Chapter 20

More On Embedding an Affine Space in a Vector Space

20.1 From Multiaffine Maps to Multilinear Maps

In this Section, we show how to homogenize multiaffine maps. The proof of Lemma 20.1.1 requires a technical result about the characterization of multiaffine maps in terms of linear maps. This result and its proof can be found in Gallier [70] and on the web site, see web page (Lemma 27.1.3).

Lemma 20.1.1 *Given any affine space E and any vector space $\overrightarrow{F}$, for any m-affine map $f\colon E^m \to \overrightarrow{F}$, there is a unique m-linear map $\widehat{f}\colon (\widehat{E})^m \to \overrightarrow{F}$ extending f, such that, if*

$$f(a_1 + v_1, \ldots, a_m + v_m) = f(a_1, \ldots, a_m) + \sum_{\substack{S \subseteq \{1,\ldots,m\},\ k=|S| \\ S=\{i_1,\ldots,i_k\},\ k\geq 1}} f_S(v_{i_1}, \ldots, v_{i_k}),$$

for all $a_1 \ldots, a_m \in E$, and all $v_1, \ldots, v_m \in \overrightarrow{E}$, where the f_S are uniquely determined multilinear maps (see the web site web page, Lemma 27.1.3), then

$$\widehat{f}(v_1 \mathbin{\widehat{+}} \lambda_1 a_1, \ldots, v_m \mathbin{\widehat{+}} \lambda_m a_m) = \lambda_1 \cdots \lambda_m f(a_1, \ldots, a_m) + \sum_{\substack{S \subseteq \{1,\ldots,m\},\ k=|S| \\ S=\{i_1,\ldots,i_k\},\ k\geq 1}} \Bigg(\prod_{\substack{j\in\{1,\ldots,m\} \\ j\notin S}} \lambda_j\Bigg) f_S(v_{i_1}, \ldots, v_{i_k}),$$

for all $a_1 \ldots, a_m \in E$, all $v_1, \ldots, v_m \in \overrightarrow{E}$, and all $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$. Furthermore, for $\lambda_i \neq 0$, $1 \leq i \leq m$, we have

$$\widehat{f}(v_1 \mathbin{\widehat{+}} \lambda_1 a_1, \ldots, v_m \mathbin{\widehat{+}} \lambda_m a_m) = \lambda_1 \cdots \lambda_m f(a_1 + \lambda_1^{-1} v_1, \ldots, a_m + \lambda_m^{-1} v_m).$$

Proof. Let us assume that $\widehat{f}$ exists. We first prove by induction on k, $1 \leq k \leq m$, that

$$\widehat{f}(a_1, \ldots, v_{i_1}, \ldots, v_{i_k}, \ldots, a_m) = f_S(v_{i_1}, \ldots, v_{i_k}),$$

for every $S \subseteq \{1, \ldots, m\}$, such that $S = \{i_1, \ldots, i_k\}$ and $k = |S|$, for all $a_1 \ldots, a_m \in E$, and all $v_1, \ldots, v_m \in \overrightarrow{E}$.

For $k = 1$, assuming for simplicity of notation that $i_1 = 1$, for any $a_1 \in E$, since $\widehat{f}$ is m-linear, we have

$$\widehat{f}(a_1 + v_1, a_2, \ldots, a_m) = \widehat{f}(a_1, a_2, \ldots, a_m) + \widehat{f}(v_1, a_2, \ldots, a_m),$$

but since $\widehat{f}$ extends f, we have

$$\widehat{f}(a_1 + v_1, a_2, \ldots, a_m) = f(a_1 + v_1, a_2, \ldots, a_m) = f(a_1, a_2, \ldots, a_m) + \widehat{f}(v_1, a_2, \ldots, a_m),$$

and using the expression of f in terms of the f_S, we also have

$$f(a_1 + v_1, a_2, \ldots, a_m) = f(a_1, a_2, \ldots, a_m) + f_{\{1\}}(v_1).$$

Thus, we have

$$\widehat{f}(v_1, a_2, \ldots, a_m) = f_{\{1\}}(v_1)$$

for all $v_1 \in \overrightarrow{E}$.

Assume that the induction hypothesis holds for all $l < k+1$, and let $S = \{i_1, \ldots, i_{k+1}\}$, with $k+1 = |S|$. Since $\widehat{f}$ is multilinear, for any $a \in E$, we have

$$\begin{aligned}\widehat{f}(a_1, \ldots, a + v_{i_1}, \ldots, a + v_{i_{k+1}}, \ldots, a_m) &= \widehat{f}(a_1, \ldots, a, \ldots, a, \ldots, a_m)\\ &+ \widehat{f}(a_1, \ldots, v_{i_1}, \ldots, v_{i_{k+1}}, \ldots, a_m) + \sum_{\substack{T=\{j_1,\ldots,j_l\}\\ T\subseteq S,\ 1\leq l\leq k}} \widehat{f}(a_1, \ldots, v_{j_1}, \ldots, v_{j_l}, \ldots, a_m).\end{aligned}$$

However, by the induction hypothesis, we have

$$\widehat{f}(a_1, \ldots, v_{j_1}, \ldots, v_{j_l}, \ldots, a_m) = f_T(v_{j_1}, \ldots, v_{j_l}),$$

for every $T = \{j_1, \ldots, j_l\}$, $1 \leq l \leq k$, and since $\widehat{f}$ extends f, we get

$$\begin{aligned}\widehat{f}(a_1, \ldots, a + v_{i_1}, \ldots, a + v_{i_{k+1}}, \ldots, a_m) &= f(a_1, \ldots, a, \ldots, a, \ldots, a_m)\\ &+ \widehat{f}(a_1, \ldots, v_{i_1}, \ldots, v_{i_{k+1}}, \ldots, a_m) + \sum_{\substack{T=\{j_1,\ldots,j_l\}\\ T\subseteq S,\ 1\leq l\leq k}} f_T(v_{j_1}, \ldots, v_{j_l}).\end{aligned}$$

Since $\widehat{f}$ extends f, also have

$$\widehat{f}(a_1, \ldots, a + v_{i_1}, \ldots, a + v_{i_{k+1}}, \ldots, a_m) = f(a_1, \ldots, a + v_{i_1}, \ldots, a + v_{i_{k+1}}, \ldots, a_m),$$

and by expanding this expression in terms of the f_T, we get

$$\begin{aligned}\widehat{f}(a_1, \ldots, a + v_{i_1}, \ldots, a + v_{i_{k+1}}, \ldots, a_m) &= f(a_1, \ldots, a, \ldots, a, \ldots, a_m)\\ &+ f_S(v_{i_1}, \ldots, v_{i_{k+1}}) + \sum_{\substack{T=\{j_1,\ldots,j_l\}\\ T\subseteq S,\ 1\leq l\leq k}} f_T(v_{j_1}, \ldots, v_{j_l}).\end{aligned}$$

Thus, we conclude that

$$\widehat{f}(a_1, \ldots, v_{i_1}, \ldots, v_{i_{k+1}}, \ldots, a_m) = f_S(v_{i_1}, \ldots, v_{i_{k+1}}).$$

