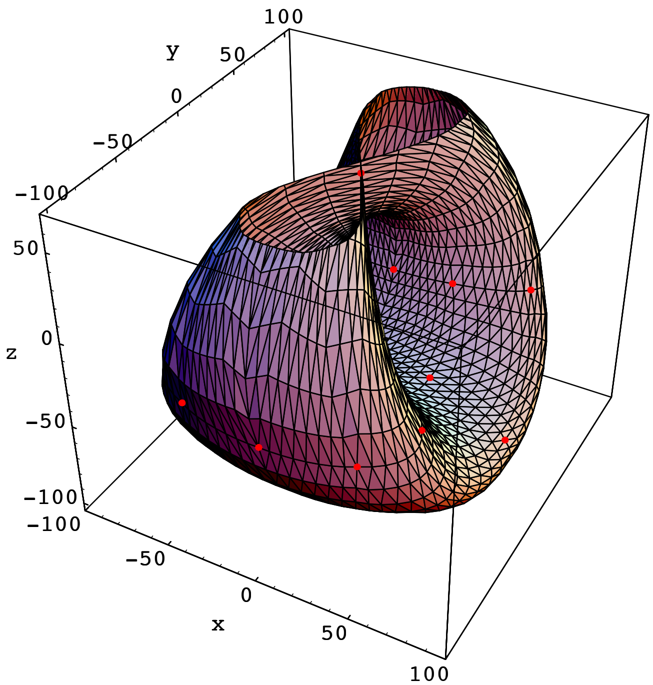

Figure 24.31: A Klein bottle, version 4

```
kleintnet = {{0, 0, 64, 1}, {0, 0, 56, 1},
   {0, 0, 248/5, 15/14}, {0, 0, 760/17, 17/14},
   {0, 0, 4144/101, 101/70}, {0, 0, 192/5, 25/14}, {0, 0, 147/4, 16/7},
   {0, 0, 36, 3}, {0, 0, 36, 4}, {20, 32, 64, 1}, {130/7, 192/7, 56, 1},
   {16, 368/15, 248/5, 15/14}, {224/17, 1936/85, 760/17, 17/14},
   {1040/101, 2208/101, 4144/101, 101/70}, {36/5, 544/25, 192/5, 25/14},
  {15/4, 45/2, 147/4, 16/7}, {0, 24, 36, 3}, {112/3, 896/15, 704/15, 15/14},
   {104/3, 256/5, 208/5, 15/14}, {3600/121, 5520/121, 4404/121, 121/105},
  {560/23, 968/23, 734/23, 46/35}, {6240/331, 13248/331, 9264/331, 331/210},
   {1080/83, 3264/83, 2016/83, 83/42}, {20/3, 40, 62/3, 18/7},
   {880/17, 1280/17, 320/17, 17/14}, {48, 5504/85, 304/17, 17/14},
   {940/23, 1314/23, 342/23, 46/35}, {1755/53, 2748/53, 604/53, 53/35},
   {1080/43, 48, 976/129, 129/70}, {180/11, 496/11, 32/11, 33/14},
   {6400/101, 7680/101, -1216/101, 101/70},
   {5920/101, 6656/101, -832/101, 101/70},
   {16320/331, 18912/331, -2784/331, 331/210},
   {5080/129, 6496/129, -1312/129, 129/70},
   {4640/161, 7104/161, -2048/161, 23/10}, {72, 64, -192/5, 25/14},
   {332/5, 1408/25, -768/25, 25/14}, {4560/83, 3936/83, -2304/83, 83/42},
   {1420/33, 1312/33, -896/33, 33/14}, {155/2, 44, -56, 16/7},
   {285/4, 40, -46, 16/7}, {520/9, 32, -40, 18/7}, {80, 64/3, -64, 3},
   {220/3, 64/3, -160/3, 3}, {80, 0, -64, 4}};
```

The result of subdividing twice, and cutting off part of the top, is shown in Figure 24.31.

The surface is also a crossed torus. We will consider one more version of the Klein bottle, which is in some sense more natural.

Example 12: Klein bottle 5.

Suppose we have an ellipse of center O in the xOy plane, given by $x = a\cos\theta$, $y = b\sin\theta$. Consider the surface generated by a circle of center M on the ellipse and of radius r, and in the plane containing the normal at M to the plane xOy, and making an angle $\theta/2$ with Ox. When $\theta = 0$, this plane is the xOz plane, and when $\theta = \pi$, it is the plane othogonal to the xOz plane and containing the point $x = -a$. One will realize that the surface starts like a torus, but when θ is close to π, the surface intersects itself and ends by glueing the two ends of the tubular torus in the the plane othogonal to the xOz plane and containing the point $x = -a$. Thus, it is indeed a Klein bottle. We leave as an exercise that this surface is defined by:

$$\begin{aligned}
x &= a\cos\theta + r\cos\varphi\cos(\theta/2),\\
x &= b\sin\theta + r\cos\varphi\sin(\theta/2),\\
z &= r\sin\varphi.
\end{aligned}$$

Switching to $u = \tan(\theta/4)$ and $v = \tan(\varphi/4)$, we get:

$$\begin{aligned}
x &= \frac{a(u^4 - 6u^2 + 1)}{(u^2+1)^2} + \frac{r(1-u^2)(v^4 - 6v^2 + 1)}{(u^2+1)(v^2+1)^2},\\
y &= \frac{4bu(1-u^2)}{(u^2+1)^2} + \frac{2ru(v^4 - 6v^2 + 1)}{(u^2+1)(v^2+1)^2},\\
z &= \frac{4rv(1-v^2)}{(v^2+1)^2}.
\end{aligned}$$

