24.7 A "Gallery" of Rational Surfaces

The purpose of this section is to illustrate the power and flexibility of the methods developed earlier, by presenting some nontrivial examples of surfaces. Our intention is to present surfaces that are not only mathematically interesting but also esthetically (even artistically) attractive. In particular, we will present several interesting models of the real projective plane and of the Klein bottle. We will also give control nets for all the nondegenerate quadrics: ellipsoids, elliptic paraboloids, hyperbolic paraboloids, hyperboloids of one sheet, and hyperboloids of two sheets. Of course, we will also present a model of the Möbius strip.

Since for degrees higher than 4, it is practically impossible to compute manually control nets from parametric specifications, we wrote some programs for polarizing polynomials in two variables, in order to obtain triangular nets or rectangular nets from parametric equations. We urge the reader to do the same.

We begin our tour with the Enneper surface. This is actually a polynomial surface, but it is an intriguing surface which has some remarkable properties.

Example 1: The Enneper surface.

Recall that it is defined by

$$
\begin{aligned}
x(u,v) &= u - \frac{u^3}{3} + uv^2\\
y(u,v) &= v - \frac{v^3}{3} + u^2v\\
z(u,v) &= u^2 - v^2.
\end{aligned}
$$

The Enneper surface is what is a called a *minimal surface*, which means that the mean curvature is zero at every point. It is also easy to see that if we rotate the surface by $\pi/2$ around the z-axis, and then reflect it about the xOy plane $z = 0$, we get the same surface. Furthermore, it has two holes in orthogonal directions, and it intersects itself along two curves. A view of the Enneper surface obtained by subdividing 3 times over $[-2, 2] \times [-2, 2]$ and revealing one of the holes, is shown in Figure 24.16.

It is possible to display the Enneper surface in a nicer way if we use a polar coordinates representation. Letting $u = \rho\cos\theta$ and $v = \rho\sin\theta$, we get

$$
\begin{aligned}
x &= (\rho + \rho^3)\cos\theta - \frac{4}{3}\,\rho^3\cos^3\theta,\\
y &= (\rho + \rho^3)\sin\theta - \frac{4}{3}\,\rho^3\sin^3\theta,\\
z &= \rho^2(\cos^2\theta - \sin^2\theta).
\end{aligned}
$$

These are not rational functions, but expressing $\cos\theta$ and $\sin\theta$ in terms of $u = \tan(\theta/2)$ and letting $v = \rho$, we get the following specification:

$$
\begin{aligned}
x &= \frac{3(v+v^3)(1-u^2)(1+u^2)^2 - 4v^3(1-u^2)^3}{3(1+u^2)^3},\\
y &= \frac{6(v+v^3)u(1+u^2)^2 - 32u^3v^3}{3(1+u^2)^3},\\
z &= \frac{3v^2(u^4-6u^2+1)(1+u^2)}{3(1+u^2)^3}.
\end{aligned}
$$

As a bipolynomial rational surface, we find the following rectangular net of degree $(6, 3)$:

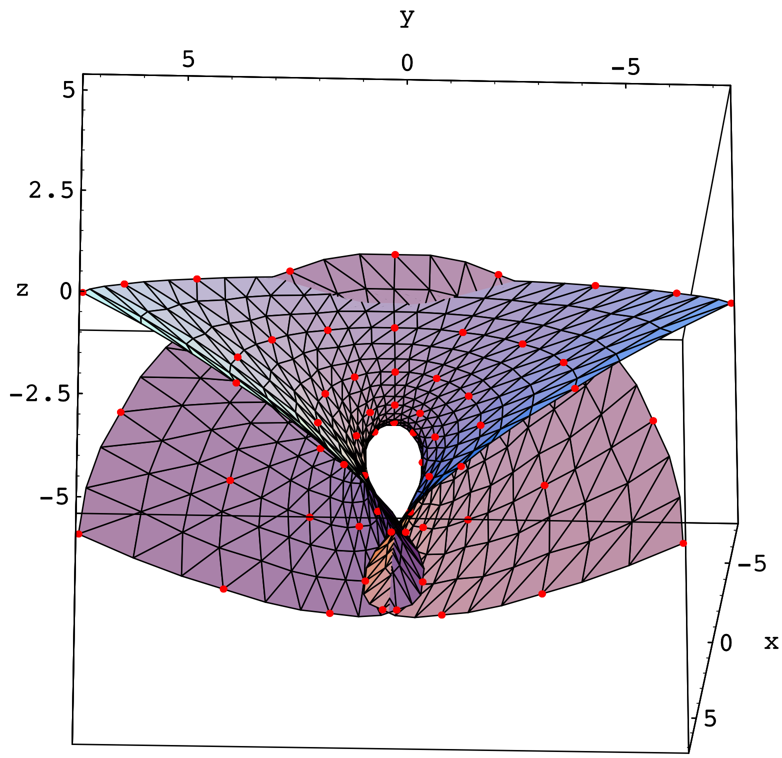

Figure 24.16: The Enneper surface

```
ennep6 =   {{0, 0, 0, 3}, {1/3, 0, 0, 3}, {2/3, 0, 1/3, 3}, {2/3, 0, 1, 3},
   {0, 0, 0, 3}, {1/3, 1/9, 0, 3}, {2/3, 2/9, 1/3, 3}, {2/3, 2/3, 1, 3},
   {0, 0, 0, 18/5}, {8/27, 5/27, 0, 18/5}, {16/27, 10/27, 5/27, 18/5},
   {8/9, 10/9, 5/9, 18/5}, {0, 0, 0, 24/5}, {1/4, 1/4, 0, 24/5},
   {1/2, 1/2, 0, 24/5}, {7/6, 7/6, 0, 24/5}, {0, 0, 0, 36/5},
   {5/27, 8/27, 0, 36/5}, {10/27, 16/27, -5/27, 36/5},
   {10/9, 8/9, -5/9, 36/5}, {0, 0, 0, 12}, {1/9, 1/3, 0, 12},
   {2/9, 2/3, -1/3, 12}, {2/3, 2/3, -1, 12}, {0, 0, 0, 24}, {0, 1/3, 0, 24},
   {0, 2/3, -1/3, 24}, {0, 2/3, -1, 24}};
```