This shows that $\widehat{f}$ is uniquely defined on $\overrightarrow{E}$, and clearly, the above defines a multilinear map. Now, assume that $\lambda_i \neq 0$, $1 \leq i \leq m$. We get

$$\widehat{f}(v_1 \,\widehat{+}\, \lambda_1 a_1, \ldots, v_m \,\widehat{+}\, \lambda_m a_m) = \widehat{f}(\lambda_1(a_1 + \lambda_1^{-1} v_1), \ldots, \lambda_m(a_m + \lambda_m^{-1} v_m)),$$

and since $\widehat{f}$ is m-linear, we get

$$\widehat{f}(\lambda_1(a_1 + \lambda_1^{-1} v_1), \ldots, \lambda_m(a_m + \lambda_m^{-1} v_m)) = \lambda_1 \cdots \lambda_m \widehat{f}(a_1 + \lambda_1^{-1} v_1, \ldots, a_m + \lambda_m^{-1} v_m).$$

Since $\widehat{f}$ extends f, we get

$$\widehat{f}(v_1 \mathbin{\widehat{+}} \lambda_1 a_1, \ldots, v_m \mathbin{\widehat{+}} \lambda_m a_m) = \lambda_1 \cdots \lambda_m f(a_1 + \lambda_1^{-1} v_1, \ldots, a_m + \lambda_m^{-1} v_m).$$

We can expand the right-hand side using the f_S, and we get

$$f(a_1 + \lambda_1^{-1} v_1, \ldots, a_m + \lambda_m^{-1} v_m) = f(a_1, \ldots, a_m) + \sum_{\substack{S \subseteq \{1,\ldots,m\},\, k=|S| \\ S=\{i_1,\ldots,i_k\},\, k\geq 1}} \lambda_{i_1}^{-1} \cdots \lambda_{i_k}^{-1} f_S(v_{i_1}, \ldots, v_{i_k}),$$

and thus, we get

$$\begin{aligned}\widehat{f}(v_1 \mathbin{\widehat{+}} \lambda_1 a_1, \ldots, v_m \mathbin{\widehat{+}} \lambda_m a_m) &= \lambda_1 \cdots \lambda_m f(a_1, \ldots, a_m) \\ &\quad + \sum_{\substack{S \subseteq \{1,\ldots,m\},\, k=|S| \\ S=\{i_1,\ldots,i_k\},\, k\geq 1}} \Bigg(\prod_{\substack{j \in \{1,\ldots,m\} \\ j \notin S}} \lambda_j \Bigg) f_S(v_{i_1}, \ldots, v_{i_k}).\end{aligned}$$

This expression agrees with the previous one when $\lambda_i = 0$ for some of the λ_i, $1 \leq i \leq m$, and this shows that $\widehat{f}$ is uniquely defined. Clearly, the above expression defines an m-linear map. Thus, the lemma is proved. □

As a corollary, we obtain the following useful lemma.

Lemma 20.1.2 *Given any two affine spaces E and F and an m-affine map $f\colon E^m \to F$, there is a unique m-linear map $\widehat{f}\colon (\widehat{E})^m \to \widehat{F}$ extending f as in the diagram below,*

$$\begin{array}{ccc} E^m & \xrightarrow{f} & F \\ {\scriptstyle j\times\cdots\times j}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ (\widehat{E})^m & \xrightarrow[\widehat{f}]{} & \widehat{F} \end{array}$$

such that, if

$$f(a_1 + v_1, \ldots, a_m + v_m) = f(a_1, \ldots, a_m) + \sum_{\substack{S \subseteq \{1,\ldots,m\},\, k=|S| \\ S=\{i_1,\ldots,i_k\},\, k\geq 1}} f_S(v_{i_1}, \ldots, v_{i_k}),$$

for all $a_1 \ldots, a_m \in E$, and all $v_1, \ldots, v_m \in \overrightarrow{E}$, where the f_S are uniquely determined multilinear maps, then

$$\begin{aligned}\widehat{f}(v_1 \mathbin{\widehat{+}} \lambda_1 a_1, \ldots, v_m \mathbin{\widehat{+}} \lambda_m a_m) &= \lambda_1 \cdots \lambda_m f(a_1, \ldots, a_m) \\ &\quad \mathbin{\widehat{+}} \sum_{\substack{S \subseteq \{1,\ldots,m\},\, k=|S| \\ S=\{i_1,\ldots,i_k\},\, k\geq 1}} \Bigg(\prod_{\substack{j \in \{1,\ldots,m\} \\ j \notin S}} \lambda_j \Bigg) f_S(v_{i_1}, \ldots, v_{i_k}),\end{aligned}$$

for all $a_1 \ldots, a_m \in E$, all $v_1, \ldots, v_m \in \overrightarrow{E}$, and all $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$. Furthermore, for $\lambda_i \neq 0$, $1 \leq i \leq m$, we have

$$\widehat{f}(v_1 \mathbin{\widehat{+}} \lambda_1 a_1, \ldots, v_m \mathbin{\widehat{+}} \lambda_m a_m) = \lambda_1 \cdots \lambda_m f(a_1 + \lambda_1^{-1} v_1, \ldots, a_m + \lambda_m^{-1} v_m).$$

Proof. Immediate from lemma 20.1.1 (see the proof of lemma 4.4.3 from lemma 4.4.2). □

The homogenized version $\widehat{f}$ of an m-affine map f is weight-multiplicative, in the sense that

$$\omega(\widehat{f}(z_1, \ldots, z_m)) = \omega(z_1) \cdots \omega(z_m),$$

for all $z_1, \ldots, z_m \in \widehat{E}$.

From a practical point of view,

$$\widehat{f}(v_1 \mathbin{\widehat{+}} \lambda_1 a_1, \ldots, v_m \mathbin{\widehat{+}} \lambda_m a_m) = \lambda_1 \cdots \lambda_m f(a_1 + \lambda_1^{-1} v_1, \ldots, a_m + \lambda_m^{-1} v_m),$$

shows us that f is recovered from $\widehat{f}$ by setting $\lambda_i = 1$, for $1 \leq i \leq m$. We can use this formula to find the homogenized version $\widehat{f}$ of the map f. For example, if we consider the affine space $\mathbb{A}$ with its canonical affine frame (the origin is $\overline{0}$, and the basis consists of the single vector 1), if $f\colon \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ is the biaffine map defined such that

$$f(x_1, x_2) = a x_1 x_2 + b x_1 + c x_2 + d,$$

the bilinear map $\widehat{f}\colon \widehat{\mathbb{A}} \times \widehat{\mathbb{A}} \to \widehat{\mathbb{A}}$, is given by

$$\begin{aligned}\widehat{f}((x_1, \lambda_1), (x_2, \lambda_2)) &= (\lambda_1 \lambda_2 \big[a(x_1 \lambda_1^{-1})(x_2 \lambda_2^{-1}) + b x_1 \lambda_1^{-1} + c x_2 \lambda_2^{-1} + d\big],\ \lambda_1 \lambda_2)\\ &= (a x_1 x_2 + b x_1 \lambda_2 + c x_2 \lambda_1 + d \lambda_1 \lambda_2,\ \lambda_1 \lambda_2),\end{aligned}$$

where we choose the basis $(1, \overline{0})$, in $\widehat{\mathbb{A}}$.

