# Chapter 27

# Appendix 2

## 27.1 Multiaffine Maps

The purpose of this section is to prove a lemma showing how a multiaffine map can be expressed in terms of some uniquely determined multilinear maps. For the reader's convenience, we recall the definition of a multilinear map. Let $E_1, \ldots, E_m$, and $F$, be vector spaces over $\mathbb{R}$, where $m \geq 1$.

**Definition 27.1.1** A function $f\colon E_1 \times \ldots \times E_m \to F$ is a *multilinear map (or an $m$-linear map)*, iff it is linear in each argument, holding the others fixed. More explicitly, for every $i$, $1 \leq i \leq m$, for all $x_1 \in E_1 \ldots$, $x_{i-1} \in E_{i-1}$, $x_{i+1} \in E_{i+1}, \ldots$, $x_m \in E_m$, for every family $(y_j)_{j \in J}$ of vectors in $E_i$, for every family $(\lambda_j)_{j \in J}$ of scalars,

$$f(x_1,\, \ldots,\, x_{i-1},\, \sum_{j \in J} \lambda_j y_j,\, x_{i+1},\, \ldots,\, x_n) = \sum_{j \in J} \lambda_j f(x_1,\, \ldots,\, x_{i-1},\, y_j,\, x_{i+1},\, \ldots,\, x_n).$$

Having reviewed the definition of a multilinear map, we define multiaffine maps. Let $E_1, \ldots, E_m$, and $F$, be affine spaces over $\mathbb{R}$, where $m \geq 1$.

**Definition 27.1.2** A function $f\colon E_1 \times \ldots \times E_m \to F$ is a *multiaffine map (or an $m$-affine map)*, iff it is affine in each argument, that is, for every $i$, $1 \leq i \leq m$, for all $a_1 \in E_1, \ldots, a_{i-1} \in E_{i-1}, a_{i+1} \in E_{i+1}, \ldots, a_m \in E_m, a \in E_i$, the map $a \mapsto f(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_m)$ is an affine map, i.e. iff it preserves barycentric combinations. More explicitly, for every family $(b_j)_{j \in J}$ of points in $E_i$, for every family $(\lambda_j)_{j \in J}$ of scalars such that $\sum_{j \in J} \lambda_j = 1$, we have

$$f(a_1, \ldots, a_{i-1}, \sum_{j \in J} \lambda_j b_j, a_{i+1}, \ldots, a_m) = \sum_{j \in J} \lambda_j f(a_1, \ldots, a_{i-1}, b_j, a_{i+1}, \ldots, a_m).$$

An arbitrary function $f\colon E^m \to F$ is symmetric (where $E$ and $F$ are arbitrary sets, not just vector spaces or affine spaces), iff

$$f(x_{\pi(1)}, \ldots, x_{\pi(m)}) = f(x_1, \ldots, x_m),$$

for every permutation $\pi\colon \{1, \ldots, m\} \to \{1, \ldots, m\}$.

It is immediately verified that a multilinear map is also a multiaffine map (viewing a vector space as an affine space).

The next lemma will show that an $n$-affine form can be expressed as the sum of $2^n - 1$ $k$-linear forms, where $1 \leq k \leq n$, plus a constant. Thus, we see that the main difference between multilinear forms and multiaffine forms, is that multilinear forms are *homogeneous* in their arguments, whereas multiaffine forms are not, but they are sums of homogeneous forms. A good example of $n$-affine forms is the elementary symmetric functions. Given $n$ variables $x_1, \ldots, x_n$, for each $k$, $0 \leq k \leq n$, we define the $k$-th *elementary symmetric function* $\sigma_k(x_1, \ldots, x_n)$, for short, $\sigma_k$, as follows:

$\sigma_0 = 1$;

$\sigma_1 = x_1 + \cdots + x_n$;

$\sigma_2 = x_1x_2 + x_1x_3 + \cdots + x_1x_n + x_2x_3 + \cdots + x_{n-1}x_n$;

$\sigma_k = \sum_{1 \leq i_1 < \ldots < i_k \leq n} x_{i_1} \cdots x_{i_k}$;

$\sigma_n = x_1x_2 \cdots x_n$.

