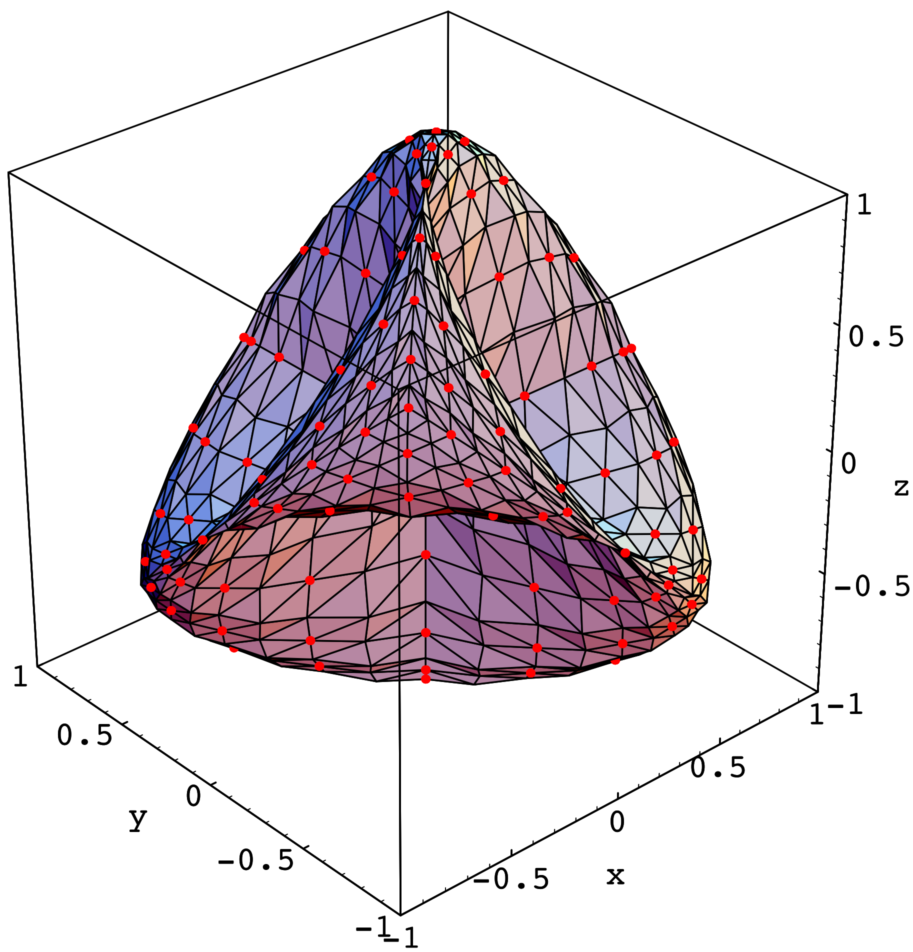
Chapter 23

Rational Surfaces



23.1 Rational Surfaces and Multiprojective Maps

In this chapter, rational surfaces are investigated. We begin by exploring the possibility of defining rational surfaces in terms of polar forms. As in the case of rational curves, polar forms are actually multilinear, and rational surfaces are defined in terms of multiprojective maps. A rational surface is defined by fractions of polynomials $F_i(u,v)$ in two variables u,v. If we first homogenize the polynomials $F_i(u,v)$ with respect to the total degree m (replacing u by u/z and v by v/z) and then polarize with respect to (u,v,z) as a whole, we can view a rational surface as a map

$$F\colon \widetilde{\mathcal{P}} \to \widetilde{\mathcal{E}}$$

(where $\mathcal{P} = \mathbb{A}^2$ is the affine plane) induced by some symmetric multilinear map

$$f\colon (\widehat{\mathcal{P}})^m \to \widehat{\mathcal{E}}$$

such that

$$F([u,v,z]) = \mathbf{P}(f)(\underbrace{(u,v,z),\ldots,(u,v,z)}_{m}),$$

for all homogeneous coordinates $(u,v,z) \in \mathbb{R}^3$. We call such surfaces *rational total degree surfaces*, or *triangular rational surfaces*. On the other hand, if we first homogenize the polynomials $F_i(u,v)$ with respect to u and the maximum degree p in u (replacing u by u/t_1) and second with respect to v and the maximum degree q in v (replacing v by v/t_2), and then polarize separately w.r.t. (u,t_1) and (v,t_2), we can view a rational surface as a map

$$F\colon \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \to \widetilde{\mathcal{E}}$$

induced by some multilinear map

$$f\colon (\widehat{\mathbb{A}})^p \times (\widehat{\mathbb{A}})^q \to \widehat{\mathcal{E}}$$

which is symmetric in its first p arguments and in its last q arguments, and such that

$$F([u,t_1],[v,t_2]) = \mathbf{P}(f)(\underbrace{(u,t_1),\ldots,(u,t_1)}_{p},\underbrace{(v,t_2),\ldots,(v,t_2)}_{q}),$$

for all homogeneous coordinates $(u,t_1),(v,t_2) \in \mathbb{R}^2$. We call such surfaces *rational surfaces of bidegree* $\langle p,q\rangle$, or *rectangular rational surfaces*. However, unlike the affine case where $\mathbb{A}\times\mathbb{A}$ is isomorphic to the affine plane $\mathcal{P} = \mathbb{A}^2$, the Cartesian product $\widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} = \mathbb{RP}^1 \times \mathbb{RP}^1$ is *not* a projective space, and thus, it is not isomorphic to the projective plane $\widetilde{\mathcal{P}} = \mathbb{RP}^2$. For instance, topologically, the points at infinity in $\mathbb{RP}^1 \times \mathbb{RP}^1$ correspond to two intersecting circles, but the points at infinity in $\mathbb{RP}^2$ correspond to a single circle. As a consequence, triangular rational surfaces and rectangular rational surfaces are two distinct classes. This difference can be observed in terms of base points, i.e., when

$$f(\underbrace{(u,v,z),\ldots,(u,v,z)}_{m}) = 0$$

or

$$f(\underbrace{(u,t_1),\ldots,(u,t_1)}_{p},\underbrace{(v,t_2),\ldots,(v,t_2)}_{q}) = 0.$$

For example, a torus viewed as a triangular rational surface may have base points, although it does not as a rectangular rational surface. Similarly, a sphere may have base points viewed as a rectangular rational surface, although it does not as a triangular rational surface.

As in the case of rational curves, we define rectangular and triangular rational surfaces in BR-form. This allows us to handle rational surfaces in terms of control points in the hat space $\widehat{\mathcal{E}}$ obtained from $\mathcal{E}$. We give formulae for expressing triangular and rectangular rational surfaces in terms of Bernstein polynomials. We also show how the two versions of the de Casteljau algorithm for polynomial surfaces (triangular version and rectangular version) can be easily adapted to rational surfaces, as well as the subdivision methods.

Our treatment of rational surfaces will parallel rather closely the treatment of rational curves of chapter 22. We begin by defining rational surfaces in a traditional manner, and then, we show how this definition can be advantageously recast in terms of symmetric multiprojective maps. Keep in mind that rational surfaces really live in the projective completion $\widetilde{\mathcal{E}}$ of the affine space $\mathcal{E}$. This is the reason why homogeneous polynomials are involved.

We assume that some fixed affine frame $(O, (\mathbf{i_1}, \mathbf{i_2}))$ for the affine plane $\mathcal{P}$ is chosen, typically, the canonical affine frame for $\mathcal{P}$, where $O = (0, 0)$, $\mathbf{i_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $\mathbf{i_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, unless specified otherwise. We will assume that the homogenization $\widehat{\mathcal{P}}$ of the affine plane $\mathcal{P}$ is identified with the direct sum $\mathbb{R}^2 \oplus \mathbb{R}O$. Then, every element of $\widehat{\mathcal{P}}$ is of the form $(u, v, z) \in \mathbb{R}^3$. Let $\mathcal{E}$ be some affine space of finite dimension $n \geq 3$, and let $(\Omega_1, (e_1, \ldots, e_n))$ be an affine frame for $\mathcal{E}$.

A rational surface of degree m in an affine space $\mathcal{E}$ of dimension n is specified by n fractions, say

$$x_1(u, v) = \frac{F_1(u, v)}{F_{n+1}(u, v)}, \quad x_2(u, v) = \frac{F_2(u, v)}{F_{n+1}(u, v)}, \quad \ldots, \quad x_n(u, v) = \frac{F_n(u, v)}{F_{n+1}(u, v)},$$

where $F_1(X, Y), \ldots, F_{n+1}(X, Y)$ are polynomials of degree at most m. In order to deal with the case where the denominator $F_{n+1}(X, Y)$ is null, we view the rational surface as the projection of the polynomial surface defined by the polynomials $F_1(X, Y), \ldots, F_{n+1}(X, Y)$. To make this rigorous, we can view the rational surface either as a map $F\colon \widetilde{\mathcal{P}} \to \widetilde{\mathcal{E}}$, or as a map $F\colon \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \to \widetilde{\mathcal{E}}$. In the first case, we need to homogenize the polynomials $F_1(X, Y), \ldots, F_{n+1}(X, Y)$ as polynomials of the same total degree (replacing X by X/Z and Y by Y/Z) so that after polarizing w.r.t. (X, Y, Z), we get a (symmetric) multilinear map which induces a (symmetric) multiprojective map. In the second case, we homogenize first in X with respect to the maximum degree p in X (replacing X by X/T_1) and second in Y with respect to the maximum degree q in Y (replacing Y by Y/T_2), so that after polarizing separately w.r.t. (X, T_1) and (Y, T_2), we get a multilinear map symmetric in its first p arguments and in its last q arguments which also induces a multiprojective map. Following the first approach leads to the following definition.

Definition 23.1.1 A *rational surface of total degree* m is a function $F\colon \widehat{\mathcal{P}} \to \widehat{\mathcal{E}}$, such that, for all $(u, v, z) \in \mathbb{R}^3$, we have

$$F(u, v, z) = F_1(u, v, z)e_1 \mathbin{\widehat{+}} \cdots \mathbin{\widehat{+}} F_n(u, v, z)e_n \mathbin{\widehat{+}} F_{n+1}(u, v, z)\langle \Omega_1, 1\rangle,$$

where $F_1(X, Y, Z), \ldots, F_{n+1}(X, Y, Z)$ are homogeneous polynomials in $\mathbb{R}[X, Y, Z]$, each of total degree m.

It is important to require that all of the polynomials $F_1(X, Y, Z), \ldots, F_{n+1}(X, Y, Z)$ are homogeneous and of the *same total degree* m. Otherwise, we would not be able to polarize these polynomials and obtain a symmetric *multilinear map* (as opposed to a multiaffine map).

For example, the following homogeneous polynomials define a (unit) sphere:

$$\begin{aligned}
F_1(X, Y, Z) &= 2XZ \\
F_2(X, Y, Z) &= 2YZ \\
F_3(X, Y, Z) &= X^2 + Y^2 - Z^2 \\
F_4(X, Y, Z) &= X^2 + Y^2 + Z^2.
\end{aligned}$$

Note that we recover a standard parameterization of the sphere, by setting $Z = 1$. But in general, it is also possible to get points at infinity in the projective space $\widetilde{\mathcal{E}}$, or points corresponding to points at infinity in $\widehat{\mathcal{P}}$. For example, for $(X, Y, Z) = (a, b, 0)$, we get the point of homogeneous coordinates $(0, 0, 1, 1)$, the North pole. Also, note that the North pole is obtained for infinitely many parameter values $(a, b, 0)$.

