

## Chapter 24

# Approximating Closed Rational Surfaces

### 24.1 Partitioning the Parameter Domain

In this chapter, the problem of drawing a closed rational surface is considered. As in the case of rational curves, the key to drawing a closed rational surface is to partition the parameter domain into simple connected regions  $R_i$  such as triangles or rectangles, in such a way that there is some prespecified region  $R_0$  and some projectivities such that every other region is the image of the region  $R_0$  under one of the projectivities. Furthermore, if the patch associated with the region  $R_0$  is given by a control net  $\mathcal{N}_0$ , we want the control net  $\mathcal{N}_i$  associated with the region  $R_i$  to be computable very quickly from  $\mathcal{N}_0$ . However, since there are two kinds of rational surfaces, rectangular rational surfaces  $F: \tilde{\mathbb{A}} \times \tilde{\mathbb{A}} \rightarrow \tilde{\mathcal{E}}$  and triangular rational surfaces  $F: \tilde{\mathcal{P}} \rightarrow \tilde{\mathcal{E}}$ , the situation is more complicated than in the case of curves.

In the case of the real projective plane  $\mathbb{RP}^2 = \tilde{\mathcal{P}}$ , we can use the fact that  $\mathbb{RP}^2$  is obtained as the quotient of the sphere  $S^2$  after identification of antipodal points. To get a partition of the real projective plane, we can use the projection on a plane of any polyhedron inscribed in the sphere  $S^2$ . This way of dividing the real projective plane into regions is discussed quite extensively in Hilbert and Cohn-Vossen [84] (see Chapter III). As noted by Hilbert, it is better to use polyhedra with central symmetry, so that the projective plane is covered only once since vertices come in pairs of antipodal points. In particular, we can use the four Platonic solids other than the tetrahedron, but if we want rectangular or triangular regions, only the cube, the octahedron, and the icosahedron can be used. Indeed, projection of the dodecahedron yields pentagonal regions (see Hilbert and Cohn-Vossen [84], page 147-150). We can also inscribe a rhombus inside of the sphere and project it onto a plane.

If we project the cube onto one of its faces from its center, we get three rectangular regions (see section 24.3.). It is easy to find the projectivities that map the central region onto the other two. Since we are dealing with the projective plane, it is better to use triangular control nets to avoid base points, and it is necessary to split the central rectangle into two triangles. Thus, the trace of the rational surface is the union of six patches over various triangles. It is shown in section 24.3 how the control nets of the other four patches are easily (and cheaply) obtained from the control nets of the two central triangles.

If we project the octahedron onto one of its faces from its center, we get four triangular regions (see section 24.4). This time, it is a little harder to write down the projectivities that map the central triangle  $rst$  to the other three triangles  $R, S, T$ . However it is not necessary to find explicit formulae for these projectivities, and using a geometric argument, we can find very simple formulae to compute the control nets associated with the other three triangles from the control net associated with the central triangle, as shown in section 24.4.

Projecting the icosahedron onto one of its faces from its center yields ten triangular regions, but we haven't found formulae to compute control nets of the other regions from the central triangle. We leave the discovery of such formulae as an open and possibly challenging problem.

Let us now consider the problem of partitioning  $\mathbb{RP}^1 \times \mathbb{RP}^1 = \tilde{\mathbb{A}} \times \tilde{\mathbb{A}}$  into simple regions. Since the projective line  $\mathbb{RP}^1$  is topologically a circle, a very simple method is to inscribe a rectangle (or a square) in the circle and then project it. One way to do so leads to a partition of  $\mathbb{RP}^1$  into  $[-1, 1]$  and  $\mathbb{RP}^1 - ]-1, 1[$ . The corresponding projectivity is  $t \mapsto \frac{1}{t}$ . As shown in section 22.4, other projections lead to a partition of  $\mathbb{RP}^1$  into  $[r, s]$  and  $\mathbb{RP}^1 - ]r, s[$  for any affine frame  $(r, s)$ . In all cases, the torus is split into four rectangular regions, and there are very simple formulae for computing the control nets of the other three rectangular regions from the control net associated with the patch over  $[-1, 1] \times [-1, 1]$  (or more generally,  $[r_1, s_1] \times [r_2, s_2]$ ), as shown in section 24.2.

Other methods for drawing rational surfaces were investigated by DeRose [45] and Bajaj and Royappa [6, 7], and will be discussed in section 24.2 and section 24.4.

As we said earlier, there is a problem with our methods when all the numerators and the denominator vanish simultaneously. In this case, we have what is called a *base point*. We will give a new method for resolving base points (in the case of a rational surface specified by a triangular control net), using a simple “blowing-up” technique based on an idea of Warren [177].

## 24.2 Splitting Rectangular Rational Surfaces Into Four Rectangular Patches

In this section, we show that every rectangular rational surface can be obtained as the union of four rectangular patches, and that the control nets for these patches can be computed very easily from the original control net. The idea is simple: we partition  $\mathbb{RP}^1 \times \mathbb{RP}^1$  into the four regions associated with the partitioning of  $\mathbb{RP}^1$  into  $[-1, 1]$  and  $\mathbb{RP}^1 - ]-1, 1[$ . Let  $\varphi$  be the projectivity of  $\mathbb{RP}^1$  defined such that

$$\varphi(u, t) = (t, u).$$

We also define the following rectangular surfaces.

**Definition 24.2.1** Given an affine space  $\mathcal{E}$  of dimension  $\geq 3$ , for every rectangular rational surface  $F: \tilde{\mathbb{A}} \times \tilde{\mathbb{A}} \rightarrow \tilde{\mathcal{E}}$  of bidegree  $\langle p, q \rangle$  specified by some  $\langle p, q \rangle$ -symmetric multilinear map  $f: (\hat{\mathbb{A}})^p \times (\hat{\mathbb{A}})^q \rightarrow \hat{\mathcal{E}}$ , define the three  $\langle p, q \rangle$ -symmetric multilinear maps  $f_i: (\hat{\mathbb{A}})^p \times (\hat{\mathbb{A}})^q \rightarrow \hat{\mathcal{E}}$ ,  $i = 1, 2, 3$ , such that

$$\begin{aligned} f_1((u_1, t_1), \dots, (u_p, t_p), (v_1, s_1), \dots, (v_q, s_q)) &= f(\varphi(u_1, t_1), \dots, \varphi(u_p, t_p), (v_1, s_1), \dots, (v_q, s_q)), \\ f_2((u_1, t_1), \dots, (u_p, t_p), (v_1, s_1), \dots, (v_q, s_q)) &= f((u_1, t_1), \dots, (u_p, t_p), \varphi(v_1, s_1), \dots, \varphi(v_q, s_q)), \\ f_3((u_1, t_1), \dots, (u_p, t_p), (v_1, s_1), \dots, (v_q, s_q)) &= f(\varphi(u_1, t_1), \dots, \varphi(u_p, t_p), \varphi(v_1, s_1), \dots, \varphi(v_q, s_q)). \end{aligned}$$

The following lemma shows that provided that there are no base points, a rectangular rational surface is the union of four rectangular patches, and that given a rectangular net  $\alpha$  w.r.t.  $(-1, 1) \times (-1, 1)$ , the other three nets can be obtained very easily from  $\alpha$ .

**Lemma 24.2.2** Given an affine space  $\mathcal{E}$  of dimension  $\geq 3$ , for every rectangular rational surface  $F: \tilde{\mathbb{A}} \times \tilde{\mathbb{A}} \rightarrow \tilde{\mathcal{E}}$  of bidegree  $\langle p, q \rangle$  specified by some  $\langle p, q \rangle$ -symmetric multilinear map  $f: (\hat{\mathbb{A}})^p \times (\hat{\mathbb{A}})^q \rightarrow \hat{\mathcal{E}}$ , if  $f_1, f_2, f_3$  are the  $\langle p, q \rangle$ -symmetric multilinear maps of definition 24.2.1, except for the base points (if any), the trace  $F_1([-1, 1] \times [-1, 1])$  is the trace of  $F$  over  $\varphi([-1, 1]) \times [-1, 1]$ , the trace  $F_2([-1, 1] \times [-1, 1])$  is the trace of  $F$  over  $[-1, 1] \times \varphi([-1, 1])$ , and the trace  $F_3([-1, 1] \times [-1, 1])$  is the trace of  $F$  over  $\varphi([-1, 1]) \times \varphi([-1, 1])$ . Furthermore, if the control net (in  $\hat{\mathcal{E}}$ ) of the rectangular surface  $F$  w.r.t.  $(-1, 1) \times (-1, 1)$  is

$$\alpha = (\alpha_{i,j})_{0 \leq i \leq p, 0 \leq j \leq q},$$

the control nets  $\theta^1$ ,  $\theta^2$ , and  $\theta^3$  (in  $\widehat{\mathcal{E}}$ ) of the rectangular surfaces  $F_1, F_2, F_3$  w.r.t.  $(-1, 1) \times (-1, 1)$  is are given by the equations

$$\begin{aligned}\theta_{i,j}^1 &= (-1)^{p-i} \alpha_{i,j}, \\ \theta_{i,j}^2 &= (-1)^{q-j} \alpha_{i,j}, \\ \theta_{i,j}^3 &= (-1)^{p+q-i-j} \alpha_{i,j}.\end{aligned}$$

The proof is left as an exercise (see the Problems). In view of Lemma 22.4.2, the same result applies to surfaces specified by a rectangular net over  $[r_1, s_1] \times [r_2, s_2]$  for any affine frames  $(r_1, s_1)$  and  $(r_2, s_2)$ , since we can use the projectivity

$$\varphi(t) = \frac{(s+r)t - 2rs}{2t - (s+r)}$$

that maps  $[r, s]$  onto  $\mathbb{RP}^1 - ]r, s[$ . The upshot is that in order to draw a closed rational surface specified by a rectangular net  $\alpha$  w.r.t.  $(r_1, s_1) \times (r_2, s_2)$ , we simply have to compute the nets  $\theta^1, \theta^2, \theta^3$ , which is very cheap, and draw the corresponding rectangular patches. For example, a torus can be defined by the following rectangular net of bidegree  $\langle 2, 2 \rangle$  w.r.t.  $(-1, 1) \times (-1, 1)$ :

```
tornet4 = {{0, -(a + b), 0, 4}, {0, 0, 4c, 0}, {0, (-a + b), 0, 4},
           {4(a + b), 0, 0, 0}, {0, 0, 0, 0}, {4(a - b), 0, 0, 0},
           {0, a + b, 0, 4}, {0, 0, 4c, 0}, {0, a - b, 0, 4}}
```

For  $a = 2, b = 1, c = 1$ , we get

```
tornet4 = {{0, -3, 0, 4}, {0, 0, 4, 0}, {0, -1, 0, 4},
           {12, 0, 0, 0}, {0, 0, 0, 0}, {4, 0, 0, 0},
           {0, 3, 0, 4}, {0, 0, 4, 0}, {0, 1, 0, 4}}
```

The result of subdividing the patches associated with  $F, F_1, F_2$  and  $F_3$  is shown in Figure 24.1.

On the other hand, the method applied to a rectangular net of bidegree  $\langle 2, 2 \rangle$  for an ellipsoid yields base points. For example, it can be shown that a control net of bidegree  $\langle 2, 2 \rangle$  w.r.t.  $(-1, 1) \times (-1, 1)$  for an ellipsoid is given by:

```
reclnet3 = {{-8/3, -2, 2/3, 3}, {-8, 0, -2, 1}, {-8/3, 2, 2/3, 3},
            {0, -6, -2, 1}, {0, 0, 6, -1}, {0, 6, -2, 1},
            {8/3, -2, 2/3, 3}, {8, 0, -2, 1}, {8/3, 2, 2/3, 3}}
```

Unfortunately, the patch corresponding to  $F_3$  has a base point. The same thing happens for the Steiner roman surface. This is not surprising since neither the sphere nor the real projective plane are of the same topological type as the torus. It can be shown that a control net of bidegree  $\langle 2, 2 \rangle$  w.r.t.  $(-1, 1) \times (-1, 1)$  for the Steiner roman surface is given by:

```
sqstein3 = {{-2/3, -2/3, 2/3, 3}, {0, -2, 0, 1}, {2/3, -2/3, -2/3, 3},
            {-2, 0, 0, 1}, {0, 0, 0, -1}, {2, 0, 0, 1},
            {-2/3, 2/3, -2/3, 3}, {0, 2, 0, 1}, {2/3, 2/3, 2/3, 3}};
```

Again, the patch corresponding to  $F_3$  has base points.

Another method for drawing rectangular rational surfaces was investigated by DeRose [45] who credits Patterson [135] for the original idea behind the method. Basically, the method consists in using the homogeneous Bernstein polynomials  $\binom{p}{i} \binom{q}{j} u^i t_1^{p-i} v^j t_2^{q-j}$ , and to view a rectangular rational surface as a

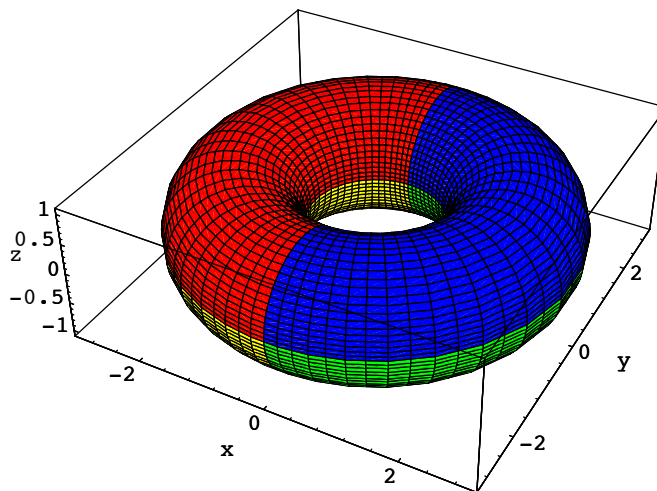


Figure 24.1: A torus

rational map from  $\mathbb{RP}^1 \times \mathbb{RP}^1$ . Then, by using any  $2D$  model of the projective line, it is possible to draw a closed rational surface in one piece.

In general, it is not easy to remove base points. This involves a technique from algebraic geometry known as “blowing-up” (see Fulton [67] or Harris [83]). In section 24.5, we will present a method for resolving base points in the case of triangular rational surfaces. However, we have not worked out the resolution of base points in the case of rectangular rational surfaces. We leave this problem as an interesting challenge to the reader.

### 24.3 Splitting Triangular Rational Surfaces Into Six Triangular Patches

As we explained in section 24.1, if we project a cube onto one of its faces from its center, we obtain a partition of the projective plane  $\tilde{\mathcal{P}}$  into three rectangular regions, in such a way that there exist simple projectivities  $\varphi$  and  $\psi$  between the square  $[-1, 1] \times [-1, 1]$  and the other two regions.

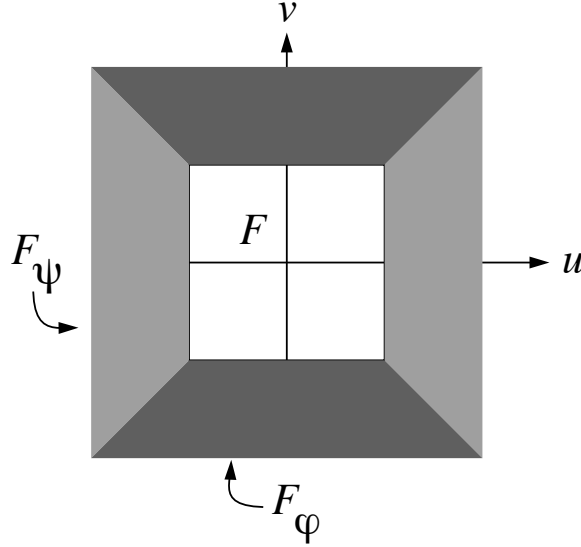


Figure 24.2: Dividing the projective plane into 3 rectangular regions

The projectivity  $\varphi$  is induced by the linear isomorphism of  $\mathbb{R}^3$  given by

$$(u, v, w) \mapsto (v, w, u).$$

The projectivity  $\psi$  is induced by the linear isomorphism of  $\mathbb{R}^3$  given by

$$(u, v, w) \mapsto (w, u, v).$$

Provided that  $F$  has no base points, the trace of the rational surface  $F$  can be obtained as the union of the traces of three rational surfaces  $F(u, v)$ ,  $F_\varphi(u, v) = F\left(\frac{v}{u}, \frac{1}{u}\right)$ , and  $F_\psi(u, v) = F\left(\frac{1}{v}, \frac{u}{v}\right)$ , over the square  $[-1, 1] \times [-1, 1]$ . It is important to note that the degree of the two new patches is **not** raised.