A concise way to express $\sigma_k$ is as follows:

$$\sigma_k = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I|=k}} \left( \prod_{i \in I} x_i \right).$$

Note that $\sigma_k$ consists of a sum of $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ terms of the form $x_{i_1} \cdots x_{i_k}$. As a consequence,

$$\sigma_k(x, x, \ldots, x) = \binom{n}{k} x^k.$$

Clearly, each $\sigma_k$ is symmetric.

We will prove a generalization of lemma 2.7.2, characterizing multiaffine maps in terms of multilinear maps. The proof is more complicated than might be expected, but luckily, an adaptation of Cartan's use of "successive differences" allows us to overcome the complications.

In order to understand where the proof of the next lemma comes from, let us consider the special case of a biaffine map $f \colon E^2 \to F$, where $F$ is a vector space. Because $f$ is biaffine, note that

$$f(a_1 + v_1, a_2 + v_2) - f(a_1, a_2 + v_2) = g(v_1, a_2 + v_2)$$

is a linear map in $v_1$, and as a difference of affine maps in $a_2 + v_2$, it is affine in $a_2 + v_2$. But then, we have

$$g(v_1, a_2 + v_2) = g(v_1, a_2) + h_1(v_1, v_2),$$

where $h_1(v_1, v_2)$ is linear in $v_2$. Thus, we have

$$f(a_1 + v_1, a_2 + v_2) - f(a_1, a_2 + v_2) = f(a_1 + v_1, a_2) - f(a_1, a_2) + h_1(v_1, v_2),$$

that is,

$$f(a_1 + v_1, a_2 + v_2) = f(a_1, a_2) + h_1(v_1, v_2) + f(a_1, a_2 + v_2) - f(a_1, a_2) + f(a_1 + v_1, a_2) - f(a_1, a_2).$$

Since

$$g(v_1, a_2 + v_2) - g(v_1, a_2) = h_1(v_1, v_2),$$

where both $g(v_1, a_2 + v_2)$ and and $g(v_1, a_2)$ are linear in $v_1$, $h_1(v_1, v_2)$ is also linear in $v_1$, and since we already know that $h_1(v_1, v_2)$ is linear in $v_2$, then $h_1$ is bilinear. But $f(a_1, a_2 + v_2) - f(a_1, a_2)$ is linear in $v_2$, and $f(a_1 + v_1, a_2) - f(a_1, a_2)$ is linear in $v_1$, which shows that we can write

$$f(a_1 + v_1, a_2 + v_2) = f(a_1, a_2) + h_1(v_1, v_2) + h_2(v_1) + h_3(v_2),$$

where $h_1$ is bilinear, and $h_2$ and $h_3$ are linear. The uniqueness of $h_1$ is clear, and as a consequence, the uniqueness of $h_2$ and $h_3$ follows easily.

The above argument uses the crucial fact that the expression

$$f(a_1 + v_1, a_2 + v_2) - f(a_1, a_2 + v_2) - f(a_1 + v_1, a_2) + f(a_1, a_2) = h_1(v_1, v_2),$$

is bilinear. Thus, we are led to consider differences of the form

$$\Delta_{v_1} f(a_1, a_2) = f(a_1 + v_1, a_2) - f(a_1, a_2).$$

The slight trick is that if we compute the difference

$$\Delta_{v_2}\Delta_{v_1} f(a_1, a_2) = \Delta_{v_1} f(a_1, a_2 + v_2) - \Delta_{v_1} f(a_1, a_2),$$

where we incremented the **second** argument instead of the first argument as in the previous step, we get

$$\Delta_{v_2}\Delta_{v_1} f(a_1, a_2) = f(a_1 + v_1, a_2 + v_2) - f(a_1, a_2 + v_2) - f(a_1 + v_1, a_2) + f(a_1, a_2),$$

which is precisely the bilinear map $h_1(v_1, v_2)$. This idea of using successive differences (where at each step, we move from argument $k$ to argument $k + 1$) will be central to the proof of the next lemma.

