Shelling (Bruggesser-Mani 1971) and Ranking

Let $P = \{x \in \mathbb{R}^d : A_i\, x \leq 1,\ i = 1, 2, \ldots, m\}$ be a polytope. P has such a representation iff it contains the origin in its interior. For a generic $c \in \mathbb{R}^d$, sort the inequalities so that $A_1\, c > A_2\, c > \cdots > A_m\, c$.

(= a ranking of vertices A_i's in the dual by a linear function).

Geometrically, the line $L(\lambda) = \{\lambda\, c \mid \lambda \in \mathbb{R}\}$ meets each hyperplane $A_i\, x = 1$. Let λ_i denotes the parameter value at the intersection. Thus,

$$z_i = \lambda_i\, c \quad \text{and} \quad A_i\, z_i = 1.$$

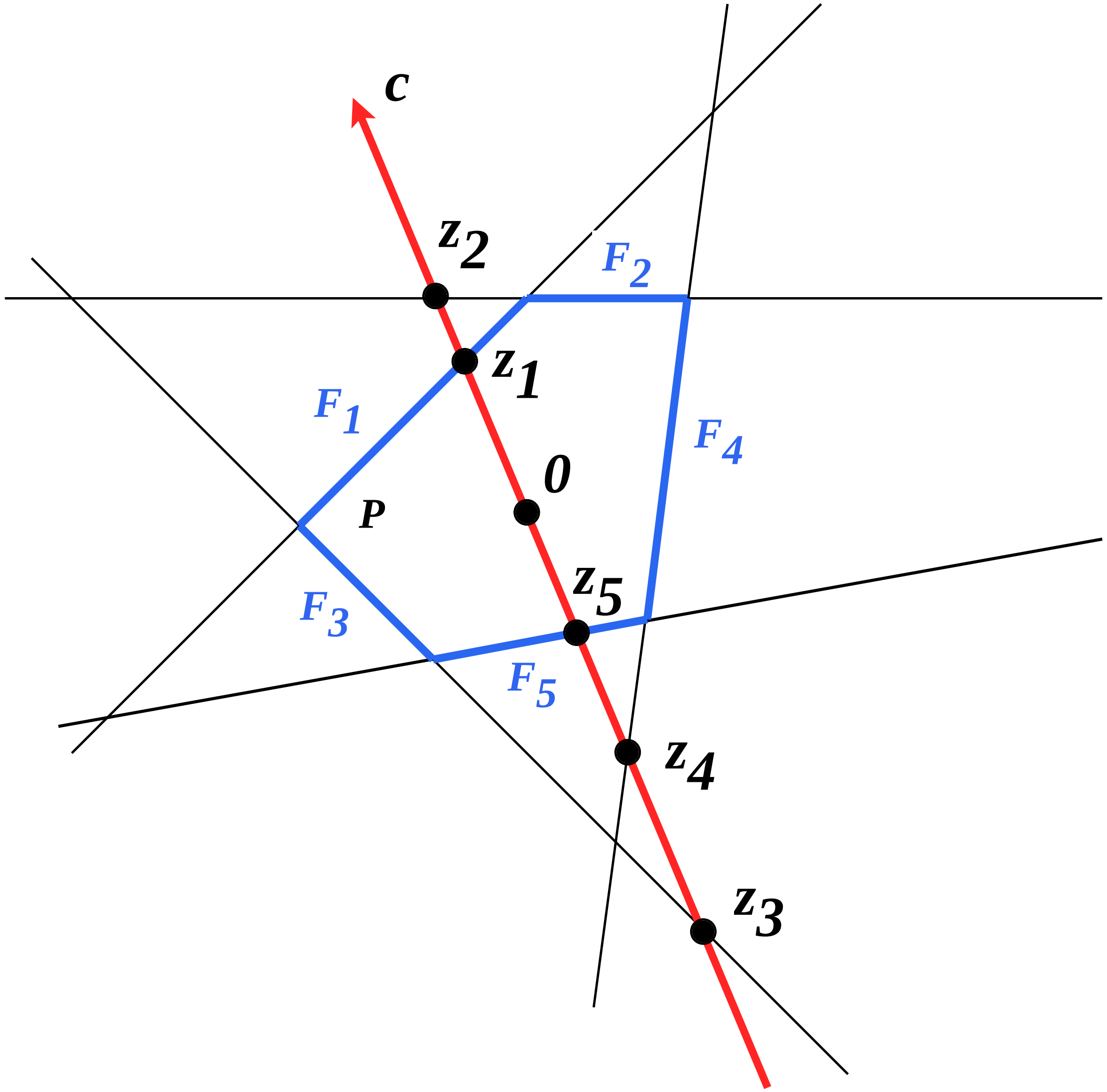
Consequently:

$$1/\lambda_1 > 1/\lambda_2 > \cdots > 1/\lambda_m.$$

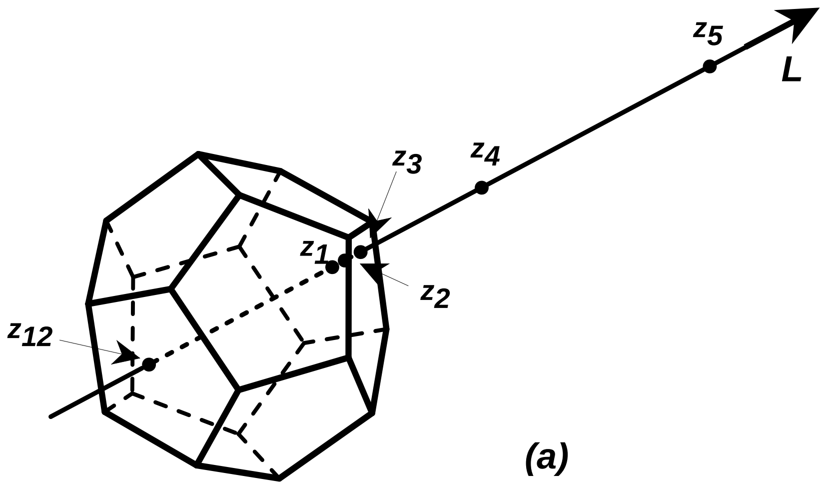
This ordering induces a shelling of P:

> $(\cup_{i=1}^{k-1} F_i) \cap F_k$ is a topological $(d-2)$-ball for each $2 \leq k \leq m-1$.

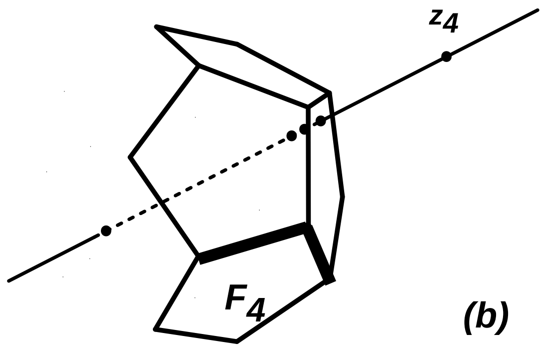
Shelling and Ranking (cont.)