An example of the rational surfaces  $F_\varphi$  and  $F_\psi$ , if  $F$  is the sphere defined by the fractions

$$x(u, v) = \frac{2u}{u^2 + v^2 + 1}, \quad y(u, v) = \frac{2v}{u^2 + v^2 + 1}, \quad z(u, v) = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1},$$

under the map

$$\varphi: (u, v) \mapsto (v/u, 1/u),$$

we get the rational surface  $F_\varphi$  specified by

$$x(u, v) = \frac{2uv}{u^2 + v^2 + 1}, \quad y(u, v) = \frac{2u}{u^2 + v^2 + 1}, \quad z(u, v) = \frac{v^2 - u^2 + 1}{u^2 + v^2 + 1},$$

and under the map

$$\psi(u, v) \mapsto (1/v, u/v),$$

we get the rational surface  $F_\psi$  specified by

$$x(u, v) = \frac{2v}{u^2 + v^2 + 1}, \quad y(u, v) = \frac{2uv}{u^2 + v^2 + 1}, \quad z(u, v) = \frac{u^2 - v^2 + 1}{u^2 + v^2 + 1}.$$

It is easily verified that  $F_\varphi$  and  $F_\psi$  define the unit sphere, but these parameterizations are quite different from that of  $F$ .

Actually, it turns out that the method of this section holds for any region defined by a nondegenerate quadrilateral  $(a, b, c, d)$ , i.e. when  $(a, b, c, d)$  is a projective frame. However, the details are a bit messy, and for simplicity, we restrict our attention to a rectangular region  $[r_1, s_1] \times [r_2, s_2]$ . Since we are dealing with triangular surfaces, it will be necessary to split the rectangle  $[r_1, s_1] \times [r_2, s_2]$  into two triangles, and thus, we will obtain the trace of a rational surface as the union of 6 patches over various triangles in the rectangle  $[r_1, s_1] \times [r_2, s_2]$ . Letting  $a, b, c, d$  be the vertices of the rectangle  $[r_1, s_1] \times [r_2, s_2]$  defined such that

$$a = (s_1, s_2), \quad b = (r_1, s_2), \quad c = (r_1, r_2), \quad d = (s_1, r_2),$$

as shown below

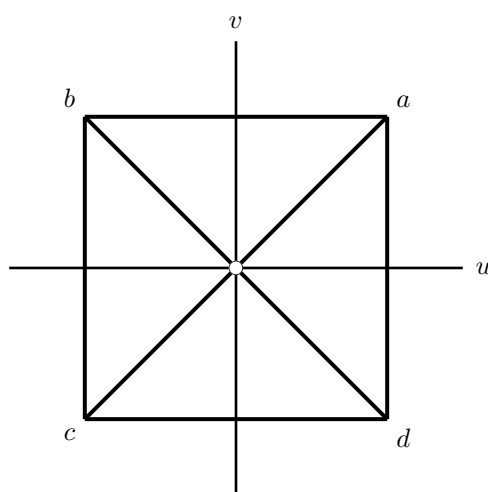


Figure 24.3: Some affine frames associated with the rectangle  $(a, b, c, d)$

we will consider the following affine frames

$$\Delta bca = ((r_1, s_2), (r_1, r_2), (s_1, s_2)),$$

$$\Delta dac = ((s_1, r_2), (s_1, s_2), (r_1, r_2)),$$

$$\Delta bad = ((r_1, s_2), (s_1, s_2), (s_1, r_2)).$$

In particular, a rectangular surface patch defined over the rectangle  $[r_1, s_1] \times [r_2, s_2]$  will be treated as the union of two triangular surface patches defined over the triangles  $\Delta bca$  and  $\Delta dac$ . It is somewhat unfortunate that a control net over the third frame  $\Delta bad$  needs to be computed, but that is what the proof of lemma 24.3.2 shows. In any case, such a control net can be computed very cheaply from a control net over  $\Delta bca$  (or  $\Delta dac$ ).

There is simple geometric explanation of the partitioning method in terms of the usual model of the real projective plane  $\tilde{\mathcal{P}} = \mathbb{RP}^2$  in  $\mathbb{R}^3$ . Recall that in this model, the real projective plane  $\mathbb{RP}^2$  consists of the points in the plane  $z = 1$  corresponding to the lines through the origin not in the plane  $z = 0$ , and of the points at infinity corresponding to the lines through the origin in the plane  $z = 0$ . We view the vertices of the rectangle  $(a, b, c, d)$  defined above as points in the plane  $z = 1$ , in which case their coordinates are  $(s_1, s_2, 1)$ ,  $(r_1, s_2, 1)$ ,  $(r_1, r_2, 1)$ , and  $(s_1, r_2, 1)$ . Then, we have the parallelepiped  $(a, b, c, d, -a, -b, -c, -d)$ . There is a unique projectivity  $\mathbf{P}(\varphi)$  such that

$$\mathbf{P}(\varphi)(a) = a, \quad \mathbf{P}(\varphi)(b) = c, \quad \mathbf{P}(\varphi)(c) = d, \quad \mathbf{P}(\varphi)(d) = b.$$

For instance, it is induced by the unique linear map  $\varphi$  such that

$$\varphi(a) = a, \quad \varphi(b) = -c, \quad \varphi(c) = -d.$$

Since  $d = -b + a + c$ , we get

$$\varphi(d) = -\varphi(b) + \varphi(a) + \varphi(c) = c + a - d = (r_1, s_2, 1) = b.$$

The linear map  $\varphi$  transforms the top face  $(a, b, c, d)$  of the parallelepiped to the back face  $(a, -c, -d, b)$ . When a line  $L$  through the origin and passing through a point of the face  $(a, -c, -d, b)$  varies, the intersection of  $L$  with the plane  $z = 1$  varies in  $\varphi([r_1, s_1] \times [r_2, s_2])$ . We can define a rhombus  $(a, e, f, g, -a, -e, -f, -g)$  inscribed in the sphere of center  $O = (0, 0, 0)$  and of radius  $R = \sqrt{s_1^2 + s_2^2 + 1}$  passing through  $a$ , as follows: the points  $e, f, g$  are on the upper half-sphere and they are determined by the intersection of the lines  $(O, b)$ ,  $(O, c)$  and  $(O, d)$  with the sphere. Then, it is obvious that under the central projection of center  $O$  onto the plane  $z = 1$ , the top face  $(a, e, f, g)$  of the rhombus projects onto the face  $(a, b, c, d)$  of the parallelepiped, and that the projection of the rhombus onto the plane  $z = 1$  yields the desired partitioning of  $\mathbb{R}P^2$ .

Similarly, there is a unique projectivity  $\mathbf{P}(\psi)$  such that

$$\mathbf{P}(\psi)(a) = a, \quad \mathbf{P}(\psi)(b) = d, \quad \mathbf{P}(\psi)(c) = b, \quad \mathbf{P}(\psi)(d) = c.$$

It is induced by the unique linear map  $\psi$  such that

$$\psi(a) = a, \quad \psi(b) = d, \quad \psi(c) = -b.$$

Since  $d = -b + a + c$ , we get

$$\psi(d) = -\psi(b) + \psi(a) + \psi(c) = -d + a - b = (-r_1, -r_2, -1) = -c.$$

The linear map  $\psi$  transforms the top face  $(a, b, c, d)$  of the parallelepiped to the right face  $(a, d, -b, -c)$ . When a line  $L$  through the origin and passing through a point of the face  $(a, d, -b, -c)$  varies, the intersection of  $L$  with the plane  $z = 1$  varies in  $\psi([r_1, s_1] \times [r_2, s_2])$ . Again, it is obvious that under the central projection of center  $O$  onto the plane  $z = 1$ , the top face  $(a, e, f, g)$  of the rhombus projects onto the face  $(a, b, c, d)$  of the parallelepiped, and that the projection of the rhombus onto the plane  $z = 1$  yields the desired partitioning of  $\mathbb{R}P^2$ . Figure 24.4 shows the parallelepiped  $(a, b, c, d, -a, -b, -c, -d)$  and the rhombus  $(a, e, f, g, -a, -e, -f, -g)$ .

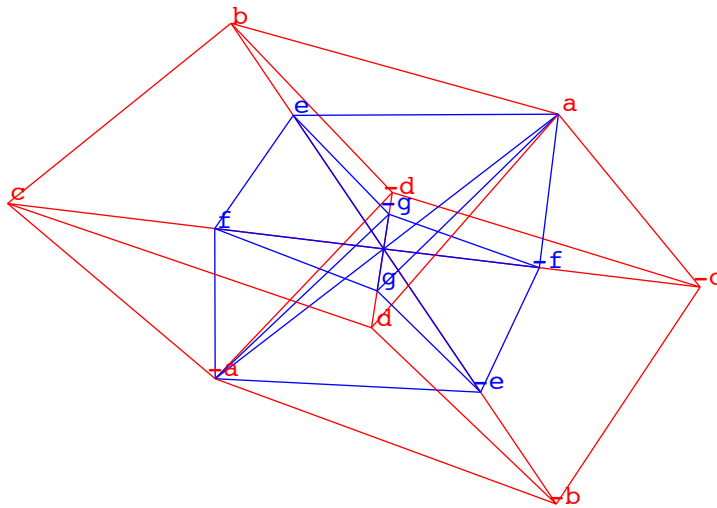
We will now use the maps  $\varphi$  and  $\psi$  to show how the trace of a rational surface  $F$  can be obtained as the union of the traces of three rational surfaces over the rectangle  $[r_1, s_1] \times [r_2, s_2]$ .<sup>1</sup> The first of these surfaces is  $F$  itself, and the two other rational surfaces  $F_\varphi$  and  $F_\psi$  are easily obtained from  $F$ . However, depending on the multilinear map  $f$  defining  $F$ , the surface  $F$  (and thus,  $F_\varphi$  and  $F_\psi$ ) may have *base points*, that is, we may have

$$f(\underbrace{(u, v, z), \dots, (u, v, z)}_m) = 0$$

for some  $(u, v, z) \neq (0, 0, 0)$ . We will show how to deal with this situation later on.

In order to render the trace of  $F$ , we will use the fact that it is the union of the six traces  $F(\Delta bca)$ ,  $F(\Delta dac)$ ,  $F(\varphi(\Delta bca))$ ,  $F(\varphi(\Delta dac))$ ,  $F(\psi(\Delta bca))$ , and  $F(\psi(\Delta dac))$ . Furthermore, the last four traces are also obtained as traces of  $F_\varphi$  and  $F_\psi$  over some appropriate choice of affine frames among  $\Delta bca$ ,  $\Delta dac$ , and  $\Delta bad$ .

<sup>1</sup>While reading Appell's *Treatise of Rational Mechanics*, we stumbled on the fact that the change of variable  $(u, v) \mapsto (1/v, u/v)$  was used by Appell in his solution to a problem of Bertrand (see [3], Tome I, Part III, Chapter XI, page 422-423). Appell explains that he found this "homographic transformation" in 1889. The problem of Bertrand is to find all central force laws depending only on the position of a moving particle, so that the trajectory of the particle is a conic for every choice of initial conditions.

Figure 24.4: Parallelepiped and rhombus associated with  $(a, b, c, d)$ 

We now show how  $F_\varphi$  and  $F_\psi$  are defined, and how their control points can be computed very simply from the control points of  $F$  (computed with respect to the affine frames  $\Delta bca$ ,  $\Delta dac$ , and  $\Delta bad$ ). We will assume that the homogenization  $\widehat{\mathcal{P}}$  of the affine plane  $\mathcal{P}$  is identified with the direct sum  $\mathbb{R}^2 \oplus \mathbb{R}O$ , where  $O = (0, 0)$ . Then, every element of  $\widehat{\mathcal{P}}$  is of the form  $(u, v, z) \in \mathbb{R}^3$ .

**Definition 24.3.1** Given an affine space  $\mathcal{E}$  of dimension  $\geq 3$ , for every rational surface  $F: \widehat{\mathcal{P}} \rightarrow \widehat{\mathcal{E}}$  of degree  $m$  specified by some symmetric multilinear map  $f: (\widehat{\mathcal{P}})^m \rightarrow \widehat{\mathcal{E}}$ , the symmetric multilinear maps  $f_\varphi: (\widehat{\mathcal{P}})^m \rightarrow \widehat{\mathcal{E}}$  and  $f_\psi: (\widehat{\mathcal{P}})^m \rightarrow \widehat{\mathcal{E}}$  are defined such that

$$\begin{aligned} f_\varphi((u_1, v_1, w_1), \dots, (u_m, v_m, w_m)) &= f(\varphi(u_1, v_1, w_1), \dots, \varphi(u_m, v_m, w_m)), \\ f_\psi((u_1, v_1, w_1), \dots, (u_m, v_m, w_m)) &= f(\psi(u_1, v_1, w_1), \dots, \psi(u_m, v_m, w_m)). \end{aligned}$$

Let  $F_\varphi: \widehat{\mathcal{P}} \rightarrow \widehat{\mathcal{E}}$  be the rational surface specified by  $f_\varphi: (\widehat{\mathcal{P}})^m \rightarrow \widehat{\mathcal{E}}$ , and let  $F_\psi: \widehat{\mathcal{P}} \rightarrow \widehat{\mathcal{E}}$  be the rational surface specified by  $f_\psi: (\widehat{\mathcal{P}})^m \rightarrow \widehat{\mathcal{E}}$ .