Following the second approach for homogenizing and polarizing leads to the following definition. First, let us say that a multilinear map $f\colon (\widehat{\mathbb{A}})^p \times (\widehat{\mathbb{A}})^q \to \widehat{\mathcal{E}}$ is *$\langle p, q\rangle$-symmetric* if it symmetric in its first p arguments and in its last q arguments.

Definition 23.1.2 A *rational surface of bidegree* $\langle p, q\rangle$ is a function $F\colon \widehat{\mathbb{A}} \times \widehat{\mathbb{A}} \to \widehat{\mathcal{E}}$, such that, for all $(u, t_1), (v, t_2) \in \mathbb{R}^2$, we have

$$F((u, t_1), (v, t_2)) = F_1(u, t_1, v, t_2)e_1 \mathbin{\widehat{+}} \cdots \mathbin{\widehat{+}} F_n(u, t_1, v, t_2)e_n \mathbin{\widehat{+}} F_{n+1}(u, t_1, v, t_2)\langle \Omega_1, 1\rangle,$$

where $F_1(U, T_1, V, T_2), \ldots, F_{n+1}(U, T_1, V, T_2)$ are polynomials in $\mathbb{R}[U, T_1, V, T_2]$, each homogeneous of degree p in U, T_1, and homogeneous of degree q in V, T_2 (in particular, each $F_i(U, T_1, V, T_2)$ is homogeneous and of total degree $p + q$).

For example, the following polynomials define a (unit) sphere:

$$\begin{aligned}
F_1(U, T_1, V, T_2) &= 2UT_1T_2^2\\
F_2(U, T_1, V, T_2) &= 2T_1^2VT_2\\
F_3(U, T_1, V, T_2) &= U^2T_2^2 + V^2T_1^2 - T_1^2T_2^2\\
F_4(U, T_1, V, T_2) &= U^2T_2^2 + V^2T_1^2 + T_1^2T_2^2.
\end{aligned}$$

Observe that there is a base point for $T_1 = T_2 = 0$.

Using a Lemma in Gallier [70], (see Lemma 27.2.1), for each homogeneous polynomial $F_i(X, Y, Z)$ of total degree m, there is a unique symmetric multilinear map $f_i\colon (\mathbb{R}^3)^m \to \mathbb{R}$, such that

$$F_i(u, v, z) = f_i(\underbrace{(u, v, z), \ldots, (u, v, z)}_{m}),$$

for all $(u, v, z) \in \mathbb{R}^3$, and together, these $n + 1$ maps define a symmetric multilinear map

$$f\colon (\widehat{\mathcal{P}})^m \to \widehat{\mathcal{E}},$$

such that

$$F(u, v, z) = f(\underbrace{(u, v, z), \ldots, (u, v, z)}_{m}),$$

for all $(u, v, z) \in \mathbb{R}^3$. But then, we get a symmetric multiprojective map

$$\mathbf{P}(f)\colon (\widetilde{\mathcal{P}})^m \to \widetilde{\mathcal{E}},$$

and we define the projective rational surface

$$\widetilde{F}\colon \widetilde{\mathcal{P}} \to \widetilde{\mathcal{E}},$$

such that

$$\widetilde{F}([u, v, z]) = \mathbf{P}(f)(\underbrace{[u, v, z], \ldots, [u, v, z]}_{m})$$

for all $(u, v, z) \in \mathbb{R}^3$, where we view (u, v, z) as homogeneous coordinates of a point in $\widetilde{\mathcal{P}}$, that is, where $(u, v, z) \neq (0, 0, 0)$.

Definition 23.1.3 Given an affine space $\mathcal{E}$ of dimension ≥ 3, a *projective rational surface* F *of degree* m or *triangular rational surface of degree* m is a (partial) projective map $F\colon \widetilde{\mathcal{P}} \to \widetilde{\mathcal{E}}$, such that there is some symmetric multilinear map $f\colon (\widehat{\mathcal{P}})^m \to \widehat{\mathcal{E}}$, with

$$F([u, v, z]) = \mathbf{P}(f)(\underbrace{[u, v, z], \ldots, [u, v, z]}_{m}),$$

for all $[u, v, z] \in \widetilde{\mathcal{P}}$ (i.e., all homogeneous coordinates $(u, v, z) \in \mathbb{R}^3$). The symmetric multilinear map f is also called a *polar form* of F. The *trace of* F is the subset $F(\widetilde{\mathcal{P}})$ of the projective completion $\widetilde{\mathcal{E}}$ of the affine space $\mathcal{E}$.

Since $\widetilde{\mathcal{P}} = \mathbb{RP}^2$, a triangular rational surface F of degree m is a rational map $F\colon \mathbb{RP}^2 \to \widetilde{\mathcal{E}}$ defined as the diagonal of a symmetric multiprojective map $\mathbf{P}(f)\colon (\mathbb{RP}^2)^m \to \widetilde{\mathcal{E}}$.

From the discussion just before definition 23.1.3, we note that every rational surface F as in definition 23.1.1 defines a triangular rational surface $\widetilde{F}$ of degree m.

The converse is also true when $\mathcal{E}$ is of finite dimension. Let G be a projective rational surface defined by the symmetric multilinear map $g\colon (\widehat{\mathcal{P}})^m \to \widehat{\mathcal{E}}$. Given an affine frame $(\Omega_1, (e_1, \ldots, e_n))$ for $\mathcal{E}$, and the corresponding basis $(e_1, \ldots, e_n, \langle \Omega_1, 1\rangle)$ for $\widehat{\mathcal{E}}$, a Lemma in Gallier [70] (see Lemma 27.1.5) shows that a symmetric multilinear map $g\colon (\widehat{\mathcal{P}})^m \to \widehat{\mathcal{E}}$ defines a homogeneous polynomial map F of degree m in three variables, with coefficients in $\widehat{\mathcal{E}}$. Expressing these coefficients over the basis $(e_1, \ldots, e_n, \langle \Omega_1, 1\rangle)$, we see that the polynomial map F is defined by some homogeneous polynomials $F_1(X, Y, Z), \ldots, F_{n+1}(X, Y, Z)$ in $\mathbb{R}[X, Y, Z]$, each of total degree m, such that, for all $(u, v, z) \in \mathbb{R}^3$, we have

$$F(u, v, z) = F_1(u, v, z)e_1 \mathbin{\widehat{+}} \cdots \mathbin{\widehat{+}} F_n(u, v, z)e_n \mathbin{\widehat{+}} F_{n+1}(u, v, z)\langle \Omega_1, 1\rangle.$$

Thus F is a rational surface, and clearly, $\widetilde{F} = G$. Therefore, when $\mathcal{E}$ is of finite dimension, definition 23.1.3 and definition 23.1.1 define the same class of rational surfaces. Definition 23.1.3 will be preferred, since it leads to a treatment of rational surfaces in terms of control points rather similar to the treatment of Bézier surfaces (it also works for any dimension, even infinite).

Remarks: (1) Contrary to the case of rational curves, it is not always possible to extend the definition of a rational surface by continuity when (u, v, z) is a common zero of the polynomials $F_1(X, Y, Z), \ldots, F_{n+1}(X, Y, Z)$. Consider the rational surface defined such that

$$\begin{aligned}
F_1(X, Y, Z) &= YZ\\
F_2(X, Y, Z) &= X^2 + 2XZ\\
F_3(X, Y, Z) &= XZ\\
F_4(X, Y, Z) &= XZ + YZ.
\end{aligned}$$

Since $(0, 0, 1)$ is a common zero of F_1, F_2, F_3, F_4, the surface F is undefined at $(0, 0, 1)$. In non-homogeneous coordinates, F is defined by

$$\begin{aligned}
x(u, v) &= \frac{v}{u + v}\\
y(u, v) &= \frac{u^2 + 2u}{u + v}\\
z(u, v) &= \frac{u}{u + v}.
\end{aligned}$$

If $v = \lambda u$, for $\lambda \neq 0$, when the point (u, v) approaches the origin in $\mathcal{P}$, the point $(x(u, v), y(u, v), z(u, v))$ converges to the point

$$\left(\frac{\lambda}{1 + \lambda}, \frac{2}{1 + \lambda}, \frac{1}{1 + \lambda}\right),$$

which depends on λ. Therefore, it is impossible to define $F(0,0,1)$ by continuity. However, methods of algebraic geometry can be used ("blowing-up").

(2) The set of points for which the curve F is undefined is the zero locus of the homogeneous polynomials $F_1(X,Y,Z),\ldots,F_{n+1}(X,Y,Z)$, and thus, it is an algebraic curve in the (real) projective plane $\widetilde{\mathcal{P}}$.

Given a rectangular rational surface of bidegree $\langle p,q\rangle$, using a Lemma in Gallier [70] (see Lemma 27.2.1), for each polynomial $F_i(U,T_1,V,T_2)$ homogeneous of degree p in U,T_1 and homogeneous of degree q in V,T_2, there is a unique $\langle p,q\rangle$-symmetric multilinear map $f_i\colon (\mathbb{R}^2)^p \times (\mathbb{R}^2)^q \to \mathbb{R}$, such that

$$F_i((u,t_1),(v,t_2)) = f_i(\underbrace{(u,t_1),\ldots,(u,t_1)}_{p},\underbrace{(v,t_2),\ldots,(v,t_2)}_{q}),$$

for all $(u,t_1),(v,t_2)\in\mathbb{R}^2$, and together, these $n+1$ maps define a $\langle p,q\rangle$-symmetric multilinear map

$$f\colon (\widehat{\mathbb{A}})^p \times (\widehat{\mathbb{A}})^q \to \widehat{\mathcal{E}},$$

such that

$$F((u,t_1),(v,t_2)) = f(\underbrace{(u,t_1),\ldots,(u,t_1)}_{p},\underbrace{(v,t_2),\ldots,(v,t_2)}_{q}),$$

for all $(u,t_1),(v,t_2)\in\mathbb{R}^2$. But then, we get a multiprojective map

$$F\colon \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \to \widetilde{\mathcal{E}}$$

induced by some $\langle p,q\rangle$-symmetric multilinear map

$$f\colon (\widehat{\mathbb{A}})^p \times (\widehat{\mathbb{A}})^q \to \widehat{\mathcal{E}},$$

and such that

$$F([u,t_1],[v,t_2]) = \mathbf{P}(f)(\underbrace{(u,t_1),\ldots,(u,t_1)}_{p},\underbrace{(v,t_2),\ldots,(v,t_2)}_{q}),$$

for $(u,t_1),(v,t_2)\in\mathbb{R}^2$, where we view $(u,t_1),(v,t_2)$ as homogeneous coordinates of a point in $\widetilde{\mathbb{A}}$, that is, where $(u,t_1)\neq(0,0)$ and $(v,t_2)\neq(0,0)$.

Definition 23.1.4 Given an affine space $\mathcal{E}$ of dimension ≥ 3, a *projective rational surface F of bidegree $\langle p,q\rangle$* or *rectangular rational surface of bidegree $\langle p,q\rangle$* is a (partial) projective map $F\colon \widetilde{\mathbb{A}}\times\widetilde{\mathbb{A}}\to\widetilde{\mathcal{E}}$, such that there is some $\langle p,q\rangle$-symmetric multilinear map

$$f\colon (\widehat{\mathbb{A}})^p \times (\widehat{\mathbb{A}})^q \to \widehat{\mathcal{E}}$$

with

$$F([u,t_1],[v,t_2]) = \mathbf{P}(f)(\underbrace{(u,t_1),\ldots,(u,t_1)}_{p},\underbrace{(v,t_2),\ldots,(v,t_2)}_{q}),$$

for all $[u,t_1],[v,t_2]\in\widetilde{\mathbb{A}}$ (i.e., all homogeneous coordinates $(u,t_1),(v,t_2)\in\mathbb{R}^2$). The multilinear map f is also called a *polar form* of F. The *trace of* F is the subset $F(\widetilde{\mathbb{A}}\times\widetilde{\mathbb{A}})$ of the projective completion $\widetilde{\mathcal{E}}$ of the affine space $\mathcal{E}$.