Shelling in $\mathbb{R}^3$ (Launching a Space Shuttle)

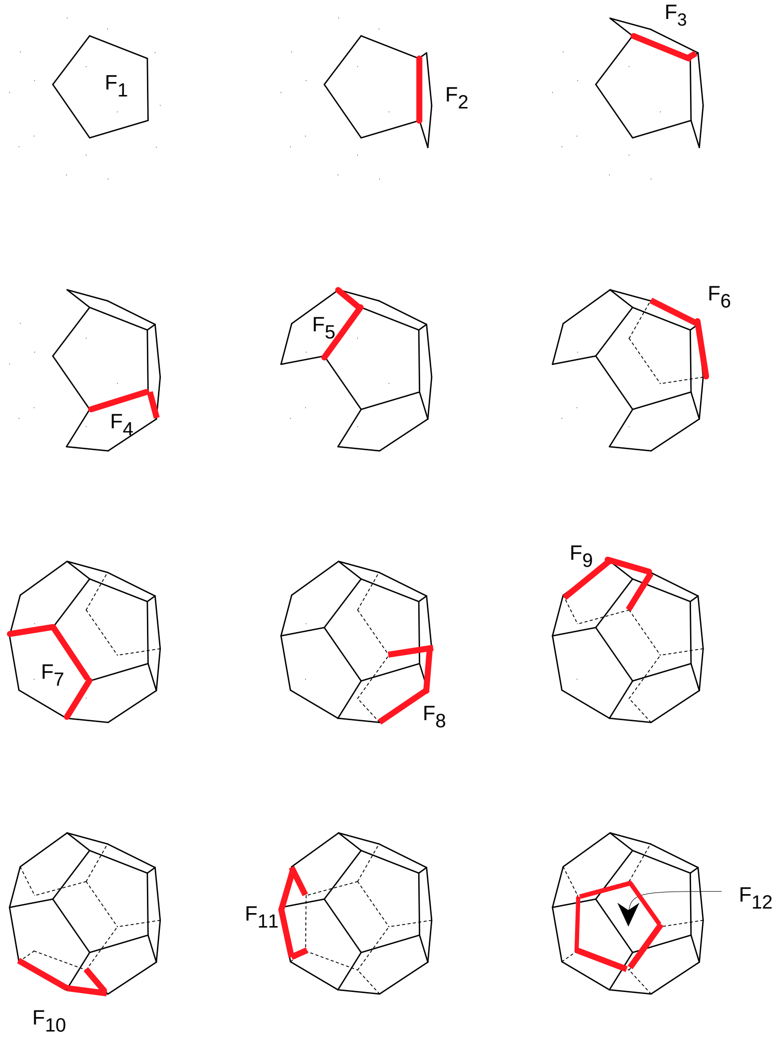


(a)



(b)

Shelling in $\mathbb{R}^3$ (cont.)

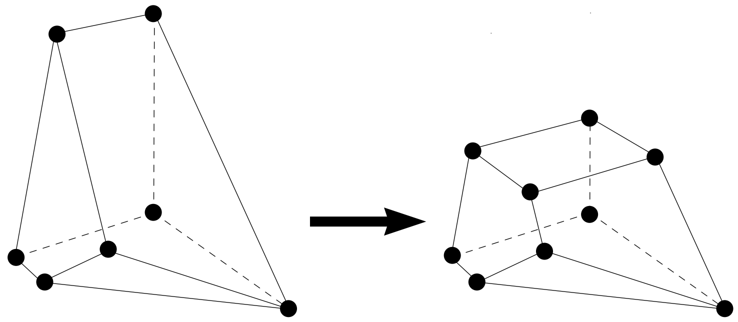


The Double Description Method: Complexity?

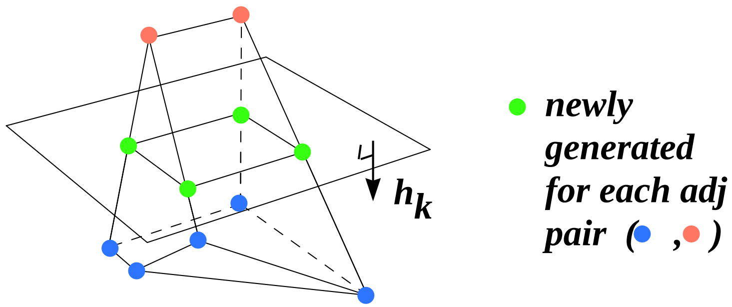
P : an H-polytope represented by m halfspaces $h_1, \ldots, h_m$ in $\mathbb{R}^d$.

$P_k = \cap_{i=1}^{k} h_i$: kth polytope ($P = P_m$).

$V_k = V(P_k)$: the vertex set computed at kth step.



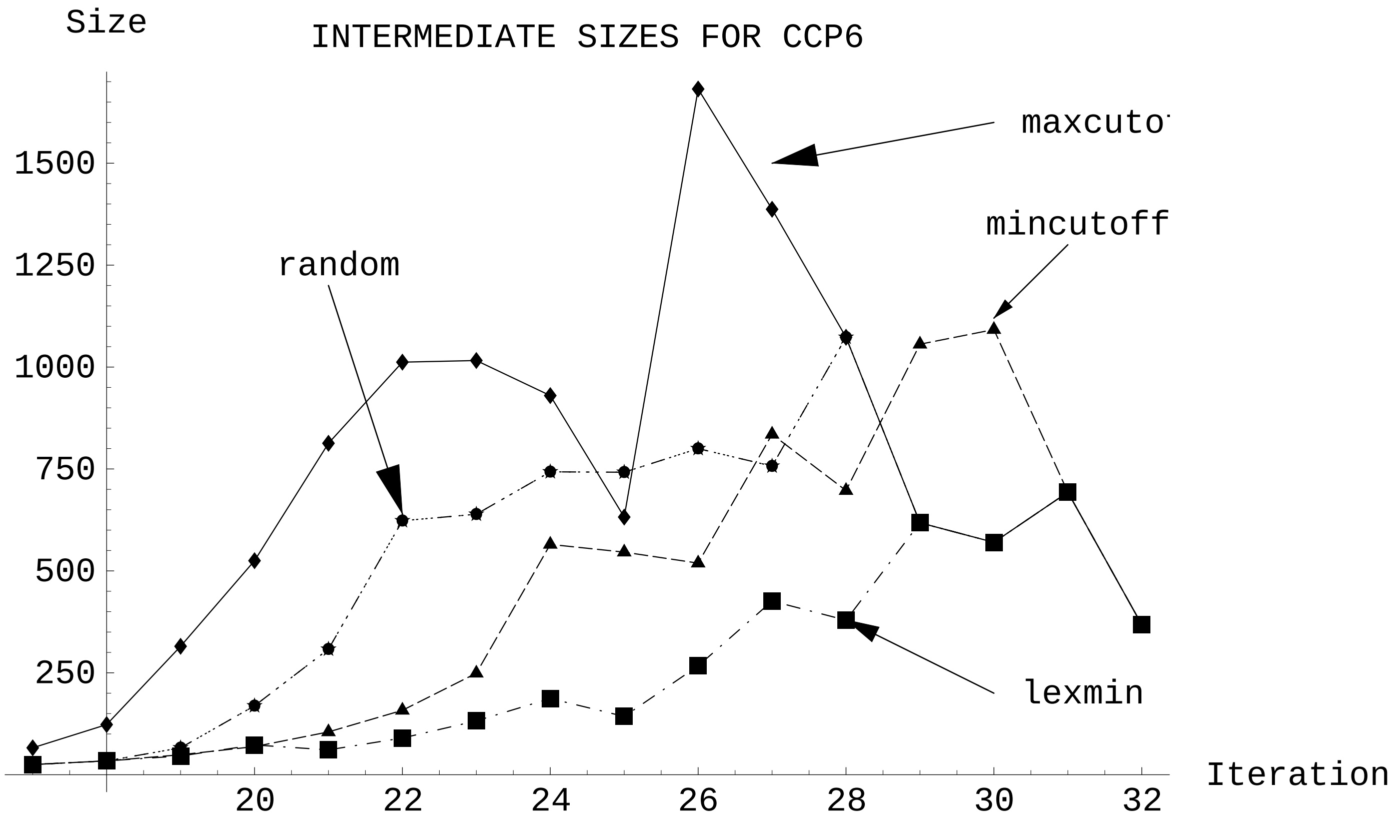
(P_{k-1}, V_{k-1}) (P_k, V_k)