Observe that the base points of  $F_\varphi$ , if any, have coordinates  $(u, v, w) \neq (0, 0, 0)$  such that

$$f(\varphi(u, v, w), \dots, \varphi(u, v, w)) = 0,$$

and that the base points of  $F_\psi$ , if any, have coordinates  $(u, v, w) \neq (0, 0, 0)$  such that

$$f(\psi(u, v, w), \dots, \psi(u, v, w)) = 0.$$

**Lemma 24.3.2** Given an affine space  $\mathcal{E}$  of dimension  $\geq 3$ , for every rational surface  $F: \widehat{\mathcal{P}} \rightarrow \widehat{\mathcal{E}}$  of degree  $m$  specified by some symmetric multilinear map  $f: (\widehat{\mathcal{P}})^m \rightarrow \widehat{\mathcal{E}}$ , if  $f_\varphi$  and  $f_\psi$  are the symmetric multilinear maps of definition 24.3.1, except for base points,  $F$ ,  $F_\varphi$  and  $F_\psi$  have the same trace. The trace of  $F_\varphi$  over  $\Delta bca$  is the trace of  $F$  over  $\varphi(\Delta bca)$ , the trace of  $F_\varphi$  over  $\Delta dac$  is the trace of  $F$  over  $\varphi(\Delta dac)$ , the trace of  $F_\psi$



over  $\Delta bca$  is the trace of  $F$  over  $\psi(\Delta bca)$ , and the trace of  $F_\psi$  over  $\Delta dac$  is the trace of  $F$  over  $\psi(\Delta dac)$ . Furthermore, if the control nets (in  $\widehat{\mathcal{E}}$ ) of the surface  $F$  w.r.t. the affine frames  $\Delta bca$ ,  $\Delta dac$ , and  $\Delta bad$ , are respectively

$$\begin{aligned} \alpha &= (\alpha_{i,j,k})_{(i,j,k) \in \Delta_m}, \\ \beta &= (\beta_{i,j,k})_{(i,j,k) \in \Delta_m}, \\ \gamma &= (\gamma_{i,j,k})_{(i,j,k) \in \Delta_m}, \end{aligned}$$

the control nets  $\theta^1$  and  $\theta^2$  (in  $\widehat{\mathcal{E}}$ ) of the surface  $F_\varphi$  w.r.t. the affine frames  $\Delta bca$  and  $\Delta dac$ , and the control nets  $\rho^1$  and  $\rho^2$  (in  $\widehat{\mathcal{E}}$ ) of the surface  $F_\psi$  w.r.t. the affine frame  $\Delta bca$  and  $\Delta dac$ , are given by the equations

$$\begin{aligned} \theta_{i,j,k}^1 &= (-1)^{i+j} \beta_{j,k,i}, \\ \theta_{i,j,k}^2 &= (-1)^k \gamma_{i,j,k}, \\ \rho_{i,j,k}^1 &= (-1)^j \gamma_{j,k,i}, \\ \rho_{i,j,k}^2 &= (-1)^{i+k} \alpha_{k,i,j}. \end{aligned}$$

*Proof.* We have

$$f_\varphi((u_1, v_1, w_1), \dots, (u_m, v_m, w_m)) = f(\varphi(u_1, v_1, w_1), \dots, \varphi(u_m, v_m, w_m)),$$

and thus

$$\mathbf{P}(f_\varphi)([u_1, v_1, w_1], \dots, [u_m, v_m, w_m]) = \mathbf{P}(f)([\varphi(u_1, v_1, w_1)], \dots, [\varphi(u_m, v_m, w_m)]).$$

In view of the properties of  $\varphi$ , it is clear that  $F$  and  $F_\varphi$  have the same trace (except for base points), and that the trace of  $F_\varphi$  over  $\Delta bca$  is the trace of  $F$  over  $\varphi(\Delta bca)$ , and the trace of  $F_\varphi$  over  $\Delta dac$  is the trace of  $F$  over  $\varphi(\Delta dac)$ . A similar argument applies to  $F$  and  $F_\psi$ . The formulae for computing the control points of  $F_\varphi$  w.r.t. the triangle  $\Delta bca$  are obtained by computing

$$f_\varphi(\underbrace{b, \dots, b}_i, \underbrace{c, \dots, c}_j, \underbrace{a, \dots, a}_k).$$

Since

$$f_\varphi((u_1, v_1, w_1), \dots, (u_m, v_m, w_m)) = f(\varphi(u_1, v_1, w_1), \dots, \varphi(u_m, v_m, w_m)),$$

$\varphi(b) = -c$ ,  $\varphi(c) = -d$ , and  $\varphi(a) = a$ , we have

$$f_\varphi(\underbrace{b, \dots, b}_i, \underbrace{c, \dots, c}_j, \underbrace{a, \dots, a}_k) = f(\underbrace{-c, \dots, -c}_i, \underbrace{-d, \dots, -d}_j, \underbrace{a, \dots, a}_k),$$

that is

$$f_\varphi(\underbrace{b, \dots, b}_i, \underbrace{c, \dots, c}_j, \underbrace{a, \dots, a}_k) = (-1)^{i+j} f(\underbrace{c, \dots, c}_i, \underbrace{d, \dots, d}_j, \underbrace{a, \dots, a}_k),$$

and since the control points  $\beta_{i,j,k}$  are computed w.r.t. the triangle  $\Delta dac$ , we get

$$\theta_{i,j,k}^1 = (-1)^{i+j} \beta_{j,k,i}.$$

The formulae for computing the control points of  $F_\varphi$  w.r.t. the triangle  $\Delta dac$  are obtained by computing

$$f_\varphi(\underbrace{d, \dots, d}_i, \underbrace{a, \dots, a}_j, \underbrace{c, \dots, c}_k).$$

Since

$$f_\varphi((u_1, v_1, w_1), \dots, (u_m, v_m, w_m)) = f(\varphi(u_1, v_1, w_1), \dots, \varphi(u_m, v_m, w_m)),$$

$\varphi(d) = b$ ,  $\varphi(c) = -d$ , and  $\varphi(a) = a$ , we have

$$f_\varphi(\underbrace{d, \dots, d}_i, \underbrace{a, \dots, a}_j, \underbrace{c, \dots, c}_k) = f(\underbrace{b, \dots, b}_i, \underbrace{a, \dots, a}_j, \underbrace{-d, \dots, -d}_k),$$

that is

$$f_\varphi(\underbrace{d, \dots, d}_i, \underbrace{a, \dots, a}_j, \underbrace{c, \dots, c}_k) = (-1)^k f(\underbrace{b, \dots, b}_i, \underbrace{a, \dots, a}_j, \underbrace{d, \dots, d}_k),$$

and since the control points  $\gamma_{i,j,k}$  are computed w.r.t. the triangle  $\Delta bad$ , we get

$$\theta_{i,j,k}^2 = (-1)^k \gamma_{i,j,k}.$$

The formulae for computing the control points of  $F_\psi$  w.r.t. the triangle  $\Delta bca$  are obtained by computing

$$f_\psi(\underbrace{b, \dots, b}_i, \underbrace{c, \dots, c}_j, \underbrace{a, \dots, a}_k).$$

Since

$$f_\psi((u_1, v_1, w_1), \dots, (u_m, v_m, w_m)) = f(\psi(u_1, v_1, w_1), \dots, \psi(u_m, v_m, w_m)),$$

$\psi(b) = d$ ,  $\psi(c) = -b$ , and  $\psi(a) = a$ , we have

$$f_\psi(\underbrace{b, \dots, b}_i, \underbrace{c, \dots, c}_j, \underbrace{a, \dots, a}_k) = f(\underbrace{d, \dots, d}_i, \underbrace{-b, \dots, -b}_j, \underbrace{a, \dots, a}_k),$$

that is

$$f_\psi(\underbrace{b, \dots, b}_i, \underbrace{c, \dots, c}_j, \underbrace{a, \dots, a}_k) = (-1)^j f(\underbrace{d, \dots, d}_i, \underbrace{b, \dots, b}_j, \underbrace{a, \dots, a}_k),$$

and since the control points  $\gamma_{i,j,k}$  are computed w.r.t. the triangle  $\Delta bad$ , we get

$$\rho_{i,j,k}^1 = (-1)^j \gamma_{j,k,i}.$$

Finally, the formulae for computing the control points of  $F_\psi$  w.r.t. the triangle  $\Delta dac$  are obtained by computing

$$f_\psi(\underbrace{d, \dots, d}_i, \underbrace{a, \dots, a}_j, \underbrace{c, \dots, c}_k).$$

Since

$$f_\psi((u_1, v_1, w_1), \dots, (u_m, v_m, w_m)) = f(\psi(u_1, v_1, w_1), \dots, \psi(u_m, v_m, w_m)),$$

$\psi(d) = -c$ ,  $\psi(c) = -b$ , and  $\psi(a) = a$ , we have

$$f_\psi(\underbrace{d, \dots, d}_i, \underbrace{a, \dots, a}_j, \underbrace{c, \dots, c}_k) = f(\underbrace{-c, \dots, -c}_i, \underbrace{a, \dots, a}_j, \underbrace{-b, \dots, -b}_k),$$

that is

$$f_\psi(\underbrace{d, \dots, d}_i, \underbrace{a, \dots, a}_j, \underbrace{c, \dots, c}_k) = (-1)^{i+k} f(\underbrace{c, \dots, c}_i, \underbrace{a, \dots, a}_j, \underbrace{b, \dots, b}_k),$$

and since the control points  $\alpha_{i,j,k}$  are computed w.r.t. the triangle  $\Delta bca$ , we get

$$\rho_{i,j,k}^2 = (-1)^{i+k} \alpha_{k,i,j}.$$

□

The above calculations show that  $\varphi$  and  $\psi$  can be defined as above provided that  $d = -b + a + c$ , or equivalently  $b + d = a + c$ , which means that  $(a, b, c, d)$  is a parallelogram. Actually, lemma 24.3.2 also holds in the more general situation where  $(a, b, c, d)$  is a projective frame, i.e. a quadrilateral whose vertices are in general position. However, the definition of the linear maps  $\varphi$  and  $\psi$  is a little more messy. As before, we identify  $a, b, c, d$  with points in the plane  $z = 1$ , and we let  $a = (a_1, a_2, 1)$ ,  $b = (b_1, b_2, 1)$ ,  $c = (c_1, c_2, 1)$ , and  $d = (d_1, d_2, 1)$ . To find a linear map  $\varphi$  inducing the unique projectivity  $\mathbf{P}(\varphi)$  such that

$$\mathbf{P}(\varphi)(a) = a, \quad \mathbf{P}(\varphi)(b) = c, \quad \mathbf{P}(\varphi)(c) = d, \quad \mathbf{P}(\varphi)(d) = b,$$

we let  $d = \lambda a + \mu b + \nu c$  and  $b = \lambda' a + \mu' c + \nu' d$ , where  $\lambda + \mu + \nu = 1$  and  $\lambda' + \mu' + \nu' = 1$ , and  $\varphi$  is the unique linear map such that

$$\varphi(\lambda a) = \lambda' a, \quad \varphi(\mu b) = \mu' c, \quad \varphi(\nu c) = \nu' d.$$

Then,  $\varphi(d) = b$ , as desired. The linear map  $\psi$  can be defined in a similar way. The proof still goes through since the maps involved are multilinear, and thus not disturbed by scalar multiples.

Lemma 24.3.2 shows that in order to render a rational surface, provided that it does not have base points, we just need to compute the control nets  $\alpha, \beta, \gamma$  for the surface  $F$  w.r.t. the affine frames  $\Delta bca$ ,  $\Delta dac$ , and  $\Delta bad$ , since then, the control nets  $\theta^1$  and  $\theta^2$  (in  $\widehat{\mathcal{E}}$ ) of the surface  $F_\varphi$  w.r.t. the affine frames  $\Delta bca$  and  $\Delta dac$ , and the control nets  $\rho^1$  and  $\rho^2$  (in  $\widehat{\mathcal{E}}$ ) of the surface  $F_\psi$  w.r.t. the affine frame  $\Delta bca$  and  $\Delta dac$ , are obtained at trivial cost.

*Remark:* It should be noted that the surface patches associated with the control nets  $\alpha, \beta, \theta^1, \theta^2, \rho^1$ , and  $\rho^2$ , may overlap in more than boundaries. In fact, we will see examples where  $\alpha$  and  $\beta$  determine the entire surface, and other examples in which  $\theta^1, \theta^2, \rho^1$ , and  $\rho^2$ , determine the entire surface.

We now describe an implementation of this method in *Mathematica*. The function `net123` computes the nets `net1`, `net2`, and `net3`, over the reference triangles

$$\begin{aligned} \text{reftrig1} &= ((-1, 1, 1), (-1, -1, 3), (1, 1, -1)) \\ \text{reftrig2} &= ((1, -1, 1), (1, 1, -1), (-1, -1, 3)) \\ \text{reftrig3} &= ((-1, 1, 1), (1, 1, -1), (1, -1, 1)) \end{aligned}$$

from the original net, using `newcnet3`. The function `newcnet3`, as well as other functions, can be found in Gallier [70].

```
(* Computes the new nets net1, net2, net3, in order to compute *)
(* theta1, theta2, rho1, rho2 *)
```

```
net123[{oldnet__}, mm_, debug_] :=
Block[
{net = {oldnet}, net0, net1, net2, net3, res, reftrig1,
reftrig2, reftrig3, stop},
reftrig1 = {{-1, 1, 1}, {-1, -1, 3}, {1, 1, -1}};
reftrig2 = {{1, -1, 1}, {1, 1, -1}, {-1, -1, 3}};
reftrig3 = {{-1, 1, 1}, {1, 1, -1}, {1, -1, 1}};
stop = testm[net, mm];
If[stop === 1, Return["*** Unable to Run ***"]];
net1 = newcnet3[net0, mm, reftrig1];
Print["New Net 1: ", net1];
net2 = newcnet3[net0, mm, reftrig2];
```

```

Print["New Net 2: ", net2];
net3 = newcnet3[net0,mm,reftrig3];
Print["New Net 3: ", net3];
res = Join[{net1}, {net2}];
res = Join[res, {net3}];
res
];

(* To test that a triangular net really has size
(m + 1)(m + 2)/2, when the polar degree is m *)

testm[cnet__, m_] := Block[
{cc = cnet, ll, s, i, stop, res},
{ll = Length[cc]; s = N[Sqrt[2 * ll]]; i = 1; stop = 1; res = 1;
While[i <= s && stop === 1,
If[(res === ll) && (i === m + 1), stop = 0, i = i + 1;
res = res + i]
];
If[stop === 1,
Print["*** Net Size: ", ll,
" Inconsistent with Surface Degree: ", m, " ***"]];
stop
)
];

```

Using the nets `net1`, `net2`, and `net3`, the nets `theta1`, `theta2`, `rho1`, and `rho2`, are computed as follows:

```

(* Computes the new nets theta1 and theta2 *)

thet Janet[{inlnet__}, mm_, debug_] :=
Block[
{net1, net2, net3, lnet = {inlnet}, anet, theta1, theta2,
pt, i, j, k, res},
net1 = lnet[[1]]; net2 = lnet[[2]]; net3 = lnet[[3]];
anet = convtomat[net2,mm];
theta1 = {};
Do[
Do[
pt = anet[[j + 1, mm - i - j + 1]];
If[OddQ[i + j], pt = -pt];
theta1 = Append[theta1, pt], {j, 0, mm - i}
], {i, 0, mm}
];
anet = convtomat[net3,mm];
theta2 = {};
Do[
Do[
pt = anet[[i + 1, j + 1]];
If[OddQ[mm - i - j], pt = -pt];
theta2 = Append[theta2, pt], {j, 0, mm - i}
], {i, 0, mm}
];
];

```

```

];
Print["New Net theta1: ", theta1];
Print["New Net theta2: ", theta2];
res = Join[{theta1}, {theta2}];
res
];
(* Computes the new nets rho1 and rho2 *)

rhonet[{inlnet_}, mm_, debug_] :=
Block[
{net1, net2, net3, lnet = {inlnet}, anet, rho1, rho2,
pt, i, j, k, res},
net1 = lnet[[1]]; net2 = lnet[[2]]; net3 = lnet[[3]];
anet = convtomat[net3,mm];
rho1 = {};
Do[
Do[
pt = anet[[j + 1, mm - i - j + 1]];
If[OddQ[j], pt = -pt];
rho1 = Append[rho1, pt], {j, 0, mm - i}
], {i, 0, mm}
];
Print["New Net rho1: ", rho1];
anet = convtomat[net1,mm];
rho2 = {};
Do[
Do[
pt = anet[[mm - i - j + 1, i + 1]];
If[OddQ[mm - j], pt = -pt];
rho2 = Append[rho2, pt], {j, 0, mm - i}
], {i, 0, mm}
];
Print["New Net rho2: ", rho2];
res = Join[{rho1}, {rho2}];
res
];

```

The following function displays the two patches corresponding to the control nets `theta1` and `theta2`. A similar function can be written for `rho1` and `rho2`, and all six patches can also be displayed.

```

(* Computes the new nets 3 and 4 involved in showing all the patches *)
(* and displays the closed surface *)
(* if light = 1, uses light triangulation trianglight, *)
(* else ltriang *)
(* if fcnet = 1, displays control net, otherwise skip it *)

render34[{net_}, mm_, n_, light_, fcnet_, lx_, mx_,ly_,my_,lz_,mz_,
debug_] :=
Block[
{net0, net1, net2, net3, anet, theta1, theta2, rho1, rho2, lnet,

```

```

surf1, surf2, surf3, surf4, surf5, surf6, surf, image, stop, theta,
i, j, k},
    stop = testm[{net}, mm];
    If[stop === 1, Return["*** Unable to Run ***"]];
    net0 = maptohat[{net}];
    lnet = net123[{net}, mm, debug];
    theta = thetanet[lnet, mm, debug];
    theta1 = theta[[1]]; theta2 = theta[[2]];
    surf3 = rsubdiv4[theta1, mm, n, debug];
    If[debug === 1,
        Print["Third patch: ", surf3]
    ];
    If[light === 1, surf3 = trianglight[surf3,mm],
        surf3 = ltriang[surf3,mm]];
    Print["Third patch done! "];
    surf4 = rsubdiv4[theta2, mm, n, debug];
    If[debug === 1,
        Print["Fourth patch: ", surf4]
    ];
    If[light === 1, surf4 = trianglight[surf4,mm],
        surf4 = ltriang[surf4,mm]];
    Print["Fourth patch done! "];
    surf = Join[surf3, surf4];
    Print["Ready to display! "];
    If[fcnet === 1, image = {surf, controlnet3D[{net},mm]},
        image = surf
    ];
    Show[
    Graphics3D[{Thickness[0.0006], image}],
    PlotRange -> {{lx, mx}, {ly, my}, {lz, mz}},
    Axes -> True,
    AxesLabel -> {"x", "y", "z"},
    DisplayFunction -> $DisplayFunction];
];

```

The purpose of the functions `ltriang` and `ltrianglight` is to build a triangulation of each control net in a list of nets. The function `ltriang` builds a list of solid triangles, using the *Mathematica* construct `Polygon`. The advantage of this construct is that polygons are treated as nontransparent objects, and when a surface is displayed, hidden parts are automatically removed. It is also very easy to use the shading options of *Mathematica*, or color the polygons as desired. There is even a way of coloring differently both sides of each polygon. The function `ltrianglight` only builds the edges on the triangular sides of the net. Such functions are easily written and are left as an exercise.