**Lemma 27.1.3** *For every $m$-affine map $f\colon E^m \to F$, there are $2^m - 1$ unique multilinear maps $f_S\colon E^k \to \overrightarrow{F}$, where $S \subseteq \{1, \ldots, m\}$, $k = |S|$, $S \neq \emptyset$, such that*

$$f(a_1 + v_1, \ldots, a_m + v_m) = f(a_1, \ldots, a_m) + \sum_{\substack{S\subseteq\{1,\ldots,m\},\ k=|S| \\ S=\{i_1,\ldots,i_k\},\ k\geq 1}} f_S(v_{i_1}, \ldots, v_{i_k}),$$

*for all $a_1 \ldots, a_m \in E$, and all $v_1, \ldots, v_m \in \overrightarrow{E}$.*

*Proof*. First, we show that we can restrict our attention to multiaffine maps $f\colon E^m \to F$, where $F$ is a vector space. Pick any $b \in F$, and define $h\colon E^m \to \overrightarrow{F}$, where $h(a) = \mathbf{bf(a)}$ for every $a = (a_1, \ldots, a_m) \in E^m$, so that $f(a) = b + h(a)$. We claim that $h$ is multiaffine. For every $i$, $1 \leq i \leq m$, for every $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m \in E$, let $f_i\colon E \to F$ be the map

$$a_i \mapsto f(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_m),$$

and let $h_i\colon E \to \overrightarrow{F}$ be the map

$$a_i \mapsto h(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_m).$$

Since $f$ is multiaffine, we have

$$h_i(a_i + u) = \mathbf{b(f(a)} + \overrightarrow{f_i}(\mathbf{u})) = \mathbf{bf(a)} + \overrightarrow{f_i}(u),$$

where $a = (a_1, \ldots, a_m)$, and where $\overrightarrow{f_i}$ is the linear map associated with $f_i$, which shows that $h_i$ is an affine map with associated linear map $\overrightarrow{f_i}$.

Thus, we now assume that $F$ is a vector space. Given an $m$-affine map $f\colon E^m \to F$, for every $(v_1, \ldots, v_m) \in \overrightarrow{E}^m$, we define

$$\Delta_{v_m}\Delta_{v_{m-1}} \cdots \Delta_{v_1} f$$

inductively as follows: for every $a = (a_1, \ldots, a_m) \in E^m$,

$\Delta_{v_1} f(a) = f(a_1 + v_1, a_2, \ldots, a_m) - f(a_1, a_2, \ldots, a_m),$

and generally, for all $i, 1 \leq i \leq m$,

$\Delta_{v_i} f(a) = f(a_1, \ldots, a_{i-1}, a_i + v_i, a_{i+1}, \ldots, a_m) - f(a_1, a_2, \ldots, a_m);$

Thus, we have

$\Delta_{v_{k+1}} \Delta_{v_k} \cdots \Delta_{v_1} f(a) = \Delta_{v_k} \cdots \Delta_{v_1} f(a_1, \ldots, a_{k+1} + v_{k+1}, \ldots, a_m) - \Delta_{v_k} \cdots \Delta_{v_1} f(a),$
where $1 \leq k \leq m-1$.

We claim that the following properties hold:

(1) Each $\Delta_{v_k} \cdots \Delta_{v_1} f(a)$ is $k$-linear in $v_1, \ldots, v_k$ and $(m-k)$-affine in $a_{k+1}, \ldots, a_m$;

(2) We have

$$\Delta_{v_m} \cdots \Delta_{v_1} f(a) = \sum_{k=0}^{m} (-1)^{m-k} \sum_{1 \leq i_1 < \ldots < i_k \leq m} f(a_1, \ldots, a_{i_1} + v_{i_1}, \ldots, a_{i_k} + v_{i_k}, \ldots, a_m).$$

Properties (1) and (2) are proved by induction on $k$. We prove (1), leaving (2) as an easy exercise. Since $f$ is $m$-affine, it is affine in its first argument, and so,

$$\Delta_{v_1} f(a) = f(a_1 + v_1, a_2, \ldots, a_m) - f(a_1, a_2, \ldots, a_m)$$

is a linear map in $v_1$, and since it is the difference of two multiaffine maps in $a_2, \ldots, a_m$, it is $(m-1)$-affine in $a_2, \ldots, a_m$.