The Double Description Method: Complexity? (cont.)

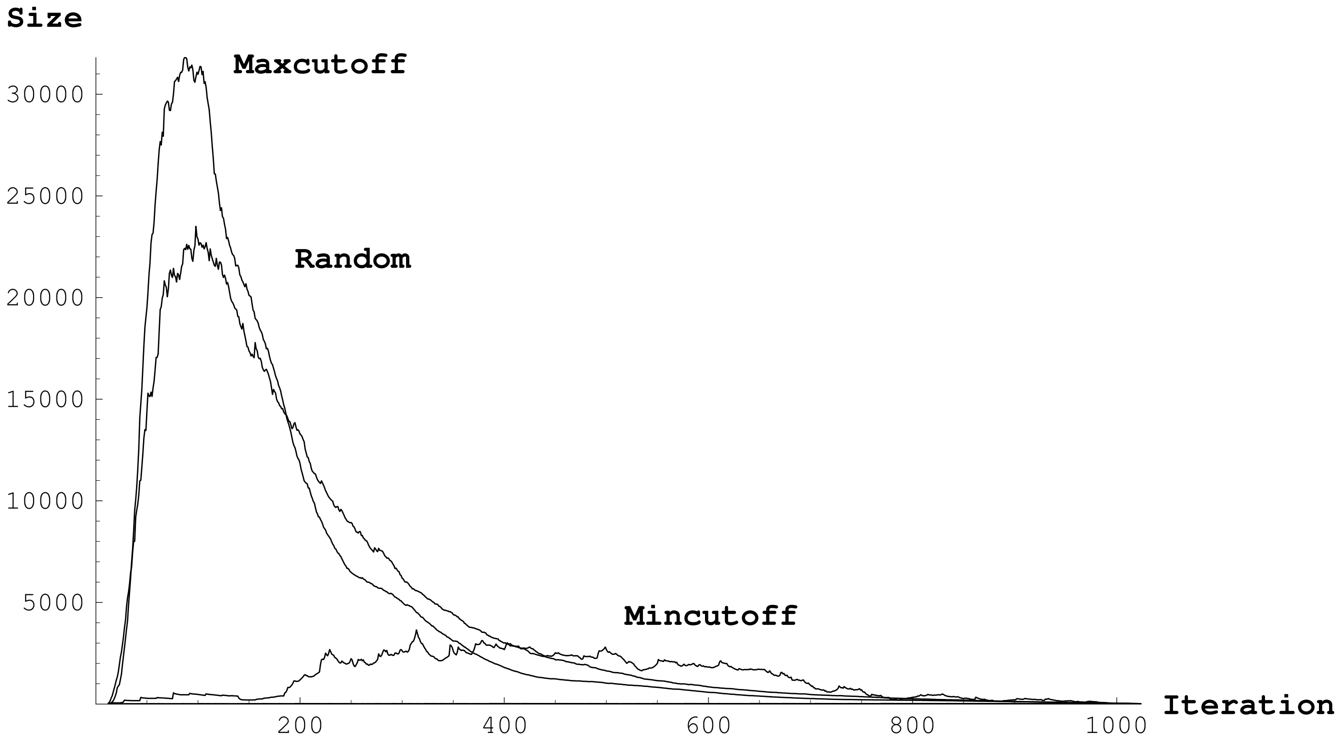
- It is an incremental method, dual to the Beneath-Beyond Method.
- Practical for low dimensions and highly degenerate inputs.
- For highly degenerate inputs, the sizes of intermediate polytopes are very sensitive to the ordering of halfspaces. For example, the maxcutoff ordering ("the deepest cut") may provoke extremely high intermediate sizes.
- It is hard to estimate its complexity in terms of and the sizes of input and output. The main reason is that the intermediate polytopes P_k can become very complex relative to the original polytope $P = P_m$.
- D. Bremner (1999) proved that there is a class of polytopes for which the double description method (and the beneath-beyond) method is exponential.

How Intermediate Sizes Fluctuate with Different Orderings



The input is a 15-dimensional polytope with 32 facets. The output is a list of 368 vertices. The lexmin is a sort of shelling ordering.

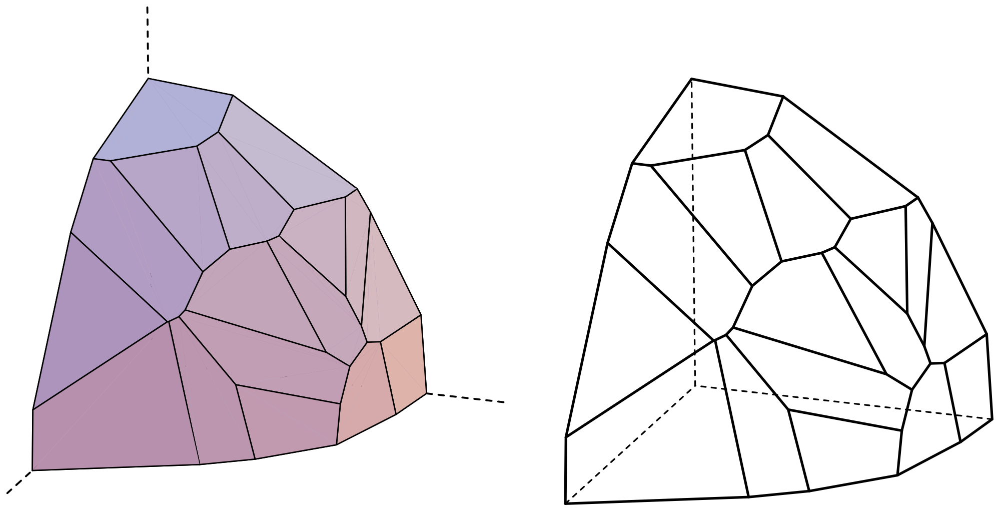
How Intermediate Sizes Fluctuate with Different Orderings



The input is a 10-dimensional cross polytope with 2^{10} facets. The output is a list of 20 vertices. The highest peak is attained by maxcutoff ordering, following by random and mincutoff. Lexmin is the best among all and the peak intermediate size is less than 30. (Too small too see it above.)