As a triangular patch, we find the following net of degree 9.:

```
ennep9 = {{0, 0, 0, 3}, {1/9, 0, 0, 3},
   {2/9, 0, 1/36, 3}, {83/252, 0, 1/12, 3},
   {3/7, 0, 1/6, 3}, {65/126, 0, 5/18, 3}, {37/63, 0, 5/12, 3},
   {23/36, 0, 7/12, 3}, {2/3, 0, 7/9, 3}, {2/3, 0, 1, 3}, {0, 0, 0, 3},
   {1/9, 1/36, 0, 3}, {2/9, 1/18, 1/36, 3}, {83/252, 11/126, 1/12, 3},
   {3/7, 8/63, 1/6, 3}, {65/126, 5/28, 5/18, 3}, {37/63, 31/126, 5/12, 3},
   {23/36, 1/3, 7/12, 3}, {2/3, 4/9, 7/9, 3}, {0, 0, 0, 13/4},
   {29/273, 2/39, 0, 13/4}, {58/273, 4/39, 16/819, 13/4},
   {29/91, 44/273, 16/273, 13/4}, {116/273, 64/273, 32/273, 13/4},
   {145/273, 30/91, 160/819, 13/4}, {58/91, 124/273, 80/273, 13/4},
   {29/39, 8/13, 16/39, 13/4}, {0, 0, 0, 15/4}, {31/315, 23/315, 0, 15/4},
   {62/315, 46/315, 2/315, 15/4}, {19/63, 71/315, 2/105, 15/4},
   {44/105, 20/63, 4/105, 15/4}, {5/9, 3/7, 4/63, 15/4},
   {226/315, 178/315, 2/21, 15/4}, {0, 0, 0, 32/7}, {7/80, 3/32, 0, 32/7},
   {7/40, 3/16, -1/96, 32/7}, {131/480, 9/32, -1/32, 32/7},
   {47/120, 3/8, -1/16, 32/7}, {13/24, 15/32, -5/48, 32/7}, {0, 0, 0, 41/7},
   {3/41, 14/123, 0, 41/7}, {6/41, 28/123, -11/369, 41/7},
   {28/123, 1/3, -11/123, 41/7}, {40/123, 52/123, -22/123, 41/7},
   {0, 0, 0, 55/7}, {3/55, 2/15, 0, 55/7}, {6/55, 4/15, -8/165, 55/7},
   {9/55, 64/165, -8/55, 55/7}, {0, 0, 0, 11}, {1/33, 5/33, 0, 11},
   {2/33, 10/33, -2/33, 11}, {0, 0, 0, 16}, {0, 1/6, 0, 16}, {0, 0, 0, 24}};
```

A particularly nice view is obtained by subdividing twice over $[-1,1]\times[-2,2]$, as shown in Figure 24.17.

We now consider several ways of viewing the real projective plane in $\mathbb{A}^3$. Recall that as a topological space, the projective plane $\mathbb{RP}^2$ is the quotient of the 2-sphere S^2 (in $\mathbb{A}^3$) by the equivalence relation that identifies antipodal points. In Hilbert and Cohn-Vossen [84] (and also do Carmo [51]), an interesting map $\mathcal{H}$ from $\mathbb{A}^3$ to $\mathbb{A}^4$ is defined as

$$(x,y,z)\mapsto (xy, yz, xz, x^2-y^2).$$

This map has the remarkable property that when restricted to S^2, we have $\mathcal{H}(x,y,z)=\mathcal{H}(x',y',z')$ iff $(x',y',z')=(x,y,z)$ or $(x',y',z')=(-x,-y,-z)$. In other words, the inverse image of every point in $\mathcal{H}(S^2)$ consists of two antipodal points. Thus, the map $\mathcal{H}$ induces an injective map from the projective plane onto $\mathcal{H}(S^2)$, which is obviously continuous, and since the projective plane is compact, it is a homeomorphism. Thus, the map $\mathcal{H}$ allows us to realize concretely the projective plane in $\mathbb{A}^4$, by choosing any parameterization of the sphere S^2, and applying the map $\mathcal{H}$ to it. Actually, it turns out to be more convenient to use the map $\mathcal{A}$ defined such that

$$(x,y,z)\mapsto (2xy, 2yz, 2xz, x^2-y^2),$$

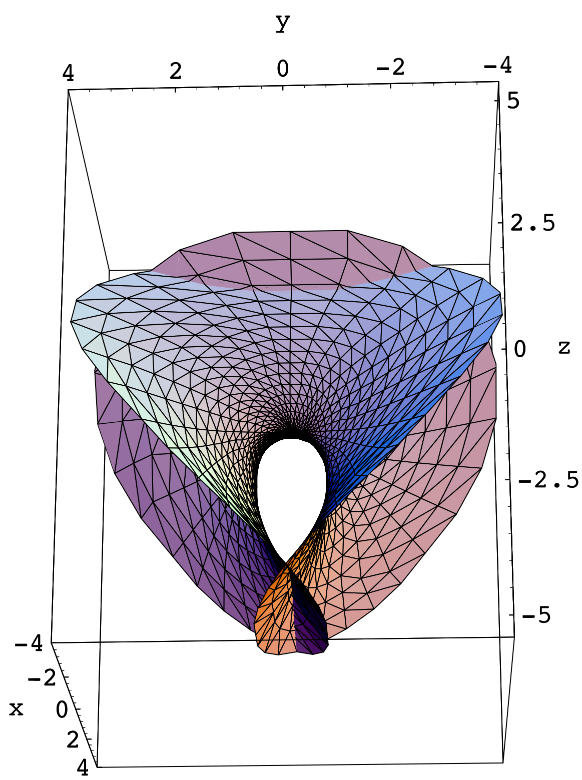

Figure 24.17: Another view of the Enneper surface

because it yields nicer parameterizations. For example, using the stereographic representation where

$$\begin{aligned} x(u,v) &= \frac{2u}{u^2+v^2+1}, \\ y(u,v) &= \frac{2v}{u^2+v^2+1}, \\ z(u,v) &= \frac{u^2+v^2-1}{u^2+v^2+1}, \end{aligned}$$

we obtain the following four fractions parameterizing the projective plane in $\mathbb{A}^4$:

$$\begin{aligned} x(u,v) &= \frac{8uv}{(u^2+v^2+1)^2}, \\ y(u,v) &= \frac{4v(u^2+v^2-1)}{(u^2+v^2+1)^2}, \\ z(u,v) &= \frac{4u(u^2+v^2-1)}{(u^2+v^2+1)^2}, \\ t(u,v) &= \frac{4(u^2-v^2)}{(u^2+v^2+1)^2}. \end{aligned}$$

Of course, we don't know how to visualize this surface in $\mathbb{A}^4$, but we can visualize its four projections in $\mathbb{A}^3$, using parallel projections with respect to the four axes, which amounts to dropping one of the four coordinates. The resulting surfaces turn out to be very interesting. Only two distinct surfaces are obtained, both very well known to topologists. Indeed, the surface obtained by dropping y or z is known as the cross-cap surface, and the surface obtained by dropping x or t is known as the Steiner roman surface. We will display all of these surfaces using our tools. We begin with the Steiner surface.