Since $\widetilde{\mathbb{A}}=\mathbb{RP}^1$, a rectangular rational surface F of degree $\langle p,q\rangle$ is a birational map $F\colon\mathbb{RP}^1\times\mathbb{RP}^1\to\widetilde{\mathcal{E}}$ defined as the diagonal of a $\langle p,q\rangle$-symmetric multiprojective map $\mathbf{P}(f)\colon(\mathbb{RP}^1)^p\times(\mathbb{RP}^1)^q\to\widetilde{\mathcal{E}}$.

It is a simple exercise to show that every rational surface F as in definition 23.1.2 defines a rectangular rational surface $\widetilde{F}$ of bidegree $\langle p,q\rangle$, and that the converse is true when $\mathcal{E}$ has finite dimension.

If $F: \widetilde{\mathcal{P}} \to \widetilde{\mathcal{E}}$ is a triangular rational surface defined by some symmetric multilinear map $f: (\widehat{\mathcal{P}})^m \to \widehat{\mathcal{E}}$, with

$$F([u, v, z]) = \mathbf{P}(f)(\underbrace{[u, v, z], \ldots, [u, v, z]}_{m}),$$

for all $[u, v, z] \in \widetilde{\mathcal{P}}$, by lemma 21.3.2, the restriction $g: \mathcal{P}^m \to \widehat{\mathcal{E}}$ of $f: (\widehat{\mathcal{P}})^m \to \widehat{\mathcal{E}}$ to $\mathcal{P}^m$, is a multiaffine map such that $\mathbf{P}(f) = \Pi\Omega \circ \widetilde{g}$. The multiaffine map $g: \mathcal{P}^m \to \widehat{\mathcal{E}}$ defines a polynomial surface $G: \mathcal{P} \to \widehat{\mathcal{E}}$, and the multilinear map $\widehat{g}: (\widehat{\mathcal{P}})^m \to \widehat{\widehat{\mathcal{E}}}$ induces a triangular rational surface $\widetilde{G}: \widetilde{\mathcal{P}} \to \widetilde{\widehat{\mathcal{E}}}$, and $\mathbf{P}(f) = \Pi\Omega \circ \widetilde{g}$ shows that the triangular rational surface F is the projection under $\Pi\Omega$ of the triangular rational surface $\widetilde{G}$, which itself, can be considered as the projective completion of the polynomial surface G. This suggests the following definition.

Definition 23.1.5 For a polynomial surface $F: \mathcal{P} \to \mathcal{E}$ defined by the polar form $f: \mathcal{P}^m \to \mathcal{E}$, the symmetric multilinear map $\widehat{f}: (\widehat{\mathcal{P}})^m \to \widehat{\mathcal{E}}$ induces a triangular rational surface $\widetilde{F}: \widetilde{\mathcal{P}} \to \widetilde{\mathcal{E}}$, called the *projective completion* of the polynomial surface F.

Clearly the restriction of the surface $\widetilde{F}$ to $\mathcal{P}$ agrees with the polynomial surface F. The projective completion $\widetilde{F}$ of F can still be considered as a polynomial surface, except that it is also defined for the points at infinity in $\widetilde{\mathcal{P}}$.

As in the triangular case, if $F: \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \to \widetilde{\mathcal{E}}$ is a rectangular rational surface defined by some $\langle p, q\rangle$-symmetric multilinear map $f: (\widehat{\mathbb{A}})^p \times (\widehat{\mathbb{A}})^q \to \widehat{\mathcal{E}}$, by lemma 21.3.2, the restriction $g: \mathbb{A}^p \times \mathbb{A}^q \to \widehat{\mathcal{E}}$ of f to $\mathbb{A}^p \times \mathbb{A}^q$ is a $\langle p, q\rangle$-symmetric multiaffine map such that $\mathbf{P}(f) = \Pi\Omega \circ \widetilde{g}$. The $\langle p, q\rangle$-symmetric multiaffine map $g: \mathbb{A}^p \times \mathbb{A}^q \to \widehat{\mathcal{E}}$ defines a bipolynomial surface $G: \mathbb{A} \times \mathbb{A} \to \widehat{\mathcal{E}}$, and the $\langle p, q\rangle$-symmetric multilinear map $\widehat{g}: (\widehat{\mathbb{A}})^p \times (\widehat{\mathbb{A}})^q \to \widehat{\mathcal{E}}$ induces a rectangular rational surface $\widetilde{G}: \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \to \widetilde{\widehat{\mathcal{E}}}$, and $\mathbf{P}(f) = \Pi\Omega \circ \widetilde{g}$ shows that the rectangular rational surface F is the projection under $\Pi\Omega$ of the rectangular rational surface $\widetilde{G}$, which itself, can be considered as the projective completion of the bipolynomial surface G. This suggests the following definition.

Definition 23.1.6 For a bipolynomial surface $F: \mathbb{A} \times \mathbb{A} \to \mathcal{E}$ defined by the polar form $f: \mathbb{A}^p \times \mathbb{A}^q \to \mathcal{E}$, the $\langle p, q\rangle$-symmetric multilinear map $\widehat{f}: (\widehat{\mathbb{A}})^p \times (\widehat{\mathbb{A}})^q \to \widehat{\mathcal{E}}$ induces a rectangular rational surface $\widetilde{F}: \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \to \widetilde{\mathcal{E}}$, called the *projective completion* of the bipolynomial surface F.

Clearly the restriction of the surface $\widetilde{F}$ to $\mathbb{A} \times \mathbb{A}$ agrees with the bipolynomial surface F. The projective completion $\widetilde{F}$ of F can still be considered as a polynomial surface, except that it is also defined for the points at infinity in $\widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}}$.

Now, given a triangular rational surface F with polar form $f: (\widehat{\mathcal{P}})^m \to \widehat{\mathcal{E}}$, the discussion just before definition 23.1.5 shows that F is completely determined by the polynomial surface G of polar form $g: \mathcal{P}^m \to \widehat{\mathcal{E}}$, the restriction of $f: (\widehat{\mathcal{P}})^m \to \widehat{\mathcal{E}}$ to $\mathcal{P}^m$, and the projection $\Pi\Omega$. Similarly, given a rectangular rational surface F with polar form $f: (\widehat{\mathbb{A}})^p \times (\widehat{\mathbb{A}})^q \to \widehat{\mathcal{E}}$, the discussion just before definition 23.1.6 shows that F is completely determined by the bipolynomial surface G of polar form $g: \mathbb{A}^p \times \mathbb{A}^q \to \widehat{\mathcal{E}}$, the restriction of $f: (\widehat{\mathbb{A}})^p \times (\widehat{\mathbb{A}})^q \to \widehat{\mathcal{E}}$ to $\mathbb{A}^p \times \mathbb{A}^q$, and the projection $\Pi\Omega$. This leads to the definition of triangular and rectangular BR-surfaces.

23.2 Triangular and Rectangular BR-Surfaces

We begin with the definition of triangular BR-surfaces.

Definition 23.2.1 Given any affine space $\mathcal{E}$ of dimension $n \geq 3$, a *triangular rational surface in Bézier form of degree m, or triangular BR-surface*, is a map $F: \widetilde{\mathcal{P}} \to \widetilde{\mathcal{E}}$ such that there is a polynomial surface $G: \mathcal{P} \to \widehat{\mathcal{E}}$

of degree m defined by a triangular control net $\mathcal{N} = (\theta_{i,\,j,\,k})_{(i,j,k)\in\Delta_m}$ in $\widehat{\mathcal{E}}$ and such that if $g\colon \mathcal{P}^m \to \widehat{\mathcal{E}}$ is the (unique) symmetric polar form of G, then

$$F(a) = \begin{cases} \Pi(G(a)) & \text{if } a \in \mathcal{P}, \text{ not a point at infinity;} \\ \Pi\Omega([\widehat{g}((u,v,0),\ldots,(u,v,0))]_{\approx}) & \text{when } a = (u,v,0) \in \widetilde{\mathcal{P}} - \mathcal{P}. \end{cases}$$

The *trace of* F is the subset $F(\widetilde{\mathcal{P}})$ of the projective completion $\widetilde{\mathcal{E}}$ of the affine space $\mathcal{E}$.

Note that in practice, we can assume that F is only defined on $\mathcal{P}$.

The following lemma shows that when $\mathcal{E}$ is of finite dimension, the class of triangular rational surfaces and the class of triangular BR-surfaces are identical.

Lemma 23.2.2 *Let $\mathcal{E}$ be any affine space of finite dimension. Every triangular rational surface $F\colon \widetilde{\mathcal{P}} \to \widetilde{\mathcal{E}}$ with polar form $f\colon (\widehat{\mathcal{P}})^m \to \widehat{\mathcal{E}}$ is the image under $\Pi\Omega$ of the projective completion $\widetilde{G}\colon \widetilde{\mathcal{P}} \to \widetilde{\widehat{\mathcal{E}}}$ of the polynomial surface $G\colon \mathcal{P} \to \widehat{\mathcal{E}}$ associated with the restriction $g\colon \mathcal{P}^m \to \widehat{\mathcal{E}}$ of f to $\mathcal{P}^m$. Conversely, given a polynomial surface $G\colon \mathcal{P} \to \widehat{\mathcal{E}}$ with polar form $g\colon \mathcal{P}^m \to \widehat{\mathcal{E}}$, the image under $\Pi\Omega$ of the projective completion $\widetilde{G}\colon \widetilde{\mathcal{P}} \to \widetilde{\widehat{\mathcal{E}}}$ of G is a triangular rational surface. As a consequence, the class of triangular rational surfaces and the class of triangular BR-surfaces are identical, when $\mathcal{E}$ is of finite dimension.*

Proof. The proof follows from lemma 21.3.2 and lemma 21.3.2. Details are left as an exercise. □

Next, we define rectangular BR-surfaces.

Definition 23.2.3 Given any affine space $\mathcal{E}$ of dimension $n \geq 3$, a *rectangular rational surface in Bézier form of bidegree $\langle p, q\rangle$, or rectangular BR-surface*, is a map $F\colon \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \to \widetilde{\mathcal{E}}$ such that there is a bipolynomial surface $G\colon \mathbb{A} \times \mathbb{A} \to \widehat{\mathcal{E}}$ of bidegree $\langle p, q\rangle$ defined by a rectangular control net $\mathcal{N} = (\theta_{i,\,j})_{0\leq i\leq p,\,0\leq j\leq q}$ in $\widehat{\mathcal{E}}$ and such that if $g\colon \mathbb{A}^p \times \mathbb{A}^q \to \widehat{\mathcal{E}}$ is the (unique) $\langle p, q\rangle$-symmetric polar form of G, then

$$F(a,b) = \begin{cases} \Pi(G(a,b)) & \text{if } a, b \in \mathbb{A},\ a, b \neq \infty; \\ \Pi\Omega([\widehat{g}((u,t_1),\ldots,(u,t_1),(v,t_2),\ldots,(v,t_2))]_{\approx}) & \text{if } a = (u,t_1), b = (v,t_2), \\ & \text{and } t_1 = 0 \text{ or } t_2 = 0. \end{cases}$$

The *trace of* F is the subset $F(\widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}})$ of the projective completion $\widetilde{\mathcal{E}}$ of the affine space $\mathcal{E}$.

Again, in practice, we can assume that F is only defined on $\mathbb{A} \times \mathbb{A}$.