Going back to example 1 from section 23.4 (an ellipsoid), iterating the subdivision algorithm 3 times on the nets `net1` and `net2` yields the surface shown in Figure 24.5.

Iterating the subdivision algorithm 3 times on the nets `theta1` and `theta2` yields the surface shown in Figure 24.6.

Iterating the subdivision algorithm 3 times on the nets `rho1` and `rho2` yields the surface shown in Figure 24.7.

The result of putting all these patches together is the entire ellipsoid, showed in Figure 24.8.

Of course, we could have taken advantage of symmetries, and our point is to illustrate the algorithm.

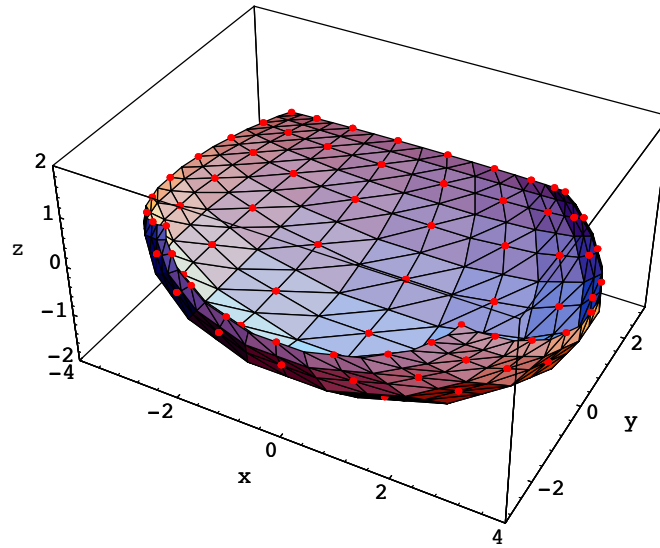


Figure 24.5: Patches 1, 2, of an ellipsoid

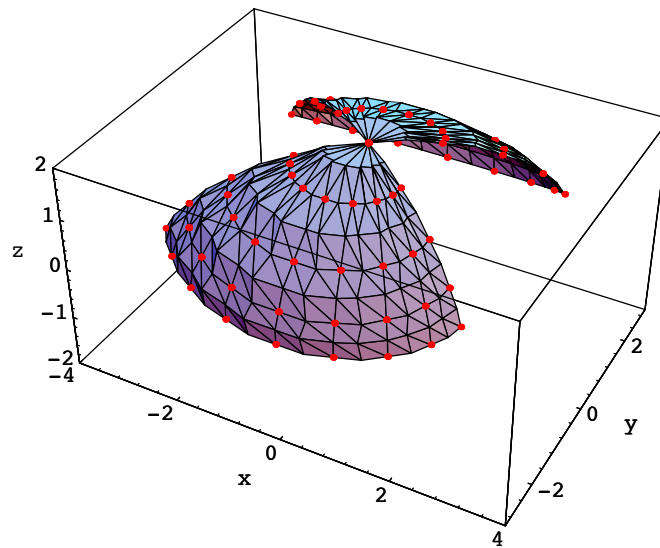


Figure 24.6: Patches 3, 4, of an ellipsoid

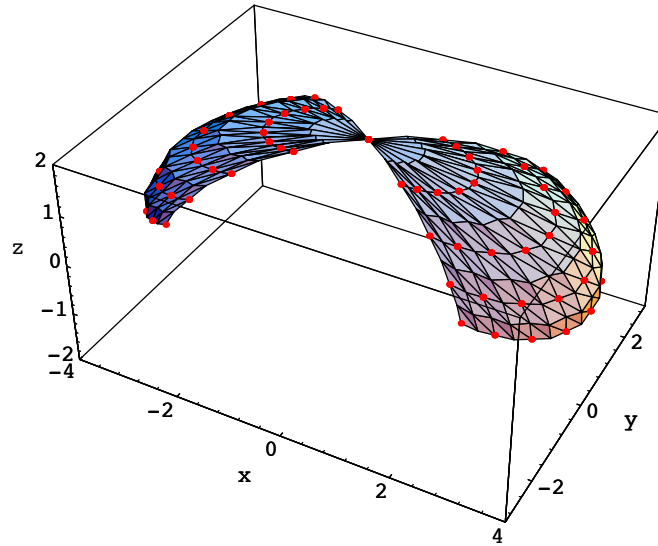


Figure 24.7: Patches 5, 6, of an ellipsoid

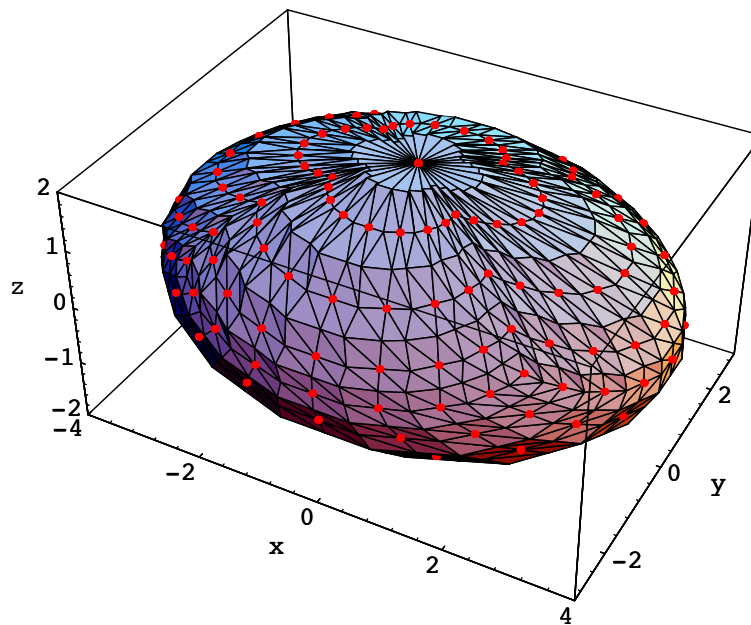


Figure 24.8: An entire ellipsoid



## 24.4 Splitting Triangular Rational Surfaces Into Four Triangular Patches

As explained in section 24.1, we obtain a partition of the real projective plane  $\mathbb{RP}^2$  into four triangles if we project an octahedron onto one of its faces from its center. We sketch such a method, leaving the details as an exercise.

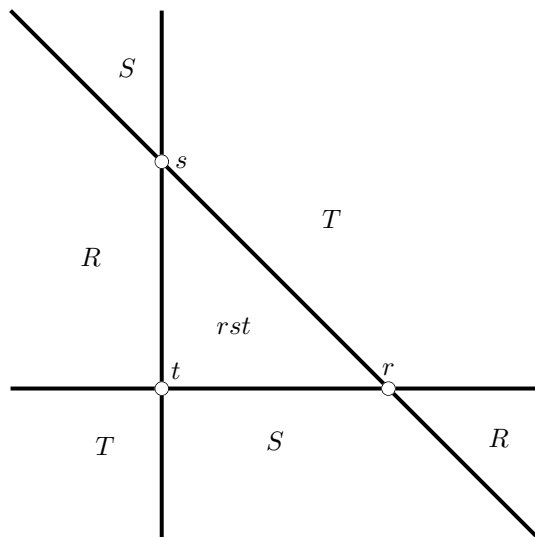


Figure 24.9: Splitting  $\mathbb{RP}^2$  into four triangles

We let  $r, s, t$  be the vertices of the central triangle. The four triangles defined by the lines  $\langle r, s \rangle$ ,  $\langle s, t \rangle$ , and  $\langle r, t \rangle$  are denoted as  $rst$ ,  $R$ ,  $S$ , and  $T$ , where  $R, S, T$  contain points at infinity. It is easy to find three projectivities  $\varphi_i: \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$ ,  $i = 1, 2, 3$ , such that  $\varphi_1(rst) = R$ ,  $\varphi_2(rst) = S$ , and  $\varphi_3(rst) = T$ . Then, we get some rational surfaces  $F_i = F \circ \varphi_i$ ,  $i = 1, 2, 3$ . Indeed, if we use the model of  $\mathbb{RP}^2$  in  $\mathbb{R}^3$  where the points  $r, s, t$  are considered as being in the plane  $z = 1$ , it is immediately verified that the linear maps

$$\begin{aligned} (r, s, t) &\mapsto (-r, s, t), \\ (r, s, t) &\mapsto (r, -s, t), \\ (r, s, t) &\mapsto (r, s, -t), \end{aligned}$$

induce  $\varphi_1, \varphi_2, \varphi_3$ . Furthermore, if the control net (in  $\widehat{\mathcal{E}}$ ) of the triangular surface  $F$  w.r.t. the affine frame  $\Delta rst$  is  $\alpha = (\alpha_{i,j,k})_{(i,j,k) \in \Delta_m}$ , it can be shown that the control nets  $\theta^1, \theta^2$ , and  $\theta^3$  of the surfaces  $F_1, F_2, F_3$  w.r.t.  $\Delta rts$  are given by the formulae

$$\begin{aligned} \theta_{i,j,k}^1 &= (-1)^i \alpha_{i,j,k}, \\ \theta_{i,j,k}^2 &= (-1)^j \alpha_{i,j,k}, \\ \theta_{i,j,k}^3 &= (-1)^k \alpha_{i,j,k}. \end{aligned}$$

Provided that there are no base points, the traces of  $F, F_1, F_2, F_3$  over  $\Delta rst$  cover the entire trace of  $F$  (over  $\mathbb{RP}^2$ ). The upshot is that in order to draw a closed rational surface given by a triangular net  $\alpha$  over  $\Delta rst$ , we simply have to draw the four patches specified by  $\alpha, \theta^1, \theta^2$ , and  $\theta^3$ , over  $\Delta rst$ .

For example, we can apply the above method to the Steiner roman surface specified by the following triangular net over  $((1, 0), (0, 1), (0, 0))$ :

```
stein1 = {{0, 0, 0, 1}, {1, 0, 0, 1}, {1, 0, 0, 2},
          {0, 1, 0, 1}, {1, 1, 1, 1},
          {0, 1, 0, 2}};
```

It turns out that the patch  $F_3$  is quite distorted. Applying the method to a net over a bigger triangle helps reduce the distortion. In particular, we can send  $r$  and  $s$  to infinity, in which case the method ends up being equivalent to a method due to Bajaj and Royappa [6, 7]. Indeed, if we project the octahedron onto a plane containing one of its vertices and perpendicular to the line from this vertex to the center, we obtain the following partition of  $\mathbb{RP}^2$  into four quadrants:

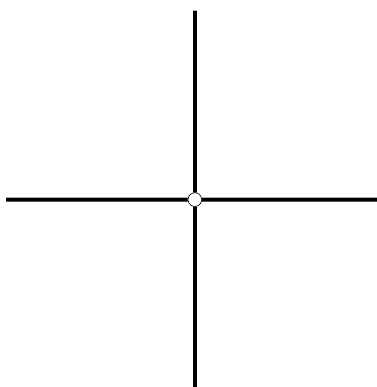


Figure 24.10: Splitting  $\mathbb{RP}^2$  into four quadrants

This way of partitioning  $\mathbb{RP}^2$  corresponds to the method investigated by Bajaj and Royappa [6, 7]. The four maps

$$(u, v) \mapsto \left( \frac{\sigma_1 u}{1 - u - v}, \frac{\sigma_2 v}{1 - u - v} \right),$$

where  $\sigma_i \in \{-1, 1\}$  for  $i = 1, 2$ , map the triangle  $((1, 0), (0, 1), (0, 0))$  bijectively onto the four quadrants of the plane respectively. However, Bajaj and Royappa do not consider the problem of computing the control nets of the surfaces

$$F \left( \frac{\sigma_1 u}{1 - u - v}, \frac{\sigma_2 v}{1 - u - v} \right).$$

There is a method for computing control points for these patches, but this method is more complicated than ours. For more details, see the problems.

In general, given a projectivity  $\varphi: \tilde{\mathcal{P}} \rightarrow \tilde{\mathcal{P}}$  (of the projective plane) defined by some invertible matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix},$$

given any rational surface specified by a symmetric multilinear map  $f: (\hat{\mathcal{P}})^m \rightarrow \hat{\mathcal{E}}$ , we can define the multilinear map  $g: (\hat{\mathcal{P}})^m \rightarrow \hat{\mathcal{E}}$  such that

$$g((u_1, v_1, w_1), \dots, (u_m, v_m, w_m)) = f(\varphi(u_1, v_1, w_1), \dots, \varphi(u_m, v_m, w_m)),$$

that is,

$$g((u_1, v_1, w_1), \dots, (u_m, v_m, w_m)) \\ = f((a_1u_1 + b_1v_1 + c_1w_1, a_2u_1 + b_2v_1 + c_2w_1, a_3u_1 + b_3v_1 + c_3w_1), \\ \dots, (a_1u_m + b_1v_m + c_1w_m, a_2u_m + b_2v_m + c_2w_m, a_3u_m + b_3v_m + c_3w_m)).$$

The multilinear map  $g$  defines the rational surface obtained by the “change of variable”  $\varphi$ . For any affine frame  $\Delta = (r = (r_1, r_1), s = (s_1, s_2), t = (t_1, t_2))$ , the control points

$$g(\underbrace{r, \dots, r}_i, \underbrace{s, \dots, s}_j, \underbrace{t, \dots, t}_k)$$

can be computed in terms of some appropriate polar values of  $f$  (see the Problems).

Another method for drawing triangular rational surfaces was also investigated by DeRose [45] who credits Patterson [135] for the original idea behind the method. Basically, the method consists in using the homogeneous Bernstein polynomials  $\binom{m}{i \ j \ k} u^i v^j w^k$ , where  $i + j + k = m$ , and to view a triangular rational surface as a rational map from the real projective plane. Then, by using any 3D model of the projective plane, it is possible to draw a closed rational surface in one piece.

## 24.5 Resolving Base Points of Triangular Nets

We now consider the case in which  $F_\varphi$  and  $F_\psi$  (as defined in section 24.3) have base points. An example for which this happens is the torus.