Example 2: the Steiner roman surface.

This is the surface obtained by dropping t. Going back to the map $\mathcal{A}$ and renaming x, y, z as α, β, γ, if

$$x = 2\alpha\beta, \quad y = 2\beta\gamma, \quad z = 2\alpha\gamma, \quad t = \alpha^2 - \beta^2,$$

it is easily seen that

$$\begin{aligned} xy &= 2z\beta^2, \\ yz &= 2x\gamma^2, \\ xz &= 2y\alpha^2. \end{aligned}$$

If (α, β, γ) is on the sphere, we have $\alpha^2+\beta^2+\gamma^2 = 1$, and thus we get the following implicit equation for the Steiner roman surface:

$$x^2y^2 + y^2z^2 + x^2z^2 = 2xyz.$$

It is easily verified that the following parameterization works:

$$\begin{aligned} x(u,v) &= \frac{2v}{u^2+v^2+1}, \\ y(u,v) &= \frac{2u}{u^2+v^2+1}, \\ z(u,v) &= \frac{2uv}{u^2+v^2+1}. \end{aligned}$$

Thus, amazingly, the Steiner roman surface can be specified by fractions of quadratic polynomials! It can be shown that every quadric surface can be defined as a rational surface of degree 2, but other surfaces can also be defined, as showed by the Steiner roman surface. Rational patches of degree 2 were investigated by Steiner, see Sederberg and Anderson [154]. It can be shown that the Steiner roman surface is contained inside the tetrahedron defined by the planes

$$\begin{aligned} -x+y+z &= 1, \\ x-y+z &= 1, \\ x+y-z &= 1, \\ -x-y-z &= 1, \end{aligned}$$

with $-1 \leq x, y, z \leq 1$. The surface touches these four planes along ellipses, and at the middle of the six edges of the tetrahedron, it has sharp edges. Furthermore, the surface is self-intersecting along the axes, and is has four closed chambers. A more extensive discussion can be found in Hilbert and Cohn-Vossen [84], in particular, its relationship to the heptahedron. A triangular control net is easily obtained:

```
stein1 = {{0, 0, 0, 1}, {1, 0, 0, 1}, {1, 0, 0, 2},
       {0, 1, 0, 1}, {1, 1, 1, 1},
       {0, 1, 0, 2}};
```

We can display the entire surface using the algorithm of section 24.3. Indeed, all six patches are needed to obtain the entire surface. One view of the surface obtained by subdividing 3 times is shown below. Patches 1 and 2 are colored blue, patches 3 and 4 are colored red, and patches 5 and 6 are colored green. A closer look reveals that the three colored patches are identical under appropriate rigid motions, and fit perfectly.

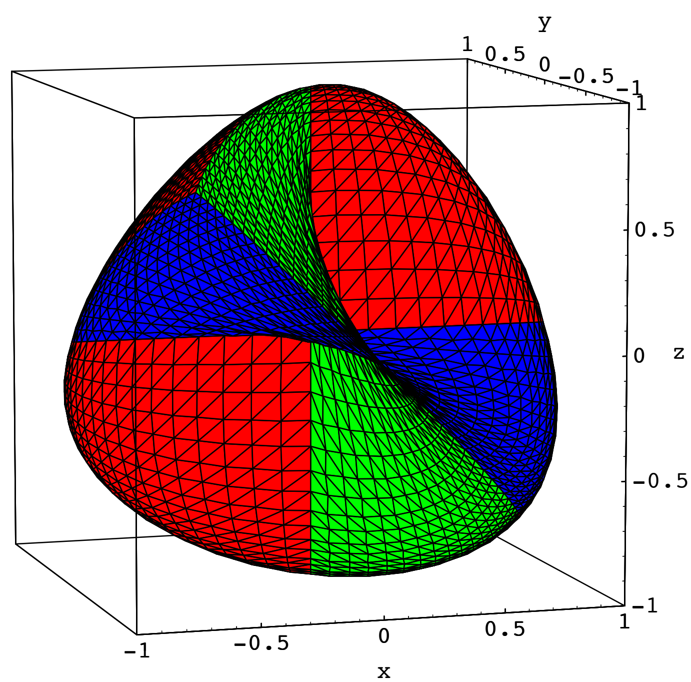

Figure 24.18: The Steiner roman surface

Another revealing view is obtained by cutting off a top portion of the surface.

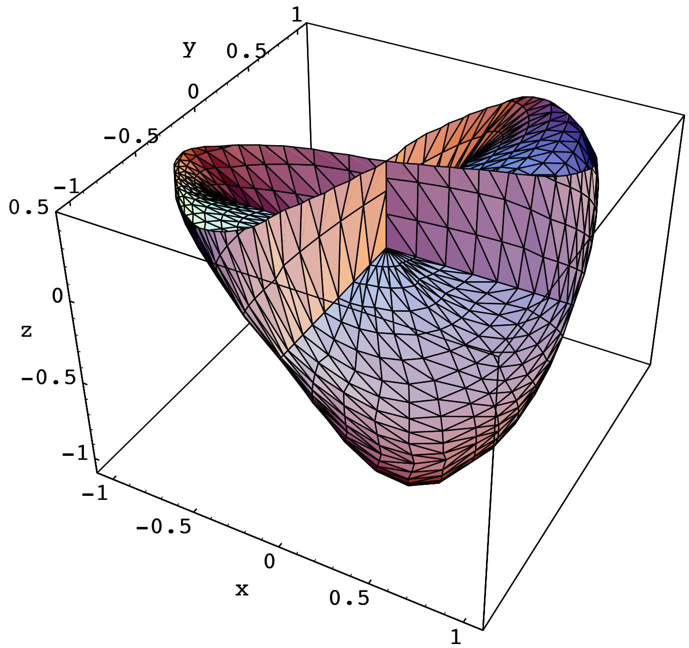

Figure 24.19: A cut of the Steiner roman surface

In the above picture, it is clear that the surface has chambers. We now consider the cross-cap surface.