Note that $f(x_1, x_2)$ is indeed recovered from $\widehat{f}$ by setting $\lambda_1 = \lambda_2 = 1$. Since multiaffine maps can be homogenized, polynomial maps can also be homogenized. This is very useful in practice. In fact, using the characterization of multiaffine maps $f\colon E^m \to F$ when E is of finite dimension given in Gallier [70] and on the web site, see
web page (Lemma 27.1.6), we can get an explicit formula for the homogenized version $\widehat{f}$ of f, generalizing our previous example.

If $(a, (\overrightarrow{e_1}, \ldots, \overrightarrow{e_n}))$ is an affine frame for E, we know that for any m vectors

$$v_j = v_{1,\,j} e_1 + \cdots + v_{n,\,j} e_n \in \overrightarrow{E},$$

we have

$$f(a + v_1, \ldots, a + v_m) = b + \sum_{1 \leq p \leq m} \sum_{\substack{I_1 \cup \ldots \cup I_n = \{1, \ldots, p\} \\ I_i \cap I_j = \emptyset,\, i \neq j \\ 1 \leq i, j \leq n}} \Bigg(\prod_{i_1 \in I_1} v_{1,\, i_1}\Bigg) \cdots \Bigg(\prod_{i_n \in I_n} v_{n,\, i_n}\Bigg)\, w_{|I_1|, \ldots, |I_n|},$$

for some $b \in F$, and some $w_{|I_1|, \ldots, |I_n|} \in \overrightarrow{F}$, and since $\widehat{E} = \overrightarrow{E} \oplus \mathbb{R}a$, with respect to the basis $(e_1, \ldots, e_n, \langle a, 1\rangle)$ of $\widehat{E}$, we have

$$\begin{aligned}\widehat{f}(v_1 \mathbin{\widehat{+}} \lambda_1 a, \ldots, v_m \mathbin{\widehat{+}} \lambda_m a) &= \lambda_1 \cdots \lambda_m b\\ &\mathbin{\widehat{+}} \sum_{1 \leq p \leq m} \sum_{\substack{I_1 \cup \ldots \cup I_n = \{1, \ldots, p\} \\ I_i \cap I_j = \emptyset,\, i \neq j \\ 1 \leq i, j \leq n}} \Bigg(\prod_{i_1 \in I_1} v_{1,\, i_1}\Bigg) \cdots \Bigg(\prod_{i_n \in I_n} v_{n,\, i_n}\Bigg) \Bigg(\prod_{\substack{j \in \{1, \ldots, m\} \\ j \notin (I_1 \cup \ldots \cup I_n)}} \lambda_j\Bigg)\, w_{|I_1|, \ldots, |I_n|}.\end{aligned}$$

In other words, we obtain the expression for $\widehat{f}$ by homogenizing the polynomials which are the coefficients of the $w_{|I_1|, \ldots, |I_n|}$. For the homogenized version $\widehat{h}$ of the affine polynomial h associated with f, we get:

$$\widehat{h}(v \mathbin{\widehat{+}} \lambda a) = \lambda^m b \mathbin{\widehat{+}} \sum_{1 \leq p \leq m} \sum_{\substack{k_1 + \cdots + k_n = p \\ 0 \leq k_i,\, 1 \leq i \leq n}} v_1^{k_1} \cdots v_n^{k_n} \lambda^{m-p}\, w_{k_1, \ldots, k_n}.$$

Remark: Recall that homogenizing a polynomial $P(X_1, \ldots, X_n) \in \mathbb{R}[X_1, \ldots, X_n]$ is done as follows. If $P(X_1, \ldots, X_n)$ is of total degree p, and we want to find a homogeneous polynomial $Q(X_1, \ldots, X_n, Z)$ of total degree $m \geq p$, such that

$$P(X_1, \ldots, X_n) = Q(X_1, \ldots, X_n, 1),$$

we let

$$Q(X_1, \ldots, X_n, Z) = Z^m P\Bigg(\frac{X_1}{Z}, \ldots, \frac{X_n}{Z}\Bigg).$$

Chapter 21

Complements of Projective Geometry

21.1 Multiprojective Maps

In order to deal with rational functions, we also need to extend definition 5.5.1 to multilinear maps.

Given a multilinear map $f\colon E^m \to F$, let

$$\mathrm{Ker}\, f = \{(u_1, \ldots, u_m) \in E^m \mid f(u_1, \ldots, u_m) = 0\}$$

be the kernel of f.

Beware that $\mathrm{Ker}\, f$ is not necessarily a subspace of E^m. Also,

$$\mathbf{P}(E)^m = \underbrace{\mathbf{P}(E) \times \cdots \times \mathbf{P}(E)}_{m}$$

is not a projective space.

Then, for any $(u_1, \ldots, u_m), (v_1, \ldots, v_m) \in (E^m - \mathrm{Ker}\, f)$, if $u_i \sim v_i$, for $1 \le i \le m$, then $v_i = \lambda_i u_i$, for some $\lambda_i \in K - \{0\}$, where $1 \le i \le m$, and since f is multilinear, we get

$$f(v_1, \ldots, v_m) = \lambda_1 \cdots \lambda_m f(u_1, \ldots, u_m).$$

Thus, if $u_i \sim v_i$, for $1 \le i \le m$, we have $f(u_1, \ldots, u_m) \sim f(v_1, \ldots, v_m)$, which shows that $\mathbf{P}(f)$ can be defined on equivalence classes modulo $\sim$, by

$$\mathbf{P}(f)([u_1]_\sim, \ldots, [u_m]_\sim) = [f(u_1, \ldots, u_m)]_\sim.$$

Definition 21.1.1 Given two nontrivial vector spaces E and F, a multilinear map $f\colon E^m \to F$ induces a partial map $\mathbf{P}(f)\colon \mathbf{P}(E)^m \to \mathbf{P}(F)$, called a *multiprojective map*, such that $\mathbf{P}(f)\colon (\mathbf{P}(E)^m - p \times \cdots \times p(\mathrm{Ker}\, f)) \to \mathbf{P}(F)$ is a total map, as in the following diagram:

$$\begin{array}{ccc}
E^m - \mathrm{Ker}\, f & \xrightarrow{f} & F - \{0\} \\
{\scriptstyle p\times\cdots\times p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\
\mathbf{P}(E)^m - p \times \cdots \times p(\mathrm{Ker}\, f) & \xrightarrow[\mathbf{P}(f)]{} & \mathbf{P}(F)
\end{array}$$

If f is injective, i.e. when $\mathrm{Ker}\, f = \{(0, \ldots, 0)\}$, then $\mathbf{P}(f)\colon \mathbf{P}(E)^m \to \mathbf{P}(F)$ is a total function called a *multiprojective transformation*.

Lemma 5.5.3 can be extended to multiprojective maps. The proof is left as an exercise to the reader.