The following lemma shows that when $\mathcal{E}$ is of finite dimension, the class of rectangular rational surfaces and the class of rectangular BR-surfaces are identical.

Lemma 23.2.4 *Let $\mathcal{E}$ be any affine space of finite dimension. Every rectangular rational surface $F\colon \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \to \widetilde{\mathcal{E}}$ with polar form $f\colon (\widehat{\mathbb{A}})^p \times (\widehat{\mathbb{A}})^q \to \widehat{\mathcal{E}}$ is the image under $\Pi\Omega$ of the projective completion $\widetilde{G}\colon \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \to \widetilde{\widehat{\mathcal{E}}}$ of the bipolynomial surface $G\colon \mathbb{A} \times \mathbb{A} \to \widehat{\mathcal{E}}$ associated with the restriction $g\colon \mathbb{A}^p \times \mathbb{A}^q \to \widehat{\mathcal{E}}$ of f to $\mathbb{A}^p \times \mathbb{A}^q$. Conversely, given a bipolynomial surface $G\colon \mathbb{A} \times \mathbb{A} \to \widehat{\mathcal{E}}$ with polar form $g\colon \mathbb{A}^p \times \mathbb{A}^q \to \widehat{\mathcal{E}}$, the image under $\Pi\Omega$ of the projective completion $\widetilde{G}\colon \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \to \widetilde{\widehat{\mathcal{E}}}$ of G is a rectangular rational surface. As a consequence, the class of rectangular rational surfaces and the class of rectangular BR-surfaces are identical, when $\mathcal{E}$ is of finite dimension.*

Proof. The proof follows from lemma 21.3.2 and lemma 21.3.2. Details are left as an exercise. □

If a triangular rational surface $F\colon \widetilde{\mathcal{P}} \to \widetilde{\mathcal{E}}$ is determined by some homogeneous polynomials $F_1(X, Y, Z), \ldots, F_{n+1}(X, Y, Z)$ in $\mathbb{R}[X, Y, Z]$, each of total degree m, the polynomial surface $G\colon \mathcal{P} \to \widehat{\mathcal{E}}$ associated with the restriction $g\colon \mathcal{P}^m \to \widehat{\mathcal{E}}$ of the polar form $f\colon (\widehat{\mathcal{P}})^m \to \widehat{\mathcal{E}}$ of F to $\mathcal{P}^m$ is simply determined by the polynomials $F_1(X, Y, 1), \ldots, F_{n+1}(X, Y, 1)$ in $\mathbb{R}[X, Y]$, obtained by setting $Z = 1$. If we homogenize these polynomials (assuming total degree m) we get the polynomials $F_1(X, Y, Z), \ldots, F_{n+1}(X, Y, Z)$ back. Similarly, if a rectangular rational surface $F\colon \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \to \widetilde{\mathcal{E}}$ is determined by polynomials $F_1(X, T_1, Y, T_2), \ldots, F_{n+1}(X, T_1, Y, T_2)$ in $\mathbb{R}[X, T_1, Y, T_2]$, each homogeneous of degree p in X, T_1 and homogeneous of degree q in Y, T_2, the polynomial surface $G\colon \mathbb{A} \times \mathbb{A} \to \widehat{\mathcal{E}}$ associated with the restriction $g\colon \mathbb{A}^p \times \mathbb{A}^q \to \widehat{\mathcal{E}}$ of the polar form $f\colon (\widehat{\mathbb{A}})^p \times (\widehat{\mathbb{A}})^q \to \widehat{\mathcal{E}}$ of F is simply determined by the polynomials $F_1(X, 1, Y, 1), \ldots, F_{n+1}(X, 1, Y, 1)$ in $\mathbb{R}[X, Y]$, obtained by setting $T_1 = T_2 = 1$. If we homogenize these polynomials separately w.r.t. X and Y (assuming maximum degree p in X and q in Y) we get the polynomials $F_1(X, T_1, Y, T_2), \ldots, F_{n+1}(X, T_1, X, T_2)$ back.

Thus, in practice, when defining rational surfaces (triangular or rectangular), it is enough to specify $n+1$ polynomials $F_1(X, Y), \ldots, F_{n+1}(X, Y)$ in $\mathbb{R}[X, Y]$. Such polynomials can be viewed as either defining a bipolynomial surface $G\colon \mathbb{A} \times \mathbb{A} \to \widehat{\mathcal{E}}$ or as defining a polynomial surface $G\colon \mathcal{P} \to \widehat{\mathcal{E}}$. In both cases, the surface G can be specified in terms of control points in $\widehat{\mathcal{E}}$. In the first case, it is a bipolynomial surface of degree $\langle p, q\rangle$ defined by a rectangular control net $\mathcal{N} = (\theta_{i,\,j})_{0\le i\le p,\, 0\le j\le q}$, and in the second case, it is a triangular surface of degree m defined by a triangular control net $\mathcal{N} = (\theta_{i,\,j,\,k})_{(i,j,k)\in\Delta_m}$.

The control points $\theta \in \widehat{\mathcal{E}}$ are either *weighted points* of the form $\langle a, w\rangle$, where $w \neq 0$ is called the *weight of* $a \in \mathcal{E}$, or *control vectors* $u \in \overrightarrow{\mathcal{E}}$. To determine the control points of the surface G given by $n+1$ polynomials, we compute the polar form g of G, either as a $\langle p, q\rangle$-symmetric affine map, or as a symmetric m-affine map. As in the case of curves, we have to remember that the coordinates of these control points will be with respect to the basis $(e_1, \ldots, e_n, \langle \Omega_1, 1\rangle)$ of $\widehat{\mathcal{E}}$, where $(\Omega_1, (e_1, \ldots, e_n))$ is the affine frame for $\mathcal{E}$. Thus, for every $\theta = \langle a, \lambda\rangle \in \widehat{\mathcal{E}}$, to find the coordinates of the point $a \in \mathcal{E}$, if θ has coordinates $(x_1, \ldots, x_n, \lambda)$ over $(e_1, \ldots, e_n, \langle \Omega_1, 1\rangle)$, then $a \in E$ has coordinates

$$\left(\frac{x_1}{\lambda}, \ldots, \frac{x_n}{\lambda}\right),$$

over $(\Omega_1, (e_1, \ldots, e_n))$, as explained in lemma 4.2.1. Let us give some examples.

Example 1. Consider the sphere given by the polynomials

$$\begin{aligned}
G_1(X, Y) &= 2X\\
G_2(X, Y) &= 2Y\\
G_3(X, Y) &= X^2 + Y^2 - 1\\
G_4(X, Y) &= X^2 + Y^2 + 1.
\end{aligned}$$

If we want to view the sphere as a bipolynomial surface of degree $\langle 2, 2\rangle$, we polarize the above polynomials separately in X and Y, obtaining:

$$\begin{aligned}
g_1(x_1, x_2, y_1, y_2) &= x_1 + x_2\\
g_2(x_1, x_2, y_1, y_2) &= y_1 + y_2\\
g_3(x_1, x_2, y_1, y_2) &= x_1x_2 + y_1y_2 - 1\\
g_4(x_1, x_2, y_1, y_2) &= x_1x_2 + y_1y_2 + 1.
\end{aligned}$$

With respect to the affine frame $(\overline{0}, \overline{1})$ of $\mathbb{A}$, we obtain the following nine control points in $\widehat{\mathbb{A}^3}$:

$\theta_{i,j}$	$\overline{0}\,\overline{0}$	$\overline{0}\,\overline{1}$	$\overline{1}\,\overline{1}$
$\overline{0}\,\overline{0}$	$(0, 0, -1, 1)$	$(0, 1, -1, 1)$	$(0, 2, 0, 2)$
$\overline{0}\,\overline{1}$	$(1, 0, -1, 1)$	$(1, 1, -1, 1)$	$(1, 2, 0, 2)$
$\overline{1}\,\overline{1}$	$(2, 0, 0, 2)$	$(2, 1, 0, 2)$	$(2, 2, 1, 3)$

Expressed in terms of points in $\mathbb{A}^3$, we have the following control points:

$$
\begin{array}{cccc}
\theta_{i,j} & \overline{0}\,\overline{0} & \overline{0}\,\overline{1} & \overline{1}\,\overline{1} \\
\overline{0}\,\overline{0} & ((0,0,-1),1) & ((0,1,-1),1) & ((0,1,0),2) \\
\overline{0}\,\overline{1} & ((1,0,-1),1) & ((1,1,-1),1) & ((\frac{1}{2},1,0),2) \\
\overline{1}\,\overline{1} & ((1,0,0),2) & ((1,\frac{1}{2},0),2) & ((\frac{2}{3},\frac{2}{3},\frac{1}{3}),3)
\end{array}
$$

Example 2. Again, consider the sphere given by the polynomials

$$
\begin{aligned}
G_1(X,Y) &= 2X \\
G_2(X,Y) &= 2Y \\
G_3(X,Y) &= X^2+Y^2-1 \\
G_4(X,Y) &= X^2+Y^2+1.
\end{aligned}
$$

This time, we want to view the sphere as a total degree surface of degree 2, and we polarize the above polynomials w.r.t X and Y, obtaining:

$$
\begin{aligned}
g_1(x_1,y_1,x_2,y_2) &= x_1+x_2 \\
g_2(x_1,y_1,x_2,y_2) &= y_1+y_2 \\
g_3(x_1,y_1,x_2,y_2) &= x_1x_2+y_1y_2-1 \\
g_4(x_1,y_1,x_2,y_2) &= x_1x_2+y_1y_2+1.
\end{aligned}
$$

With respect to the reference triangle $(r,s,t)=((1,0,0),(0,1,0),(0,0,1))$ in $\mathcal{P}$, we obtain the following six control points θ_{ijk} in $\widehat{\mathbb{A}^3}$, where $i+j+k=2$. It is important to recall that we always list the control points θ_{ijk} assuming that i is the index of the rows, starting from the left lower corner, and that j is the index of the columns, also starting from the left lower corner.

$$
\begin{array}{ccccc}
 & & (2,0,0,2) & & \\
 & (1,0,-1,1) & & (1,1,-1,1) & \\
(0,0,-1,1) & & (0,1,-1,1) & & (0,2,0,2)
\end{array}
$$

Expressed in terms of points in $\mathbb{A}^3$, we have the following control points:

$$
\begin{array}{ccccc}
 & & ((1,0,0),2) & & \\
 & ((1,0,-1),1) & & ((1,1,-1),1) & \\
((0,0,-1),1) & & ((0,1,-1),1) & & ((0,1,0),2)
\end{array}
$$

Example 3. The sphere can also be parameterized in terms of two angles θ and φ, where θ is the angle between the axis $\Omega_1 x$ and any line D_1 passing through Ω_1 in the $x\Omega_1 y$ plane, and φ is the angle between the axis $\Omega_1 z$ and any line D_2 passing through Ω_1 and in the plane determined by $\Omega_1 z$ and the line D_1. We have

$$
\begin{aligned}
x &= \cos\theta\sin\varphi \\
y &= \sin\theta\sin\varphi \\
z &= \cos\varphi.
\end{aligned}
$$

This is not a rational parameterization, but using the fact that

$$
\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \quad \text{and} \quad \sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}},
$$

letting $u = \tan\frac{\theta}{2}$ and $v = \tan\frac{\varphi}{2}$, we get

$$x = \frac{2(1-u^2)v}{(1+u^2)(1+v^2)}$$
$$y = \frac{4uv}{(1+u^2)(1+v^2)}$$
$$z = \frac{1-v^2}{1+v^2},$$

and thus, the sphere can be defined by the following polynomials:

$$G_1(X,Y) = 2(1-X^2)Y$$
$$G_2(X,Y) = 4XY$$
$$G_3(X,Y) = (1+X^2)(1-Y^2)$$
$$G_4(X,Y) = (1+X^2)(1+Y^2).$$

If we polarize these polynomials separately in X and Y, we get

$$g_1(x_1,x_2,y_1,y_2) = (1-x_1x_2)(y_1+y_2)$$
$$g_2(x_1,x_2,y_1,y_2) = (x_1+x_2)(y_1+y_2)$$
$$g_3(x_1,x_2,y_1,y_2) = (1+x_1x_2)(1-y_1y_2)$$
$$g_4(x_1,x_2,y_1,y_2) = (1+x_1x_2)(1+y_1y_2).$$