Example 3: the cross-cap surface.

This is the surface obtained by dropping either the y coordinate, or the z coordinate. Let us first consider the surface obtained by dropping y. Its implicit equation is obtained by eliminating α, β, γ in the equations

$$x = 2\alpha\beta, \quad z = 2\alpha\gamma, \quad t = \alpha^2 - \beta^2,$$

and $\alpha^2 + \beta^2 + \gamma^2 = 1$. We leave as an exercise to show that we get

$$(2x^2 + z^2)^2 = 4(x^2 + t(x^2 + z^2))(1 - t).$$

If we now consider the surface obtained by dropping z, the implicit equation is obtained by eliminating α, β, γ in the equations

$$x = 2\alpha\beta, \quad y = 2\beta\gamma, \quad t = \alpha^2 - \beta^2,$$

and $\alpha^2 + \beta^2 + \gamma^2 = 1$. We leave as an exercise to show that we get

$$(2x^2 + y^2)^2 = 4(x^2 - t(x^2 + y^2))(1 + t).$$

Note that the second implicit equation is obtained from the first by substituting y for z and $-t$ for t. This shows that the two implicit equations define the same surface.

As explained in Hilbert and Cohn-Vossen [84], one way of obtaining a model of this surface is to proceed as follows. Take a sphere made of rubber, cut a small rectangular piece from it, and assuming that the four vertices of this rectangular piece are, in the order in which they are encountered following a closed path, A, B, C, D (which means that the sides AB and CD are parallel and oriented in opposite directions, and similarly for DA and BC), and then glue the edges AB and CD together, as well as the edges DA and

CD. This can be accomplished by lowering B and D and raising A and C, and then doing the glueing. The resulting surface intersects itself along the line segment $AB = CD$.

Since an explicit parameterization of the surface obtained by dropping z is

$$\begin{aligned} x(u,v) &= \frac{8uv}{(u^2+v^2+1)^2}, \\ y(u,v) &= \frac{4v(u^2+v^2-1)}{(u^2+v^2+1)^2}, \\ z(u,v) &= \frac{4(u^2-v^2)}{(u^2+v^2+1)^2}, \end{aligned}$$

we can polarize the polynomials involved, and we get the following triangular net of degree 4:

```
projnet2 =  {{0, 0, 0, 1}, {0, -1, 0, 1}, {0, -3/2, -1/2, 4/3},
    {0, -1, -1, 2},  {0, 0, -1, 4}, {0, 0, 0, 1}, {2/3, -1, 0, 1},
    {1, -3/2, -1/2, 4/3}, {1, -1, -1, 2}, {0, 0, 1/2, 4/3},
    {1, -1/2, 1/2, 4/3}, {4/3, -2/3, 0, 2},
    {0, 0, 1, 2}, {1, 0, 1, 2}, {0, 0, 1, 4}};
```

We can display the entire surface using the algorithm of section 24.3. Actually, this time, it turns out that patches 1 and 2 already determine the entire surface. One view of the surface obtained by subdividing 3 times over $[-1,1] \times [-1,1]$, and cutting off part of the top of the surface to have a better view of the self intersection, is shown Figure 24.20:

A closer look reveals that the two patches corresponding to the triangles `reftrig1` and `reftrig2` overlap a little bit, which yields a slightly unpleasant rendering of the overlaping areas. It is possible to get a better rendering by switching to a kind of polar coordinates. We let $u = \rho\cos\theta$ and $v = \rho\sin\theta$, and obtain

$$\begin{aligned} x &= \frac{4\rho^2}{(\rho^2+1)^2}\sin 2\theta, \\ y &= \frac{4\rho(\rho^2-1)}{(\rho^2+1)^2}\sin\theta, \\ z &= \frac{4\rho^2}{(\rho^2+1)^2}\cos 2\theta. \end{aligned}$$

Then, expressing the trigonometric functions in terms of $u = \tan(\theta/2)$, letting $v = \rho$, we get

$$\begin{aligned} x &= \frac{16uv^2(1-u^2)}{(u^2+1)^2(v^2+1)^2}, \\ y &= \frac{8uv(u^2+1)(v^2-1)}{(u^2+1)^2(v^2+1)^2}, \\ z &= \frac{4v^2(u^4-6u^2+1)}{(u^2+1)^2(v^2+1)^2}. \end{aligned}$$

The following triangular net of degree 8 is obtained:

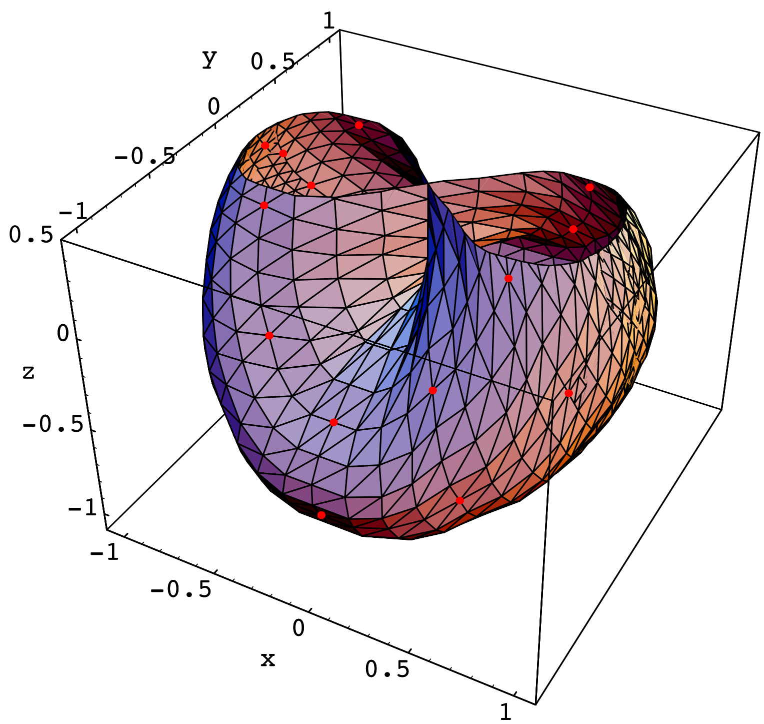

Figure 24.20: A cut of the cross-cap surface

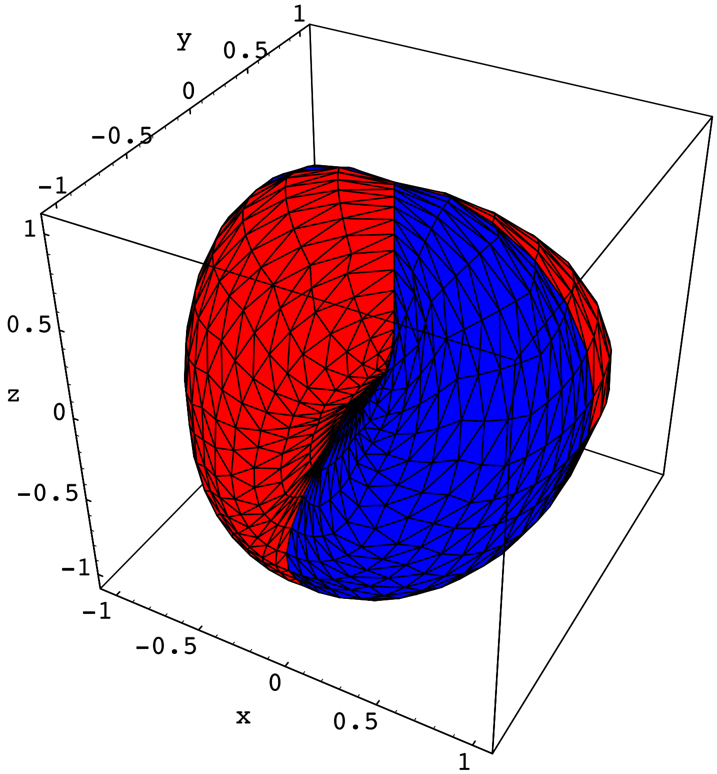

Figure 24.21: The cross-cap surface

```
projnet4 =
   {{0, 0, 0, 1}, {0, 0, 0, 1}, {0, 0, 2/15, 15/14}, {0, 0, 6/17, 17/14},
   {0, 0, 60/101, 101/70}, {0, 0, 4/5, 25/14}, {0, 0, 15/16, 16/7},
   {0, 0, 1, 3}, {0, 0, 1, 4}, {0, 0, 0, 1}, {0, -1/7, 0, 1},
   {4/45, -4/15, 2/15, 15/14}, {4/17, -28/85, 6/17, 17/14},
   {40/101, -32/101, 60/101, 101/70}, {8/15, -6/25, 4/5, 25/14},
   {5/8, -1/8, 15/16, 16/7}, {2/3, 0, 1, 3}, {0, 0, 0, 15/14},
   {0, -4/15, 0, 15/14}, {20/121, -60/121, 9/121, 121/105},
   {10/23, -14/23, 9/46, 46/35}, {240/331, -192/331, 108/331, 331/210},
   {80/83, -36/83, 36/83, 83/42}, {10/9, -2/9, 1/2, 18/7}, {0, 0, 0, 17/14},
   {0, -32/85, 0, 17/14}, {9/46, -16/23, -1/46, 46/35},
   {27/53, -89/106, -3/53, 53/35}, {36/43, -100/129, -4/43, 129/70},
   {12/11, -6/11, -4/33, 33/14}, {0, 0, 0, 101/70}, {0, -48/101, 0, 101/70},
   {56/331, -288/331, -40/331, 331/210}, {56/129, -44/43, -40/129, 129/70},
   {16/23, -144/161, -80/161, 23/10}, {0, 0, 0, 25/14},
   {0, -14/25, 0, 25/14}, {8/83, -84/83, -16/83, 83/42},
   {8/33, -38/33, -16/33, 33/14}, {0, 0, 0, 16/7}, {0, -5/8, 0, 16/7},
   {0, -10/9, -2/9, 18/7}, {0, 0, 0, 3}, {0, -2/3, 0, 3}, {0, 0, 0, 4}}
```

Again, the entire surface is obtained by subdividing twice over $[-1, 1] \times [-1, 1]$, as shown in Figure 24.21

From the last specification, it is easy to see that the cross-cap has a self-intersection iff $\tan 2\theta_1 = \tan 2\theta_2$, that is, if $2\theta_1 = 2\theta_2 + \pi$. But then, it is easy to show that we must have $\theta_1 = 0$ and $\theta_2 = \pi$. Thus, the cross-cap intersects itself along the portion of the z axis corresponding to $0 \leq z \leq 1$.

Example 4: the Steiner roman surface, again.

The last projection of the projective plane is obtained by dropping the x coordinate. Its implicit equation is obtained by eliminating α, β, γ in the equations

$$y = 2\beta\gamma, \quad z = 2\alpha\gamma, \quad t = \alpha^2 - \beta^2,$$

and $\alpha^2 + \beta^2 + \gamma^2 = 1$. We leave as an exercise to show that we get

$$4(y^2 + z^2)t^2 = (z^2 - y^2)(y^2 - z^2 + 4t).$$

This time, it is not so obvious that it corresponds to the Steiner roman surface. However, if we perform the rotation of the y, z plane by $\pi/4$, we have

$$\begin{aligned} y &= \frac{\sqrt{2}}{2}Y - \frac{\sqrt{2}}{2}Z, \\ z &= \frac{\sqrt{2}}{2}Y + \frac{\sqrt{2}}{2}Z, \end{aligned}$$

and we have $y^2 + z^2 = Y^2 + Z^2$ and $z^2 - y^2 = 2YZ$. Thus, the implicit equation becomes

$$(Y^2 + Z^2)t^2 = 2YZ(-2YZ + 4t),$$

which simplifies to

$$Y^2t^2 + Z^2t^2 + Y^2Z^2 = 2YZt,$$

which is exactly the equation of the Steiner roman surface.