Assuming that $\Delta_{v_k} \cdots \Delta_{v_1} f(a)$ is $k$-linear in $v_1, \ldots, v_k$ and $(m-k)$-affine in $a_{k+1}, \ldots, a_m$, since it is affine in $a_{k+1}$,

$$\Delta_{v_{k+1}} \Delta_{v_k} \cdots \Delta_{v_1} f(a) = \Delta_{v_k} \cdots \Delta_{v_1} f(a_1, \ldots, a_{k+1} + v_{k+1}, \ldots, a_m) - \Delta_{v_k} \cdots \Delta_{v_1} f(a)$$

is linear in $v_{k+1}$, and since it is the difference of two $k$-linear maps in $v_1, \ldots, v_k$, it is $(k+1)$-linear in $v_1, \ldots, v_{k+1}$, and since it is the difference of two $(m-k-1)$-affine maps in $a_{k+2} \ldots, a_m$, it is $(m-k-1)$-affine in $a_{k+2} \ldots, a_m$. This concludes the induction.

As a consequence of (1), $\Delta_{v_m} \cdots \Delta_{v_1} f$ is a $m$-linear map. Then, in view of (2), we can write

$$f(a_1 + v_1, \ldots, a_m + v_m) = \\ \Delta_{v_m} \cdots \Delta_{v_1} f(a) + \sum_{k=0}^{m-1} (-1)^{m-k-1} \sum_{1 \leq i_1 < \ldots < i_k \leq m} f(a_1, \ldots, a_{i_1} + v_{i_1}, \ldots, a_{i_k} + v_{i_k}, \ldots, a_m),$$

and since every

$$f(a_1, \ldots, a_{i_1} + v_{i_1}, \ldots, a_{i_k} + v_{i_k}, \ldots, a_m)$$

in the above sum contains at most $m-1$ of the $v_1, \ldots, v_m$, we can apply the induction hypothesis, which gives us sums of $k$-linear maps, for $1 \leq k \leq m-1$, and of $2^m - 1$ terms of the form $(-1)^{m-k-1} f(a_1, \ldots, a_m)$, which all cancel out except for a single $f(a_1, \ldots, a_m)$, which proves the existence of multilinear maps $f_S$ such that

$$f(a_1 + v_1, \ldots, a_m + v_m) = f(a_1, \ldots, a_m) + \sum_{\substack{S \subseteq \{1, \ldots, m\},\, k = |S| \\ S = \{i_1, \ldots, i_k\},\, k \geq 1}} f_S(v_{i_1}, \ldots, v_{i_k}),$$

for all $a_1 \ldots, a_m \in E$, and all $v_1, \ldots, v_m \in \overrightarrow{E}$.

We still have to prove the uniqueness of the linear maps in the sum. This can be done using the $\Delta_{v_m} \cdots \Delta_{v_1} f$. We claim the following slightly stronger property, that can be shown by induction on $m$: if

$$g(a_1 + v_1, \ldots, a_m + v_m) = f(a_1, \ldots, a_m) + \sum_{\substack{S \subseteq \{1,\ldots,m\},\, k=|S| \\ S=\{i_1,\ldots,i_k\},\, k \geq 1}} f_S(v_{i_1}, \ldots, v_{i_k}),$$

for all $a_1 \ldots, a_m \in E$, and all $v_1, \ldots, v_m \in \overrightarrow{E}$, then

$$\Delta_{v_{j_n}} \cdots \Delta_{v_{j_1}} g(a) = f_{\{j_1,\ldots,j_n\}}(v_{j_1}, \ldots, v_{j_n}),$$

where $\{j_1, \ldots, j_n\} \subseteq \{1, \ldots, m\}$, $j_1 < \ldots < j_n$, and $a = (a_1, \ldots, a_m)$. We can now show the uniqueness of the $f_S$, where $S \subseteq \{1, \ldots, n\}$, $S \neq \emptyset$, by induction. Indeed, from above, we get

$$\Delta_{v_m} \cdots \Delta_{v_1} f = f_{\{1,\ldots,m\}}.$$

But $g - f_{\{1,\ldots,m\}}$ is also $m$-affine, and it is a sum of the above form, where $n = m - 1$, so we can apply the induction hypothesis, and conclude the uniqueness of all the $f_S$. $\square$