Pivoting Algorithms for Vertex Enumeration

Basic Idea: Search the connected graph of an H-polytope P by pivoting operations to list all vertices.



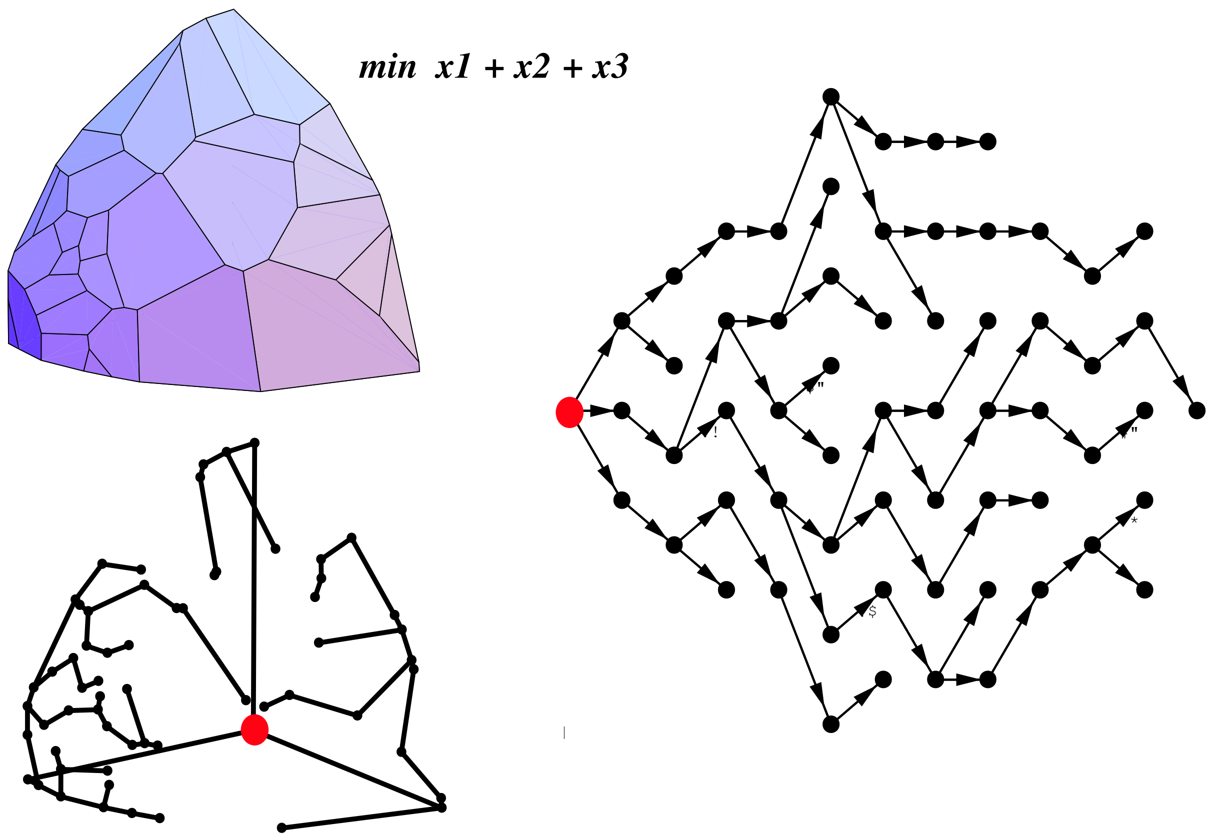
A polytope P and its graph (1-skeleton)

Advantage: Under the usual nondegeneracy (i.e. no points in P lie on more than d facets), it is polynomial in the input size and the output size.

Space Complexity: Depends on the search technique. The standard depth-first search requires to store all vertices found.

Memory Free: Reverse Search for Vertex Enumeration

Key idea: Reverse the simplex method from the optimal vertex in all possible ways:



Complexity: $O(mdf_0)$-time and $O(md)$-space (under nondegeneracy).

Reverse Search: General Description

Two functions f and adj define the search:

A finite local search f for a graph $G = (V, E)$ with a special node $s \in V$ is a function: $V \setminus \{s\} \to V$ satisfying

(L1) $\{v, f(v)\} \in E$ for each $v \in V \setminus \{s\}$, and

(L2) for each $v \in V \setminus \{s\}$, $\exists k > 0$ such that $f^k(v) = s$.

Example:

- Let $P = \{x \in \mathbb{R}^d : A\,x \leq b\}$ be a simple polytope, and $c^T x$ be any generic linear objective function. Let V be the set of all vertices of P, s the unique optimal, and $f(v)$ be the vertex adjacent to v selected by the (deterministic) simplex method.

Reverse Search: General Description

A adjacency oracle adj for a graph $G = (V, E)$ is a function (where δ a upper bound for the maximum degree of G) satisfying:

(i) for each vertex v and each number k with $1 \leq k \leq \delta$ the oracle returns $adj(v, k)$, a vertex adjacent to v or extraneous 0 (zero),

(ii) if $adj(v, k) = adj(v, k') \neq 0$ for some $v \in V$, k and k', then $k = k'$,

(iii) for each vertex v, $\{adj(v, k) : adj(v, k) \neq 0, 1 \leq k \leq \delta\}$ is exactly the set of vertices adjacent to v.

Example:

- Let $P = \{x \in \mathbb{R}^d : A\,x \leq b\}$ be a simple polytope. Let V be the set of all vertices of P, δ be the number of nonbasic variables and $adj(v, k)$ be the vertex adjacent to v obtained by pivoting on the kth nonbasic variable at v.

Reverse Search: General Description

procedure ReverseSearch(adj,δ,s,f);

$v := s$; $j := 0$; (* j: neighbor counter *)

repeat

while $j < \delta$ **do**

$j := j + 1$;

(r1) $next := adj(v, j)$;

if $next \neq 0$ **then**

(r2) **if** $f(next) = v$ **then** (* reverse traverse *)

$v := next$; $j := 0$

endif

endif

endwhile;

if $v \neq s$ **then** (* forward traverse *)

(f1) $u := v$; $v := f(v)$;

(f2) $j := 0$; **repeat** $j := j + 1$ **until** $adj(v, j) = u$ (* restore j *)

endif

until $v = s$ and $j = \delta$

Pivoting Algorithm vs Incremental Algorithm

- Pivoting algorithms, in particular the reverse search algorithm (lrs, lrslib), work well for high dimensional cases.

- Incremental algorithms work well for low (up to 10) dimensional cases and highly degenerate cases. For example, the codes cdd/cddlib and porta are implemented for highly degenerate cases and the code qhull for low (up to 10) dimensional cases.

- The reverse search algorithm seems to be the only method that scales very efficiently in massively parallel environment.