With respect to the affine frame $(\overline{0},\overline{1})$ of $\mathbb{A}$, we obtain the following nine control points in $\widehat{\mathbb{A}^3}$:

$\theta_{i,j}$	$\overline{0}\,\overline{0}$	$\overline{0}\,\overline{1}$	$\overline{1}\,\overline{1}$
$\overline{0}\,\overline{0}$	$(0,0,1,1)$	$(1,0,1,1)$	$(2,0,0,2)$
$\overline{0}\,\overline{1}$	$(0,0,1,1)$	$(1,1,1,1)$	$(2,2,0,2)$
$\overline{1}\,\overline{1}$	$(0,0,2,2)$	$(0,2,2,2)$	$(0,4,0,4)$

Letting $A = (0,0,1)$, $B = (0,1,0)$, $C = (0,0,1)$, $D = (0,1,1)$, $E = (1,0,1)$, $F = (1,1,0)$, $G = (1,1,1)$, observe that the points $\Omega_1, A, B, C,\ D, E, F, G$ are the vertices of the unit cube built on Ω_1, A, B, C, and expressed in terms of points in $\mathbb{A}^3$, we have the following nine control points:

$\theta_{i,j}$	$\overline{0}\,\overline{0}$	$\overline{0}\,\overline{1}$	$\overline{1}\,\overline{1}$
$\overline{0}\,\overline{0}$	$(C,1)$	$(E,1)$	$(A,2)$
$\overline{0}\,\overline{1}$	$(C,1)$	$(G,1)$	$(F,2)$
$\overline{1}\,\overline{1}$	$(C,2)$	$(D,2)$	$(B,4)$

For $0 \le u \le 1$ and $0 \le v \le 1$, we obtain the eighth of sphere determined by Ω_1, A, B, C.

Example 4. Consider the torus obtained by rotating the circle of center $(a,0,0)$ and of radius $b < a$ situated in the plane $x\Omega_1 z$, around the axis $\Omega_1 z$. This torus can be parameterized in terms of two angles θ and φ, where θ is the angle between the axis $\Omega_1 x$ and any line D_1 passing through Ω_1 in the $x\Omega_1 y$ plane, and φ is the angle between the line D_1 and any line D_2 passing through the center of the rotating circle and in the plane determined by $\Omega_1 z$ and the line D_1. We have

$$x = (a + b\cos\varphi)\cos\theta$$
$$y = (a + b\cos\varphi)\sin\theta$$
$$z = b\sin\varphi.$$

This is not a rational parameterization, but using the trick involving $\tan\frac{\theta}{2}$ and $\tan\frac{\varphi}{2}$, as in example 3, we get

$$\begin{aligned} x &= \frac{(1-u^2)(a(1+v^2)+b(1-v^2))}{(1+u^2)(1+v^2)} \\ y &= \frac{2u(a(1+v^2)+b(1-v^2))}{(1+u^2)(1+v^2)} \\ z &= \frac{2bv}{1+v^2}, \end{aligned}$$

and thus, the torus can be defined by the following polynomials:

$$\begin{aligned} G_1(X,Y) &= (1-X^2)(a(1+Y^2)+b(1-Y^2)) \\ G_2(X,Y) &= 2X(a(1+Y^2)+b(1-Y^2)) \\ G_3(X,Y) &= 2b(1+X^2)Y \\ G_4(X,Y) &= (1+X^2)(1+Y^2). \end{aligned}$$

Note that for $a = 0$, we get another representation of the sphere (of radius b). If we polarize these polynomials separately in X and Y, we get

$$\begin{aligned} g_1(x_1,x_2,y_1,y_2) &= (1-x_1x_2)(a(1+y_1y_2)+b(1-y_1y_2)) \\ g_2(x_1,x_2,y_1,y_2) &= (x_1+x_2)(a(1+y_1y_2)+b(1-y_1y_2)) \\ g_3(x_1,x_2,y_1,y_2) &= b(1+x_1x_2)(y_1+y_2) \\ g_4(x_1,x_2,y_1,y_2) &= (1+x_1x_2)(1+y_1y_2). \end{aligned}$$

With respect to the affine frame $(\overline{0},\overline{1})$ of $\mathbb{A}$, we obtain the following nine control points in $\widehat{\mathbb{A}^3}$:

$\theta_{i,j}$	$\overline{0}\,\overline{0}$	$\overline{0}\,\overline{1}$	$\overline{1}\,\overline{1}$
$\overline{0}\,\overline{0}$	$(a+b,0,0,1)$	$(a+b,0,b,1)$	$(2a,0,2b,2)$
$\overline{0}\,\overline{1}$	$(a+b,a+b,0,1)$	$(a+b,a+b,b,1)$	$(2a,2a,2b,2)$
$\overline{1}\,\overline{1}$	$(0,2(a+b),0,2)$	$(0,2(a+b),2b,2)$	$(0,4a,4b,4)$

Expressed in terms of points in $\mathbb{A}^3$, we have the following nine control points:

$\theta_{i,j}$	$\overline{0}\,\overline{0}$	$\overline{0}\,\overline{1}$	$\overline{1}\,\overline{1}$
$\overline{0}\,\overline{0}$	$((a+b,0,0),1)$	$((a+b,0,b),1)$	$((a,0,b),2)$
$\overline{0}\,\overline{1}$	$((a+b,a+b,0),1)$	$((a+b,a+b,b),1)$	$((a,a,b),2)$
$\overline{1}\,\overline{1}$	$((0,a+b,0),2)$	$((0,a+b,b),2)$	$((0,a,b),4)$

For $0 \leq u \leq 1$ and $0 \leq v \leq 1$, we obtain a sixteenth of the torus, the "front end" of the upper quarter of the torus. To obtain the "back end", we can assume that the torus is parameterized in terms of the angles θ and $\pi - \varphi$, which yields the following polynomials:

$$\begin{aligned} G_1'(X,Y) &= (1-X^2)(a(1+Y^2)-b(1-Y^2)) \\ G_2'(X,Y) &= 2X(a(1+Y^2)-b(1-Y^2)) \\ G_3'(X,Y) &= 2b(1+X^2)Y \\ G_4'(X,Y) &= (1+X^2)(1+Y^2). \end{aligned}$$

It is easily verified that we find the following control points:

$\theta_{i,j}$	$\overline{0}\,\overline{0}$	$\overline{0}\,\overline{1}$	$\overline{1}\,\overline{1}$
$\overline{0}\,\overline{0}$	$((a-b,0,0),1)$	$((a-b,0,b),1)$	$((a,0,b),2)$
$\overline{0}\,\overline{1}$	$((a-b,a-b,0),1)$	$((a-b,a-b,b),1)$	$((a,a,b),2)$
$\overline{1}\,\overline{1}$	$((0,a-b,0),2)$	$((0,a-b,b),2)$	$((0,a,b),4)$

We note the symmetry with respect to the circle of center Ω_1 and radius $a > b$. Now, we can construct one eighth of a torus, and by symmetry with respect to the plane $x\Omega_1 y$, $y\Omega_1 z$, and $x\Omega_1 z$, we obtain the entire torus.

Example 5. A Moëbius strip can be defined parametrically as

$$\begin{aligned} x &= (2 + v\sin\frac{\theta}{2})\cos\theta \\ y &= (2 + v\sin\frac{\theta}{2})\sin\theta \\ z &= v\cos\frac{\theta}{2}. \end{aligned}$$

Geometrically, the above formulae can be explained as follows. Consider the point $C(\theta)$ on the circle of center Ω_1 and radius 2 situated in the plane $x\Omega_1 y$, where $\theta \in [0, 2\pi]$ is the angle between $\Omega_1 x$ and $\Omega_1 C(\theta)$, and the point $M_1(\theta)$ on the semi-circle of center $C(\theta)$ and radius 1 situated in the plane determined by $\Omega_1 z$ and $\Omega_1 C(\theta)$, such that the angle between $\Omega_1 C$ and $\Omega_1 M_1$ is $\frac{\pi}{2} - \frac{\theta}{2}$. Then, the point $m(v, \theta)$ defined such that

$$\mathbf{Cm} = v\,\mathbf{CM_1},$$

is on a Moëbius strip, when $\theta \in [0, 2\pi]$ and $v \in [-1, 1]$. The above is not a rational parameterization, but using the trick of example 4, we can express everything in terms of $u = \tan\frac{\theta}{4}$, and we get

$$\begin{aligned} x &= \frac{2((1-u^2)^2 - 4u^2)(1+u^2+uv)}{(1+u^2)^3} \\ y &= \frac{8u(1-u^2)(1+u^2+uv)}{(1+u^2)^3} \\ z &= \frac{(1-u^2)v}{1+u^2}, \end{aligned}$$

and thus, the Moëbius strip can be defined by the following polynomials:

$$\begin{aligned} G_1(X,Y) &= 2((1-X^2)^2 - 4X^2)(1+X^2+XY) \\ G_2(X,Y) &= 8X(1-X^2)(1+X^2+XY) \\ G_3(X,Y) &= (1-X^2)(1+X^2)^2 Y \\ G_4(X,Y) &= (1+X^2)^3. \end{aligned}$$

It is easily verified that these polynomials can be written as

$$\begin{aligned} G_1(X,Y) &= 2[1 - 5X^2 - 5X^4 + X^6 + (X - 6X^3 + X^5)Y] \\ G_2(X,Y) &= 8[X - X^5 + (X^2 - X^4)Y] \\ G_3(X,Y) &= (1 + X^2 - X^4 - X^6)Y \\ G_4(X,Y) &= 1 + 3X^2 + 3X^4 + X^6. \end{aligned}$$

Perseverant readers can polarize the above polynomials separately in X and Y, and then compute 14 control points that form a rectangular control net for the Moëbius strip:

$\theta_{i,j}$	$j=0$	$j=1$
$i=0$	$((2,0,0),1)$	$((2,0,1),1)$
$i=1$	$((2,\frac{4}{3},0),1)$	$((\frac{7}{3},\frac{4}{3},1),1)$
$i=2$	$((\frac{10}{9},\frac{20}{9},0),\frac{6}{5})$	$((\frac{5}{3},\frac{8}{3},\frac{8}{9}),\frac{6}{5})$
$i=3$	$((0,\frac{5}{2},0),\frac{8}{5})$	$((\frac{1}{4},\frac{7}{2},\frac{3}{4}),\frac{8}{5})$
$i=4$	$((-\frac{10}{9},\frac{20}{9},0),\frac{12}{5})$	$((-\frac{14}{9},\frac{10}{3},\frac{5}{9}),\frac{12}{5})$
$i=5$	$((-2,\frac{4}{3},0),4)$	$((-3,2,\frac{1}{3}),4)$
$i=6$	$((-2,0,0),8)$	$((-3,0,0),8)$

Remark: With the above control net, the Moëbius strip is obtained over $[-1,1]\times[-1,1]$. Using the change of parameter

$$\begin{aligned} u &= \frac{u'}{1-u'} \\ v &= 2v'-1, \end{aligned}$$

it is possible to find a rectangular control net for the Moëbius strip, such that the entire strip is obtained for $(u',v')\in[0,1]\times[0,1]$. In the next chapter, we will also give a triangular control net.