When $f\colon E^m \to F$ is a symmetric $m$-affine map, we can obtain a more precise characterization in terms of $m$ symmetric $k$-linear maps, $1 \leq k \leq m$.

**Lemma 27.1.4** *For every symmetric $m$-affine map $f\colon E^m \to F$, there are $m$ unique symmetric multilinear maps $f_k\colon E^k \to \overrightarrow{F}$, where $1 \leq k \leq m$, such that*

$$f(a_1 + v_1, \ldots, a_m + v_m) = f(a_1, \ldots, a_m) + \sum_{k=1}^{m} \sum_{1 \leq i_1 < \ldots < i_k \leq m} f_k(v_{i_1}, \ldots, v_{i_k}),$$

*for all $a_1 \ldots, a_m \in E$, and all $v_1, \ldots, v_m \in \overrightarrow{E}$.*

*Proof*. Since $f$ is symmetric, for every $k$, $1 \leq k \leq m$, for every sequences $\langle i_1 \ldots, i_k \rangle$ and $\langle j_1 \ldots, j_k \rangle$ such that $1 \leq i_1 < \ldots < i_k \leq m$ and $1 \leq j_1 < \ldots < j_k \leq m$, there is a permutation $\pi$ such that $\pi(i_1) = j_1, \ldots, \pi(i_k) = j_k$, and since

$$f(x_{\pi(1)}, \ldots, x_{\pi(m)}) = f(x_1, \ldots, x_m),$$

by the uniqueness of the sum given by lemma 27.1.3, we must have

$$f_{\{i_1,\ldots,i_k\}}(v_{j_1}, \ldots, v_{j_k}) = f_{\{j_1,\ldots,j_k\}}(v_{j_1}, \ldots, v_{j_k}),$$

which shows that,

$$f_{\{i_1,\ldots,i_k\}} = f_{\{j_1,\ldots,j_k\}},$$

and then that each $f_{\{i_1,\ldots,i_k\}}$ is symmetric, and thus, letting $f_k = f_{\{1,\ldots,k\}}$, we have

$$f(a_1 + v_1, \ldots, a_m + v_m) = f(a_1, \ldots, a_m) + \sum_{k=1}^{m} \sum_{\substack{S \subseteq \{1,\ldots,m\} \\ S=\{i_1,\ldots,i_k\}}} f_k(v_{i_1}, \ldots, v_{i_k}),$$

for all $a_1 \ldots, a_m \in E$, and all $v_1, \ldots, v_m \in \overrightarrow{E}$. $\square$

Thus, a symmetric $m$-affine map is obtained by making symmetric in $v_1, \ldots v_m$, the sum $f_m + f_{m-1} + \cdots + f_1$ of $m$ symmetric $k$-linear maps, $1 \leq k \leq m$. The above lemma shows that it is equivalent to deal with symmetric $m$-affine maps, or with symmetrized sums $f_m + f_{m-1} + \cdots + f_1$ of symmetric $k$-linear maps, $1 \leq k \leq m$.

When $\overrightarrow{E}$ is a vector space of finite dimension $n$, and $\overrightarrow{F}$ is a vector space, we obtain the following characterization of multilinear maps (readers who are nervous, may assume for simplicity that $\overrightarrow{F} = \mathbb{R}$). Let $(e_1, \ldots, e_n)$ be a basis of $E$.