21.2 More on Projective Completions and Frames

Lemma 5.6.3 can be extended to deal with multiprojective maps. The proof is left as an exercise (the tricky part is to show the uniqueness of $\widetilde{f}$). To state the new result, we need a definition. Given a multiaffine map $f\colon E^m \to F$, we know from Gallier [70] and the web site, see web page (Lemma 27.1.3) that f has a unique decomposition as a sum of multilinear maps of the form

$$f(a+u_1,\ldots,a+u_m) = f(a,\ldots,a) \mathbin{\widehat{+}} \sum_{\substack{S\subseteq\{1,\ldots,m\},\, k=|S|\\ S=\{i_1,\ldots,i_k\},\, k\geq 1}} f_S(u_{i_1},\ldots,u_{i_k}).$$

For every nonempty subset $S=\{i_1,\ldots,i_k\}$ of $\{1,2,\ldots,m\}$, let $\overrightarrow{i_S}\colon (\overrightarrow{E})^k \to (\overrightarrow{E})^m$ be the injection defined such that for all $u_1,\ldots,u_k \in \overrightarrow{E}$,

$$\overrightarrow{i_S}(u_1,\ldots,u_k)_j = \begin{cases} u_h & \text{if } j\in S \text{ and } j=i_h,\\ 0 & \text{if } j\notin S. \end{cases}$$

Lemma 21.2.1 *Given any affine space $(E,\overrightarrow{E})$, for every projective space $\mathbf{P}(F)$, every hyperplane H in F, and every $f\colon E^m \to \mathbf{P}(F)$ such that $f(E^m)\subseteq F_H$ and f is multiaffine, there is a unique multiprojective map $\widetilde{f}\colon (\widetilde{E})^m \to \mathbf{P}(F)$ such that*

$$f = \widetilde{f}\circ i \quad \textit{and} \quad \mathbf{P}(f_S) = \widetilde{f}\circ \mathbf{P}(\overrightarrow{i_S})$$

for every nonempty subset S of $\{1,\ldots,m\}$, as in the following diagram:

$$\begin{array}{ccccc} E^m & \xrightarrow{i} & E^m \subseteq (\widetilde{E})^m \supseteq \mathbf{P}(\overrightarrow{E})^m & \xleftarrow{\mathbf{P}(\overrightarrow{i_S})} & \mathbf{P}(\overrightarrow{E})^k \\ & f\searrow & \Big\downarrow \widetilde{f} & \swarrow \mathbf{P}(f_S) & \\ & & F_H \subseteq \mathbf{P}(F) \supseteq \mathbf{P}(H) & & \end{array}$$

The lemma also holds for symmetric multiaffine maps and symmetric multiprojective maps.

As a corollary of the previous two lemmas, we obtain the following lemma.

Lemma 21.2.2 *Given any two affine spaces E and F and an affine map $f\colon E\to F$, there is a unique projective map $\widetilde{f}\colon \widetilde{E}\to\widetilde{F}$ extending f and such that the restriction of $\widetilde{f}$ to $\mathbf{P}(\overrightarrow{E})$ agrees with $\mathbf{P}(f)$, as in the diagram below:*

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ {\scriptstyle i}\Big\downarrow & & \Big\downarrow{\scriptstyle i} \\ \widetilde{E} & \xrightarrow[\widetilde{f}]{} & \widetilde{F} \end{array}$$

Given a multiaffine map $f\colon E^m \to F$, there is a unique multiprojective map $\widetilde{f}\colon (\widetilde{E})^m \to \widetilde{F}$ extending f and such that the restriction of $\widetilde{f}$ to $\mathbf{P}(\overrightarrow{E})^m$ agrees with $\mathbf{P}(f_S)$ for every nonempty subset S of $\{1,\ldots,m\}$ (more precisely, $\mathbf{P}(f_S) = \widetilde{f}\circ\mathbf{P}(\overrightarrow{i_S})$), as in the diagram below:

$$\begin{array}{ccc} E^m & \xrightarrow{f} & F \\ {\scriptstyle i}\Big\downarrow & & \Big\downarrow{\scriptstyle i} \\ (\widetilde{E})^m & \xrightarrow[\widetilde{f}]{} & \widetilde{F} \end{array}$$

The lemma also holds for symmetric multiaffine maps and symmetric multiprojective maps. In both cases, the map $\widetilde{f}$ is called the projective completion of f.

Proof. Given the affine map $f: E \to F$, the composition $i \circ f: E \to \widetilde{F}$ is an affine map to the affine space $\widehat{F}_F = F$, complement of the projective hyperplane $\mathbf{P}(\overrightarrow{F})$ in $\widetilde{F} = \mathbf{P}(\widehat{F})$, and by lemma 5.6.3, there is a unique projective map $\widetilde{f}: \widetilde{E} \to \widetilde{F}$ extending $i \circ f$, i.e. extending f. For the second part, apply lemma 21.2.1. $\square$

It is often useful to define a projective frame of the projective completion $\widetilde{\mathcal{E}}$ of an affine space $\mathcal{E}$ from an affine frame of $\mathcal{E}$, and this is object of the next definitions.

Definition 21.2.3 Given an affine space $\mathcal{E}$ and an affine frame $(\Omega_1, (e_1, \ldots, e_n))$ for $\mathcal{E}$, where Ω_1 is the chosen origin of $\mathcal{E}$, we associate the projective frame

$$(e_{1\infty}, \ldots, e_{n\infty}, \Omega_1, \Omega_1 + e_1 + \cdots + e_n),$$

corresponding to the basis $(e_1, \ldots, e_n, \Omega_1)$ of $\widehat{\mathcal{E}}$. If $a \in \mathcal{E}$ has coordinates $(x_1, \ldots, x_n)$ over the frame $(\Omega_1, (e_1, \ldots, e_n))$ of $\mathcal{E}$, then $a \in \widetilde{\mathcal{E}}$ has homogeneous coordinates $(X_1, \ldots, X_{n+1})$ proportional to $(x_1, \ldots, x_n, 1)$, and thus, $X_{n+1} \neq 0$, and $x_i = \frac{X_i}{X_{n+1}}$, for $1 \leq i \leq n$. If $u_\infty \in \widetilde{\mathcal{E}}$ is a point at infinity, and $u \in \overrightarrow{\mathcal{E}}$ has coordinates $(x_1, \ldots, x_n)$ over the affine basis of $\mathcal{E}$, then $u_\infty \in \widetilde{\mathcal{E}}$ has homogeneous coordinates $(X_1, \ldots, X_n, 0)$ proportional to $(x_1, \ldots, x_n, 0)$. Thus, points at infinity of $\widetilde{\mathcal{E}}$ are characterized by $X_{n+1} = 0$.