As in the case of rational curves, it is sometimes convenient to have an explicit formula giving the current point $F(a)$ (or $F(\overline{u},\overline{v})$) on a rational surface specified as a BR-surface. There are two cases, depending whether the control net specifying F is rectangular or triangular.

23.3 Rational Surfaces and Bernstein Polynomials

Let us assume that F is specified by the rectangular control net $\mathcal{N}=(\theta_{i,\,j})_{0\leq i\leq p,\,0\leq j\leq q}$, and that H and K are the sets of pairs of indices such that, $H\cup K=\{0,\ldots,p\}\times\{0,\ldots,q\}$, $H\cap K=\emptyset$, and for every $(i,j)\in H$,

$$\theta_{i,j}=\langle a_{i,j},w_{i,j}\rangle$$

is a weighted point, where $a_{i,j}\in\mathcal{E}$ and $w_{i,j}\neq 0$, and for every $(i,j)\in K$,

$$\theta_{i,j}=u_{i,j}$$

is a control vector. Then, with respect to the affine frame $(\overline{0},\overline{1})$ in $\mathbb{A}$, the bipolynomial surface $G[\mathcal{N}]$ defined (in $\widehat{\mathcal{E}}$) by the rectangular control net $\mathcal{N}=(\theta_{i,\,j})_{0\leq i\leq p,\,0\leq j\leq q}$ is given in terms of the Bernstein polynomials as

$$G[\mathcal{N}](\overline{u},\overline{v})=\sum_{\substack{0\leq i\leq p,\\ 0\leq j\leq q}} B_i^p(u)B_j^q(v)\,\theta_{i,j}.$$

Now, the above formula gives $G[\mathcal{N}](\overline{u},\overline{v})$ as a barycenter in $\widehat{\mathcal{E}}$, which, by lemma 4.1.2, can be expressed more explicitly. The explicit form of the barycenter depends on the quantity

$$w(u,v)=\sum_{(i,j)\in H} w_{i,j}B_i^p(u)B_j^q(v).$$

If $w(u, v) \neq 0$, then

$$G[\mathcal{N}](\overline{u}, \overline{v}) = \left\langle \sum_{(i,j)\in H} \frac{w_{i,j} B_i^p(u) B_j^q(v)}{w(u,v)}\, a_{i,j} + \sum_{(i,j)\in K} \frac{B_i^p(u) B_j^q(v)}{w(u,v)}\, u_{i,j}, w(u,v) \right\rangle,$$

else if $w(u, v) = 0$, then

$$G[\mathcal{N}](\overline{u}, \overline{v}) = \sum_{(i,j)\in H} w_{i,j} B_i^p(u) B_j^q(v)\, a_{i,j} + \sum_{(i,j)\in K} B_i^p(u) B_j^q(v)\, u_{i,j},$$

where

$$\sum_{(i,j)\in H} w_{i,j} B_i^p(u) B_j^q(v)\, a_{i,j} = \sum_{(i,j)\in H} w_{i,j} B_i^p(u) B_j^q(v)\, \mathbf{ba_{i,j}}$$

for any $b \in \mathcal{E}$, which, by lemma 2.4.1, is a vector independent of b.

Then, since $F(\overline{u}, \overline{v}) = \Pi(G[\mathcal{N}](\overline{u}, \overline{v}))$, we have the following:

Letting

$$w(u, v) = \sum_{(i,j)\in H} w_{i,j} B_i^p(u) B_j^q(v),$$

if $w(u, v) \neq 0$, then

$$F(\overline{u}, \overline{v}) = \sum_{(i,j)\in H} \frac{w_{i,j} B_i^p(u) B_j^q(v)}{w(u,v)}\, a_{i,j} + \sum_{(i,j)\in K} \frac{B_i^p(u) B_j^q(v)}{w(u,v)}\, u_{i,j},$$

else if $w(u, v) = 0$ and

$$\sum_{(i,j)\in H} w_{i,j} B_i^p(u) B_j^q(v)\, a_{i,j} + \sum_{(i,j)\in K} B_i^p(u) B_j^q(v)\, u_{i,j} \neq 0,$$

then

$$F(\overline{u}, \overline{v}) = \left(\sum_{(i,j)\in H} w_{i,j} B_i^p(u) B_j^q(v)\, a_{i,j} + \sum_{(i,j)\in K} B_i^p(u) B_j^q(v)\, u_{i,j} \right)_{\infty},$$

else if $w(u, v) = 0$ and

$$\sum_{(i,j)\in H} w_{i,j} B_i^p(u) B_j^q(v)\, a_{i,j} + \sum_{(i,j)\in K} B_i^p(u) B_j^q(v)\, u_{i,j} = 0,$$

then

$$F(\overline{u}, \overline{v}) = \textit{undefined}.$$

Remark: If $K = \emptyset$ and $w_{i,j} \geq 0$ for all $i, j \in H$, then the trace $F([0, 1] \times [0, 1])$ of F belongs to the convex hull of the points $a_{i,j}$.

Let us now assume that F is specified by the triangular control net $\mathcal{N} = (\theta_{i,\,j,\,k})_{(i,j,k)\in\Delta_m}$, and that H and K are the sets of pairs of indices such that, $H \cup K = \Delta_m$, $H \cap K = \emptyset$, and for every $(i, j, k) \in H$,

$$\theta_{i,j,k} = \langle a_{i,j,k}, w_{i,j,k}\rangle$$

is a weighted point, where $a_{i,j,k} \in \mathcal{E}$ and $w_{i,j,k} \neq 0$, and for every $(i, j, k) \in K$,

$$\theta_{i,j,k} = u_{i,j,k}$$

is a control vector. Then, assuming the reference triangle Δrst in the affine plane $\mathcal{P}$, for every point $c = \lambda r + \mu s + \nu t$, where $\lambda + \mu + \nu = 1$, the point $G[\mathcal{N}](c)$ on the surface defined (in $\widehat{\mathcal{E}}$) by the triangular control net $\mathcal{N} = (\theta_{i,\,j,\,k})_{(i,j,k)\in\Delta_m}$ is given in terms of the Bernstein polynomials as

$$G[\mathcal{N}](c) = \sum_{(i,j,k)\in\Delta_m} B^m_{i,j,k}(\lambda,\mu,\nu)\,\theta_{i,j,k},$$

where, the Bernstein polynomial $B^m_{i,j,k}$ is given by

$$B^m_{i,j,k}(\lambda,\mu,\nu) = \frac{m!}{i!j!k!}\,\lambda^i\mu^j\nu^k.$$

Now, the above formula gives $G[\mathcal{N}](c)$ as a barycenter in $\widehat{\mathcal{E}}$, which, by lemma 4.1.2, can be expressed more explicitly. The explicit form of the barycenter depends on the quantity

$$w(\lambda,\mu,\nu) = \sum_{(i,j,k)\in H} w_{i,j.k}B^m_{i,j,k}(\lambda,\mu,\nu).$$

If $w(\lambda,\mu,\nu) \neq 0$, then

$$G[\mathcal{N}](c) = \left\langle \sum_{(i,j,k)\in H} \frac{w_{i,j,k}B^m_{i,j,k}(\lambda,\mu,\nu)}{w(\lambda,\mu,\nu)}\,a_{i,j,k} + \sum_{(i,j,k)\in K} \frac{B^m_{i,j,k}(\lambda,\mu,\nu)}{w(\lambda,\mu,\nu)}\,u_{i,j,k},\, w(\lambda,\mu,\nu)\right\rangle,$$

else if $w(\lambda,\mu,\nu) = 0$, then

$$G[\mathcal{N}](c) = \sum_{(i,j,k)\in H} w_{i,j,k}B^m_{i,j,k}(\lambda,\mu,\nu)\,a_{i,j,k} + \sum_{(i,j,k)\in K} B^m_{i,j,k}(\lambda,\mu,\nu)\,u_{i,j,k},$$

where

$$\sum_{(i,j,k)\in H} w_{i,j,k}B^m_{i,j,k}(\lambda,\mu,\nu)\,a_{i,j,k} = \sum_{(i,j,k)\in H} w_{i,j,k}B^m_{i,j,k}(\lambda,\mu,\nu)\,\mathbf{ba_{i,j,k}}$$

for any $b \in \mathcal{E}$, which, by lemma 2.4.1, is a vector independent of b.

Then, since $F(c) = \Pi(G[\mathcal{N}](c))$, we have the following:

Letting

$$w(\lambda,\mu,\nu) = \sum_{(i,j,k)\in H} w_{i,j,k}B^m_{i,j,k}(\lambda,\mu,\nu),$$

if $w(\lambda,\mu,\nu) \neq 0$, then

$$F(c) = \sum_{(i,j,k)\in H} \frac{w_{i,j,k}B^m_{i,j,k}(\lambda,\mu,\nu)}{w(\lambda,\mu,\nu)}\,a_{i,j,k} + \sum_{(i,j,k)\in K} \frac{B^m_{i,j,k}(\lambda,\mu,\nu)}{w(\lambda,\mu,\nu)}\,u_{i,j,k},$$

else if $w(\lambda,\mu,\nu) = 0$ and

$$\sum_{(i,j,k)\in H} w_{i,j,k}B^m_{i,j,k}(\lambda,\mu,\nu)\,a_{i,j,k} + \sum_{(i,j,k)\in K} B^m_{i.j.k}(\lambda,\mu,\nu)\,u_{i,j,k} \neq 0,$$

then

$$F(c) = \left(\sum_{(i,j,k)\in H} w_{i,j,k}B^m_{i,j,k}(\lambda,\mu,\nu)\,a_{i,j,k} + \sum_{(i,j,k)\in K} B^m_{i,j,k}(\lambda,\mu,\nu)\,u_{i,j,k}\right)_\infty,$$

else if $w(\lambda, \mu, \nu) = 0$ and

$$\sum_{(i,j,k)\in H} w_{i,j,k} B^m_{i,j.k}(\lambda, \mu, \nu)\, a_{i,j,k} + \sum_{(i,j,k)\in K} B^m_{i,j,k}(\lambda, \mu, \nu)\, u_{i,j,k} = 0,$$

then

$$F(c) = \mathit{undefined}.$$

Remark: If $K = \emptyset$ and $w_{i,j,k} \geq 0$ for all $i, j, k \in H$, then the trace $F(\Delta rst)$ of F belongs to the convex hull of the points $a_{i,j,k}$.

23.4 Subdivision Algorithms for Rational Surfaces

As in the case of curves (see Section 22.3), it is easy to adapt the subdivision algorithm described for polynomial surfaces to the rational case. This can be done in two ways. Either we subdivide in $\widehat{\mathcal{E}}$, getting polyhedral approximations determined by control nets in $\widehat{\mathcal{E}}$, and project these control nets down on $\mathcal{E}$ using Π. For a fairly complete treatment of this method, the reader is referred to Fiorot and Jeannin [60, 61]. The second method, which we advocate since it seems simpler, is to use the isomorphism $\widehat{\Omega}\colon \widehat{\mathcal{E}} \to \mathcal{F}$ of lemma 4.3.2. As we explained in the case of curves, what this means is that given a vector in $\widehat{\mathcal{E}}$ expressed as

$$u = (x_1, \ldots, x_n, w),$$

we have

$$\widehat{\Omega}(u) = (wx_1, \ldots, wx_n, w),$$

if $w \neq 0$, and

$$\widehat{\Omega}(u) = (x_1, \ldots, x_n, 0),$$

if $w = 0$.