**Lemma 27.1.5** *Given any vector space $\overrightarrow{E}$ of finite dimension $n$, and any vector space $\overrightarrow{F}$, for any basis $(e_1, \ldots, e_n)$ of $\overrightarrow{E}$, for any symmetric multilinear map $f\colon E^m \to \overrightarrow{F}$, for any $m$ vectors*

$$v_j = v_{1,\,j}e_1 + \cdots + v_{n,\,j}e_n \in \overrightarrow{E},$$

*we have*

$$f(v_1, \ldots, v_m) = \sum_{\substack{I_1 \cup \ldots \cup I_n = \{1,\ldots,m\} \\ I_i \cap I_j = \emptyset,\, i \neq j \\ 1 \leq i,j \leq n}} \left( \prod_{i_1 \in I_1} v_{1,\,i_1} \right) \cdots \left( \prod_{i_n \in I_n} v_{n,\,i_n} \right) f(\underbrace{e_1, \ldots, e_1}_{|I_1|}, \ldots, \underbrace{e_n, \ldots, e_n}_{|I_n|}),$$

*and for any $v \in \overrightarrow{E}$, the homogeneous polynomial function $h$ associated with $f$ is given by*

$$h(v) = \sum_{\substack{k_1 + \cdots + k_n = m \\ 0 \leq k_i,\, 1 \leq i \leq n}} \binom{m}{k_1, \ldots, k_n} v_1^{k_1} \cdots v_n^{k_n}\, f(\underbrace{e_1, \ldots, e_1}_{k_1}, \ldots, \underbrace{e_n, \ldots, e_n}_{k_n}).$$

*Proof*. By multilinearity of $f$, we have

$$f(v_1, \ldots, v_m) = \sum_{(i_1,\ldots,i_m) \in \{1,\ldots,n\}^m} v_{i_1,\,1} \cdots v_{i_m,\,m} f(e_{i_1}, \ldots, e_{i_m}).$$

Since $f$ is symmetric, we can reorder the basis vectors arguments of $f$, and this amounts to choosing $n$ disjoint sets $I_1, \ldots, I_n$ such that $I_1 \cup \ldots \cup I_n = \{1, \ldots, m\}$, where each $I_j$ specifies which arguments of $f$ are the basis vector $e_j$. Thus, we get

$$f(v_1, \ldots, v_m) = \sum_{\substack{I_1 \cup \ldots \cup I_n = \{1,\ldots,m\} \\ I_i \cap I_j = \emptyset,\, i \neq j \\ 1 \leq i,j \leq n}} \left( \prod_{i_1 \in I_1} v_{1,\,i_1} \right) \cdots \left( \prod_{i_n \in I_n} v_{n,\,i_n} \right) f(\underbrace{e_1, \ldots, e_1}_{|I_1|}, \ldots, \underbrace{e_n, \ldots, e_n}_{|I_n|}).$$

When we calculate $h(v) = f(\underbrace{v, \ldots, v}_{m})$, we get the same product $v_1^{k_1} \cdots v_n^{k_n}$ a multiple number of times, which is the number of ways of choosing $n$ disjoints sets $I_j$, each of cardinality $k_i$, where $k_1 + \cdots + k_n = m$, which is precisely $\binom{m}{k_1, \ldots, k_n}$, which explains the second formula. $\square$

Thus, lemma 27.1.5 shows that we can write $h(v)$ as

$$h(v) = \sum_{\substack{k_1 + \cdots + k_n = m \\ 0 \leq k_i,\, 1 \leq i \leq n}} v_1^{k_1} \cdots v_n^{k_n}\, c_{k_1,\ldots,k_n},$$

for some "coefficients" $c_{k_1,\ldots,k_n} \in \overrightarrow{F}$, which are vectors. When $\overrightarrow{F} = \mathbb{R}$, the homogeneous polynomial function $h$ of degree $m$ in $n$ arguments $v_1, \ldots, v_n$, agrees with the notion of polynomial function defined by a homogeneous polynomial. Indeed, $h$ is the homogeneous polynomial function induced by the homogeneous polynomial of degree $m$ in the variables $X_1, \ldots, X_n$,

$$\sum_{\substack{(k_1,\ldots,k_n),\, k_j \geq 0 \\ k_1+\cdots+k_n=m}} c_{k_1,\ldots,k_n} X_1^{k_1} \cdots X_n^{k_n}.$$

We also obtain the following useful characterization of multiaffine maps $f\colon E \to F$, when $E$ is of finite dimension.