To an affine (barycentric) frame $(a_1, \ldots, a_{n+1})$ for $\mathcal{E}$, we associate the projective frame

$$\left(a_1, \ldots, a_{n+1}, \frac{1}{n+1}(a_1 + \cdots + a_{n+1})\right),$$

corresponding to the basis $(a_1, \ldots, a_{n+1})$ of $\widehat{\mathcal{E}}$. If $a \in \mathcal{E}$ has barycentric coordinates $(\lambda_1, \ldots, \lambda_{n+1})$ over the frame $(a_1, \ldots, a_{n+1})$ for $\mathcal{E}$, with $\lambda_1 + \cdots + \lambda_{n+1} = 1$, then $a \in \widetilde{\mathcal{E}}$ has homogeneous coordinates $(X_1, \ldots, X_{n+1})$ proportional to $(\lambda_1, \ldots, \lambda_{n+1})$. Since $\lambda_1 + \cdots + \lambda_{n+1} = 1$, we must have $X_1 + \cdots + X_{n+1} \neq 0$, and $\lambda_i = \frac{X_i}{X_1 + \cdots + X_{n+1}}$, for $1 \leq i \leq n+1$. A vector $u \in \overrightarrow{\mathcal{E}}$ can be written uniquely as

$$u = \lambda_1 \mathbf{\Omega_1 a_1} + \cdots + \lambda_{n+1} \mathbf{\Omega_1 a_{n+1}},$$

where $\lambda_1 + \cdots + \lambda_{n+1} = 0$, the above being independent of $\Omega_1 \in \mathcal{E}$. Then, the point at infinity $u_\infty \in \widetilde{\mathcal{E}}$ has homogeneous coordinates $(X_1, \ldots, X_{n+1})$ proportional to $(\lambda_1, \ldots, \lambda_{n+1})$. Note that points are infinity are characterized by $X_1 + \cdots + X_{n+1} = 0$.

Most of the time, we use the first kind of projective frame. We are now almost ready to define rational curves, but we still need to define certain kinds of projections.

Definition 21.2.4 Given a vector space $\overrightarrow{E}$, for any hyperplane $\overrightarrow{H}$ in $\overrightarrow{E}$, and any vector u **not** in $\overrightarrow{H}$, since $\overrightarrow{E} = \overrightarrow{H} \oplus Ku$, the map $p_{u,H}: \overrightarrow{E} \to \overrightarrow{H}$, defined such that

$$p_{u,H}(h + \lambda u) = h,$$

for every $h \in \overrightarrow{H}$ and every $\lambda \in K$ is a linear map called the *(cylindric) projection from $\overrightarrow{E}$ onto $\overrightarrow{H}$ along the direction u.* We also define the projective map

$$\pi_{u,\, H}: (\mathbf{P}(\overrightarrow{E}) - \{[u]\}) \to \mathbf{P}(\overrightarrow{H}),$$

called the *central projection (or conic projection, or perspective projection) of center $[u]$ from $\mathbf{P}(\overrightarrow{E})$ onto $\mathbf{P}(\overrightarrow{H})$*, such that $\pi_{u,\, H} = \mathbf{P}(p_{u,H})$. Finally, given any affine space E with associated vector space $\overrightarrow{E}$, given any affine hyperplane H in E (with direction $\overrightarrow{H}$), given any point $\Omega \in E$ **not** in H, the *central projection (or conic projection, or perspective projection)*

$$\pi_{\Omega,\widehat{H}}: (\widetilde{E} - \{\Omega\}) \to \widetilde{H},$$

of center Ω *from* $\widetilde{E}$ *onto* $\widetilde{H}$ is the projective map induced by the projection $p_{\Omega,\widehat{H}}: \widehat{E} \to \widehat{H}$ from $\widehat{E}$ onto the hyperplane $\widehat{H}$ along the direction Ω in $\widehat{E}$ (where $\Omega = \langle \Omega, 1\rangle \in \widehat{E}$).

The reader can verify for himself that the projection $\pi_{\Omega,\widehat{H}}(\theta)$ of a point $\theta \in \widetilde{E}$ distinct from Ω is the intersection of the projective line passing through Ω and θ with the projective hyperplane $\widetilde{H}$. This is why it is called central (or conic, or perspective) projection of center Ω. It is only undefined at Ω.

It might help the reader to see what $p_{\Omega,\widehat{H}}: \widehat{E} \to \widehat{H}$ and $\pi_{\Omega,\widehat{H}}: (\widetilde{E} - \{\Omega\}) \to \widetilde{H}$ really are in terms of coordinate systems. Let $(e_1, \ldots, e_n)$ be a basis of $\widehat{H}$. Since $\Omega \notin H$, we can complete this basis into a basis $(e_1, \ldots, e_{n+1})$ of $\widehat{E}$, where $e_{n+1} = \langle \Omega, 1\rangle$. As a projective frame of $\widetilde{H}$, we choose $(a_i)_{1\leq i\leq n+1}$, where $a_i = [e_i]_\sim$ for $1 \leq i \leq n$, and $a_{n+1} = [e_1 + \cdots + e_n]_\sim$, and as a projective frame of $\widetilde{E}$, we choose $(b_i)_{1\leq i\leq n+2}$, where $b_i = [e_i]_\sim$ for $1 \leq i \leq n+1$, and $b_{n+2} = [e_1 + \cdots + e_{n+1}]_\sim$. Then, the matrix of the linear map $p_{\Omega,\widehat{H}}$ with respect to the bases $(e_1, \ldots, e_{n+1})$ and $(e_1, \ldots, e_n)$ is the $n \times (n+1)$ matrix $(I_n, 0)$, consisting of the $n \times n$ identity matrix, with an extra column consisting of n zeros. If a point $\theta \in \widetilde{E}$ has homogeneous coordinates $(x_1, \ldots, x_n, x_{n+1})$ with respect to the frame $(b_i)_{1\leq i\leq n+2}$ in $\widetilde{E}$, then $\pi_{\Omega,\widehat{H}}(\theta)$ has homogeneous coordinates $(x_1, \ldots, x_n)$ with respect to the frame $(a_i)_{1\leq i\leq n+1}$ in $\widetilde{H}$.

Thus, in terms of these coordinates systems, the central (perspective) projection $\pi_{\Omega,\widehat{H}}(\theta)$ amounts to dropping the $(n+1)$-th entry from the homogeneous coordinates $(x_1, \ldots, x_n, x_{n+1})$ for θ.

The central projection $\pi_{u,H}$ will be used in the situation where $\overrightarrow{E} = \widehat{\overrightarrow{F}}$ is the homogenized version of some vector space $\overrightarrow{F}$, viewed as an affine space, where $\overrightarrow{H} = \overrightarrow{F}$, the hyperplane in $\widehat{\overrightarrow{F}}$ corresponding to the vector space $\overrightarrow{F}$ itself, and where the direction of projection is $u = \langle \Omega, 1\rangle$ which does not belong to the hyperplane $\overrightarrow{F}$, as required (with $\Omega = 0$ the origin of $\overrightarrow{F}$). In this situation, the central projection

$$\pi_{\Omega,F}: (\widetilde{\overrightarrow{F}} - \{\Omega\}) \to \mathbf{P}(\overrightarrow{F}),$$

from the projective completion $\widetilde{\overrightarrow{F}}$ of $\overrightarrow{F}$ to $\mathbf{P}(\overrightarrow{F})$ is denoted as $\Pi\Omega: (\widetilde{\overrightarrow{F}} - \{\Omega\}) \to \mathbf{P}(\overrightarrow{F})$.

In the special case where $\overrightarrow{F} = \widehat{\mathcal{E}}$ is the homogenized version of some affine space $\mathcal{E}$, the central projection

$$\pi_{\Omega,\widehat{\mathcal{E}}}: (\widetilde{\widehat{\mathcal{E}}} - \{\Omega\}) \to \widetilde{\mathcal{E}}$$

from the projective completion $\widetilde{\widehat{\mathcal{E}}}$ of $\widehat{\mathcal{E}}$ to the projective completion $\widetilde{\mathcal{E}}$ of $\mathcal{E}$ is also denoted as $\Pi\Omega: (\widetilde{\widehat{\mathcal{E}}} - \{\Omega\}) \to \widetilde{\mathcal{E}}$.