Just for fun, we also get the parameterization

$$\begin{aligned} x(u,v) &= \frac{4v(u^2 + v^2 - 1)}{(u^2 + v^2 + 1)^2}, \\ y(u,v) &= \frac{4u(u^2 + v^2 - 1)}{(u^2 + v^2 + 1)^2}, \\ z(u,v) &= \frac{4(u^2 - v^2)}{(u^2 + v^2 + 1)^2}. \end{aligned}$$

The following triangular net of degree 4 is obtained:

```
projnet3 =  {{0, 0, 0, 1}, {-1, 0, 0, 1}, {-3/2, 0, -1/2, 4/3},
       {-1, 0, -1, 2},  {0, 0, -1, 4}, {0, -1, 0, 1}, {-1, -1, 0, 1},
       {-3/2, -1/2, -1/2, 4/3}, {-1, 0, -1, 2}, {0, -3/2, 1/2, 4/3},
       {-1/2, -3/2, 1/2, 4/3},   {-2/3, -2/3, 0, 2}, {0, -1, 1, 2},
       {0, -1, 1, 2}, {0, 0, 1, 4}};
```

Again, the entire surface is obtained by subdividing over $[-1,1] \times [-1,1]$. As in the case of the cross-cap, the two patches corresponding to `reftrig1` and `reftrig2` overlap a little, and we get a better rendering if we switch to a kind of polar coordinates. We let $u = \rho \cos\theta$ and $v = \rho \sin\theta$, and obtain

$$\begin{aligned} x &= \frac{4\rho(\rho^2 - 1)}{(\rho^2 + 1)^2} \sin\theta, \\ y &= \frac{4\rho(\rho^2 - 1)}{(\rho^2 + 1)^2} \cos\theta, \\ z &= \frac{4\rho^2}{(\rho^2 + 1)^2} \cos 2\theta. \end{aligned}$$

Then, expressing the trigonometric functions in terms of $u = \tan(\theta/2)$, letting $v = \rho$, we get

$$\begin{aligned} x &= \frac{8uv(u^2+1)(v^2-1)}{(u^2+1)^2(v^2+1)^2}, \\ y &= \frac{4v(1-u^4)(v^2-1)}{(u^2+1)^2(v^2+1)^2}, \\ z &= \frac{4v^2(u^4-6u^2+1)}{(u^2+1)^2(v^2+1)^2}. \end{aligned}$$

The following triangular net of degree 8 is obtained:

```
projnet5 =   {{0, 0, 0, 1}, {0, -1/2, 0, 1}, {0, -14/15, 2/15, 15/14},
   {0, -20/17, 6/17, 17/14}, {0, -120/101, 60/101, 101/70},
   {0, -1, 4/5, 25/14}, {0, -11/16, 15/16, 16/7}, {0, -1/3, 1, 3},
   {0, 0, 1, 4}, {0, 0, 0, 1}, {-1/7, -1/2, 0, 1},
   {-4/15, -14/15, 2/15, 15/14}, {-28/85, -20/17, 6/17, 17/14},
   {-32/101, -120/101, 60/101, 101/70}, {-6/25, -1, 4/5, 25/14},
   {-1/8, -11/16, 15/16, 16/7}, {0, -1/3, 1, 3}, {0, 0, 0, 15/14},
   {-4/15, -7/15, 0, 15/14}, {-60/121, -105/121, 9/121, 121/105},
   {-14/23, -25/23, 9/46, 46/35}, {-192/331, -360/331, 108/331, 331/210},
   {-36/83, -75/83, 36/83, 83/42}, {-2/9, -11/18, 1/2, 18/7},
   {0, 0, 0, 17/14}, {-32/85, -7/17, 0, 17/14},
   {-16/23, -35/46, -1/46, 46/35}, {-89/106, -50/53, -3/53, 53/35},
   {-100/129, -40/43, -4/43, 129/70}, {-6/11, -25/33, -4/33, 33/14},
   {0, 0, 0, 101/70}, {-48/101, -34/101, 0, 101/70},
   {-288/331, -204/331, -40/331, 331/210},
   {-44/43, -98/129, -40/129, 129/70}, {-144/161, -120/161, -80/161, 23/10},
   {0, 0, 0, 25/14}, {-14/25, -6/25, 0, 25/14},
   {-84/83, -36/83, -16/83, 83/42}, {-38/33, -6/11, -16/33, 33/14},
   {0, 0, 0, 16/7}, {-5/8, -1/8, 0, 16/7}, {-10/9, -2/9, -2/9, 18/7},
   {0, 0, 0, 3}, {-2/3, 0, 0, 3}, {0, 0, 0, 4}};
```

Subdividing twice over $[-1, 1] \times [-1, 1]$, we obtain the view of the Seiner roman surface displayed in Figure 24.22.

The Steiner roman surface contains four chambers. This is apparent if we cut off part of its top, as shown in Figure 24.23.

It is claimed in Hilbert and Cohn-Vossen ([84], page 341) that using the map

$$(x, y, z) \mapsto (xy, yz, xz, x^2 - y^2),$$

the two equations

$$y^2z^2 + y^2t^2 + z^2t^2 - yzt = 0$$

and

$$y(z^2 - t^2) - xzt = 0$$

suffice, but this is incorrect, since these equations are satisfied by all points such that $z = t = 0$. As far as I know, finding algebraic equations characterizing the version of $\mathbb{RP}^2$ that we have been discussing is still an open problem.

We can actually compute a control net in $\mathbb{A}^4$ for the real projective plane. We simply have to polarize all five polynomials! Corresponding to the parameterization