**Lemma 27.1.6** *Given any affine space $E$ of finite dimension $n$, and any affine space $F$, for any basis $(e_1, \ldots, e_n)$ of $\overrightarrow{E}$, for any symmetric multiaffine map $f\colon E^m \to F$, for any $m$ vectors*

$$v_j = v_{1,\,j} e_1 + \cdots + v_{n,\,j} e_n \in \overrightarrow{E},$$

*for any points $a_1, \ldots, a_m \in E$, we have*

$$f(a_1 + v_1, \ldots, a_m + v_m) = b + \sum_{1 \leq p \leq m} \sum_{\substack{I_1 \cup \ldots \cup I_n = \{1,\ldots,p\} \\ I_i \cap I_j = \emptyset,\, i \neq j \\ 1 \leq i,j \leq n}} \left( \prod_{i_1 \in I_1} v_{1,\,i_1} \right) \cdots \left( \prod_{i_n \in I_n} v_{n,\,i_n} \right) w_{|I_1|,\ldots,|I_n|},$$

*for some $b \in F$, and some $w_{|I_1|,\ldots,|I_n|} \in \overrightarrow{F}$, and for any $a \in E$, and $v \in \overrightarrow{E}$, the affine polynomial function $h$ associated with $f$ is given by*

$$h(a + v) = b + \sum_{1 \leq p \leq m} \sum_{\substack{k_1+\cdots+k_n=p \\ 0 \leq k_i,\, 1 \leq i \leq n}} v_1^{k_1} \cdots v_n^{k_n}\, w_{k_1,\ldots,k_n},$$

*for some $b \in F$, and some $w_{k_1,\ldots,k_n} \in \overrightarrow{F}$.*

Lemma 27.1.6 shows the crucial role played by homogeneous polynomials. We could have taken the form of an affine map given by this lemma as a definition, when $E$ is of finite dimension.

When $\overrightarrow{F}$ is a vector space of dimension greater than one, or an affine space, one should not confuse such polynomial **functions** with the polynomials defined as usual, say in Lang [107], Artin [5], or Mac Lane and Birkhoff [116]. The standard approach is to define formal polynomials whose coefficients belong to a (commutative) ring. Then, it is shown how a polynomial defines a polynomial function. In the present approach, we define directly certain functions that behave like generalized polynomial functions. Another major difference between the polynomial functions above and formal polynomials, is that formal polynomials can be added and multiplied. Although we can make sense of addition as affine combination in the case of polynomial functions with range an affine space, multiplication does not make any sense.

## 27.2 Polarizing Polynomials in One or Several Variables

We show that polynomials in one or several variables are uniquely defined by polar forms which are multiaffine maps. We first show the following simple lemma.

**Lemma 27.2.1** *(1) For every polynomial $p(X) \in \mathbb{R}[X]$, of degree $\leq m$, there is a symmetric $m$-affine form $f\colon \mathbb{R}^m \to \mathbb{R}$, such that $p(x) = f(x, x, \ldots, x)$ for all $x \in \mathbb{R}$. If $p(X) \in \mathbb{R}[X]$ is a homogeneous polynomial of degree exactly $m$, then the symmetric $m$-affine form $f$ is multilinear.*

*(2) For every polynomial $p(X_1, \ldots, X_n) \in \mathbb{R}[X_1, \ldots, X_n]$, of total degree $\leq m$, there is a symmetric $m$-affine form $f\colon (\mathbb{R}^n)^m \to \mathbb{R}$, such that $p(x_1, \ldots, x_n) = f(x, x, \ldots, x)$, for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. If $p(X_1, \ldots, X_n) \in \mathbb{R}[X_1, \ldots, X_n]$ is a homogeneous polynomial of total degree exactly $m$, then $f$ is a symmetric multilinear map $f\colon (\mathbb{R}^n)^m \to \mathbb{R}$.*

*Proof*. (1) It is enough to prove it for a monomial of the form $X^k$, $k \leq m$. Clearly,

$$f(x_1, \ldots, x_m) = \frac{k!(m-k)!}{m!}\sigma_k$$

is a symmetric $m$-affine form satisfying the lemma (where $\sigma_k$ is the $k$-th elementary symmetric function, which consists of $\binom{m}{k} = \frac{m!}{k!(m-k)!}$ terms), and when $k = m$, we get a multilinear map.