- Various comparisons of representation conversion algorithms and implementations can be found in the excellent article:

  D. Avis, D. Bremner, and R. Seidel. How good are convex hull algorithms. Computational Geometry: Theory and Applications, 7:265–302, 1997.

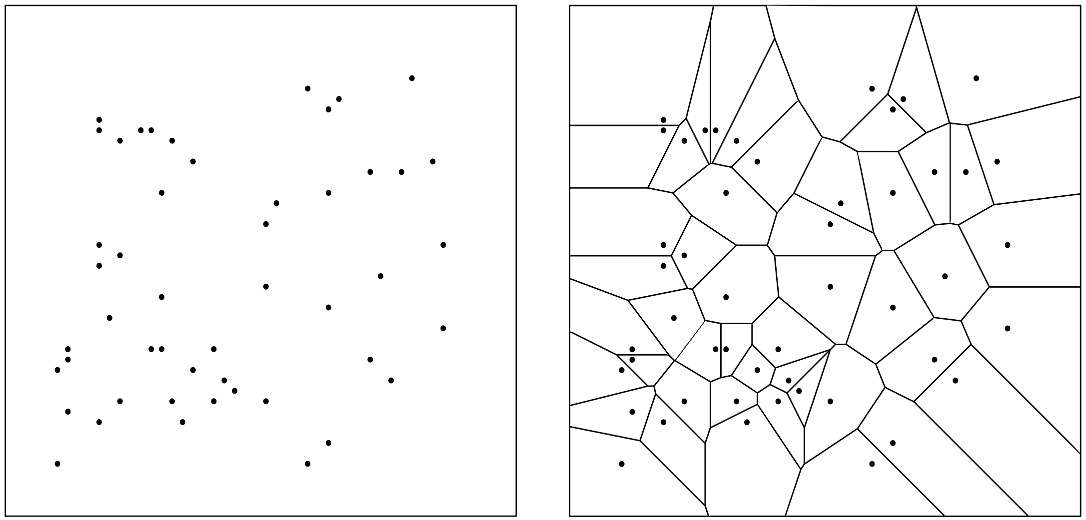
Voronoi Diagram in $\mathbb{R}^d$

S: a set of n distinct points in R^d

Voronoi diagram is the partition of R^d into n polyhedral regions:

$$vo(p) = \{x \in R^d | dist(x,p) \leq dist(x,q) \quad \forall q \in S - p\}, \text{ for } p \in S$$

where $dist$ is the Euclidean distance function. Each region $vo(p)$ is called the Voronoi cell of p.



Voronoi Diagram as Polyhedral Projection

For p in S, consider the hyperplane $h(p)$ tangent to the paraboloid $(x_{d+1} = x_1^2 + \cdots + x_d^2)$ in R^{d+1} at p:

$$\sum_{j=1}^{d} p_j^2 - \sum_{j=1}^{d} 2p_j x_j + x_{d+1} = 0.$$

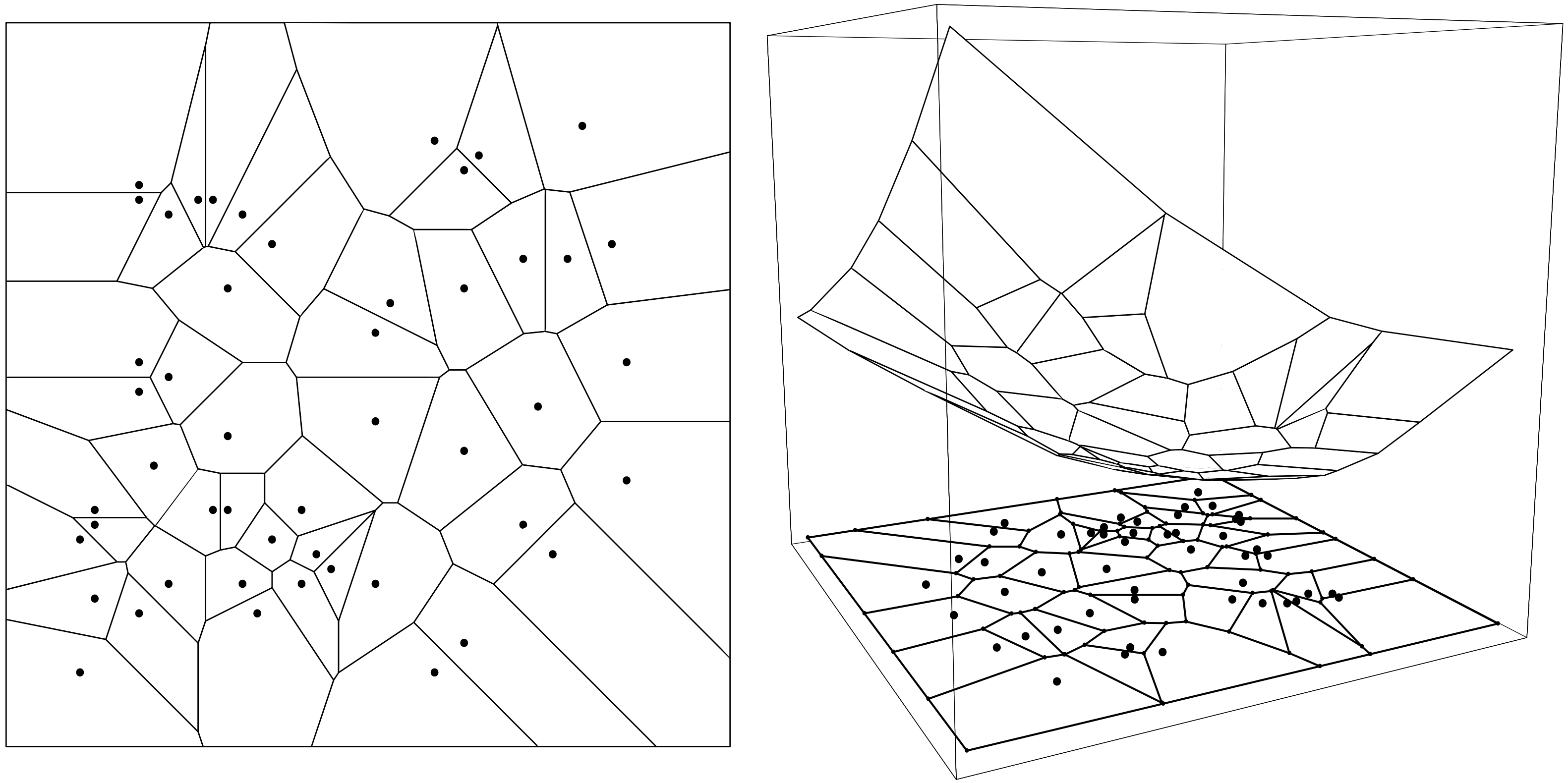
Replacing equation with inequality $\geq$ for each $p \in S$, we obtain the polyhedron

$$P = \{x \in \mathbb{R}^{d+1} \ : \ \sum_{j=1}^{d} p_j^2 - \sum_{j=1}^{d} 2p_j x_j + x_{d+1} \geq 0, \forall p \in S\}.$$

The key observation is that for two distinct points p and q, the intersection of two hyperplanes $h(p)$ and $h(q)$ is in fact the equal separator hyperplane of the two points.