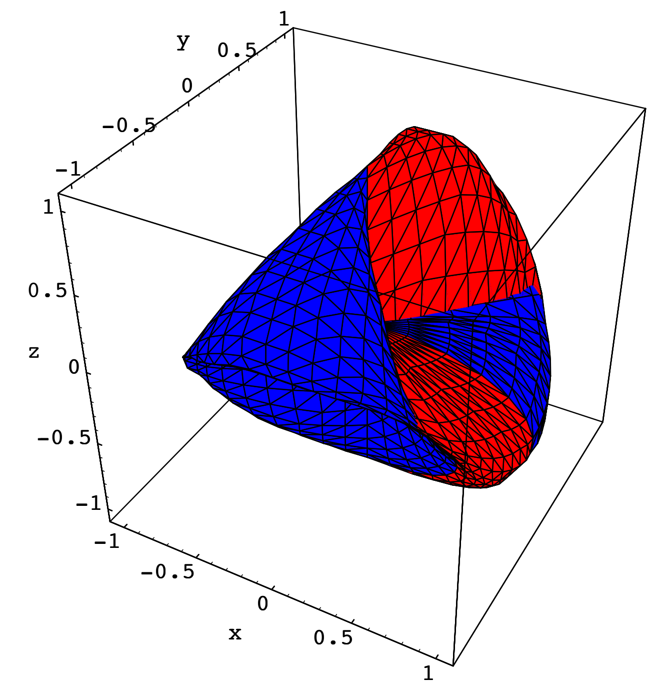

Figure 24.22: Another view of the Steiner roman surface

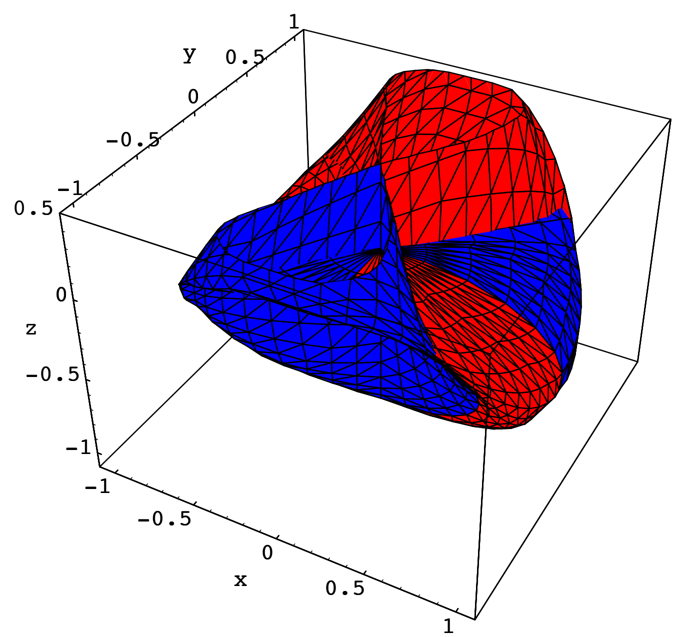

Figure 24.23: A cut of the Steiner roman surface

$$
\begin{aligned}
x(u,v) &= \frac{8uv}{(u^2+v^2+1)^2},\\
y(u,v) &= \frac{4v(u^2+v^2-1)}{(u^2+v^2+1)^2},\\
z(u,v) &= \frac{4u(u^2+v^2-1)}{(u^2+v^2+1)^2},\\
t(u,v) &= \frac{4(u^2-v^2)}{(u^2+v^2+1)^2},
\end{aligned}
$$

the following control net of degree 4 is obtained:

```
proj4net = {{0, 0, 0, 0, 1}, {0, -1, 0, 0, 1}, {0, -3/2, 0, -1/2, 4/3},
   {0, -1, 0, -1, 2}, {0, 0, 0, -1, 4}, {0, 0, -1, 0, 1},
   {2/3, -1, -1, 0, 1}, {1, -3/2, -1/2, -1/2, 4/3}, {1, -1, 0, -1, 2},
   {0, 0, -3/2, 1/2, 4/3}, {1, -1/2, -3/2, 1/2, 4/3},
   {4/3, -2/3, -2/3, 0, 2}, {0, 0, -1, 1, 2}, {1, 0, -1, 1, 2},
   {0, 0, 0, 1, 4}};
```

If we use the polar parameterization

$$
\begin{aligned}
x &= \frac{4\rho^2}{(\rho^2+1)^2}\,\sin 2\theta,\\
y &= \frac{4\rho(\rho^2-1)}{(\rho^2+1)^2}\,\sin\theta,\\
z &= \frac{4\rho(\rho^2-1)}{(\rho^2+1)^2}\,\cos\theta,\\
t &= \frac{4\rho^2}{(\rho^2+1)^2}\,\cos 2\theta,
\end{aligned}
$$

and convert it to the rational parameterization

$$
\begin{aligned}
x &= \frac{16uv^2(1-u^2)}{(u^2+1)^2(v^2+1)^2},\\
y &= \frac{8uv(u^2+1)(v^2-1)}{(u^2+1)^2(v^2+1)^2},\\
z &= \frac{4v(1-u^4)(v^2-1)}{(u^2+1)^2(v^2+1)^2},\\
t &= \frac{4v^2(u^4-6u^2+1)}{(u^2+1)^2(v^2+1)^2},
\end{aligned}
$$

the following net of degree 8 is obtained:

```
proj8net =
   {{0, 0, 0, 0, 1}, {0, 0, -1/2, 0, 1}, {0, 0, -14/15, 2/15, 15/14},
   {0, 0, -20/17, 6/17, 17/14}, {0, 0, -120/101, 60/101, 101/70},
   {0, 0, -1, 4/5, 25/14}, {0, 0, -11/16, 15/16, 16/7}, {0, 0, -1/3, 1, 3},
```

```
{0, 0, 0, 1, 4}, {0, 0, 0, 0, 1}, {0, -1/7, -1/2, 0, 1},
{4/45, -4/15, -14/15, 2/15, 15/14}, {4/17, -28/85, -20/17, 6/17, 17/14},
{40/101, -32/101, -120/101, 60/101, 101/70},
{8/15, -6/25, -1, 4/5, 25/14}, {5/8, -1/8, -11/16, 15/16, 16/7},
{2/3, 0, -1/3, 1, 3}, {0, 0, 0, 0, 15/14}, {0, -4/15, -7/15, 0, 15/14},
{20/121, -60/121, -105/121, 9/121, 121/105},
{10/23, -14/23, -25/23, 9/46, 46/35},
{240/331, -192/331, -360/331, 108/331, 331/210},
{80/83, -36/83, -75/83, 36/83, 83/42}, {10/9, -2/9, -11/18, 1/2, 18/7},
{0, 0, 0, 0, 17/14}, {0, -32/85, -7/17, 0, 17/14},
{9/46, -16/23, -35/46, -1/46, 46/35},
{27/53, -89/106, -50/53, -3/53, 53/35},
{36/43, -100/129, -40/43, -4/43, 129/70},
{12/11, -6/11, -25/33, -4/33, 33/14}, {0, 0, 0, 0, 101/70},
{0, -48/101, -34/101, 0, 101/70},
{56/331, -288/331, -204/331, -40/331, 331/210},
{56/129, -44/43, -98/129, -40/129, 129/70},
{16/23, -144/161, -120/161, -80/161, 23/10}, {0, 0, 0, 0, 25/14},
{0, -14/25, -6/25, 0, 25/14}, {8/83, -84/83, -36/83, -16/83, 83/42},
{8/33, -38/33, -6/11, -16/33, 33/14}, {0, 0, 0, 0, 16/7},
{0, -5/8, -1/8, 0, 16/7}, {0, -10/9, -2/9, -2/9, 18/7}, {0, 0, 0, 0, 3},
{0, -2/3, 0, 0, 3}, {0, 0, 0, 0, 4}};
```