An elliptic torus can be defined parametrically as follows:

$$x = (a - b \sin \varphi) \cos \theta, \\ y = (a - b \sin \varphi) \sin \theta, \\ z = c \cos \varphi.$$

As usual, we obtain a rational parameterization by expressing  $\cos t$  and  $\sin t$  in terms of  $\tan(t/2)$ , and we get the fractions

$$x = \frac{(1 - u^2)(a(1 + v^2) - 2bv)}{(1 + u^2)(1 + v^2)}, \\ y = \frac{2u(a(1 + v^2) - 2bv)}{(1 + u^2)(1 + v^2)}, \\ z = \frac{c(1 - v^2)}{1 + v^2}.$$

Thus, the torus as a rational surface  $F$  is defined by

$$x(u, v) = (1 - u^2)(a(1 + v^2) - 2bv), \\ y(u, v) = 2u(a(1 + v^2) - 2bv), \\ z(u, v) = c(1 + u^2)(1 - v^2), \\ w(u, v) = (1 + u^2)(1 + v^2).$$

Rendering  $F$  over  $[-1, 1] \times [-1, 1]$  yields one fourth of the torus, specifically, the front half of the upper half. Performing the change of variables

$$(u, v) \mapsto \left( \frac{v}{u}, \frac{1}{u} \right),$$

the rational surface  $F_\varphi$  is defined by

$$\begin{aligned}x(u, v) &= (u^2 - v^2)(a(1 + u^2) - 2bu), \\y(u, v) &= 2uv(a(1 + u^2) - 2bu), \\z(u, v) &= c(u^2 - 1)(u^2 + v^2), \\w(u, v) &= (u^2 + v^2)(u^2 + 1).\end{aligned}$$

Unfortunately,  $x(0, 0) = y(0, 0) = z(0, 0) = w(0, 0) = 0$ , and  $(0, 0)$  is a base point of  $F_\varphi$ .

Performing the change of variables

$$(u, v) \mapsto \left(\frac{1}{v}, \frac{u}{v}\right),$$

the rational surface  $F_\psi$  is defined by

$$\begin{aligned}x(u, v) &= (v^2 - 1)(a(u^2 + v^2) - 2buv), \\y(u, v) &= 2v(a(u^2 + v^2) - 2buv), \\z(u, v) &= c(v^2 - u^2)(v^2 + 1), \\w(u, v) &= (u^2 + v^2)(v^2 + 1).\end{aligned}$$

Unfortunately, we also have  $x(0, 0) = y(0, 0) = z(0, 0) = w(0, 0) = 0$ , and  $(0, 0)$  is a base point of  $F_\psi$ .

If we try to render the rational surfaces  $F_\varphi$  and  $F_\psi$  over  $[-1, 1] \times [-1, 1]$ , we discover that some regions of these surfaces are not drawn properly. In these regions, there are holes and many lines segments shooting in all directions! The problem is that  $(0, 0)$  is a discontinuity point for both surfaces, and that the limit reached when  $u$  and  $v$  approach 0 depends very much on the ratio  $v/u$ . One way to understand what happens is to let  $v = ku$ , simplify the fractions, and see what is the limit when  $u$  approaches 0. For  $F_\varphi$ , after calculations, we find that the limit when  $u$  approaches 0 is

$$\left(\frac{a(1 - k^2)}{1 + k^2}, \frac{2ak}{1 + k^2}, -c\right),$$

which corresponds to the circle of radius  $a$  in the plane  $z = -c$ . For  $F_\psi$ , after calculations, we find that the limit when  $u$  approaches 0 is

$$\left(-a + \frac{2bk}{1 + k^2}, 0, -\frac{c(1 - k^2)}{1 + k^2}\right),$$

which corresponds to an ellipse in the plane  $y = 0$ , centered at the point  $(-a, 0, 0)$ . It is indeed in the neighborhood of these two curves on the torus that  $F_\varphi$  and  $F_\psi$  are not drawn properly.

We now propose a method to resolve the singularities caused by base points. The method is inspired by a technique in algebraic geometry known as “blowing-up” (see Fulton [67] or Harris [83]). What is new is that we give formulae for computing “resolved” control nets.

In most cases, base points occur during a subdivision step in which a triangular net with a corner of zeros appears. Using a change of base triangle if necessary, it can be assumed without loss of generality that the corner of zeros has  $t$  as one of its vertices. If we display control nets (in  $\hat{\mathcal{E}}$ ) with  $F(r)$  at the top corner,  $F(s)$  as the rightmost lower corner, and  $F(t)$  as the leftmost lower corner, a control net  $\theta = (\theta_{i,j,k})_{(i,j,k) \in \Delta_m}$  of degree  $m$  has the following shape:

$$\begin{array}{ccccccc} & & & & \times & & \\ & & & & \times \times & & \\ & & & & \dots & & \\ \times & \times & \dots & \times & \times & & \end{array}$$

$$\begin{array}{c}
 0 \times \times \dots \times \times \\
 0 \ 0 \times \times \dots \times \times \\
 \dots \\
 0 \dots 0 \times \times \dots \times \times \\
 \underbrace{00 \dots 00}_n \times \times \dots \times \times \\
 \qquad \qquad \qquad \underbrace{\qquad \qquad \qquad}_{m+1-n}
 \end{array}$$

It is assumed that all entries designated as  $\times$  are nonzero. The more general case can be treated, but it is computationally too expensive to be practical.

Given an affine frame  $\Delta rst$  in the plane, recall that a rational surface  $F$  of degree  $m$  defined by the control net  $\theta = (\theta_{i,j,k})_{(i,j,k) \in \Delta_m}$  is the projection onto  $\tilde{\mathcal{E}}$  of the polynomial surface  $G$  in  $\hat{\mathcal{E}}$  defined by  $\theta$ . Also, we have

$$G(u, v) = \sum_{i+j+k=m} \theta_{i,j,k} \frac{m!}{i!j!k!} u^i v^j (1-u-v)^k,$$

for all  $u, v \in \mathbb{R}$ . It will be convenient to assume that if  $\theta_{i,j,k} \in \hat{\mathcal{E}}$  is a weighted point, then its weight is denoted as  $w_{i,j,k}$ , and if  $\theta_{i,j,k}$  is a control vector, then we assign it the weight  $w_{i,j,k} = 0$ . If we define  $w(u, v)$  as

$$w(u, v) = \sum_{i+j+k=m} w_{i,j,k} \frac{m!}{i!j!k!} u^i v^j (1-u-v)^k,$$

whenever  $w(u, v) \neq 0$ , we have

$$F(u, v) = \sum_{i+j+k=m} \theta_{i,j,k} \frac{m!}{i!j!k!} \frac{u^i v^j (1-u-v)^k}{w(u, v)},$$

for all  $u, v \in \mathbb{R}$ .

The “blowing-up” method used here relies on the following observation based on an idea of Warren [177]. Given the polynomial surface  $G$  in  $\mathcal{E}$  (and  $w$ ), we define the polynomial surface  $G_b$  and  $w_b$  as follows:

$$\begin{aligned}
 G_b(\alpha, \beta) &= G(\alpha(1-\beta), \alpha\beta), \\
 w_b(\alpha, \beta) &= w(\alpha(1-\beta), \alpha\beta).
 \end{aligned}$$

Since  $\alpha(1-\beta) + \alpha\beta = \alpha$ , we get

$$G_b(\alpha, \beta) = \sum_{i+j+k=m} \theta_{i,j,k} \frac{m!}{i!j!k!} \alpha^{i+j} (1-\alpha)^k \beta^j (1-\beta)^i,$$

and

$$w_b(\alpha, \beta) = \sum_{i+j+k=m} w_{i,j,k} \frac{m!}{i!j!k!} \alpha^{i+j} (1-\alpha)^k \beta^j (1-\beta)^i.$$

Now, if  $\theta_{i,j,k} = 0$  for  $i+j < n$  (with  $i+j+k = m$ ), we note that both  $G_b(\alpha, \beta)$  and  $w_b(\alpha, \beta)$  are divisible by  $\alpha^n$ . If we define the polynomial surface  $\tilde{G}$  (and  $\tilde{w}$ ), such that

$$\tilde{G}(\alpha, \beta) = \frac{G_b(\alpha, \beta)}{\alpha^n}$$

and

$$\tilde{w}(\alpha, \beta) = \frac{w_b(\alpha, \beta)}{\alpha^n},$$

then we have

$$\frac{G_b(\alpha, \beta)}{w_b(\alpha, \beta)} = \frac{\tilde{G}(\alpha, \beta)}{\tilde{w}(\alpha, \beta)},$$

for all  $\alpha \neq 0$ . Furthermore when  $\alpha = 0$ , we have

$$\tilde{G}(0, \beta) = \sum_{i+j=n} \theta_{i,j,m-n} \frac{m!}{i!j!(m-n)!} \beta^j (1-\beta)^i,$$

and

$$\tilde{w}(0, \beta) = \sum_{i+j=n} w_{i,j,m-n} \frac{m!}{i!j!(m-n)!} \beta^j (1-\beta)^i.$$

Thus, for all  $\beta$  for which  $\tilde{G}(0, \beta)$  and  $\tilde{w}(0, \beta)$  are not simultaneously null,

$$\frac{\tilde{G}(0, \beta)}{\tilde{w}(0, \beta)}$$

is defined, and the polynomial surface  $\tilde{G}$  defines the rational surface  $\tilde{F}$  such that

$$\tilde{F}(\alpha, \beta) = \frac{\tilde{G}(\alpha, \beta)}{\tilde{w}(\alpha, \beta)}.$$

Thus, what happens is that the triangular patch  $F$  over  $\Delta rst$  is really a four-sided patch, the point  $F(t)$  being “blown up” into the rational curve of degree  $n$  whose control points are

$$(\theta_{i,j,m-n})_{i+j=n}.$$

If this rational curve has no base points, then the rational surface patch  $\tilde{F}$  defined by the polynomial surface  $\tilde{G}$  has no base point, and it extends the surface patch  $F$  over  $\Delta rst$ . If it has base points, they are common zeros of some polynomials in  $\beta$ , and by simplifying by common factors and using continuity, we could eliminate these base points. For simplicity, we will assume that the boundary curve has no base points.

Viewing  $\tilde{G}$  as a bipolynomial surface, note that  $\tilde{G}$  has bidegree  $\langle m-n, m \rangle$ . Also observe that the function

$$(\alpha, \beta) \mapsto (\alpha(1-\beta), \alpha\beta)$$

maps the unit square with vertices

$$(0, 0), (0, 1), (1, 1), (1, 0)$$

onto the triangle  $\Delta rst = ((0, 1), (1, 0), (0, 0))$ , in such a way that the edge  $((0, 0), (0, 1))$  is mapped onto  $t$ , the vertex  $(1, 1)$  is mapped onto  $s$ , and the vertex  $(1, 0)$  is mapped onto  $r$ . Furthermore, if  $u = \alpha(1-\beta)$  and  $v = \alpha\beta$ , we get

$$\alpha = u + v$$

and

$$\beta = \frac{v}{u+v}$$

and thus, the map is invertible except on the line  $u + v = 0$ . Thus, we can think of the inverse map as “blowing up” the affine frame  $\Delta rst$  into the unit square. Specifically, the point  $t$  is “blown up” into the edge  $((0, 0), (0, 1))$ .

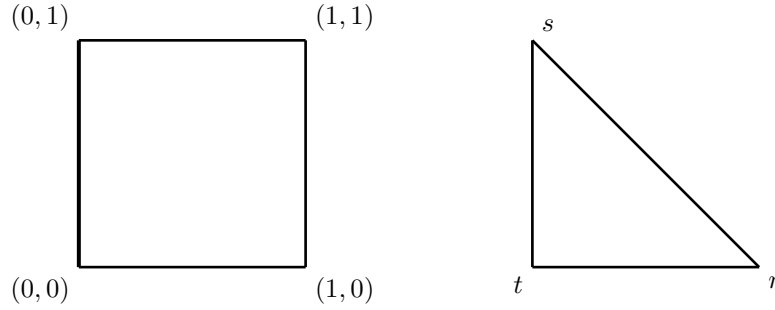


Figure 24.11: Blowing up a triangle into a square

The only remaining problem is that the above method yields a rational square patch not given by a control net, and that we often need to map the unit square to an arbitrary base triangle. The second problem is easily solved. Assume that the affine frame  $\Delta rst$  has coordinates  $((r_1, r_2), (s_1, s_2), (t_1, t_2))$ . It is easily seen that the map defined such that

$$\begin{aligned} u &= (s_1 - r_1)\alpha\beta + (r_1 - t_1)\beta + t_1, \\ v &= (s_2 - r_2)\alpha\beta + (r_2 - t_2)\beta + t_2, \end{aligned}$$

maps the unit square to the triangle  $\Delta rst$ , in such a way that the edge  $((0,0), (0,1))$  is mapped onto  $t$ , the vertex  $(1,1)$  is mapped onto  $s$ , and the vertex  $(1,0)$  is mapped onto  $r$ . Some simple calculations show that

$$\begin{aligned} \alpha &= \frac{(s_2 - r_2)(u - t_1) - (s_1 - r_1)(v - t_2)}{(r_1 - t_1)(s_2 - r_2) - (r_2 - t_2)(s_1 - r_1)}, \\ \beta &= \frac{(r_1 - t_1)(v - t_2) - (r_2 - t_2)(u - t_1)}{(s_2 - r_2)(u - t_1) - (s_1 - r_1)(v - t_2)}, \end{aligned}$$

and thus, the map is only invertible outside the line of equation

$$(s_2 - r_2)(u - t_1) - (s_1 - r_1)(v - t_2) = 0,$$

the parallel to the vector  $(s_1 - r_1, s_2 - r_2)$  through  $t$ .

Now, if  $g: (\mathcal{P})^m \rightarrow \widehat{\mathcal{E}}$  is the polar form associated with  $G$ , we can compute the polar form  $g_b: (\mathcal{P})^m \times (\mathcal{P})^m \rightarrow \widehat{\mathcal{E}}$  associated with the bipolynomial surface  $G_b$  as follows:

$$g_b(u_1, \dots, u_m, v_1, \dots, v_m) = \frac{1}{m!} \sum_{\sigma \in \mathcal{S}_m} g((u_1(1 - v_{\sigma(1)}), u_1 v_{\sigma(1)}), \dots, (u_m(1 - v_{\sigma(m)}), u_m v_{\sigma(m)})),$$

where  $\mathcal{S}_m$  denotes the group of permutations on  $\{1, \dots, m\}$ . The above formula corresponds to the case of the simple mapping  $u = \alpha(1 - \beta)$ ,  $v = \alpha\beta$ , and it is obvious how to adapt it to the more general map

$$\begin{aligned} u &= (s_1 - r_1)\alpha\beta + (r_1 - t_1)\beta + t_1, \\ v &= (s_2 - r_2)\alpha\beta + (r_2 - t_2)\beta + t_2, \end{aligned}$$

Now, over the affine basis  $(0,1)$ , the square control net  $\theta_{\square} = (\theta_{\square_{i,j}})_{0 \leq i,j \leq m}$  associated with  $g_b$  is defined such that

$$\theta_{\square_{i,j}} = g_b(\underbrace{0, \dots, 0}_{m-i}, \underbrace{1, \dots, 1}_i, \underbrace{0, \dots, 0}_{m-j}, \underbrace{1, \dots, 1}_j).$$

However, if  $\theta_{i,j,k} = 0$  for  $i + j < n$ , with  $i + j + k = m$ , then  $\theta_{\square_{i,j}} = 0$  for  $i < n$ , and thus we obtain the rectangular net  $\tilde{\theta}_{\square}$  of degree  $(m - n, m)$  associated with  $\tilde{g}$ , given by

$$\tilde{\theta}_{\square} = (\theta_{\square_{i,j}})_{n \leq i \leq m, 0 \leq j \leq m},$$

which corresponds to the rational surface defined by  $\tilde{G}$ .