It is also useful to define the following map.

Definition 21.2.5 Given any affine space E with associated vector space $\overrightarrow{E}$, the *natural projection* $\Pi: \widehat{E} \to \widetilde{E}$ *from* $\widehat{E}$ *onto* $\widetilde{E}$ is the map

$$\Pi: (\widehat{E} - \{0\}) \to \widetilde{E},$$

where

$$\Pi(\theta) = \begin{cases} a & \text{if } \theta = \langle a, \lambda\rangle, \text{ where } a \in E \text{ and } \lambda \neq 0; \\ u_\infty & \text{if } \theta = u, \text{ where } u \in (\overrightarrow{E} - \{0\}). \end{cases}$$

It is obvious that Π is surjective. Note that if $\sim$ is the equivalence relation on $\widehat{E}$ used to define $\widetilde{E} = \mathbf{P}(\widehat{E})$, since we identified $\widetilde{E}$ with the disjoint union $E \cup \mathbf{P}(\overrightarrow{E})$, then

$$\Pi\colon (\widehat{E} - \{0\}) \to \widetilde{E}$$

is just the canonical projection map of definition 5.2.1. In other words, we have $\Pi(\theta) = [\theta]_\sim$, for every $\theta \in (\widehat{E} - \{0\})$.

Let us consider again the situation where $E = \widehat{\overrightarrow{F}}$ is the homogenized version of some vector space $\overrightarrow{F}$. The following lemma shows that the projection map $p\colon \overrightarrow{F} \to \mathbf{P}(\overrightarrow{F})$ carries essentially as much information as the central projection $\Pi\Omega\colon (\widetilde{\overrightarrow{F}} - \{\Omega\}) \to \mathbf{P}(\overrightarrow{F})$. Recall that $\widetilde{\overrightarrow{F}}$ is identified with $\overrightarrow{F} \cup \mathbf{P}(\overrightarrow{F})$.

Lemma 21.2.6 *Given any vector space $\overrightarrow{F}$, the natural projection map $p\colon \overrightarrow{F} \to \mathbf{P}(\overrightarrow{F})$ is the restriction of the central projection $\Pi\Omega\colon ((\overrightarrow{F} - \{\Omega\}) \cup \mathbf{P}(\overrightarrow{F})) \to \mathbf{P}(\overrightarrow{F})$ to $(\overrightarrow{F} - \{\Omega\})$ (where $\overrightarrow{F}$ is viewed as an affine space and Ω is the origin 0 of $\overrightarrow{F}$). Furthermore, for every point $u_\infty \in \mathbf{P}(\overrightarrow{F})$ (point at infinity in $\widetilde{\overrightarrow{F}}$), we have $\Pi\Omega(u_\infty) = u_\infty = p(u)$.*

Proof. We just need to figure out what is the linear projection

$$p_{\Omega,F}\colon \widehat{\overrightarrow{F}} \to \overrightarrow{F},$$

onto the hyperplane $\overrightarrow{F}$, along the direction $\Omega = \langle \Omega, 1\rangle$ in $\widehat{\overrightarrow{F}}$. But we know that

$$\widehat{\overrightarrow{F}} = \overrightarrow{F} \oplus K\langle \Omega, 1\rangle,$$

and thus, for every $u \mathbin{\widehat{+}} \mu\langle \Omega, 1\rangle \in \widehat{\overrightarrow{F}}$, where $u \in \overrightarrow{F}$, we have

$$p_{\Omega,F}(u \mathbin{\widehat{+}} \mu\langle \Omega, 1\rangle) = u,$$

and consequently,

$$\pi_{\Omega,F}([u \mathbin{\widehat{+}} \mu\langle \Omega, 1\rangle]_\approx) = p(u) = [u]_\sim,$$

where $\approx$ is the equivalence relation on $\widehat{\overrightarrow{F}}$ inducing $\widetilde{\overrightarrow{F}}$, and $\sim$ is the equivalence relation on $\overrightarrow{F}$ inducing $\mathbf{P}(\overrightarrow{F})$. Now, $\widetilde{\overrightarrow{F}}$ is identified with the disjoint union $\overrightarrow{F} \cup \mathbf{P}(\overrightarrow{F})$. When $\mu \neq 0$, we have

$$\langle u, \mu\rangle = \langle 0 + u, \mu\rangle = \langle \Omega + u, \mu\rangle = \mu u \mathbin{\widehat{+}} \langle \Omega, \mu\rangle = \mu u \mathbin{\widehat{+}} \mu\langle \Omega, 1\rangle,$$

and so, we have

$$\pi_{\Omega,F}([\langle u, \mu\rangle]_\approx) = \pi_{\Omega,F}([\mu u \mathbin{\widehat{+}} \mu\langle \Omega, 1\rangle]_\approx) = p(\mu u) = p(u) = [u]_\sim.$$

Note that we used the fact that $\overrightarrow{F}$ is viewed as an affine space and that Ω is the origin 0 of $\overrightarrow{F}$ to conclude that $0 + u = u$. Since we have identified $[\langle u, \mu\rangle]_\approx$ with u, the above shows that p is the restriction of the central projection $\Pi\Omega$ to $(\overrightarrow{F} - \{\Omega\})$.[1] When $\mu = 0$, we still have

$$\Pi\Omega([u]_\approx) = [u]_\sim = p(u).$$

[1]Recall that $[\Omega]_\sim = [0]_\sim$ is undefined. This explains why we have to remove Ω from $\overrightarrow{F}$.

In the second case, $[u]_{\approx} = u_\infty \in \mathbf{P}(\overrightarrow{F})$ is a point at infinity in $\widetilde{\overrightarrow{F}}$, and p is not defined on such points at infinity, but $\Pi\Omega(u_\infty) = u_\infty = p(u) = [u]_\sim$, and thus, $\Pi\Omega$ is the identity on $\mathbf{P}(\overrightarrow{F})$. $\square$

If we apply lemma 21.2.6 to the special case where $\overrightarrow{F} = \widehat{\mathcal{E}}$ for some affine space $\mathcal{E}$, we note that the natural projection $\Pi\colon (\widehat{\mathcal{E}} - \{\Omega\}) \to \widetilde{\mathcal{E}}$ is the restriction of the central projection $\Pi\Omega\colon ((\widehat{\mathcal{E}} - \{\Omega\}) \cup \widetilde{\mathcal{E}}) \to \widetilde{\mathcal{E}}$ to $(\widehat{\mathcal{E}} - \{\Omega\})$ (where Ω is the origin 0 of $\widehat{\mathcal{E}}$, where $\widehat{\mathcal{E}}$ is viewed as an affine space). Furthermore, for every point $\theta_\infty \in \mathbf{P}(\widehat{\mathcal{E}}) = \widetilde{\mathcal{E}}$, we have $\Pi\Omega(\theta_\infty) = \theta_\infty = \Pi(\theta)$.