Voronoi Diagram as Polyhedral Projection

The Voronoi diagram is simply the orthogonal projection of $P \in R^{d+1}$ onto the original space $\mathbb{R}^d$.

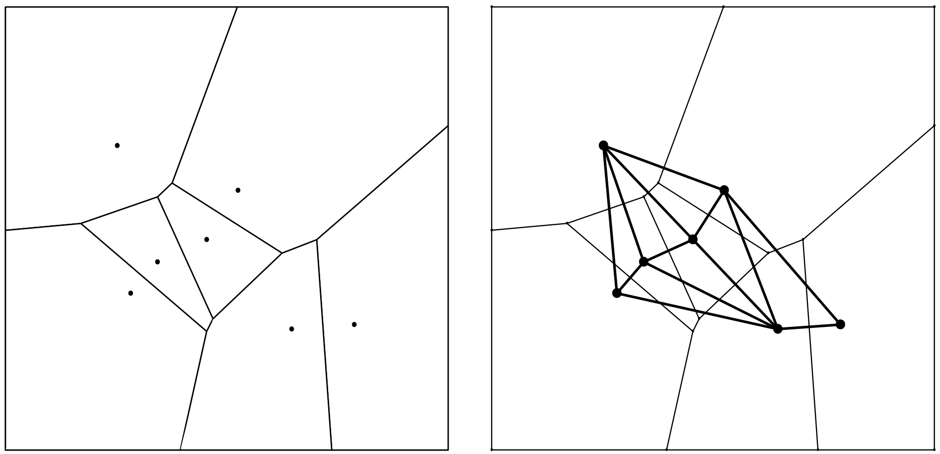


The projected vertices of P are called the Voronoi vertices.

Delaunay Triangulation in $\mathbb{R}^d$

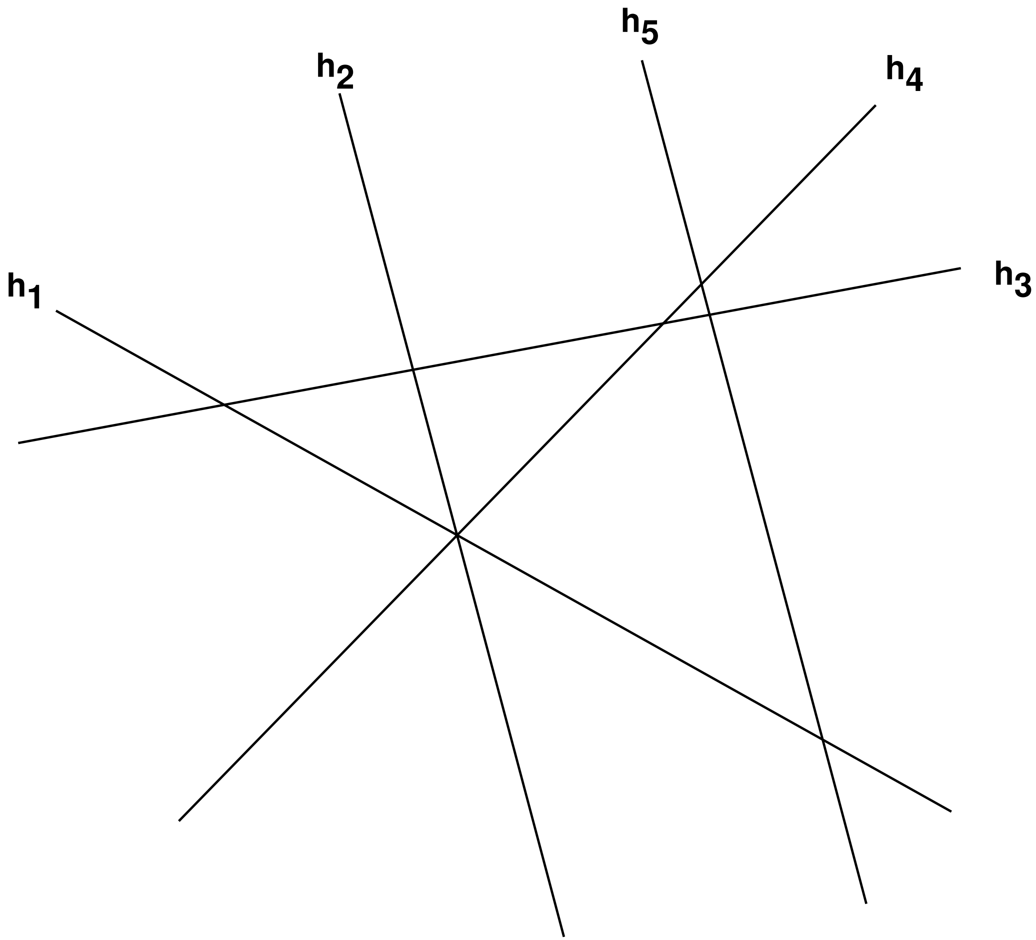
For each point $v \in R^d$, the nearest neighbor set $nb(S, v)$ of S is the set of points $p \in S$ which are closest to v in Euclidean distance.

The convex hull $conv(nb(S, v))$ of the nearest neighbor set of a Voronoi vertex v is called the Delaunay cell of v. The Delaunay triangulation of S is a partition of the convex hull $conv(S)$ into the Delaunay cells of Voronoi vertices. The one-to-one correspondence between the Voronoi vertices and the Delaunay cells is duality that .



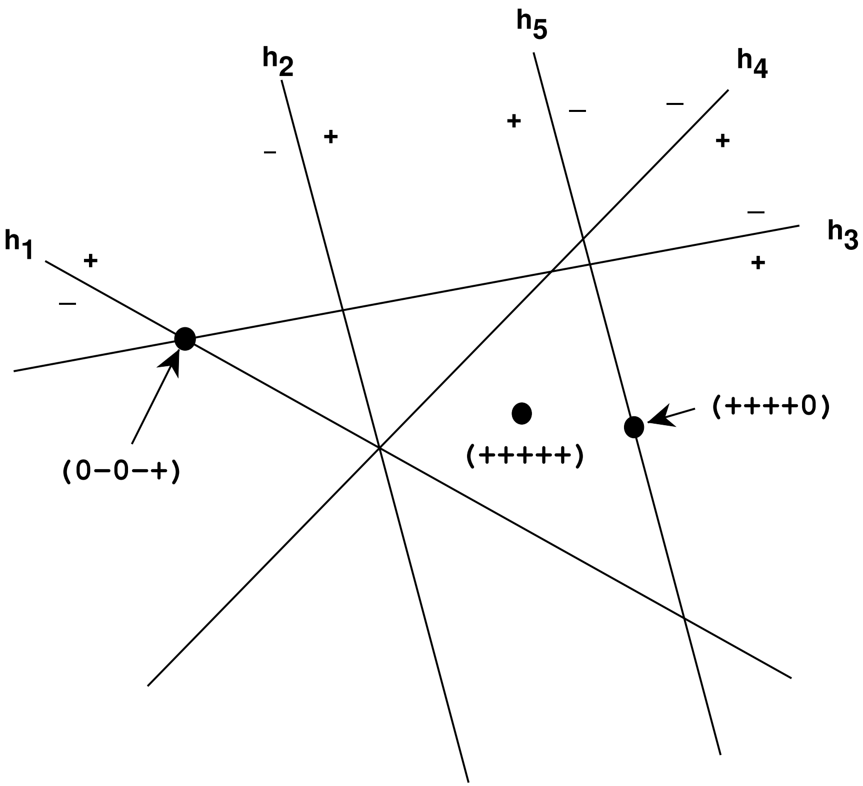
Arrangement of Hyperplanes

A finite family $\mathcal{A} = \{h_i : i = 1, 2, \ldots, m\}$ of hyperplanes in $\mathbb{R}^d$ is called an arrangement of hyperplanes.