The entire surface happens to be defined over $[-1, 1] \times [-1, 1]$. For $a = 6$, $b = 3$, and $r = 2$, the following net of degree 8 is obtained:

```
klein2 =   {{8, 0, 0, 1}, {8, 0, 1, 1}, {112/15, 0, 28/15, 15/14},
   {112/17, 0, 40/17, 17/14}, {568/101, 0, 240/101, 101/70},
   {24/5, 0, 2, 25/14}, {17/4, 0, 11/8, 16/7}, {4, 0, 2/3, 3}, {4, 0, 0, 4},
   {8, 2, 0, 1}, {8, 2, 1, 1}, {112/15, 28/15, 28/15, 15/14},
   {112/17, 28/17, 40/17, 17/14}, {568/101, 144/101, 240/101, 101/70},
   {24/5, 32/25, 2, 25/14}, {17/4, 5/4, 11/8, 16/7}, {4, 4/3, 2/3, 3},
   {94/15, 56/15, 0, 15/14}, {94/15, 56/15, 46/45, 15/14},
   {687/121, 420/121, 230/121, 121/105}, {217/46, 70/23, 109/46, 46/35},
   {1200/331, 864/331, 776/331, 331/210}, {216/83, 192/83, 158/83, 83/42},
   {16/9, 20/9, 11/9, 18/7}, {58/17, 82/17, 0, 17/14},
   {58/17, 82/17, 18/17, 17/14}, {127/46, 101/23, 45/23, 46/35},
   {91/53, 196/53, 127/53, 53/35}, {64/129, 128/43, 296/129, 129/70},
   {-8/11, 80/33, 58/33, 33/14}, {24/101, 520/101, 0, 101/70},
   {24/101, 520/101, 112/101, 101/70},
  {-132/331, 1488/331, 672/331, 331/210}, {-60/43, 448/129, 104/43, 129/70},
   {-408/161, 384/161, 352/161, 23/10}, {-64/25, 24/5, 0, 25/14},
   {-64/25, 24/5, 88/75, 25/14}, {-252/83, 324/83, 176/83, 83/42},
   {-124/33, 28/11, 80/33, 33/14}, {-73/16, 4, 0, 16/7},
   {-73/16, 4, 5/4, 16/7}, {-85/18, 26/9, 20/9, 18/7}, {-17/3, 3, 0, 3},
   {-17/3, 3, 4/3, 3}, {-6, 2, 0, 4}};
```

The result of subdividing twice, and cutting off the top half of the surface, is shown in Figure 24.32. Note that the self-intersection is particularly clear.

Another view of the entire surface is shown in Figure 24.33.

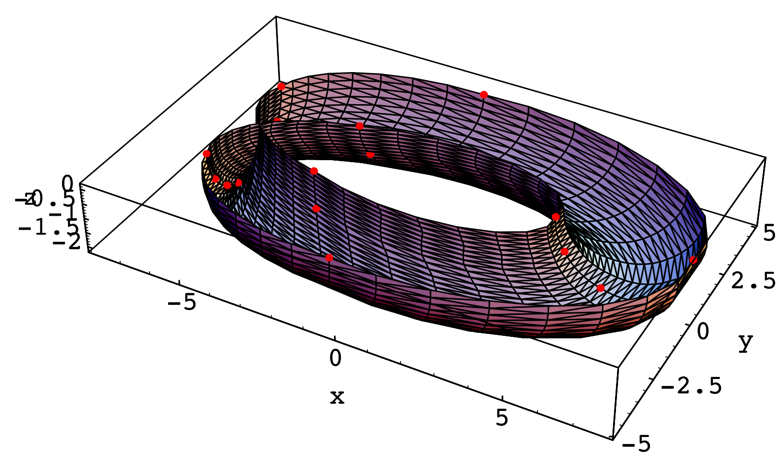

Figure 24.32: Half of a Klein bottle, version 5

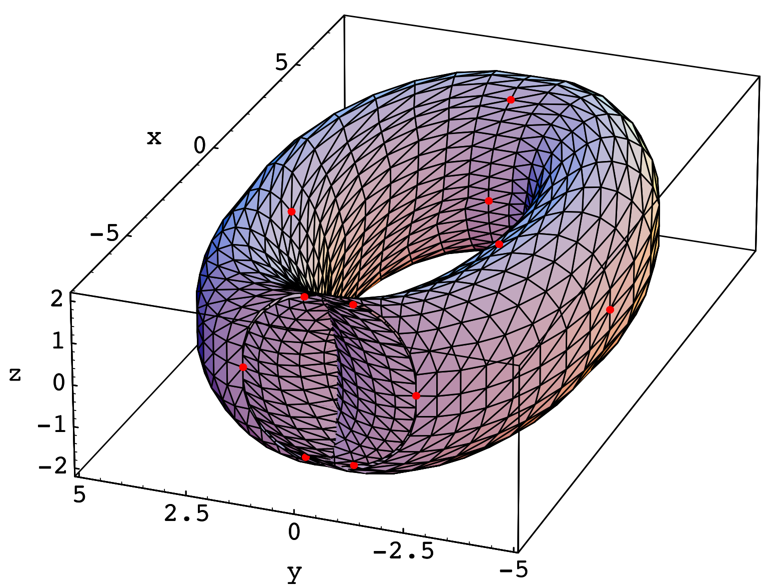

Figure 24.33: A Klein bottle, version 6