Thus, we know how to compute a rectangular net for the blown-up version  $\tilde{G}$  of  $G$ . A triangular net of degree  $2m - n$  can easily be obtained. Indeed, there is a simple way for converting the polar form  $g: (\mathcal{P})^p \times (\mathcal{P})^q \rightarrow \hat{\mathcal{E}}$  of a bipolynomial surface of degree  $(p, q)$  into a symmetric multilinear polar form  $g_{\Delta}: (\mathcal{P})^{p+q} \rightarrow \hat{\mathcal{E}}$ , using the following formula: letting  $m = p + q$ , we have

$$g_{\Delta}((u_1, v_1), \dots, (u_m, v_m)) = \frac{1}{\binom{m}{p}} \sum_{\substack{I \cup J = \{1, \dots, m\} \\ I \cap J = \emptyset \\ |I|=p, |J|=q}} g\left(\prod_{i \in I} u_i, \prod_{j \in J} v_j\right),$$

where

$$g\left(\prod_{i \in I} u_i, \prod_{j \in J} v_j\right) = g(u_{i_1}, \dots, u_{i_p}, v_{j_1}, \dots, v_{j_q}),$$

with  $I = \{i_1, \dots, i_p\}$ , and  $J = \{j_1, \dots, j_q\}$ .

Note that it is also possible to convert the polar form  $f_{\Delta}: (\mathcal{P})^m \rightarrow \hat{\mathcal{E}}$  of a surface of degree  $m$  into a symmetric  $(m, m)$ -multilinear polar form  $g: (\mathcal{P})^m \times (\mathcal{P})^m \rightarrow \hat{\mathcal{E}}$ , using the following formula:

$$g(u_1, \dots, u_m, v_1, \dots, v_m) = \frac{1}{m!} \sum_{\sigma \in \mathcal{S}_m} f((u_1, v_{\sigma(1)}), \dots, (u_m, v_{\sigma(m)})),$$

where  $\mathcal{S}_m$  denotes the group of permutations on  $\{1, \dots, m\}$ .

Thus, we have a method for blowing up a control net  $\theta$  of degree  $m$  with a corner of zeros of size  $n$ , into a triangular net  $\tilde{\theta}$  of degree  $2m - n$ , by first blowing up the triangular net  $\theta$  into a rectangular net  $\tilde{\theta}_{\square}$ , and then converting  $\tilde{\theta}_{\square}$  into a triangular net  $\tilde{\theta}$ .

The following function blows up the triangular net  $\theta$  into a rectangular net  $\tilde{\theta}_{\square}$ . The function `poldecas3` computing polar values given a triangular net can be found in Gallier [70], and the built-in *Mathematica* function `Permutation` returns the list of all permutations of a finite set.

```
(* computes a rectangular net of degree ((m - nm), m)
   from a triangular net of degree m *)
(* by "blowing up" a degenerate corner *)
(* General version, blowing up a triangular patch based on the
   triangle ((r_1, r_2), (s_1, s_2), (t_1, t_2)),
   wrt the standard frame (r, s, t) *)

sqblowup[{oldnet__}, {reftrig__}, mm_, nm_, debug_] :=
Block[
{net = {oldnet}, newnet, m = mm, rr, ss, ntrig,
 temp, izargs, jzargs, pt, u, v, w, i, j, ii, permset,
 perm, setm, pi, kk, ll, xx, lowi, r1, r2, s1, s2, t1, t2},
{rr = 0; ss = 1; lowi = nm;
 Print["In sqblowup, m = ", m, ", nm = ", nm];
 r1 = ntrig[[1, 1]]; r2 = ntrig[[1, 2]]; s1 = ntrig[[2, 1]]];
```



```

s2 = ntrig[[2, 2]];
t1 = ntrig[[3, 1]]; t2 = ntrig[[3, 2]];
Print[" r1 = ", r1, ", r2 = ", r2, ", s1 = ", s1,
      ", s2 = ", s2, ", t1 = ", t1, ", t2 = ", t2];
newnet = {}; setm = Table[ii, {ii, 1, m}];
If[debug === -2, Print["*** Left corner patch in blowup: ", net]];
Do[izargs = {};
  Do[
    izargs = Append[izargs, rr], {ii, 1, m - i}
  ];
  Do[
    izargs = Append[izargs, ss], {ii, m + 1 - i, m}
  ];
  Do[
    jzargs = {}; xx = 0;
    Do[
      jzargs = Append[jzargs, rr], {ii, 1, m - j}
    ];
    Do[
      jzargs = Append[jzargs, ss], {ii, m + 1 - j, m}
    ];
    If[debug === 2, Print[" izargs = ", izargs]];
    If[debug === 2, Print[" jzargs = ", jzargs]];
    permset = Permutations[setm]; ll = Length[permset];
    Do[
      perm = permset[[kk]]; args = {};
      If[debug === 2, Print[" perm = ", perm]];
      Do[
        pi = perm[[ii]];
        u = (s1 - r1)*izargs[[ii]]*jzargs[[pi]] +
            (r1 - t1)*izargs[[ii]] + t1;
        v = (s2 - r2)*izargs[[ii]]*jzargs[[pi]] +
            (r2 - t2)*izargs[[ii]] + t2;
        w = 1 - u - v;
        temp = {u, v, w};
        args = Append[args, temp], {ii, 1, m}
      ];
      If[debug === 2, Print[" args = ", args]];
      pt = poldecas3[net, m, args];
      xx = xx + pt, {kk, 1, ll}
    ];
    pt = xx/ll; If[debug === 2, Print[" xx/ll: ", pt]];
    newnet = Join[newnet, pt];
    Print["In sqblowup, i = ", i, ", j = ", j], {j, 0, m}
  ], {i, lowi, m}
];
If[debug === 2, Print[" newnet = ", newnet]];
newnet
)
];

```

The following function converts a rectangular net into a triangular net. It uses a function `subsets` that returns the list of subsets of size `p` of some finite set `setm`. The function `recpoldecas` computing polar values given a rectangular net can be found in Gallier [70].

```
(* converts a rectangular net of degree (p, q) to *)
(* a triangular net of degree m = p + q, wrt the standard
   frame (r, s, t), where r = (1, 0, 0), s = (0, 1, 0),
   t = (0, 0, 1) *)

rectotr[{oldnet_}, p_, q_, debug_] :=
Block[
{net = {oldnet}, i, j, newnet, Isets, I1, J1, ii, jj, l, m, setm,
pt1, pt2, pt, lambda, mu, xx, u, v, rr, ss, tt, r1, s1, r2, s2},
rr = {1, 0}; ss = {0, 1}; tt = {0, 0};
r1 = 0; r2 = 0; s1 = 1; s2 = 1;
newnet = {}; m = p + q;
(* Print["p = ", p, " q = ", q]; *)
setm = Table[i, {i, m}];
Do[
  Do[lambda = {}; mu = {}; u = {}; v = {};
    Do[
      lambda = Append[lambda, rr[[1]]];
      mu = Append[mu, rr[[2]]], {ii, 1, i}
    ];
    Do[
      lambda = Append[lambda, ss[[1]]];
      mu = Append[mu, ss[[2]]], {ii, 1, j}
    ];
    Do[
      lambda = Append[lambda, tt[[1]]];
      mu = Append[mu, tt[[2]]], {ii, 1, m - i - j}
    ];
    Isets = subsets[setm, p]; l = Length[Isets];
    pt = 0;
    Do[
      I1 = Isets[[ii]]; J1 = relcomp[setm, I1];
      If[debug === 2, Print[" I1 = ", I1]];
      If[debug === 2, Print[" J1 = ", J1]];
      u = {}; v = {};
      Do[
        kk = I1[[jj]];
        pt1 = lambda[[kk]];
        u = Append[u, pt1], {jj, 1, p}
      ];
      If[debug === 2, Print[" u = ", u]];
      Do[
        kk = J1[[jj]];
        pt2 = mu[[kk]];
        v = Append[v, pt2], {jj, 1, q}
      ];
      If[debug === 2, Print[" v = ", v]];

```

```

        xx = recpoldecas[net, p, q, r1, s1, r2, s2, u, v, debug];
        pt = pt + xx, {ii, 1, 1}
    ];
    xx = pt/Binomial[m,p];
    newnet = Join[newnet, {xx}];
    Print["In rectotr, i = ", i, ", j = ", j], {j, 0, m - i}
], {i, 0, m}
];
If[debug === 2, Print[" newnet = ", newnet]];
newnet
];

```

The following function detects whether a triangular net contains a corner of zeros.

```

(* Detects whether a corner of a net contains zero vectors *)
(* And if so, if all the elements on the top nonzero row are identical *)

zerocorner[{net__}, oldm_, debug_] :=
Block[
{cc = {net}, anet, newnet = {}, m = oldm, pt,
stop1, stop2, i, j, res, flag},
(anet = convtomat[cc, m];
(* checks it top corner is zero *)
stop1 = 0; i = m;
While[stop1 === 0 && i >= 0,
j = 0; stop2 = 0;
While[stop2 === 0 && j <= m - i,
pt = anet[[i+1, j+1]];
If[nearzero[pt] === 1, j = j + 1, stop2 = 1]
];
If[stop2 === 0, i = i - 1, stop1 = 1]
];
(* checks it top nonzero row is consistent *)
If[stop1 === 1 && i < m,
flag = 0; j = 0;
While[flag === 0 && j < m - i,
If[anet[[i + 1, j + 1]] === anet[[i + 1, j + 2]],
j = j + 1,
flag = 1
]
];
If[flag === 1, res = {i, j},
res = {-1, i, anet[[i + 1, 1]]}],
res = {-1}
];
res
)
];

```

Going back to example 2 of section 23.4, a torus, it turns out that in subdividing the nets `theta1`, `theta2`, `rho1`, and `rho2`, degenerate nets with a corner of zeros are encountered. In fact, these corners have two rows

of zeros. and thus, the blowing up method yields nets of degree 6. For example, the net corresponding to `theta1` is resolved to a triangular net, which after 3 iterations of subdivision, yields the surface shown in Figure 24.12.

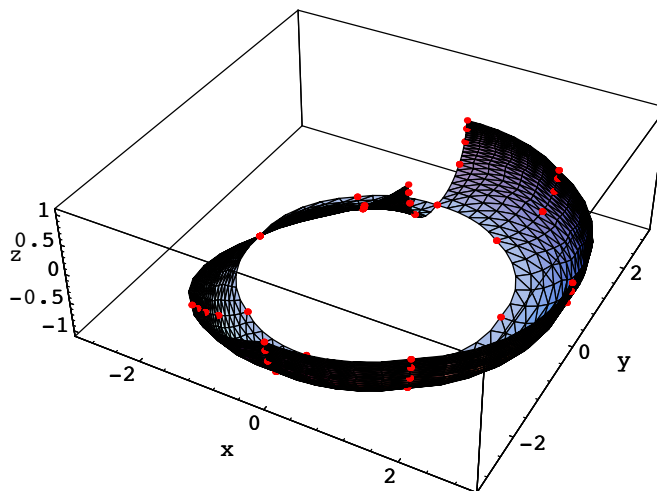


Figure 24.12: A blow up of patch 3 of a torus

The net corresponding to `theta2` is resolved to a triangular net, which after 3 iterations of subdivision, yields the surface shown in Figure 24.13.

Displaying these two pictures together, we get a shape reminiscent of a horse-shoe crab, shown in Figure 24.14!

Similarly, blowing up the nets `rho1` and `rho2` yields the surface surface shown in Figure 24.15.

Together with the patches 1 and 2 from example 2 of section 23.4, we get the entire torus. Of course, we could have taken advantage of symmetries, and our point is to illustrate the algorithm.

## 24.6 Problems

**Problem 1** (i) Consider a rational surface  $F$  of degree  $\langle p, q \rangle$  specified by the rectangular control net

$$\mathcal{N} = (\theta_{i,j})_{0 \leq i \leq p, 0 \leq j \leq q}$$

over  $(0, 1) \times (0, 1)$ , where each  $\theta_{i,j}$  is a vector in  $\hat{\mathcal{E}}$ . Show that for every pair of constant  $\rho, \sigma \neq 0$ , the control net

$$(\rho^i \sigma^j \theta_{i,j})_{0 \leq i \leq p, 0 \leq j \leq q}$$

also specifies the surface  $F$ . Show that if  $0 < \rho, \sigma \leq 1$ , the surface patch defined over  $(0, 1) \times (0, 1)$  by  $(\theta_{i,j})_{0 \leq i \leq p, 0 \leq j \leq q}$  is equal to the surface patch defined over  $(0, 1) \times (0, 1)$  by  $(\rho^i \sigma^j \theta_{i,j})_{0 \leq i \leq p, 0 \leq j \leq q}$ . What happens if  $\rho < 0$  or  $\rho > 1$  (or  $\sigma < 0$  or  $\sigma > 1$ ).

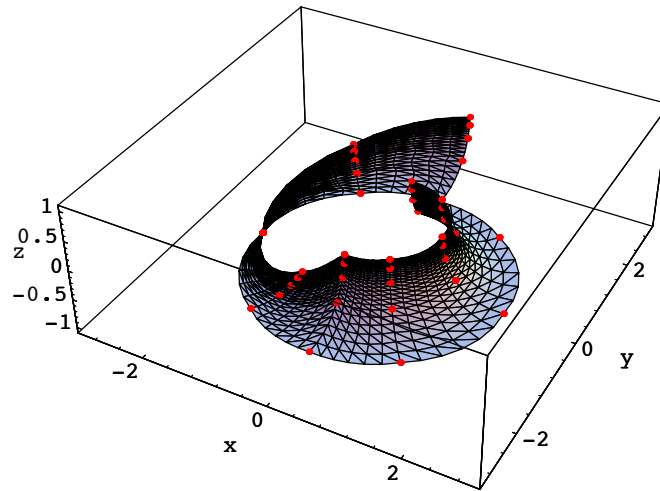


Figure 24.13: A blow up of patch 4 of a torus

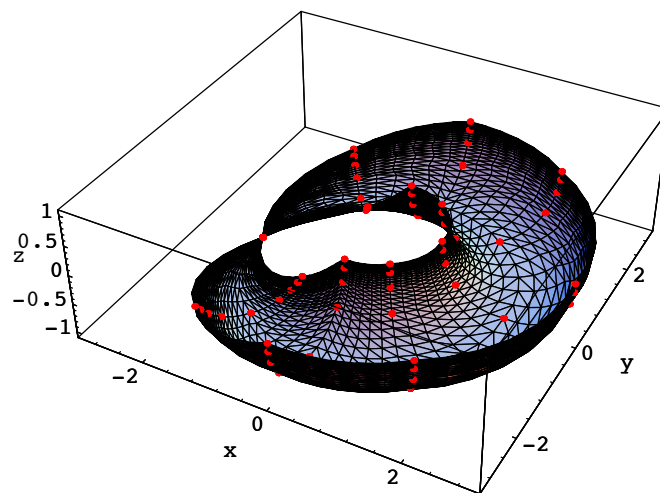


Figure 24.14: A blow up of patches 4,3 of a torus

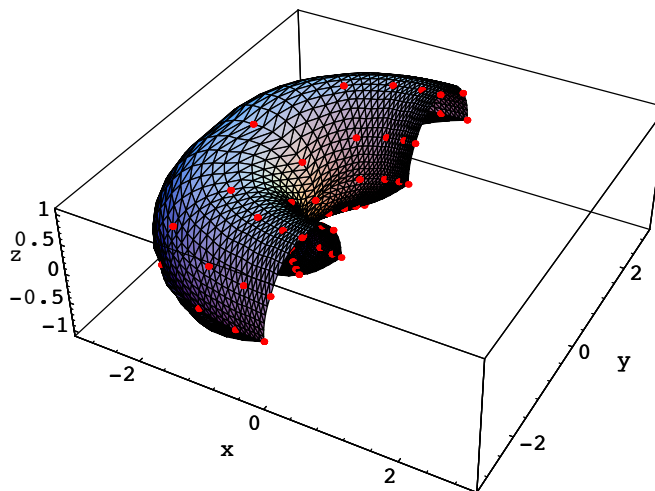


Figure 24.15: A blow up of patches 5,6 of a torus

*Hint:* Use the change of variables

$$(u, v) \mapsto \left( \frac{\rho u}{1 + (\rho - 1)u}, \frac{\sigma v}{1 + (\sigma - 1)v} \right).$$

(ii) Assuming that every  $\theta_{i,j}$  is a control point of the form  $\langle a_{i,j}, w_{i,j} \rangle$ , show that if  $w_{0,0}$ ,  $w_{p,0}$ ,  $w_{0,q}$ , and  $w_{p,q}$  have the same sign, then it is possible to choose  $\rho$  and  $\sigma$  such that three of the weights  $w_{0,0}$ ,  $w_{p,0}$ ,  $w_{0,q}$ , and  $w_{p,q}$ , are equal to 1.