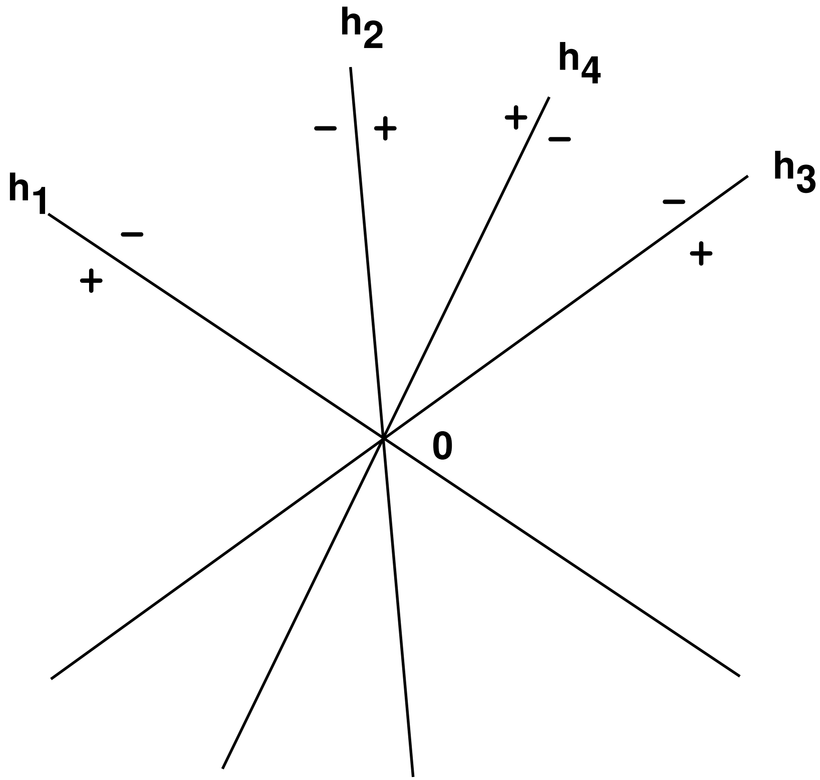
Arrangement of Hyperplanes: Representation of Faces

$$h_i^+ = \{x : A_i\, x < b_i\},\ h_i^0 = \{x : A_i\, x = b_i\},\ h_i^- = \{x : A_i\, x > b_i\}.$$

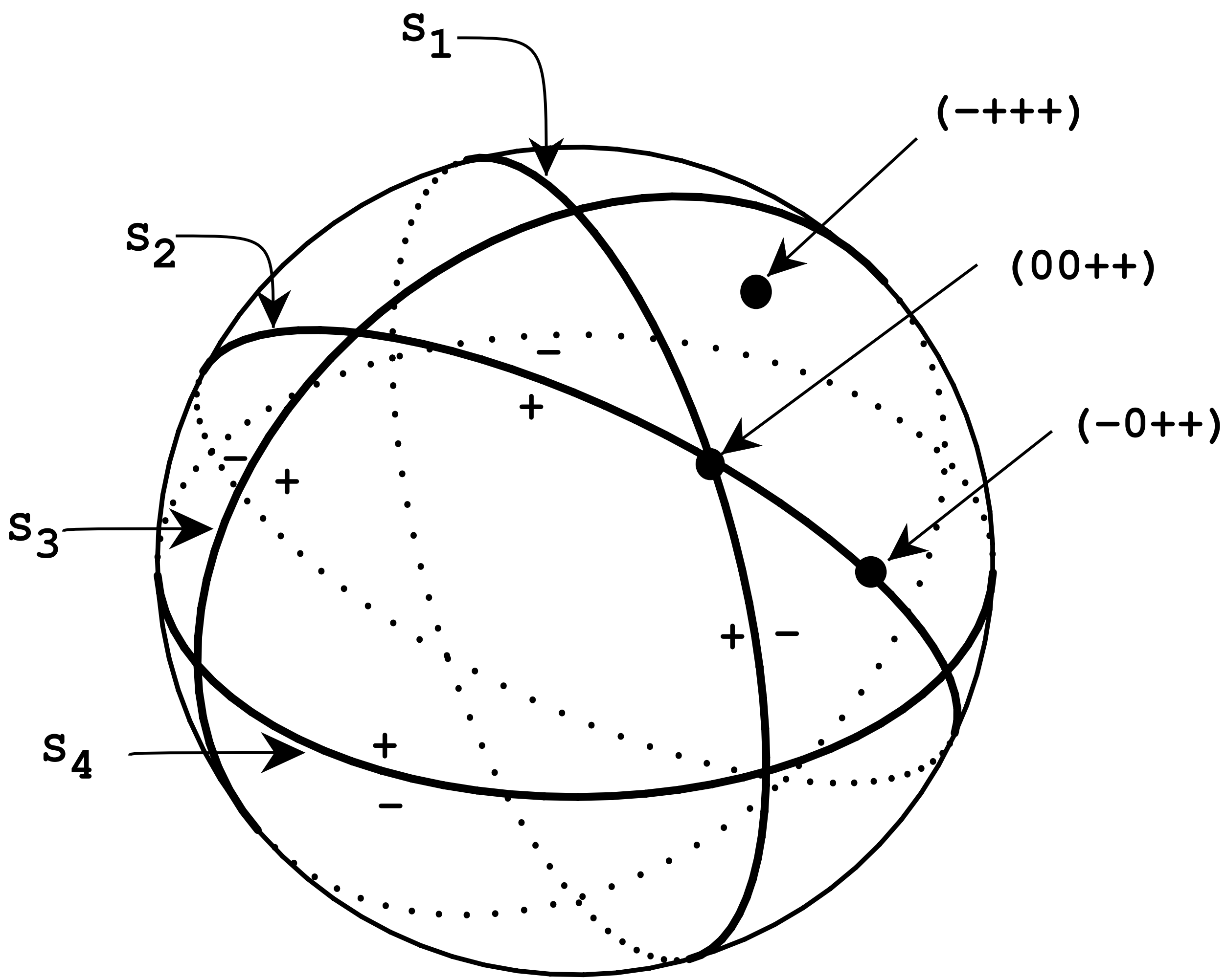


Central Arrangement of Hyperplanes

An arrangement of hyperplanes in which all its hyperplanes contain the origin 0 is called a <u>central arrangement of hyperplanes</u>.