If we project the real projective plane onto a hyperplane in $\mathbb{A}^4$, either from a center or parallel to a direction, we can see a "3D shadow" of the real projective plane in $\mathbb{A}^3$. For example, a fun thing to do is to travel along a circle in the z, t plane, to see how the Steiner roman surface evolves into the cross-cap. We leave such explorations as very challenging programming projects. Readers interested in the topology of curves and surfaces should consult the book by Francis [65], where many beautiful pictures are shown, and shadows of four dimensional surfaces are discussed extensively.

Example 5: Another cross-cap.

A very interesting surface related to the cross-cap is described in Hilbert and Cohn-Vossen [84]. This surface of implicit equation

$$(ax^2 + by^2)(x^2 + y^2 + z^2) = 2z(x^2 + y^2)$$

has a nice geometric interpretation in terms of differential geometry. Given any point p on a surface, assuming that the normal N_p to the surface at p is defined, we can consider any plane Π through p containing the normal N_p, and the curve C_Π, the intersection of the plane Π with the surface (or at least a segment of this curve in a neighborhood around p). If we let the plane rotate around the normal N_p, we get different curves, and in general, the curvature κ_Π of C_Π at p has a maximum and a minimum (these are the principal curvatures at p). Furthermore, the planes corresponding to the minimum and the maximum curvature are orthogonal. Corresponding to each curvature κ_Π, we have a circle in Π, centered on the normal N_p, and of radius $1/\kappa_\Pi$. Such circles are called *circles of normal curvature*. If the two principal curvatures have the same sign, when Π varies, the circles of normal curvatures generate a certain surface which looks like a cross-cap, and whose implicit equation was just given. A parametric specification of this surface can be obtained as follows. Inspired by stereographic projection, let

$$\begin{aligned} x &= \frac{2\rho u}{u^2 + v^2 + 1}, \\ y &= \frac{2\rho v}{u^2 + v^2 + 1}, \\ z &= \frac{\rho(u^2 + v^2 - 1)}{u^2 + v^2 + 1}. \end{aligned}$$

Substituting these expressions in

$$(ax^2 + by^2)(x^2 + y^2 + z^2) = 2z(x^2 + y^2),$$

we get

$$\rho = \frac{2(u^2 + v^2 - 1)(u^2 + v^2)}{(u^2 + v^2 + 1)(au^2 + bv^2)},$$

and thus,

$$\begin{aligned} x &= \frac{4u(u^2 + v^2 - 1)(u^2 + v^2)}{(u^2 + v^2 + 1)^2(au^2 + bv^2)}, \\ y &= \frac{4v(u^2 + v^2 - 1)(u^2 + v^2)}{(u^2 + v^2 + 1)^2(au^2 + bv^2)}, \\ z &= \frac{2(u^2 + v^2 - 1)^2(u^2 + v^2)}{(u^2 + v^2 + 1)^2(au^2 + bv^2)}. \end{aligned}$$

It is advantageous to switch to polar coordinates. Letting $u = \rho\cos\theta$ and $v = \rho\sin\theta$, we get

$$\begin{aligned} x &= \frac{4\rho(\rho^2 - 1)\cos\theta}{(\rho^2 + 1)^2(a\cos^2\theta + b\sin^2\theta)}, \\ y &= \frac{4\rho(\rho^2 - 1)\sin\theta}{(\rho^2 + 1)^2(a\cos^2\theta + b\sin^2\theta)}, \\ z &= \frac{2(\rho^2 - 1)^2}{(\rho^2 + 1)^2(a\cos^2\theta + b\sin^2\theta)}. \end{aligned}$$

Finally, switching to $v = \tan(\theta/2)$, and letting $u = \rho$, we get

$$\begin{aligned} x &= \frac{4u(u^2 - 1)(1 - v^4)}{(u^2 + 1)^2(av^4 + 2(2b - a)v^2 + a)}, \\ y &= \frac{8uv(u^2 - 1)(1 + v^2)}{(u^2 + 1)^2(av^4 + 2(2b - a)v^2 + a)}, \\ z &= \frac{2(u^2 - 1)^2(1 + v^2)^2}{(u^2 + 1)^2(av^4 + 2(2b - a)v^2 + a)}. \end{aligned}$$

It turns out that the entire surface is determined by patches 1 and 2 over $[-1, 1] \times [-1, 1]$. For $a = 1$ and $b = 1/2$, the following triangular control net is obtained:

```
hilnet =   {{0, 0, 2, 1}, {0, 0, 2, 1}, {0, 0, 15/7, 1}, {0, 0, 17/7, 1},
   {0, 0, 202/71, 71/70}, {0, 0, 10/3, 15/14}, {0, 0, 64/17, 17/14},
   {0, 0, 4, 3/2}, {0, 0, 4, 2}, {-1/2, 0, 2, 1}, {-1/2, -1/7, 2, 1},
   {-1/2, -2/7, 15/7, 1}, {-1/2, -16/35, 17/7, 1},
   {-34/71, -48/71, 202/71, 71/70}, {-2/5, -14/15, 10/3, 15/14},
   {-4/17, -20/17, 64/17, 17/14}, {0, -4/3, 4, 3/2},
   {-14/15, 0, 26/15, 15/14}, {-14/15, -4/15, 26/15, 15/14},
   {-14/15, -8/15, 416/225, 15/14}, {-14/15, -64/75, 52/25, 15/14},
   {-204/229, -288/229, 550/229, 229/210}, {-36/49, -12/7, 134/49, 7/6},
   {-8/19, -40/19, 56/19, 19/14}, {-20/17, 0, 22/17, 17/14},
   {-20/17, -28/85, 22/17, 17/14}, {-20/17, -56/85, 116/85, 17/14},
```