Remarks: (1) The reason for considering $\widetilde{\widehat{\mathcal{E}}}$ is that when we define a polynomial curve (or surface) in $\widehat{\mathcal{E}}$, in order to be rigorous, we need to specify the behavior at points at infinity before we can project this curve or surface onto $\widetilde{\mathcal{E}}$. If one is willing to ignore points at infinity, Π is all we need to define rational curves or surfaces. Unfortunately, if we want to treat points at infinity rigorously, we must use the central projection $\Pi\Omega$.

(2) Note that $\Pi\Omega(\theta_\infty) = \theta_\infty = \Pi(\theta)$ shows that every point at infinity in $\widetilde{\widehat{\mathcal{E}}}$ is projected onto itself in $\widetilde{\mathcal{E}}$, i.e., $\Pi\Omega$ is the identity on $\widetilde{\mathcal{E}}$. However, a point $u \in \widetilde{\widehat{\mathcal{E}}}$ (where $u \in \overrightarrow{\mathcal{E}}$) which is not a point at infinity is projected to the point at infinity u_∞ in $\widetilde{\mathcal{E}}$.

21.3 More on Multiprojective maps and Multilinear Maps

We will also need to relate multiprojective maps $\mathbf{P}(f)\colon (\widetilde{E})^m \to \widetilde{\mathcal{E}}$ associated with multilinear maps $f\colon (\widehat{E})^m \to \widehat{\mathcal{E}}$ to multiprojective maps $\widetilde{g}\colon (\widetilde{E})^m \to \widetilde{\widehat{\mathcal{E}}}$ associated with multiaffine maps $g\colon E^m \to \widehat{\mathcal{E}}$. The intuition is that a rational curve or surface (defined by a multilinear polar form $f\colon (\widehat{E})^m \to \widehat{\mathcal{E}}$, where $E = \mathbb{A}$ or $E = \mathbb{A}^2$) is the central projection of some polynomial curve of surface (defined by a multiaffine polar form $g\colon E^m \to \widehat{\mathcal{E}}$, where $E = \mathbb{A}$ or $E = \mathbb{A}^2$). Conversely, the central projection of a polynomial curve or surface is a rational curve or surface. To be rigorous, all of this has to be stated in terms of multiprojective maps. In fact, it will be simpler to prove a slightly more general lemma.

Lemma 21.3.1 *Given any affine space E and any vector space $\overrightarrow{F}$, for any multilinear map $f\colon (\widehat{E})^m \to \overrightarrow{F}$, the restriction $g\colon E^m \to \overrightarrow{F}$ of $f\colon (\widehat{E})^m \to \overrightarrow{F}$ to E^m is a multiaffine map such that $\mathbf{P}(f) = \Pi\Omega\circ\widetilde{g}$. Conversely, for any multiaffine map $g\colon E^m \to \overrightarrow{F}$, there is a multilinear map $f\colon (\widehat{E})^m \to \overrightarrow{F}$ such that $g\colon E^m \to \overrightarrow{F}$ is the restriction of $f\colon (\widehat{E})^m \to \overrightarrow{F}$ to E^m, and $\mathbf{P}(f) = \Pi\Omega \circ \widetilde{g}$. These two properties are expressed by the following diagram:*

$$
\begin{tikzcd}
 & E^m \arrow[r, "g"] \arrow[d, "i"'] & \overrightarrow{F} \arrow[d, "i"] \\
(\widehat{E})^m \arrow[r, "p"] \arrow[d, "f"'] & (\widetilde{E})^m \arrow[r, "\widetilde{g}"] \arrow[d, "\mathbf{P}(f)"'] & \widetilde{\overrightarrow{F}} \arrow[dl, "\Pi\Omega"] \\
\overrightarrow{F} \arrow[r, "p"'] & \mathbf{P}(\overrightarrow{F}) &
\end{tikzcd}
$$

The lemma also holds for symmetric multiaffine maps and symmetric multilinear maps.

Proof. Assume $f\colon (\widehat{E})^m \to \overrightarrow{F}$ is a multilinear map. The map $g\colon E^m \to \overrightarrow{F}$ is the restriction of $f\colon (\widehat{E})^m \to \overrightarrow{F}$ to E^m. Since f is multilinear, it is obvious that g is multiaffine.

Since $\widehat{E} = \overrightarrow{E} \oplus Ka$ for every $a \in E$, pick some origin $a \in E$. Now, since f is multilinear, we can write

$$f(a \mathbin{\widehat{+}} u_1, \ldots, a \mathbin{\widehat{+}} u_m) = \sum_{\substack{S\subseteq\{1,\ldots,m\}\\ S=\{i_1,\ldots,i_k\}\\ i_1<\ldots<i_k}} f(a, \ldots, u_{i_1}, \ldots, u_{i_k}, \ldots, a).$$

The above expression gives us the unique decomposition of the multiaffine map g as a sum of k-linear maps (where $f(a, \ldots, a)$ is treated as a point), and using lemma 20.1.2, we see that the unique multilinear map $\widehat{g}\colon (\widehat{E})^m \to \widehat{\overrightarrow{F}}$ extending g, is given by the expression

$$\widehat{g}(u_1 \mathbin{\widehat{+}} \lambda_1 a, \ldots, u_m \mathbin{\widehat{+}} \lambda_m a) = \sum_{\substack{S\subseteq\{1,\ldots,m\}\\ S=\{i_1,\ldots,i_k\}\\ i_1<\ldots<i_k}} \Bigg(\prod_{\substack{j\in\{1,\ldots,m\}\\ j\notin S}} \lambda_j \Bigg) \cdot f(a, \ldots, u_{i_1}, \ldots, u_{i_k}, \ldots, a),$$

where the symbol $\cdot$ is used to denote scalar multiplication in $\widehat{\overrightarrow{F}}$, to avoid confusion with scalar multiplication in $\overrightarrow{F}$ which is denoted as juxtaposition, and $f(a, \ldots, a)$ is treated as a point of $\widehat{\overrightarrow{F}}$. Since f is multilinear, we have

$$f(u_1 \mathbin{\widehat{+}} \lambda_1 a, \ldots, u_m \mathbin{\widehat{+}} \lambda_m a) = \sum_{\substack{S\subseteq\{1,\ldots,m\}\\ S=\{i_1,\ldots,i_k\}\\ i_1<\ldots<i_k}} \Bigg(\prod_{\substack{j\in\{1,\ldots,m\}\\ j\notin S}} \lambda_j \Bigg) f(a, \ldots, u_{i_1}, \ldots, u_{i_k}, \ldots, a).$$

If $\widehat{g}(u_1 \mathbin{\widehat{+}} \lambda_1 a, \ldots, u_m \mathbin{\widehat{+}} \lambda_m a)$ is a point in $\widehat{\overrightarrow{F}}$, then we must have $\lambda_i \neq 0$, for $1 \leq i \leq m$, and we can write

$$\begin{aligned}\widehat{g}(u_1 \mathbin{\widehat{+}} \lambda_1 a, \ldots, u_m \mathbin{\widehat{+}} \lambda_m a) &= \lambda_1 \cdots \lambda_m \cdot \sum_{\substack{S\subseteq\{1,\ldots,m\}\\ S=\{i_1,\ldots,i_k\}\\ i_1<\ldots<i_k}} \Bigg(\prod_{j\in S} \lambda_j^{-1} \Bigg) f(a, \ldots, u_{i_1}, \ldots, u_{i_k}, \ldots, a),\\ &= \lambda_1 \cdots \lambda_m \cdot \sum_{\substack{S\subseteq\{1,\ldots,m\}\\ S=\{i_1,\ldots,i_k\}\\ i_1<\ldots<i_k}} f(a, \ldots, \lambda_{i_1}^{-1} u_{i_1}, \ldots, \lambda_{i_k}^{-1} u_{i_k}, \ldots, a).\end{aligned}$$