We can then apply the subdivision methods suggested in chapter 19 (for details, see Gallier [70]), but in $\mathcal{F}$. The following function sends a control net consisting of points in $\widehat{\mathcal{E}}$ to a control net of points in $\mathcal{F}$.

```
(* To map a net in the affine space into the hat space  *)

 maptohat[{net__}] :=
 Block[
 {lnet = {net}, newnet = {},
  pt, h, w, i, l1},
 l1 = Length[lnet];
 Do[
     pt = lnet[[i]]; w = Last[pt]; h = Drop[pt, -1];
     If[w =!= 0, h = w * h; pt = Append[h, w]
       ];
     newnet = Append[newnet,pt],    {i, 1, l1}
   ];
 newnet
 ];
```

Once a list of nets (in $\mathcal{F}$) has been computed using subdivision, it is necessary to map these nets down to the original affine space $\mathcal{E}$. This is performed by the function *proj* such that,

$$\mathit{proj}[(x_1, \ldots, x_n, w)] = (x_1/w, \ldots, x_n/w),$$

if $w \neq 0$,

$$proj[(x_1, \ldots, x_n, 0)] = (x_1, \ldots, x_n),$$

if some $x_i \neq 0$, which corresponds to a point at infinity, and

$$proj[(0, \ldots, 0, 0)] = \text{undefined}.$$

Concretely, we have to be careful in dividing by w when $|w|$ is very close to zero, since due to limited numerical precision, this may cause a division by zero. As in the case of curves, we only perform division when $|w| > 10^{-20}$, and we give a warning if $|w| \leq 10^{-16}$. Another problem is that a vector $(x_1, \ldots, x_n, w)$ may be very close to the zero vector. This time, there is no simple way of dealing with zero vectors. We cannot simply discard such entries, since this yields bad control nets, and continuity cannot be used either. We will suggest a way of dealing with this situation later on. The following *Mathematica* functions implement the projection of a list of nets.

```
(* To project a point in the hat space back onto the affine space   *)

 projpt[pt_] :=
 Block[
 {h, w},
  w = Last[pt]; h = Drop[pt, -1];
      If[Abs[w] > 10^(-20),
         If[Abs[w] <= 10^(-16),
            Print["*** Warning: Point with very
                  small weight: ", pt," ***"]];
            h = h/w, Print["*** Warning:
                  Point at infinity!: ", pt, " ***"];
         h = 10^(20) * h
        ];
 h
 ];


(* To project a net in the hat space back onto the affine space   *)

 projnet[{net__}] :=
 Block[
 {snet = {net}, newnet = {}, pt, h, j, l2, flag},
  l2 = Length[snet];
     Do[
        pt = snet[[j]];  h = projpt[pt];
        newnet = Append[newnet,h],  {j, 1, l2}
       ];
 newnet
 ];


(* To project a list of nets in the hat space back onto
   the affine space   *)

 projlis[{netlis__}] :=
```

```
Block[
{slis = {netlis}, newlis = {},
 anet, newnet, j, l2},
    l2 = Length[slis];
    Do[
       anet = slis[[j]]; newnet = projnet[anet];
       newlis = Append[newlis,newnet],  {j, 1, l2}
      ];
newlis
]
```

Example 1. The subdivision algorithm is illustrated by the following example of an ellipsoid defined by the fractions

$$\begin{aligned} x(u,v) &= \frac{2au}{u^2+v^2+1}, \\ y(u,v) &= \frac{2bv}{u^2+v^2+1}, \\ z(u,v) &= \frac{c(u^2+v^2-1)}{u^2+v^2+1}. \end{aligned}$$

It is easily verified that this representation of the ellipsoid is derived from the stereographic projection from the north pole onto the plane $z = 0$. The coordinates of a point on the sphere are the coordinates of the image of a point (u, v) the xOy plane, under the inverse of stereographic projection. We leave as an exercise to show that the following triangular control net for $a = 4$, $b = 3$, $c = 2$, is obtained:

```
net = {{0, 0, -2, 1}, {0, 3, -2, 1}, {0, 3, 0, 2},
       {4, 0, -2, 1}, {4, 3, -2, 1}, {4, 0, 0, 2}}
```

Figures 23.1, 23.2, 23.3, show the result of the subdivision over the standard reference triangle Δrts, for $n = 1, 2, 3$.

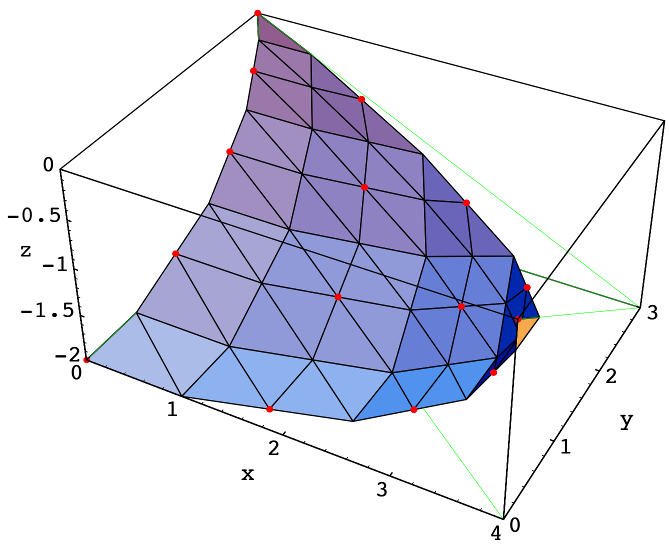


Figure 23.2: Portion of ellipsoid, 2 iterations

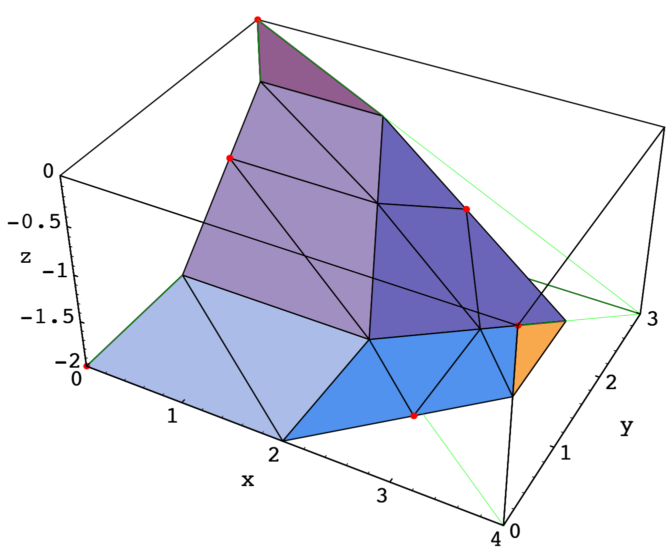


Figure 23.1: Portion of ellipsoid, 1 iteration

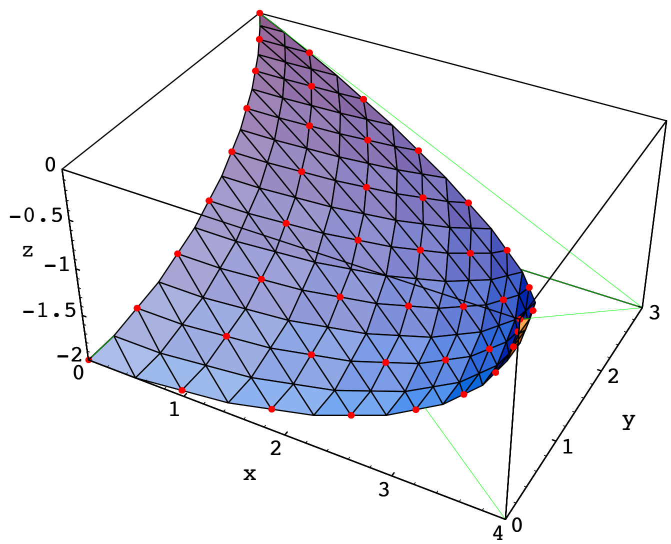


Figure 23.3: Portion of ellipsoid, 3 iterations

Example 2. As a second example, consider the torus defined by the equations

$$\begin{aligned} x &= (a - b\sin\varphi)\cos\theta \\ y &= (a - b\sin\varphi)\sin\theta \\ z &= b\cos\varphi. \end{aligned}$$

Using the trick involving $\tan\frac{\theta}{2}$ and $\tan\frac{\varphi}{2}$, we get

$$\begin{aligned} x &= \frac{(1-u^2)(a(1+v^2)-2bv)}{(1+u^2)(1+v^2)} \\ y &= \frac{2u(a(1+v^2)-2bv)}{(1+u^2)(1+v^2)} \\ z &= \frac{b(1+u^2)(1-v^2)}{(1+u^2)(1+v^2)}. \end{aligned}$$

Using a program to polarize polynomials in two variables, we obtain the following triangular net of degree 4, for $a = 2$, $b = 1$.

```
tornet =   {{2, 0, 1, 1}, {3/2, 0, 1, 1}, {8/7, 0, 5/7, 7/6},
            {1, 0, 1/3, 3/2}, {1, 0, 0, 2}, {2, 1, 1, 1},
            {3/2, 2/3, 1, 1}, {8/7, 4/7, 5/7, 7/6},  {1, 2/3, 1/3, 3/2},
            {10/7, 12/7, 1, 7/6}, {8/7, 8/7, 1, 7/6},
            {2/3, 8/9, 5/9, 3/2}, {2/3, 2, 1, 3/2}, {2/3, 4/3, 1, 3/2},
            {0, 2, 1, 2}};
```

Subdividing 3 times over the standard reference triangle Δrts, we get the picture shown in Figure 23.4.

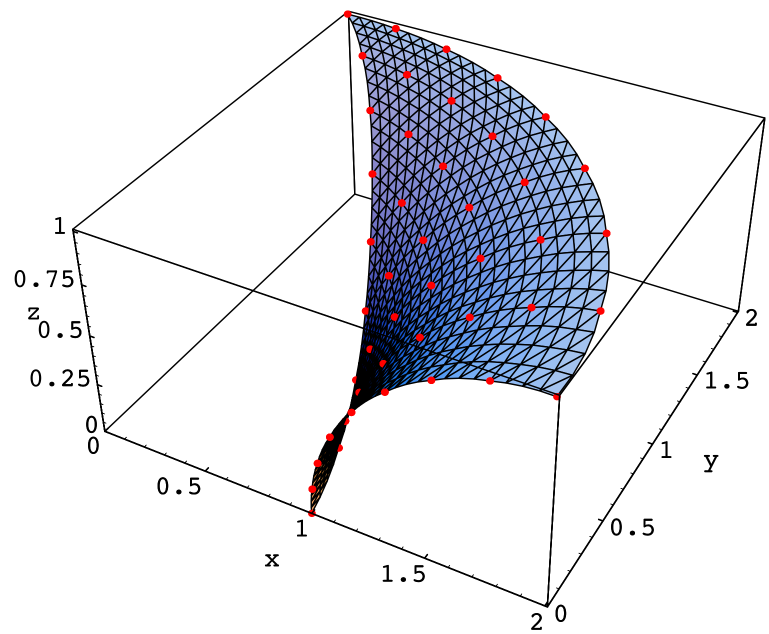


Figure 23.4: Triangular patch of torus, 3 iterations

Using `newcnet3` to compute the nets over the triangles

$$reftrig1 = ((-1, 1, 1), (-1, -1, 3), (1, 1, -1))$$

and

$$reftrig2 = ((1, -1, 1), (1, 1, -1), (-1, -1, 3)),$$

and then subdividing 3 times, we get the picture shown in Figure 23.5.

23.5 Problems

Problem 1. Prove lemma 23.2.2.

Problem 2 Prove lemma 23.2.4.