**Problem 2** (i) Consider a rational surface  $F$  of degree  $\langle p, q \rangle$  specified by the rectangular control net

$$\mathcal{N} = (\theta_{i,j})_{0 \leq i \leq p, 0 \leq j \leq q}$$

over  $(0, 1) \times (0, 1)$ , where each  $\theta_{i,j}$  is a vector in  $\widehat{\mathcal{E}}$ . The *weight points*  $p_{i,j}$  and  $q_{i,j}$  ( $0 \leq i \leq p-1, 0 \leq j \leq q-1$ ) are the points defined such that

$$p_{i,j} = \frac{w_{i,j} b_{i,j} + w_{i+1,j} b_{i+1,j}}{w_{i,j} + w_{i+1,j}},$$

$$q_{i,j} = \frac{w_{i,j} b_{i,j} + w_{i,j+1} b_{i,j+1}}{w_{i,j} + w_{i,j+1}}.$$

Prove that the four points  $p_{i,j}, p_{i,j+1}, q_{i,j}, q_{i+1,j}$  are coplanar.

(ii) Prove that if all the weights are equal, then  $(p_{i,j}, p_{i,j+1}, q_{i,j}, q_{i+1,j})$  is a parallelogram.

**Problem 3** Write your own computer program for drawing a closed rational surface.

**Problem 4** Write your own computer program for resolving base points.

**Problem 5** Prove that the polar form  $g: (\mathcal{P})^p \times (\mathcal{P})^q \rightarrow \widehat{\mathcal{E}}$  of a bipolynomial surface of degree  $\langle p, q \rangle$  can be converted to a symmetric multiaffine map  $g_\Delta: (\mathcal{P})^{p+q} \rightarrow \widehat{\mathcal{E}}$  using the following formula:

$$g_\Delta((u_1, v_1), \dots, (u_m, v_m)) = \frac{1}{\binom{m}{p}} \sum_{\substack{I \cup J = \{1, \dots, m\} \\ I \cap J = \emptyset \\ |I|=p, |J|=q}} g\left(\prod_{i \in I} u_i, \prod_{j \in J} v_j\right),$$

where  $m = p + q$  and

$$g\left(\prod_{i \in I} u_i, \prod_{j \in J} v_j\right) = g(u_{i_1}, \dots, u_{i_p}, v_{j_1}, \dots, v_{j_q}),$$

with  $I = \{i_1, \dots, i_p\}$ , and  $J = \{j_1, \dots, j_q\}$ .

**Problem 6** Write a computer program using the method of problem 5 for converting a rectangular net of degree  $\langle p, q \rangle$  to a triangular net of degree  $p + q$ .

**Problem 7** Prove that the polar form  $f_\Delta: (\mathcal{P})^m \rightarrow \widehat{\mathcal{E}}$  of a surface of degree  $m$  can be converted into a symmetric  $\langle m, m \rangle$ -multiaffine map  $g: (\mathcal{P})^m \times (\mathcal{P})^m \rightarrow \widehat{\mathcal{E}}$  using the following formula:

$$g(u_1, \dots, u_m, v_1, \dots, v_m) = \frac{1}{m!} \sum_{\sigma \in \mathcal{S}_m} f((u_1, v_{\sigma(1)}), \dots, (u_m, v_{\sigma(m)})),$$

where  $\mathcal{S}_m$  denotes the group of permutations on  $\{1, \dots, m\}$ .

**Problem 8** Write a computer program using the method of problem 7 for converting a triangular net of degree  $m$  to a rectangular net of degree  $\langle m, m \rangle$ .

**Problem 9** The following rectangular net of degree  $\langle 4, 4 \rangle$  (over  $(-1, 1) \times (-1, 1)$ ) was obtained by converting a triangular net for a torus.

```
rtornet =  {{2, 0, 1, 1}, {3/2, 0, 1, 1}, {8/7, 0, 5/7, 7/6},
            {1, 0, 1/3, 3/2}, {1, 0, 0, 2}, {2, 1, 1, 1}, {3/2, 3/4, 1, 1},
            {8/7, 4/7, 5/7, 7/6}, {1, 1/2, 1/3, 3/2}, {1, 1/2, 0, 2},
            {10/7, 12/7, 1, 7/6}, {15/14, 9/7, 1, 7/6}, {40/49, 48/49, 5/7, 49/36},
            {5/7, 6/7, 1/3, 7/4}, {5/7, 6/7, 0, 7/3}, {2/3, 2, 1, 3/2},
            {1/2, 3/2, 1, 3/2}, {8/21, 8/7, 5/7, 7/4}, {1/3, 1, 1/3, 9/4},
            {1/3, 1, 0, 3}, {0, 2, 1, 2}, {0, 3/2, 1, 2}, {0, 8/7, 5/7, 7/3},
            {0, 1, 1/3, 3}, {0, 1, 0, 4}};
```

Show that the nets  $((-1)^i \text{rtornet}_{i,j})$ ,  $((-1)^j \text{rtornet}_{i,j})$ , and  $((-1)^{i+j} \text{rtornet}_{i,j})$  have the base point  $(0, 0, 0, 0)$ . Can you explain why.

**Problem 10** Show that the formula

$$g_b(u_1, \dots, u_m, v_1, \dots, v_m) = \frac{1}{m!} \sum_{\sigma \in \mathcal{S}_m} g((u_1(1 - v_{\sigma(1)}), u_1 v_{\sigma(1)}), \dots, (u_m(1 - v_{\sigma(m)}), u_m v_{\sigma(m)}))$$

(where  $\mathcal{S}_m$  denotes the group of permutations on  $\{1, \dots, m\}$ ) can be used to compute a rectangular net for a triangular patch.

**Problem 11** Given an affine space  $\mathcal{E}$  of dimension  $\geq 3$ , let  $F: \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \rightarrow \widetilde{\mathcal{E}}$  be a rectangular rational surface of bidegree  $\langle p, q \rangle$  defined by some  $\langle p, q \rangle$ -symmetric multilinear map

$$f: (\widehat{\mathbb{A}})^p \times (\widehat{\mathbb{A}})^q \rightarrow \widehat{\mathcal{E}}$$

with

$$F([u, t], [v, z]) = \mathbf{P}(f)\left(\underbrace{[u, t], \dots, [u, t]}_p, \underbrace{[v, z], \dots, [v, z]}_q\right),$$

for all  $[u, t], [v, z] \in \tilde{\mathbb{A}}$ .

(i) Letting

$$B_i^m(u, v) = \binom{m}{i} u^i v^{m-i},$$

and assuming that the control points induced by  $f$  over  $(0, 1) \times (0, 1)$  are of the form  $\langle b_{i,j}, w_{i,j} \rangle$  where  $w_{i,j} \neq 0$ , prove that if

$$\sum_{i,j=0}^{p,q} w_{i,j} B_i^p(u, t) B_j^q(v, z) \neq 0$$

then

$$F([u, t], [v, z]) = \frac{\sum_{i,j=0}^{p,q} w_{i,j} B_i^p(u, t) B_j^q(v, z) b_{i,j}}{\sum_{i,j=0}^{p,q} w_{i,j} B_i^p(u, t) B_j^q(v, z)},$$

for all homogeneous coordinates  $[u, t], [v, z] \in \mathbb{R}^2$ .

(ii) Find a type of de Casteljau algorithm applying to these surfaces.

*Hint:* As for bipolynomial surfaces, we can view the surface as a curve of curves. Recall that in the case of a curve (see problem 20 of chapter 22), instead of computing an affine combination  $(1-u)b_i^r + u b_{i+1}^r$ , we compute the linear combination  $z b_i^r + u b_{i+1}^r$ .

(iii) As in problem 20 of chapter 22, by modeling the projective line in  $\mathbb{R}^2$  as the  $C^0$ -curve segment

$$t \mapsto (t, 1 - |t|)$$

over  $[-1, 1]$ , show how a closed biprojective rational surface can be drawn in a single piece (as the trace of  $F([u, 1 - |u|], [v, 1 - |v|])$  over  $[-1, 1] \times [-1, 1]$ ). Apply this method to draw a full torus. Experiment with other 2D models of the real projective line.

*Remark:* This problem is an adaptation of results due to DeRose [45].

**Problem 12** (i) Prove lemma 24.2.2. Prove that the same result applies to surfaces specified by a rectangular net over  $(r_1, s_1) \times (r_2, s_2)$ , where  $(r_1, s_1)$  and  $(r_2, s_2)$  are arbitrary frames.

(ii) Implement the above method. Test it on many examples, including the net of problem 9.

**Problem 13** Let  $F: \tilde{\mathcal{P}} \rightarrow \tilde{\mathcal{E}}$  be a rational surface of total degree  $m$  specified by some symmetric multilinear map  $f: (\hat{\mathcal{P}})^m \rightarrow \hat{\mathcal{E}}$ .

(i) Letting

$$B_{i,j,k}^m(u, v, w) = \binom{m}{i \ j \ k} u^i v^j w^k,$$

where  $i + j + k = m$ , and assuming that the control points induced by  $f$  over  $((1, 0), (0, 1), (0, 0))$  are of the form  $\langle b_{i,j,k}, w_{i,j,k} \rangle$  where  $w_{i,j,k} \neq 0$ , prove that if

$$\sum_{i+j+k=m} w_{i,j,k} B_{i,j,k}^m(u, v, w) \neq 0$$



then

$$F([u, v, w]) = \frac{\sum_{i+j+k=m} w_{i,j,k} B_{i,j,k}^m(u, v, w) b_{i,j,k}}{\sum_{i+j+k=m} w_{i,j,k} B_{i,j,k}^m(u, v, w)},$$

for all homogeneous coordinates  $[u, v, w] \in \mathbb{R}^3$ .

(ii) Find a type of de Casteljau algorithm computing  $F([u, v, w])$ .

(iii) By modeling the projective plane in  $\mathbb{R}^3$  as the  $C^0$ -surface patch

$$(u, v) \mapsto (u, v, 1 - |u| - |v|)$$

over the square  $S$  with vertices  $(1, 0)$ ,  $(0, 1)$ ,  $(-1, 0)$  and  $(0, -1)$ , show how a closed projective rational surface can be drawn in a single piece (as the trace of  $F([u, v, 1 - |u| - |v|])$  over the square  $S$ ). Apply this method to draw a full crosscap. Experiment with other 3D models of the real projective plane.

*Remark:* This problem is an adaptation of results due to DeRose [45].

**Problem 14** Given a projectivity  $\varphi: \tilde{\mathcal{P}} \rightarrow \tilde{\mathcal{P}}$  (of the projective plane) defined by some invertible matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix},$$

given any rational surface specified by a symmetric multilinear map  $f: (\hat{\mathcal{P}})^m \rightarrow \hat{\mathcal{E}}$ , we can define the multilinear map  $g: (\hat{\mathcal{P}})^m \rightarrow \hat{\mathcal{E}}$  such that

$$g((u_1, v_1, w_1), \dots, (u_m, v_m, w_m)) = f(\varphi(u_1, v_1, w_1), \dots, \varphi(u_m, v_m, w_m)),$$

that is,

$$\begin{aligned} g((u_1, v_1, w_1), \dots, (u_m, v_m, w_m)) &= f((a_1 u_1 + b_1 v_1 + c_1 w_1, a_2 u_1 + b_2 v_1 + c_2 w_1, a_3 u_1 + b_3 v_1 + c_3 w_1), \\ &\dots, (a_1 u_m + b_1 v_m + c_1 w_m, a_2 u_m + b_2 v_m + c_2 w_m, a_3 u_m + b_3 v_m + c_3 w_m)). \end{aligned}$$

The multilinear map  $g$  defines the rational surface obtained by the “change of variable”  $\varphi$ .

(i) For any affine frame  $\Delta = (r = (r_1, r_1), s = (s_1, s_2), t = (t_1, t_2))$ , show that the control points

$$g(\underbrace{r, \dots, r}_i, \underbrace{s, \dots, s}_j, \underbrace{t, \dots, t}_k),$$

where  $i + j + k = m$ , are the polar values

$$\begin{aligned} (a_3 r_1 + b_3 r_2 + c_3)^i (a_3 s_1 + b_3 s_2 + c_3)^j (a_3 t_1 + b_3 t_2 + c_3)^k & f \left( \underbrace{\left( \frac{a_1 r_1 + b_1 r_2 + c_1}{a_3 r_1 + b_3 r_2 + c_3}, \frac{a_2 r_1 + b_2 r_2 + c_2}{a_3 r_1 + b_3 r_2 + c_3}, 1 \right)}_i, \right. \\ & \left. \underbrace{\left( \frac{a_1 s_1 + b_1 s_2 + c_1}{a_3 s_1 + b_3 s_2 + c_3}, \frac{a_2 s_1 + b_2 s_2 + c_2}{a_3 s_1 + b_3 s_2 + c_3}, 1 \right)}_j, \underbrace{\left( \frac{a_1 t_1 + b_1 t_2 + c_1}{a_3 t_1 + b_3 t_2 + c_3}, \frac{a_2 t_1 + b_2 t_2 + c_2}{a_3 t_1 + b_3 t_2 + c_3}, 1 \right)}_k \right), \end{aligned}$$

provided that  $a_3r_1 + b_3r_2 + c_3 \neq 0$ ,  $a_3s_1 + b_3s_2 + c_3 \neq 0$ , and  $a_3t_1 + b_3t_2 + c_3 \neq 0$ .

(ii) In the case of the maps

$$(u, v) \mapsto \left( \frac{\sigma_1 u}{1 - u - v}, \frac{\sigma_2 v}{1 - u - v} \right),$$

where  $\sigma_i \in \{-1, 1\}$  for  $i = 1, 2$ , used by Bajaj and Royappa, we can compute control nets with respect to the triangle  $\Delta = ((2, 0), (0, 2), (0, 0))$  using the above formula, and then perform a change of affine frame to  $((1, 0), (0, 1), (0, 0))$ .

Show that a quarter of sphere is specified by the control net

$$\text{sphnet} = \{\{0, 0, -1, 1\}, \{0, 1, 0, 0\}, \{0, 0, 1, 1\}, \\ \{1, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 1, 1\}\};$$

Show that a quarter of the Steiner roman surface is specified by the control net

$$\text{stein2} = \{\{0, 0, 0, 1\}, \{1, 0, 0, 0\}, \{0, 0, 0, 1\}, \\ \{0, 1, 0, 0\}, \{0, 0, 1, 0\}, \{0, 0, 0, 1\}\};$$

**Problem 15** (i) Show that a control net of bidegree  $\langle 2, 2 \rangle$  w.r.t.  $(-1, 1) \times (-1, 1)$  for an ellipsoid is given by

$$\text{reelnet3} = \{\{-8/3, -2, 2/3, 3\}, \{-8, 0, -2, 1\}, \{-8/3, 2, 2/3, 3\}, \\ \{0, -6, -2, 1\}, \{0, 0, 6, -1\}, \{0, 6, -2, 1\}, \\ \{8/3, -2, 2/3, 3\}, \{8, 0, -2, 1\}, \{8/3, 2, 2/3, 3\}\}$$

(ii) Show that the patch corresponding to  $F_3$  (see lemma 24.2.2) has base points.

**Problem 16** (i) Show that a control net of bidegree  $\langle 2, 2 \rangle$  w.r.t.  $(-1, 1) \times (-1, 1)$  for the Steiner roman surface is given by

$$\text{sqstein3} = \{\{-2/3, -2/3, 2/3, 3\}, \{0, -2, 0, 1\}, \{2/3, -2/3, -2/3, 3\}, \\ \{-2, 0, 0, 1\}, \{0, 0, 0, -1\}, \{2, 0, 0, 1\}, \\ \{-2/3, 2/3, -2/3, 3\}, \{0, 2, 0, 1\}, \{2/3, 2/3, 2/3, 3\}\};$$

(ii) Show that the patch corresponding to  $F_3$  (see lemma 24.2.2) has base points.

**Problem 17** The purpose of this problem is to prove the result stated in section 24.4. Given three noncollinear points  $r, s, t$  in the projective plane, the three lines  $\langle r, s \rangle$ ,  $\langle s, t \rangle$ , and  $\langle r, t \rangle$ , delimit four triangular regions (seven in  $\mathbb{A}^2$ ). The four triangles defined by the lines  $\langle r, s \rangle$ ,  $\langle s, t \rangle$ , and  $\langle r, t \rangle$  are denoted as  $rst$ ,  $R$ ,  $S$ , and  $T$ , where  $R, S, T$  contain points at infinity.

(i) Show that there are three projectivities  $\varphi_i: \mathbb{RP}^2 \rightarrow \mathbb{RP}^2$ ,  $i = 1, 2, 3$ , such that  $\varphi_1(rst) = R$ ,  $\varphi_2(rst) = S$ , and  $\varphi_3(rst) = T$ .

*Hint:* Use the model of  $\mathbb{RP}^2$  in  $\mathbb{R}^3$  where the points  $r, s, t$  are considered as being in the plane  $z = 1$ . Then, consider the linear maps

$$\begin{aligned} (r, s, t) &\mapsto (-r, s, t), \\ (r, s, t) &\mapsto (r, -s, t), \\ (r, s, t) &\mapsto (r, s, -t). \end{aligned}$$

Show that these linear maps induce  $\varphi_1, \varphi_2, \varphi_3$ .

(ii) Consider the rational surfaces  $F_i = F \circ \varphi_i$ ,  $i = 1, 2, 3$ . Show that if the control net (in  $\widehat{\mathcal{E}}$ ) of the triangular surface  $F$  w.r.t. the affine frame  $\Delta rst$  is  $\alpha = (\alpha_{i,j,k})_{(i,j,k) \in \Delta_m}$ , then the control nets  $\theta^1, \theta^2$ , and  $\theta^3$  of the surfaces  $F_1, F_2, F_3$  w.r.t.  $\Delta rts$  are given by the formulae

$$\begin{aligned}\theta_{i,j,k}^1 &= (-1)^i \alpha_{i,j,k}, \\ \theta_{i,j,k}^2 &= (-1)^j \alpha_{i,j,k}, \\ \theta_{i,j,k}^3 &= (-1)^k \alpha_{i,j,k}.\end{aligned}$$

Show that the traces of  $F, F_1, F_2, F_3$  over  $\Delta rst$  cover the entire trace of  $F$  (over  $\mathbb{RP}^2$ ).

(iii) Show that when  $r$  and  $s$  are points at infinity, the method is equivalent to Bajaj and Royappa's method.

**Problem 18** Implement the method of problem 17. Test your program on the Steiner roman surface.

**Problem 19** In the projective space  $\mathbb{RP}^3$ , a *quadric*  $Q$  is the set of points of homogeneous coordinates  $(x, y, z, t)$  such that  $F(x, y, z, t) = 0$ , where  $F(x, y, z, t)$  is a homogeneous polynomial in  $x, y, z, t$  of total degree 2. We can write the equation of a quadric as

$$F(x, y, z, t) = X^\top AX = 0,$$

where  $X^\top = (x, y, z, t)$  and  $A$  is a symmetric  $4 \times 4$  matrix.

We say that a quadric of equation  $X^\top AX = 0$  is *nondegenerate* if  $\det(A) \neq 0$  and *degenerate* if  $\det(A) = 0$ .

An *affine quadric* in  $\mathbb{A}^3$  is defined as the set of points of coordinates  $(x, y, z)$  such that

$$X^\top AX = 0,$$

where  $X^\top = (x, y, z, 1)$  and  $A$  is a symmetric  $4 \times 4$  matrix.

(i) Assuming that the quadric  $Q$  is nondegenerate, given any point  $c$  on  $Q$ , show that every line  $L$  through  $c$  either intersects  $Q$  in a single point  $p$  other than  $c$ , or intersects  $Q$  twice at  $c$ , in which case the line  $L$  is tangent to  $Q$  to  $c$ . More generally, show that the result holds provided that  $c$  is a *nonsingular point*, which means that  $F_x, F_y, F_z, F_t$  do not vanish simultaneously at  $c$ . In the rest of this problem, assume that  $c$  is nonsingular.

(ii) Any line  $L$  through  $c$  can be parameterized by picking a plane  $H$  not containing  $c$ , and by defining  $L$  in terms of the intersection of  $L$  and  $H$ . Then, a line through  $c$  is parameterized by a triple of homogeneous coordinates  $(u, v, w) \in \mathbb{R}^3$ . Compute the coordinates of the second intersection of  $L$  with  $Q$ . Conclude that a nondegenerate quadric can be parameterized rationally in terms of fractions involving polynomials of degree 2. What about the affine case?

(iii) Let  $b_{0,0,2}, b_{0,2,0}$  and  $b_{2,0,0}$  be three distinct points on the quadric, all distinct from  $c$ . The three planes containing  $c$  and any two of  $b_{0,0,2}, b_{0,2,0}, b_{2,0,0}$  intersect the quadric  $Q$  in three conics all passing through  $c$  and having coplanar tangents at  $c$ .

Show that the conic  $C_{1,0,1}$  passing through  $b_{0,0,2}$  and  $b_{2,0,0}$  is determined by a third point  $b_{1,0,1}$  which is the intersection of the plane  $(c, b_{0,0,2}, b_{2,0,0})$  and the tangent planes to  $Q$  at  $b_{0,0,2}$  and  $b_{2,0,0}$ , and by the condition that this conic is expressed as

$$C_{1,0,1}(t) = (1-t)^2 b_{0,0,2} + 2t(1-t) b_{1,0,1} + t^2 b_{2,0,0},$$

with  $C_{1,0,1}(\infty) = c$ . A similar result holds for the other two conics, yielding two points  $b_{1,1,0}$  and  $b_{0,1,1}$ . Show that the six control points

$$b_{0,0,2}, b_{0,2,0}, b_{2,0,0}, b_{1,0,1}, b_{1,1,0}, b_{0,1,1}$$

in  $\widehat{\mathbb{A}^3}$  define  $Q$  as a rational triangular surface of total degree 2.

(iv) Show that a rational triangular patch of degree 2 is a quadric iff the following three conditions hold:

- (1) The three boundary curves intersect in a common point  $c$ .
- (2) The boundary curves have coplanar tangents at  $c$ .
- (3) The point  $c$  on each boundary curve corresponds to  $t = \pm\infty$ .

**Problem 20** Write a computer program in order to visualize the evolution of a Steiner roman surface into a cross-cap, as we move around the real projective plane in  $\mathbb{A}^4$ .

**Problem 21** Show that the result of polarizing the following definition of a Klein bottle in  $\mathbb{A}^4$

$$\begin{aligned} x &= \frac{(u^4 - 6u^2 + 1)((a + r)v^4 + 2(a - 3r)v^2 + a + r)}{(u^2 + 1)^2(v^2 + 1)^2}, \\ y &= \frac{4u(1 - u^2)((a + r)v^4 + 2(a - 3r)v^2 + a + r)}{(u^2 + 1)^2(v^2 + 1)^2}, \\ z &= \frac{4rv(1 - u^4)(1 - v^2)}{(u^2 + 1)^2(v^2 + 1)^2}, \\ t &= \frac{8ruv(1 + u^2)(1 - v^2)}{(u^2 + 1)^2(v^2 + 1)^2}, \end{aligned}$$

yields the following control net when  $r = 1$  and  $a = 2$ .

```
kleinet4 = {{3, 0, 0, 0, 1}, {3, 0, 1/2, 0, 1}, {41/15, 0, 14/15, 0, 15/14},
{39/17, 0, 20/17, 0, 17/14}, {183/101, 0, 120/101, 0, 101/70},
{7/5, 0, 1, 0, 25/14}, {9/8, 0, 11/16, 0, 16/7}, {1, 0, 1/3, 0, 3},
{1, 0, 0, 0, 4}, {3, 3/2, 0, 0, 1}, {3, 3/2, 1/2, 1/7, 1},
{41/15, 61/45, 14/15, 4/15, 15/14}, {39/17, 19/17, 20/17, 28/85, 17/14},
{183/101, 88/101, 120/101, 32/101, 101/70}, {7/5, 52/75, 1, 6/25, 25/14},
{9/8, 5/8, 11/16, 1/8, 16/7}, {1, 2/3, 1/3, 0, 3},
{11/5, 14/5, 0, 0, 15/14}, {11/5, 14/5, 7/15, 4/15, 15/14},
{243/121, 305/121, 105/121, 60/121, 121/105},
{39/23, 95/46, 25/23, 14/23, 46/35},
{441/331, 528/331, 360/331, 192/331, 331/210},
{81/83, 104/83, 75/83, 36/83, 83/42}, {2/3, 10/9, 11/18, 2/9, 18/7},
{15/17, 60/17, 0, 0, 17/14}, {15/17, 60/17, 7/17, 32/85, 17/14},
{19/23, 291/92, 35/46, 16/23, 46/35},
{39/53, 273/106, 50/53, 89/106, 53/35},
{25/43, 84/43, 40/43, 100/129, 129/70}, {1/3, 16/11, 25/33, 6/11, 33/14},
{-57/101, 360/101, 0, 0, 101/70},
{-57/101, 360/101, 34/101, 48/101, 101/70},
{-151/331, 1052/331, 204/331, 288/331, 331/210},
{-37/129, 332/129, 98/129, 44/43, 129/70},
{-29/161, 304/161, 120/161, 144/161, 23/10}, {-9/5, 3, 0, 0, 25/14},
{-9/5, 3, 6/25, 14/25, 25/14}, {-127/83, 221/83, 36/83, 84/83, 83/42},
{-37/33, 71/33, 6/11, 38/33, 33/14}, {-21/8, 33/16, 0, 0, 16/7},
{-21/8, 33/16, 1/8, 5/8, 16/7}, {-20/9, 11/6, 2/9, 10/9, 18/7},
{-3, 1, 0, 0, 3}, {-3, 1, 0, 2/3, 3}, {-3, 0, 0, 0, 4}}
```

Write a computer program in order to visualize the evolution of a crossed-torus into a Klein bottle, as we move around the Klein bottle in  $\mathbb{A}^4$ .

**Problem 22** A surface of revolution  $S$  is obtained by rotating a plane curve  $C$  about a fixed line  $A$  called the *axis of rotation*, such that the plane containing  $C$  also contains  $A$ . The curve  $C$  is called the *generating curve of  $S$* . The circles obtained by intersecting  $S$  with planes orthogonal to the axis  $A$  are the *parallels* of  $S$ , and the various positions of  $C$  obtained by intersecting  $S$  with a plane containing the axis  $A$  are called the *meridians* of  $S$ . Taking the axis  $A$  to be the  $z$ -axis, a surface of revolution  $S$  is defined by

$$\begin{aligned}x &= R(v) \cos u, \\y &= R(v) \sin u, \\z &= H(v).\end{aligned}$$

A generating curve of  $S$  is the curve (in the  $(x, z)$  plane) defined by

$$\begin{aligned}x &= R(v), \\z &= H(v),\end{aligned}$$

(i) Assume that  $C$  is a rational curve of degree  $m$  defined over  $[0, 1]$  by the control points

$$((b_0, v_0), \dots, (b_m, v_m)),$$

where  $v_i \neq 0$ , and let  $b_i = (x_i, 0, z_i)$ . If  $S$  is defined by

$$\begin{aligned}x &= R(v) \frac{1 - u^2}{1 + u^2}, \\y &= R(v) \frac{2u}{1 + u^2}, \\z &= H(v),\end{aligned}$$

show that as a rational rectangular surface,  $S$  is defined over  $[0, 1] \times [0, 1]$  by a rectangular control net  $((d_{i,j}, w_{i,j}))_{0 \leq i \leq 2, 0 \leq j \leq m}$ , where

$$\begin{aligned}d_{0,j} &= (x_j, 0, z_j), \\d_{1,j} &= (x_j, x_j, z_j), \\d_{2,j} &= (0, x_j, z_j), \\w_{0,j} &= v_j, \\w_{1,j} &= v_j, \\w_{2,j} &= 2v_j.\end{aligned}$$

Does the above result hold when there are control vectors (when  $v_j = 0$  for some  $j$ )? Compute a control net for  $S$  over  $[-1, 1] \times [0, 1]$ .

(ii) Implement a method for drawing a surface of revolution whose generating curve is a rational curve defined by control points. Try it on a number of surfaces, including a sphere, an ellipsoid, and a torus.

**Problem 23**

The purpose of this problem is to study some surfaces known as the *cyclides of Dupin*.

Given any three distinct nonconcentric spheres  $S, S', S''$ , consider the set of all spheres tangent simultaneously to  $S, S'$  and  $S''$  and exterior to  $S, S'$  and  $S''$ . By definition, the envelope of this family of spheres is a *cyclide of Dupin*. The purpose of this problem is to characterize the cyclides of Dupin in terms of the torus and the circular cone, and the circular cylinder (Dupin (1822) [54]). This problem is adapted from *Traité d'Analyse*, Vol. I (1942), by Emile Picard [138].

Consider the circles  $C', C'', C'''$ , sections of  $S, S', S''$  by the plane containing the centers of  $S, S', S''$  (if these centers are collinear, consider the section by any plane containing the centers).

(a) If the centers of  $C', C'', C'''$  are not collinear, let  $\omega$  be the point that has equal power w.r.t.  $C, C'$ , and  $C''$  (see problem 1(c)). If the power of  $\omega$  is positive, show that by a suitable inversion,  $S, S'$  and  $S''$  are transformed into spheres whose centers are on a line  $\Delta$  (image of the circle orthogonal to  $C, C'$  and  $C''$ ).

Since inversion preserves tangency, show that the image of the cyclide of Dupin is a torus of revolution with axis  $\Delta$ .

(b) If the power of  $\omega$  is zero or negative, show that  $S, S'$  and  $S''$  intersect in two points (possibly a double point) on the line through  $\omega$  perpendicular to the plane of  $C, C'$  and  $C''$ . Prove that by a suitable inversion,  $S, S'$  and  $S''$  are transformed into planes.

Since inversion preserves tangency, show that the image of the cyclide of Dupin is a circular cone or a circular cylinder.

*Remark:* It can be shown that inversions preserve curvature lines. The previous argument can be used to show that the lines of curvature of the Dupin cyclides are circles.

In summary, you proved that a Dupin cyclide is the image by inversion of a torus of revolution, or of a circular cone, or of a circular cylinder. This characterization is apparently due to J. Liouville (1847) [114]. Another characterization is due to J.C. Maxwell (1867) [123].

(c) *Extra credit:* Prove that the locus of the center of the spheres tangent to three distinct spheres  $S, S'$ , and  $S''$  is a conic.

#### Problem 24

(a) Given a parameterization of the torus, compute a parameterization of a cyclide of Dupin. Do the same thing for a circular cone and a circular cylinder.

(b) Compute rectangular control nets from the above parameterizations.

(c) Implement a method for drawing Dupin cyclides and experiment with it, depending on the choice of the center of inversion. You may want to implement the De Casteljau algorithm for rectangular surfaces and use my method to draw a closed rectangular rational surface in four patches.