(2) It is enough to prove it for a homogeneous monomial of the form $X_1^{k_1} \cdots X_n^{k_n}$, where $k_i \geq 0$, and $k_1 + \cdots + k_n = d \leq m$. Let $f$ be defined such that

$$f((x_{1,1}, \ldots, x_{n,1}), \ldots, (x_{1,m}, \ldots, x_{n,m})) = \frac{k_1! \cdots k_n!(m-d)!}{m!} \sum_{\substack{I_1 \cup \ldots \cup I_n \subseteq \{1, \ldots, m\} \\ I_i \cap I_j = \emptyset,\, i \neq j,\, |I_j| = k_j}} \left(\prod_{i_1 \in I_1} x_{1,i_1}\right) \cdots \left(\prod_{i_n \in I_n} x_{n,i_n}\right).$$

The idea is to split any subset of $\{1, \ldots, m\}$ consisting of $d \leq m$ elements into $n$ disjoint subsets $I_1, \ldots, I_n$, where $I_j$ is of size $k_j$ (and with $k_1 + \cdots + k_n = d$). There are

$$\frac{m!}{k_1! \cdots k_n!(m-d)!} = \binom{m}{k_1, \ldots, k_n, m-d}$$

such families of $n$ disjoint sets, where $k_1 + \cdots + k_n = d \leq m$. Indeed, this is the number of ways of choosing $n+1$ disjoint subsets of $\{1, \ldots, m\}$ consisting respectively of $k_1, \ldots, k_n$, and $m-d$ elements, where $k_1 + \cdots + k_n = d$. One can also argue as follows: There are $\binom{m}{k_1}$ choices for the first subset $I_1$ of size $k_1$, and then $\binom{m-k_1}{k_2}$ choices for the second subset $I_2$ of size $k_2$, etc, and finally, $\binom{m-(k_1+\cdots+k_{n-1})}{k_n}$ choices for the last subset $I_n$ of size $k_n$. After some simple arithmetic, the number of such choices is indeed

$$\frac{m!}{k_1! \cdots k_n!(m-d)!} = \binom{m}{k_1, \ldots, k_n, m-d}.$$

It is clear that $f$ is symmetric $m$-affine in $x_1, \ldots, x_m$, where $x_j = (x_{1,j}, \ldots, x_{n,j})$, and that

$$f(\underbrace{x, \ldots, x}_{m}) = x_1^{k_1} \cdots x_n^{k_n},$$

for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Also, when $d = m$, it is easy to see that $f$ is multilinear. $\square$

As an example, if

$$p(X) = X^3 + 3X^2 + 5X - 1,$$

we get

$$f(x_1, x_2, x_3) = x_1x_2x_3 + x_1x_2 + x_1x_3 + x_2x_3 + \frac{5}{3}(x_1 + x_2 + x_3) - 1.$$

When $n = 2$, which corresponds to the case of surfaces, we can give an expression which is easier to understand. Writing $U = X_1$ and $V = X_2$, to minimize the number of subscripts, given the monomial $U^h V^k$, with $h + k = d \leq m$, we get

$$f((u_1, v_1), \ldots, (u_m, v_m)) = \frac{h!k!(m-(h+k))!}{m!} \sum_{\substack{I \cup J \subseteq \{1,\ldots,m\} \\ I \cap J = \emptyset \\ |I|=h,\, |J|=k}} \left(\prod_{i \in I} u_i\right)\left(\prod_{j \in J} v_j\right).$$

For a concrete example involving two variables, if

$$p(U, V) = UV + U^2 + V^2,$$

we get

$$f((u_1, v_1), (u_2, v_2)) = \frac{u_1 v_2 + u_2 v_1}{2} + u_1 u_2 + v_1 v_2.$$