We will now use lemma 21.2.6, which tells us that

$$\Pi\Omega([\langle u, \lambda\rangle]_{\approx}) = p(u) = [u]_{\sim},$$

for $u \in \overrightarrow{F} - \{\Omega\}$ (where Ω is the origin 0 in $\overrightarrow{F}$), and $\Pi\Omega(u_\infty) = p(u)$. By lemma 21.2.6, we have

$$\Pi\Omega(\widetilde{g}([u_1 \mathbin{\widehat{+}} \lambda_1 a], \ldots, [u_m \mathbin{\widehat{+}} \lambda_m a])) = p\Bigg(\sum_{\substack{S\subseteq\{1,\ldots,m\}\\ S=\{i_1,\ldots,i_k\}\\ i_1<\ldots<i_k}} f(a, \ldots, \lambda_{i_1}^{-1} u_{i_1}, \ldots, \lambda_{i_k}^{-1} u_{i_k}, \ldots, a) \Bigg).$$

We also have

$$f(u_1 \mathbin{\widehat{+}} \lambda_1 a, \ldots, u_m \mathbin{\widehat{+}} \lambda_m a) = \lambda_1 \cdots \lambda_m \sum_{\substack{S\subseteq\{1,\ldots,m\}\\ S=\{i_1,\ldots,i_k\}\\ i_1<\ldots<i_k}} f(a, \ldots, \lambda_{i_1}^{-1} u_{i_1}, \ldots, \lambda_{i_k}^{-1} u_{i_k}, \ldots, a),$$

and so

$$\mathbf{P}(f)([u_1 \mathbin{\widehat{+}} \lambda_1 a], \ldots, [u_m \mathbin{\widehat{+}} \lambda_m a]) = p\Bigg(\sum_{\substack{S \subseteq \{1,\ldots,m\} \\ S=\{i_1,\ldots,i_k\} \\ i_1 < \ldots < i_k}} f(a, \ldots, \lambda_{i_1}^{-1} u_{i_1}, \ldots, \lambda_{i_k}^{-1} u_{i_k}, \ldots, a) \Bigg).$$

Thus, we have shown that

$$\mathbf{P}(f)([u_1 \mathbin{\widehat{+}} \lambda_1 a], \ldots, [u_m \mathbin{\widehat{+}} \lambda_m a]) = \Pi\Omega(\widetilde{g}([u_1 \mathbin{\widehat{+}} \lambda_1 a], \ldots, [u_m \mathbin{\widehat{+}} \lambda_m a])).$$

If $\widehat{g}(u_1 \mathbin{\widehat{+}} \lambda_1 a, \ldots, u_m \mathbin{\widehat{+}} \lambda_m a)$ is a vector in $\widehat{\overrightarrow{F}}$, and thus in $\overrightarrow{F}$, then since $\Pi\Omega(u_\infty) = p(u)$ for $u \in \overrightarrow{F}$, we have

$$\mathbf{P}(f)([u_1 \mathbin{\widehat{+}} \lambda_1 a], \ldots, [u_m \mathbin{\widehat{+}} \lambda_m a]) = \Pi\Omega(\widetilde{g}([u_1 \mathbin{\widehat{+}} \lambda_1 a], \ldots, [u_m \mathbin{\widehat{+}} \lambda_m a])).$$

We still have to prove the converse. Given a multiaffine map $g\colon E^m \to \overrightarrow{F}$, we know that g has a unique decomposition as a sum of k-linear maps, of the form

$$g(a + u_1, \ldots, a + u_m) = g(a, \ldots, a) \mathbin{\widehat{+}} \sum_{\substack{S \subseteq \{1,\ldots,m\},\, k=|S| \\ S=\{i_1,\ldots,i_k\},\, k \geq 1}} g_S(u_{i_1}, \ldots, u_{i_k}).$$

Let us define the map $f\colon (\widehat{E})^m \to \overrightarrow{F}$, such that

$$\begin{aligned} f(u_1 \mathbin{\widehat{+}} \lambda_1 a, \ldots, u_m \mathbin{\widehat{+}} \lambda_m a) = {} & \lambda_1 \cdots \lambda_m g(a, \ldots, a) \\ & \mathbin{\widehat{+}} \sum_{\substack{S \subseteq \{1,\ldots,m\},\, k=|S| \\ S=\{i_1,\ldots,i_k\},\, k \geq 1}} \Bigg(\prod_{\substack{j \in \{1,\ldots,m\} \\ j \notin S}} \lambda_j \Bigg) g_S(u_{i_1}, \ldots, u_{i_k}). \end{aligned}$$

It is immediately verified that f is multilinear, and clearly,

$$g(a_1, \ldots, a_m) = f(a_1, \ldots, a_m),$$

for all $a_1, \ldots, a_m \in E$. To conclude, we apply the previous part of the proof, which shows that $\mathbf{P}(f) = \Pi\Omega \circ \widetilde{g}$.
$\square$

Applying lemma 21.3.1 to the special case where $\overrightarrow{F} = \widehat{\mathcal{E}}$ for some affine space $\mathcal{E}$, we obtain the following corollary.

Lemma 21.3.2 *Given any two affine spaces E, $\mathcal{E}$, for any multilinear map $f\colon (\widehat{E})^m \to \widehat{\mathcal{E}}$, the restriction $g\colon E^m \to \widehat{\mathcal{E}}$ of $f\colon (\widehat{E})^m \to \widehat{\mathcal{E}}$ to E^m is a multiaffine map such that $\mathbf{P}(f) = \Pi\Omega \circ \widetilde{g}$. Conversely, for any multiaffine map $g\colon E^m \to \widehat{\mathcal{E}}$, there is a multilinear map $f\colon (\widehat{E})^m \to \widehat{\mathcal{E}}$ such that $g\colon E^m \to \widehat{\mathcal{E}}$ is the restriction of $f\colon (\widehat{E})^m \to \widehat{\mathcal{E}}$ to E^m, and $\mathbf{P}(f) = \Pi\Omega \circ \widetilde{g}$. These two properties are expressed by the following diagram:*

$$\begin{tikzcd} & E^m \arrow[r, "g"] \arrow[d, "i"'] & \widehat{\mathcal{E}} \arrow[d, "i"] \\ (\widehat{E})^m \arrow[r, "p"] \arrow[d, "f"'] & (\widetilde{E})^m \arrow[r, "\widetilde{g}"] \arrow[d, "\mathbf{P}(f)"'] & \widetilde{\widehat{\mathcal{E}}} \arrow[dl, "\Pi\Omega"] \\ \widehat{\mathcal{E}} \arrow[r, "p"'] & \widetilde{\mathcal{E}} & \end{tikzcd}$$

The lemma also holds for symmetric multiaffine maps and symmetric multilinear maps.