Problem 3 Consider the *ellipsoid* defined such that

$$\begin{aligned} x &= \frac{2a(1-u^2)v}{(1+u^2)(1+v^2)} \\ y &= \frac{4buv}{(1+u^2)(1+v^2)} \\ z &= \frac{c(1-v^2)}{1+v^2}. \end{aligned}$$

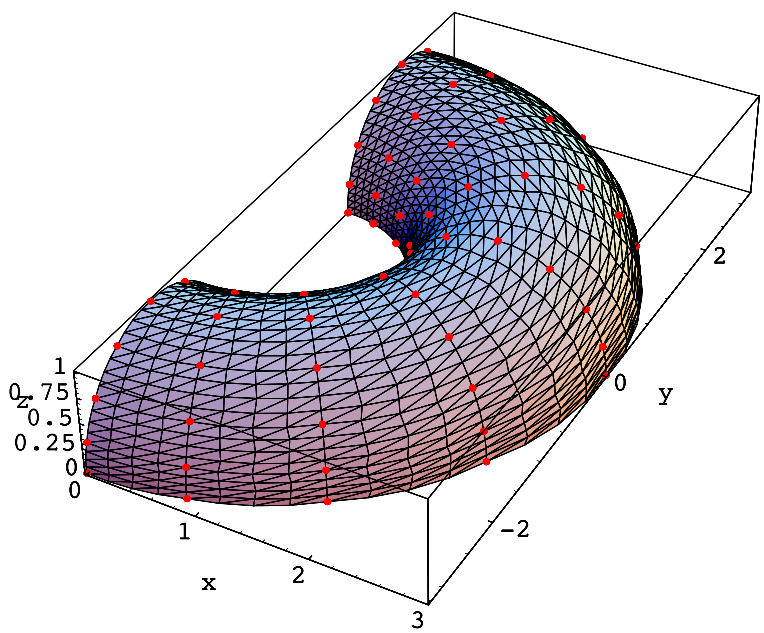


Figure 23.5: Portion of torus, 3 iterations

(i) Compute a rectangular control net w.r.t. $(0, 1) \times (0, 1)$.

(ii) Compute a triangular control net w.r.t. $((1, 0), (0, 1), (0, 0))$.

Problem 4 Consider the *right conoid* defined such that

$$\begin{aligned} x &= \frac{v(1-u^2)}{1+u^2} \\ y &= \frac{2uv}{1+u^2} \\ z &= \frac{4u}{1+u^2}. \end{aligned}$$

(i) Compute a rectangular control net w.r.t. $(0, 1) \times (0, 1)$.

(ii) Compute a triangular control net w.r.t. $((1, 0), (0, 1), (0, 0))$.

Problem 5 Consider the surface defined such that

$$\begin{aligned} x &= \frac{2v(1-u^2)(1+u^2)}{(1+u^4)(1+v^2)} \\ y &= \frac{4uv(1+u^2)}{(1+u^4)(1+v^2)} \\ z &= \frac{2v^2(1+u^2)^2}{(1+u^4)(1+v^2)}. \end{aligned}$$

(i) Compute a rectangular control net w.r.t. $(0, 1) \times (0, 1)$.

(ii) Compute a triangular control net w.r.t. $((1, 0), (0, 1), (0, 0))$.

Problem 6 Consider the *hyperboloid of one sheet* defined such that

$$\begin{aligned} x &= \frac{a(1-u^2)(1+v^2)}{(1+u^2)(1-v^2)} \\ y &= \frac{2bu(1+v^2)}{(1+u^2)(1-v^2)} \\ z &= \frac{2cv(1+u^2)}{(1+u^2)(1-v^2)}. \end{aligned}$$

(i) Compute a rectangular control net w.r.t. $(0, 1) \times (0, 1)$.

(ii) Compute a triangular control net w.r.t. $((1, 0), (0, 1), (0, 0))$.

Problem 7 Consider the *hyperboloid of two sheets* defined such that

$$\begin{aligned} x &= \frac{2av(1-u^2)}{(1+u^2)(1-v^2)} \\ y &= \frac{4buv}{(1+u^2)(1-v^2)} \\ z &= \frac{c(1+u^2)(1+v^2)}{(1+u^2)(1-v^2)}. \end{aligned}$$

(i) Compute a rectangular control net w.r.t. $(0, 1) \times (0, 1)$.

(ii) Compute a triangular control net w.r.t. $((1,0),(0,1),(0,0))$.

Problem 8 Consider the surface defined such that

$$x = v\cos\theta,$$
$$y = v\sin\theta.$$
$$z = a\cos\theta - bv.$$

(i) Prove that the above surface also has the following rational definition:

$$x = \frac{(u^4 - 6u^2 + 1)v}{(1+u^2)^2}$$
$$y = \frac{4uv(1-u^2)}{(1+u^2)^2}$$
$$z = \frac{a(u^4 - 6u^2 + 1) - bv(u^2+1)^2}{(1+u^2)^2}.$$

(ii) Compute a triangular control net w.r.t. $((1,0),(0,1),(0,0))$.

Problem 9 Write your own program to draw a rational surface patch. Use your program to draw the following two patches over $[-1,1] \times [-1,1]$.

```
hilnetb1 = {{0, 0, 0, 1},{1/3, 0, 0, 1},
   {5/8, 0, 1/8, 16/15}, {5/6, 0, 1/3, 6/5},
   {20/21, 0, 4/7, 7/5}, {1, 0, 4/5, 5/3}, {1, 0, 1, 2}, {0, 0, 0, 1},
   {1/3, 2/15, 0, 1}, {5/8, 1/4, 1/8, 16/15}, {5/6, 1/3, 1/3, 6/5},
   {20/21, 8/21, 4/7, 7/5}, {1, 2/5, 4/5, 5/3}, {0, 0, 0, 1},
   {1/3, 4/15, 0, 1}, {5/8, 1/2, 1/6, 16/15}, {5/6, 2/3, 4/9, 6/5},
   {20/21, 16/21, 16/21, 7/5}, {0, 0, 0, 1}, {1/3, 7/15, 0, 1},
   {5/8, 7/8, 1/4, 16/15}, {5/6, 7/6, 2/3, 6/5}, {0, 0, 0, 16/15},
   {1/4, 3/4, 0, 16/15}, {4/9, 4/3, 4/9, 6/5}, {0, 0, 0, 4/3},
   {0, 1, 0, 4/3}, {0, 0, 0, 2}};

hilnetb2 = {{0, 0, 2, 1},{1/3, 0, 2, 1},
   {5/8, 0, 15/8, 16/15}, {5/6, 0, 5/3, 6/5},
   {20/21, 0, 10/7, 7/5}, {1, 0, 6/5, 5/3}, {1, 0, 1, 2}, {0, 0, 2, 1},
   {1/3, 2/15, 2, 1}, {5/8, 1/4, 15/8, 16/15}, {5/6, 1/3, 5/3, 6/5},
   {20/21, 8/21, 10/7, 7/5}, {1, 2/5, 6/5, 5/3}, {0, 0, 34/15, 1},
   {1/3, 4/15, 34/15, 1}, {5/8, 1/2, 17/8, 16/15}, {5/6, 2/3, 17/9, 6/5},
   {20/21, 16/21, 34/21, 7/5}, {0, 0, 14/5, 1}, {1/3, 7/15, 14/5, 1},
   {5/8, 7/8, 21/8, 16/15}, {5/6, 7/6, 7/3, 6/5}, {0, 0, 7/2, 16/15},
   {1/4, 3/4, 7/2, 16/15}, {4/9, 4/3, 28/9, 6/5}, {0, 0, 4, 4/3},
   {0, 1, 4, 4/3}, {0, 0, 4, 2}};
```

Problem 10 Consider the rational surface K_1 defined such that

$$x = \frac{a(u^4 - 6u^2 + 1)(1+v^2)^2 + r(v^4 - 6v^2 + 1)(1-u^4)}{(1+u^2)^2(1+v^2)^2},$$
$$y = \frac{4bu(1-u^2)(1+v^2)^2 + 2ru(v^4 - 6v^2 + 1)(1+u^2)}{(1+u^2)^2(1+v^2)^2},$$
$$z = \frac{4rv(1-v^2)(1+u^2)^2}{(1+u^2)^2(1+v^2)^2}.$$

(i) Draw K_1 (try several values for a, b, r).

(ii) Expand the numerator of x, and define the surface K_2 obtained by changing the term $-12au^2v^2$ to $12au^2v^2$.

(iii) Draw K_2 over $[-1,1] \times [-1,1]$.

Problem 11 Consider the torus defined such that

$$
\begin{aligned}
x &= \frac{(1-u^2)(a(1+v^2)-2bv)}{(1+u^2)(1+v^2)} \\
y &= \frac{2u(a(1+v^2)-2bv)}{(1+u^2)(1+v^2)} \\
z &= \frac{b(1+u^2)(1-v^2)}{(1+u^2)(1+v^2)}.
\end{aligned}
$$

(i) Show that the above torus is defined by the following rectangular net of bidegree $\langle 2,2 \rangle$ w.r.t. $(-1,1) \times (-1,1)$:

```
tornet4 = {{0, -(a + b), 0, 4}, {0, 0, 4c, 0}, {0, (-a + b), 0, 4},
   {4(a + b), 0, 0, 0}, {0, 0, 0, 0}, {4(a - b), 0, 0, 0},
   {0, a + b, 0, 4}, {0, 0, 4c, 0}, {0, a - b, 0, 4}}
```

(ii) Show that the result of homogenizing separately w.r.t. u and v yields:

$$
\begin{aligned}
x &= (t_1^2 - u^2)(a(t_2^2+v^2) - 2bvt_2), \\
y &= 2ut_1(a(t_2^2+v^2) - 2bvt_2), \\
z &= b(t_1^2+u^2)(t_2^2-v^2), \\
w &= (t_1^2+u^2)(t_2^2+v^2).
\end{aligned}
$$

Conclude that there are no base points.

(iii) Show that the above torus is defined by the following triangular net w.r.t. ((1, 0), (0, 1), (0, 0)):

```
tornet =  {{a, 0, c, 1}, {a - b/2, 0, c, 1},
   {(6((7a)/6 - b))/7, 0, (5c)/7, 7/6},
   {(2((3a)/2 - (3b)/2))/3, 0, c/3, 3/2}, {(2a - 2b)/2, 0, 0, 2},
   {a, a/2, c, 1}, {a - b/2, a/2 - b/3, c, 1},
   {(6((7a)/6 - b))/7, (6((2a)/3 - (2b)/3))/7, (5c)/7, 7/6},
   {(2((3a)/2 - (3b)/2))/3, (2(a - b))/3, c/3, 3/2},
   {(5a)/7, (6a)/7, c, 7/6},
   {(6((5a)/6 - b/3))/7, (6(a - (2b)/3))/7, c, 7/6},
   {(2((5a)/6 - (2b)/3))/3, (2((4a)/3 - (4b)/3))/3, (5c)/9, 3/2},
   {a/3, a, c, 3/2}, {a/3, (2((3a)/2 - b))/3, c, 3/2},
   {0, a, c, 2}};
```

(iv) Show that the result of homogenizing u, v w.r.t. the degree 4 yields:

$$
\begin{aligned}
x(u,v,t) &= (t^2-u^2)(a(t^2+v^2) - 2bvt), \\
y(u,v,t) &= 2ut(a(t^2+v^2) - 2bvt), \\
z(u,v,t) &= c(t^2+u^2)(t^2-v^2), \\
w(u,v,t) &= (t^2+u^2)(t^2+v^2).
\end{aligned}
$$

Show that there are two base points corresponding to the points $(0,1,0)$ and $(1,0,0)$ on the line at infinity.