Central Arrangement and Sphere Arrangement

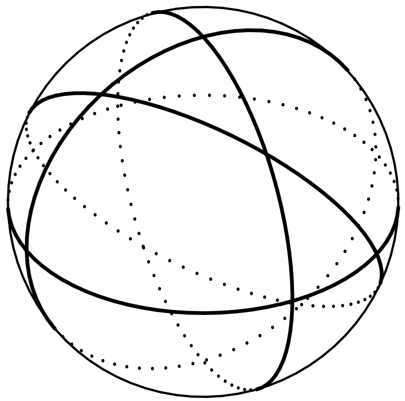


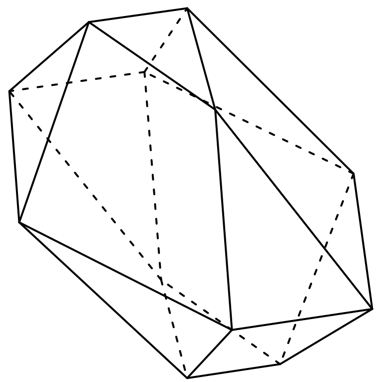
Polyhedral Realization of Arrangements

Let $\mathcal{A}$ be an arrangement of hyperplanes represented by a matrix A, i.e, $s_i = \{x : A_i \, x = 0\}, \forall i = 1, \ldots, m$.

Consider the following polytope:

$$P_A = \{x : y^T \; A \, x \leq 1, \forall \, y \in \{-1, +1\}^m\}$$





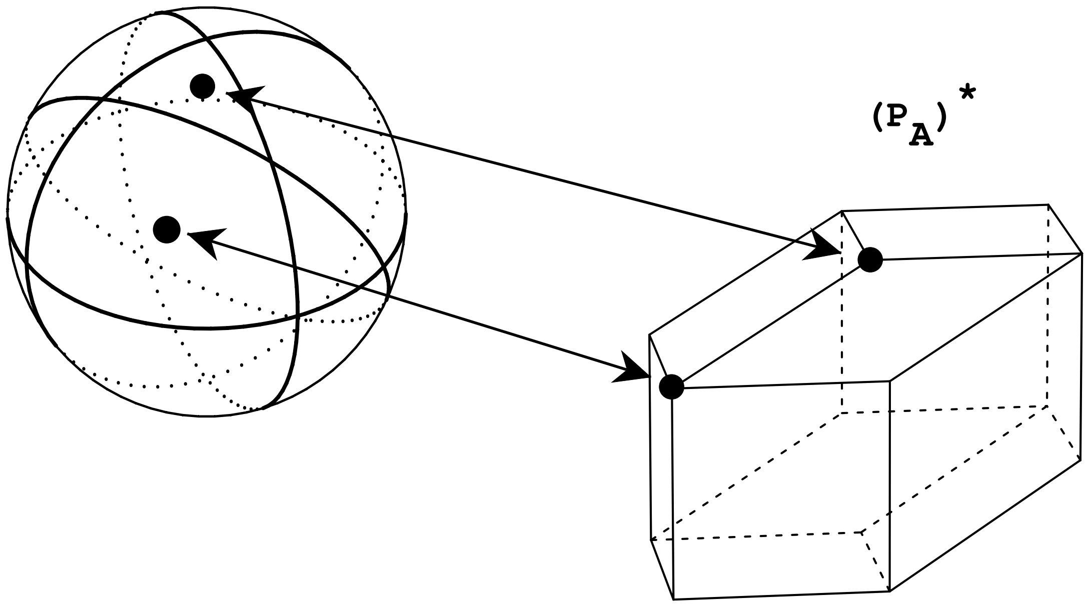
Theorem 0.14. The face lattice of $\mathcal{A}$ is isomorphic to the face lattice of the polytope P_A.

Dual of P_A: A Zonotope

The polar of the polytope P_A is the polytope (called <u>zonotope</u>)

$$
\begin{aligned}
(P_A)^* &= conv\{y^T A \in \mathbb{R}^d : y \in \{-1,+1\}^m\} \\
&= \{y^T A \in \mathbb{R}^d : y \in [-1,+1]^m\} \\
&= L_1 + L_2 + \cdots + L_m,
\end{aligned}
$$

where each <u>generator</u> L_i is the line segment $[-A_i, A_i]$.



Dual of P_A: A Zonotope

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$$\begin{aligned}(P_A)^* &= conv\{y^T A \in \mathbb{R}^d : y \in \{-1,+1\}^m\} \\ &= \{y^T A \in \mathbb{R}^d : y \in [-1,+1]^m\} \\ &= L_1 + L_2 + \cdots + L_m,\end{aligned}$$

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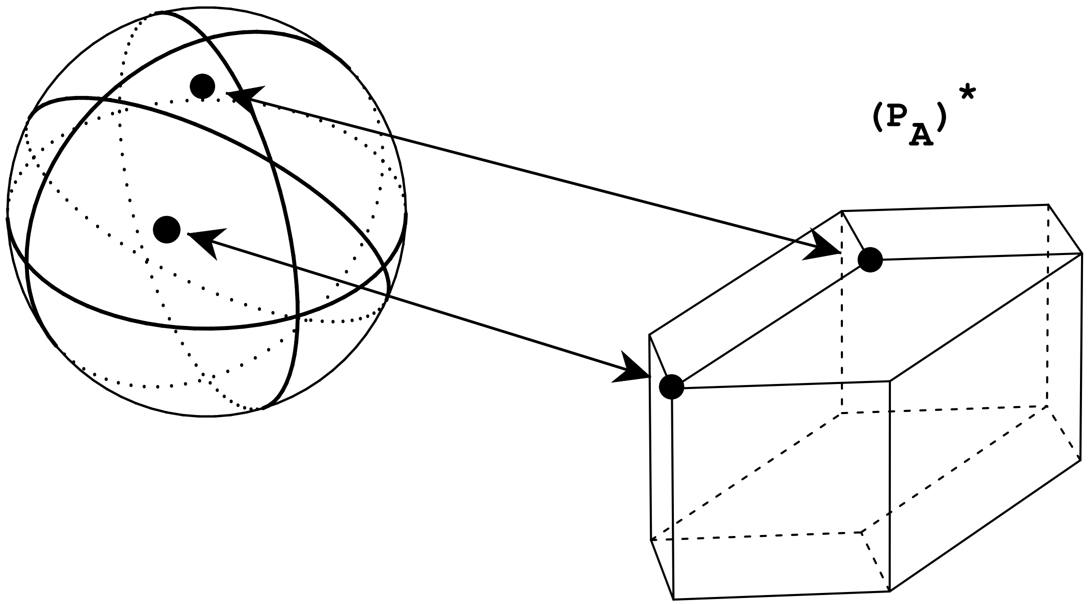

Examples of Zonotopes in $\mathbb{R}^3$

"Random" zonotopes with 5 and 10 generators: