Random Monotone Paths on Polyhedra

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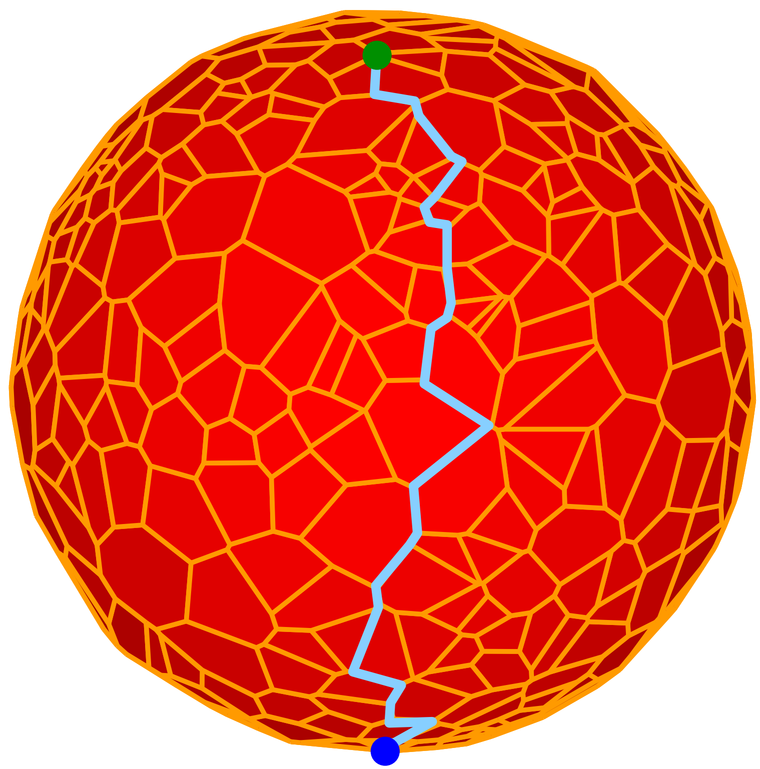

Joint work with Volker Kaibel, Rafael Mechtel, and Micha Sharir

arXiv:math.CO/0309351

Given $A \in \mathbb{R}^{n \times d}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^d$, solve

$$\min \left\{ c^t x \ : \ Ax \leq b \right\}$$

Given a linear inequality system (a polytope)
and a linear objective function (“height”),
find the “lowest” vertex

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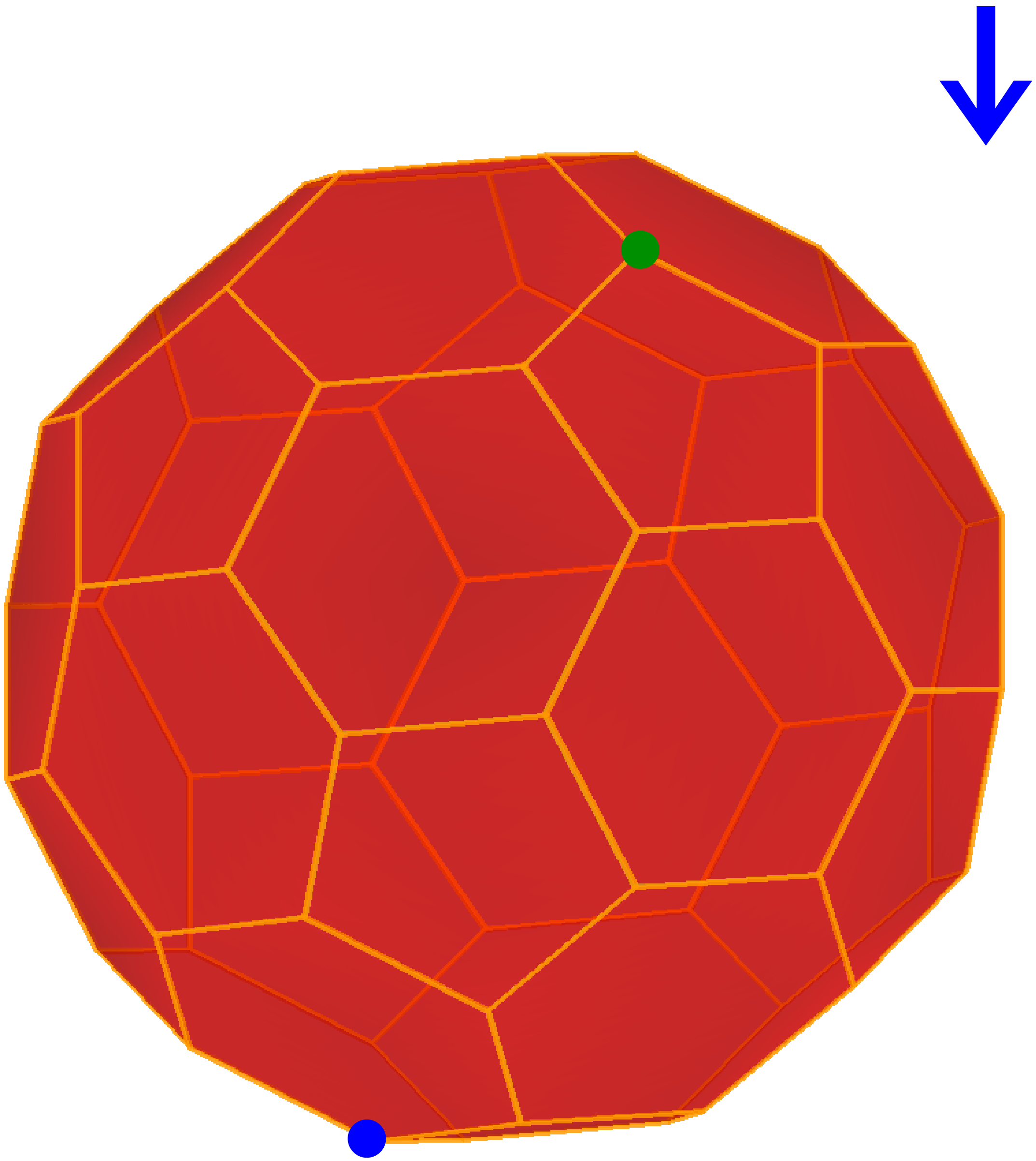
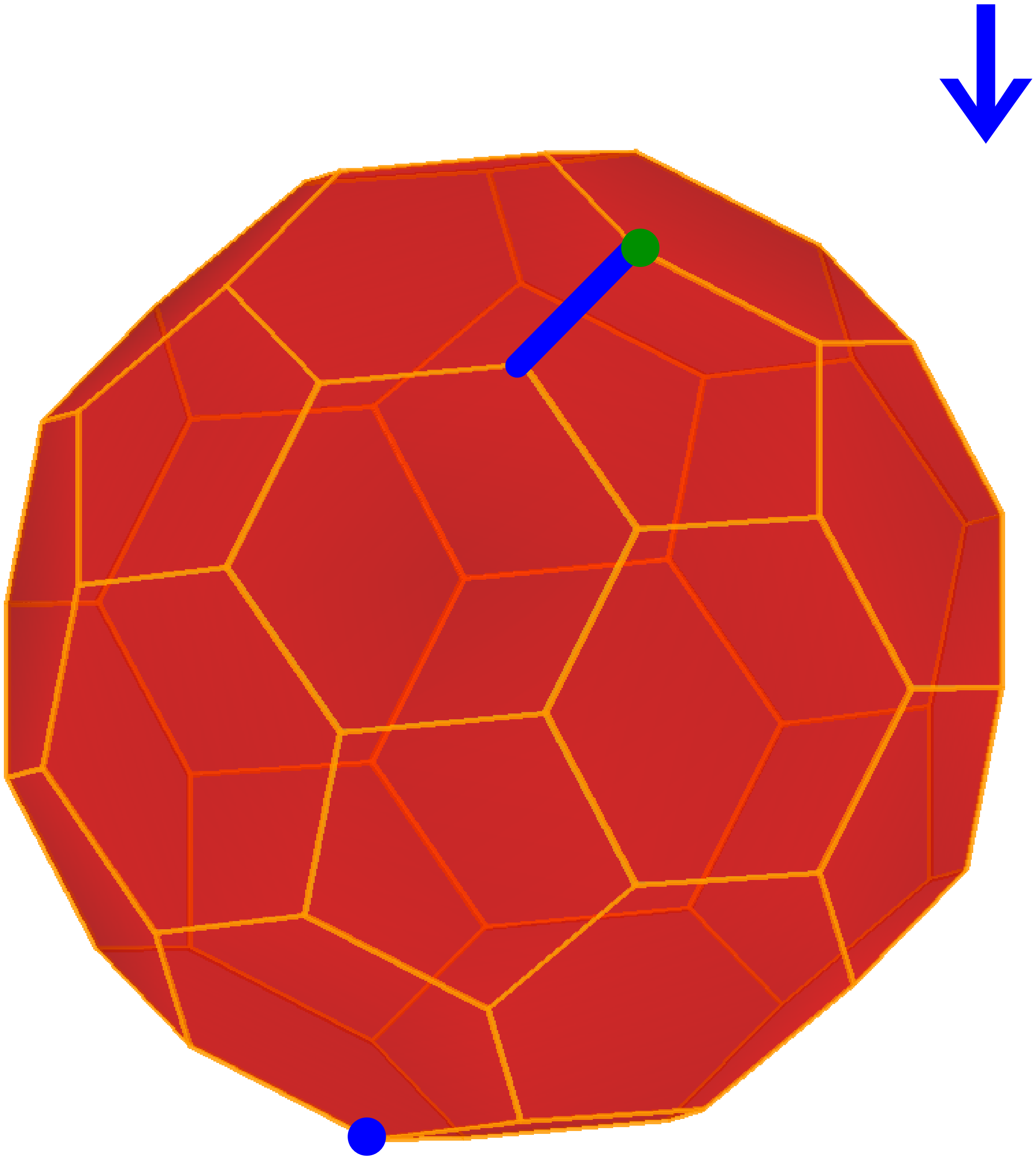
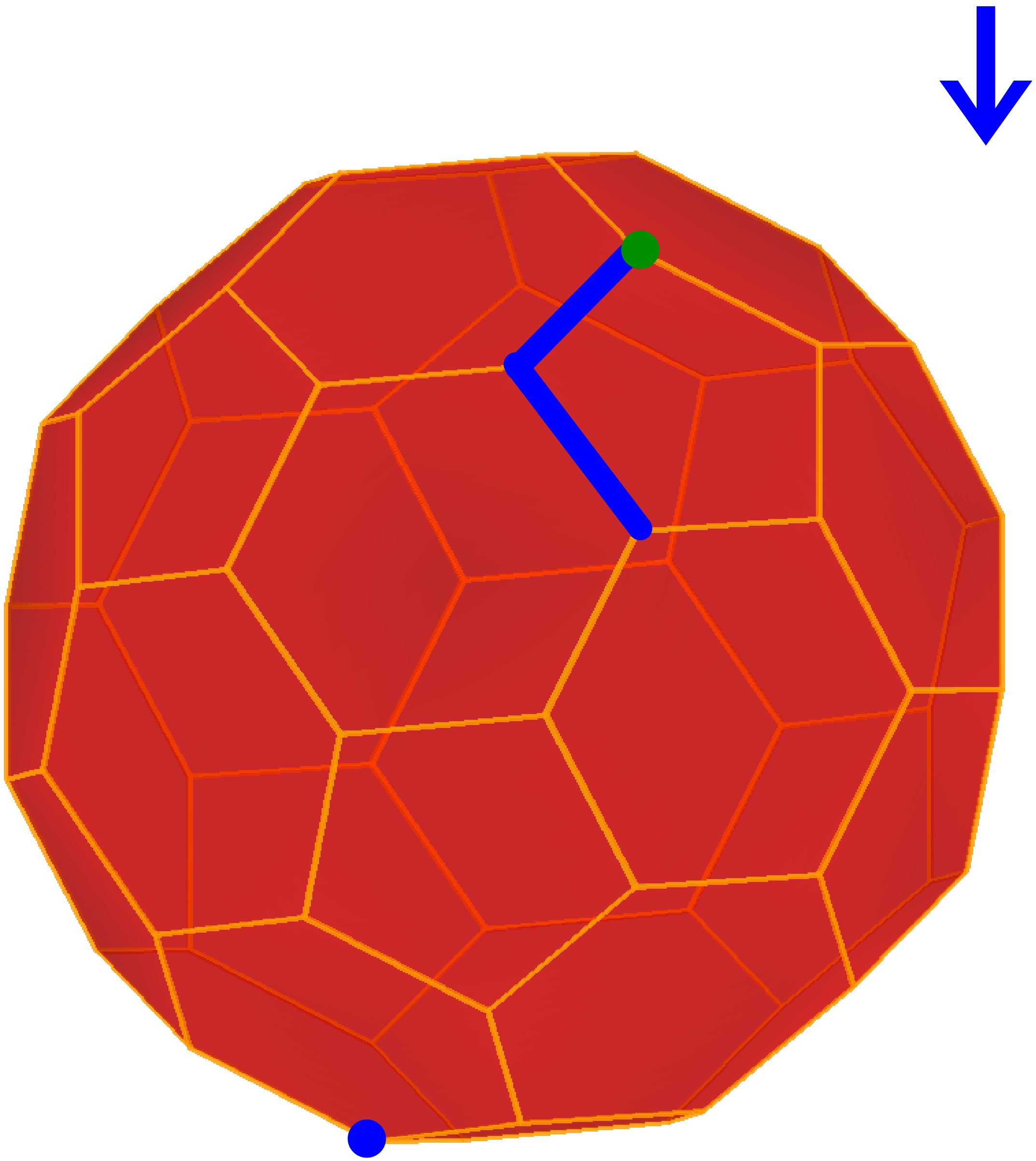
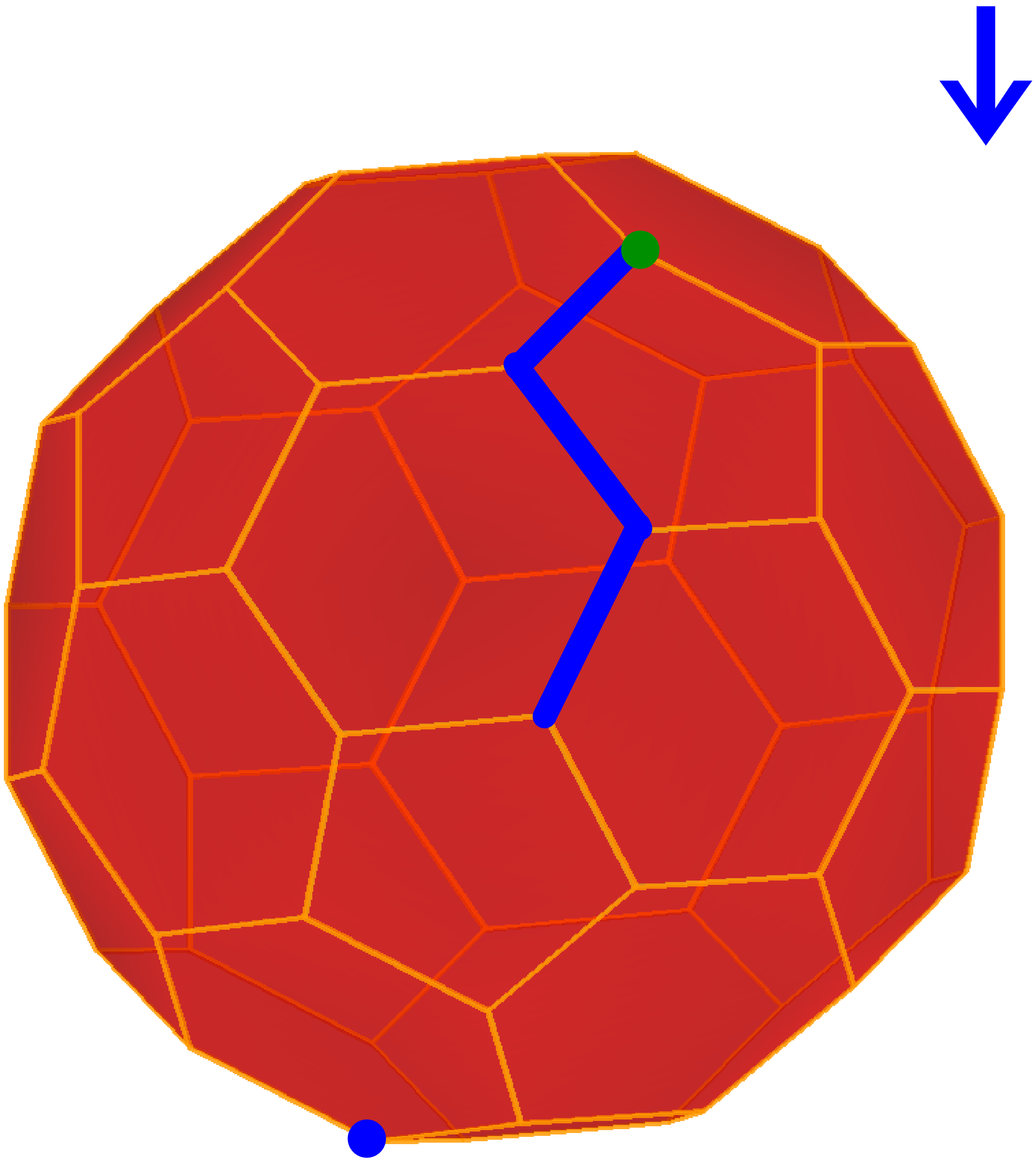
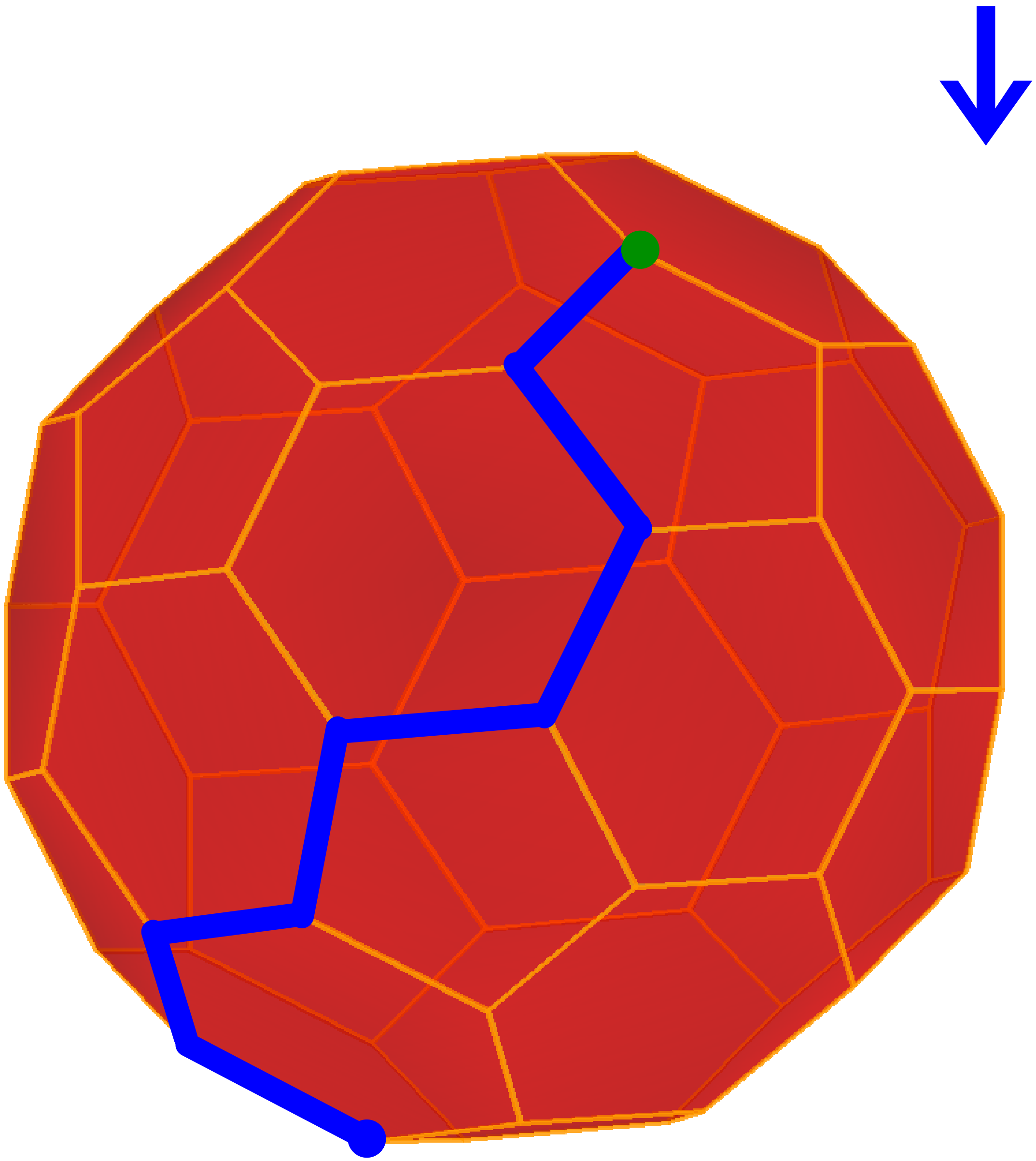
- Simplex Algorithm — DANTZIG 1947
- Ellipsoid Method: POLYNOMIAL — KHACHIYAN 1979
- Interior-Point Method: POLYNOMIAL — KARMARKAR 1984
- Algorithm in fixed dimension: LINEAR — MEGIDDO 1984

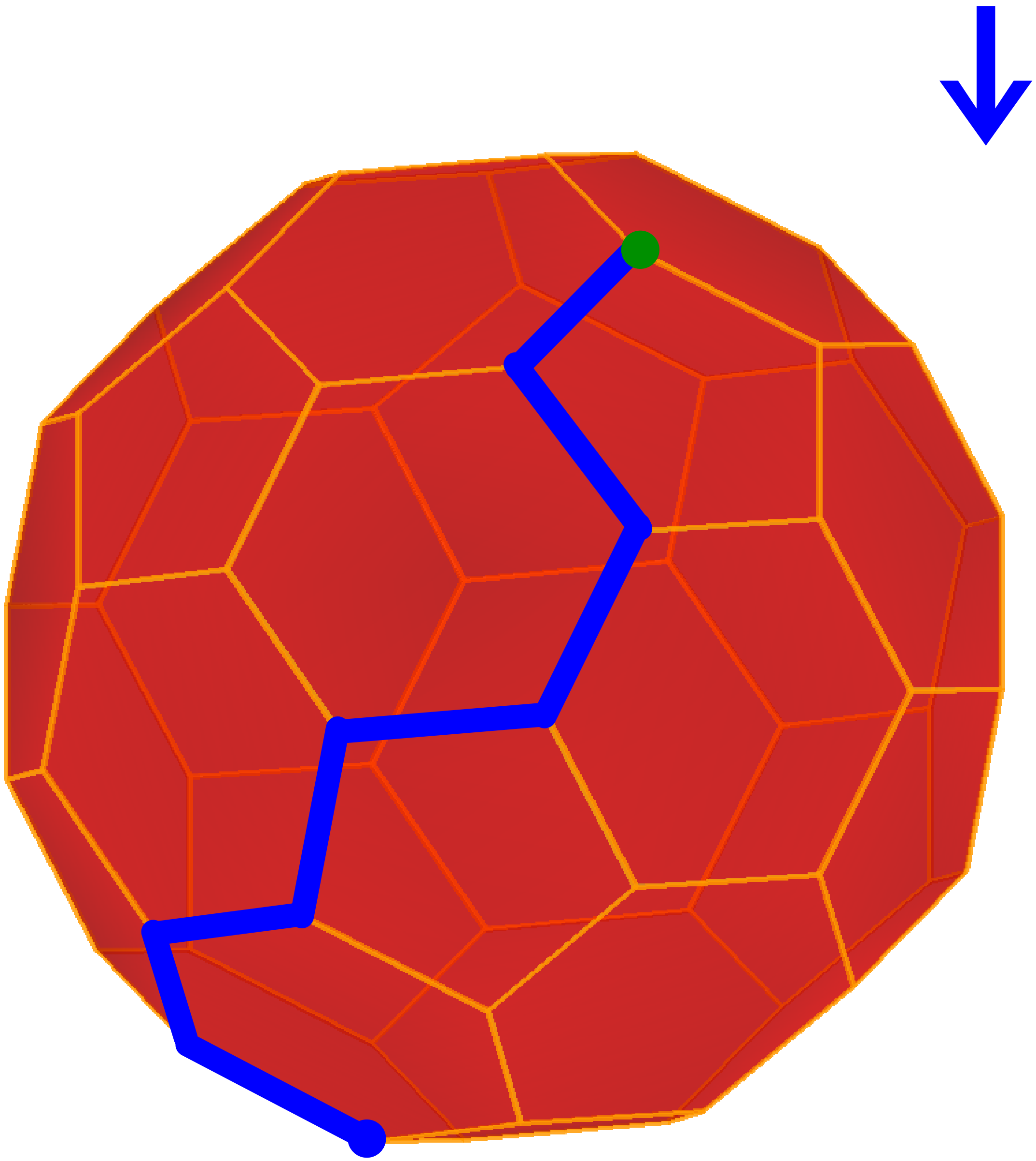












- simple polytope, generic linear objective function
- *pivoting* in a *tableau*
- which path: *pivot rules*
- successful in practice
- (*strongly*) polynomial?

- **Maximal number $f(d,n)$ of vertices** for a d-polytope with n facets? Motzkin 1957

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- **Upper Bound Theorem:**

$$f(d,n) \;=\; \binom{n-\lceil\frac{d}{2}\rceil}{\lfloor\frac{d}{2}\rfloor} + \binom{n-1-\lceil\frac{d-1}{2}\rceil}{\lfloor\frac{d-1}{2}\rfloor}$$

$$= \; O(n^{\lfloor d/2\rfloor}) \text{ for fixed } d$$

MCMULLEN 1970

- Equality achieved e. g. by duals of cyclic polytopes

- **Maximal number $M(d, n)$ of vertices on a monotone path** for a d-polytope with n facets? KLEE 1966

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- The duals of cyclic polytopes have Hamilton paths. KLEE 1965

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- The duals of cyclic polytopes have Hamilton paths. KLEE 1965
- *Monotone upper bounds*:

$$\begin{aligned} M(3,n) &= f(3,n) \\ M(4,n) &= f(4,n) \end{aligned}$$

PFEIFLE 2003

but $M(6,9) < f(6,9)$ PFEIFLE & Z. 2003

- Indeed, $27 \leq M(6,9) < f(6,9) = 30$.

- **Is there always a short path to the minimal vertex?** HIRSCH 1959
- *(Monotone) Hirsch Conjecture:* (monotone) diameter $\leq n - d$?

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- $d = 3$: monotone Hirsch Conjecture **holds** KLEE 1965
  $d > 3$: monotone Hirsch Conjecture **false** TODD 1980
- $d > 3$: strict monotone Hirsch Conjecture **open**
  (start from highest vertex) Z. 1995

- **Is there a pivot rule that always finds a short path?**

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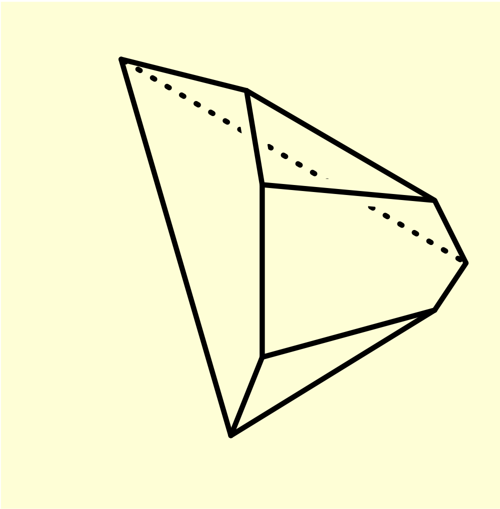
- Worst-case analysis:

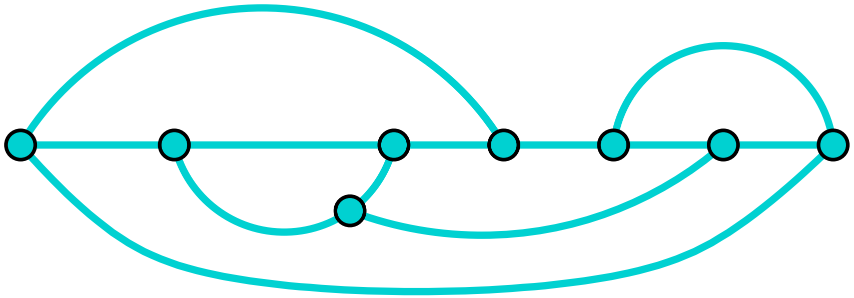
Dantzig's rule	exponential	KLEE & MINTY 1970
Greatest decrease	exponential	
Bland's Least index	exponential	
Steepest decrease	exponential	
Shadow vertex	exponential	
Smallest decrease	exponential	
etc.	exponential	AMENTA & Z. 1998
Zadeh's Least entered	?	
Random facet	subexponential	KALAI 1992
Random edge	**???**	

Random Monotone Paths on Polyhedra

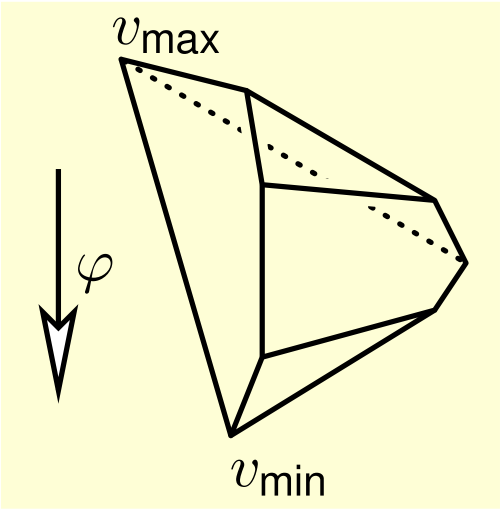
- A combinatorial model (for dimension three)
- Worst-case analysis: “coefficient of linearity”
  - **Random edge** — lower and upper bound
  - Other pivot rules:
    Greatest decrease
    Random facet
    Least entered
    Bland, Dantzig
    Steepest decrease
    Shadow vertex
- More remarks on Random edge

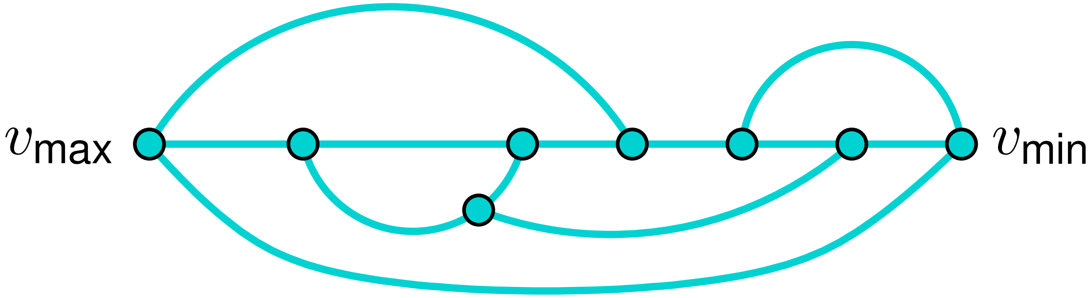


▷ 3-polytope P



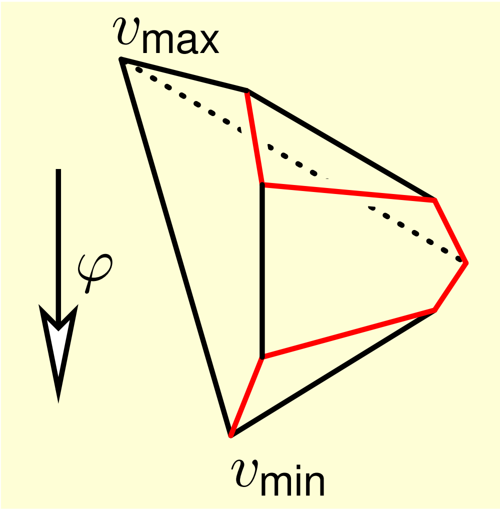
▷ planar, 3-connected graph G
(no loops, no parallel edges)

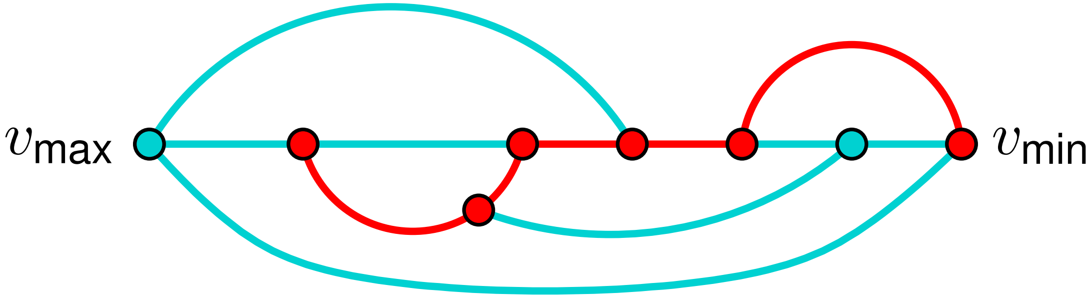




- 3-polytope P
- generic linear function $\varphi : \mathbb{R}^3 \to \mathbb{R}$

- 3-polytopal graph G STEINITZ
- acyclic orientation with unique sink, source in every face (AOF)
- three vertex-disjoint monotone paths from v_{max} to v_{min}





- 3-polytope P
- generic linear function $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}$

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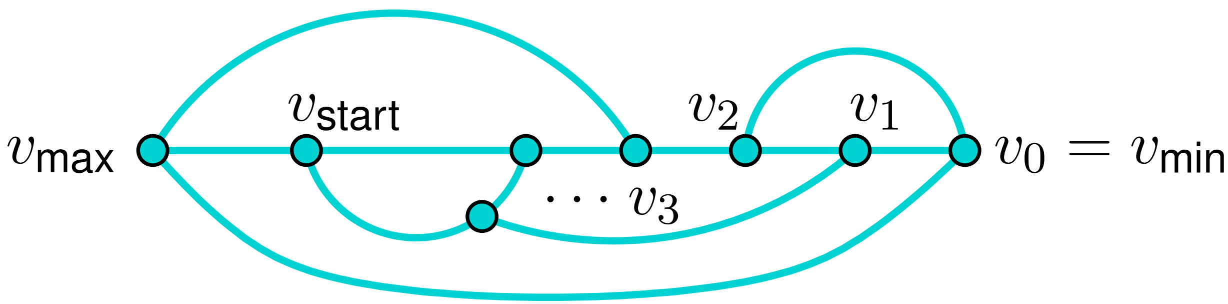
$P \subset \mathbb{R}^3$ simple polytope, n facets, $\varphi : \mathbb{R}^3 \to \mathbb{R}$ generic linear function



embedded planar graph, 3-regular, n faces
vertices decreasingly ordered from left to right
unique sinks, three disjoint v_{max}-v_{min} paths

- $2n - 4$ vertices (Euler's formula)
- # (out-degree-) 1-vertices $= n - 3$
- # (out-degree-) 2-vertices $= n - 3$

For a pivot rule $\mathcal{R}$:

$$\Lambda(\mathcal{R}) \quad := \quad \limsup_{n\to\infty} \frac{\text{worst-case running time on polyhedron with } n \text{ facets}}{n}$$

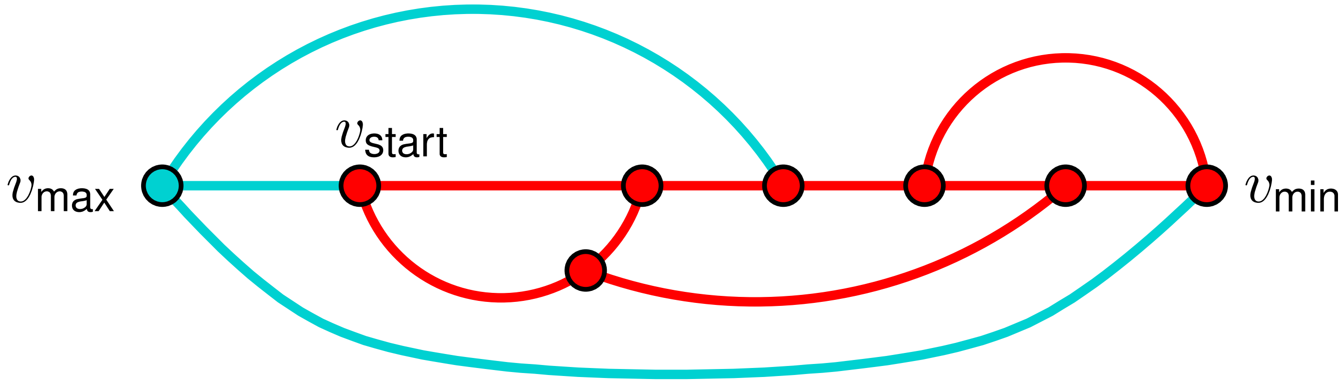
$$= \quad \min\{\Lambda : \mathcal{R} \text{ needs } \quad \Lambda n + \text{const. steps in worst case}\}$$

$$=: \quad \text{the } \textit{linearity coefficient} \text{ of } \mathcal{R}$$

$1 \ \leq \ \Lambda(R) \ \leq \ 2$ for every pivot rule

pivot rule $\mathcal{R}$	$\boldsymbol{\Lambda}(\mathcal{R})$	dimension $\boldsymbol{d}$	
Random edge	≥ 1.3473 ≤ 1.4943	unknown, could be quadratic!	
Greatest decrease	1.5	exponential	
Random facet	2	subexponential	KALAI 1992
Least entered	2	unknown: $1,000 reward	ZADEH 1980
Bland's Least index	2	exponential	
Dantzig's rule	2	exponential	
Steepest decrease	2	exponential	
Shadow vertex	2	exponential	
Smallest decrease	2	exponential: long paths!	

At any non-optimal vertex, the *random edge* pivot rule takes a step to one of its improving neighbors, chosen uniformly at random.



THEOREM

$$1.3473 \leq \Lambda(\text{RE}) \leq 1.4943$$

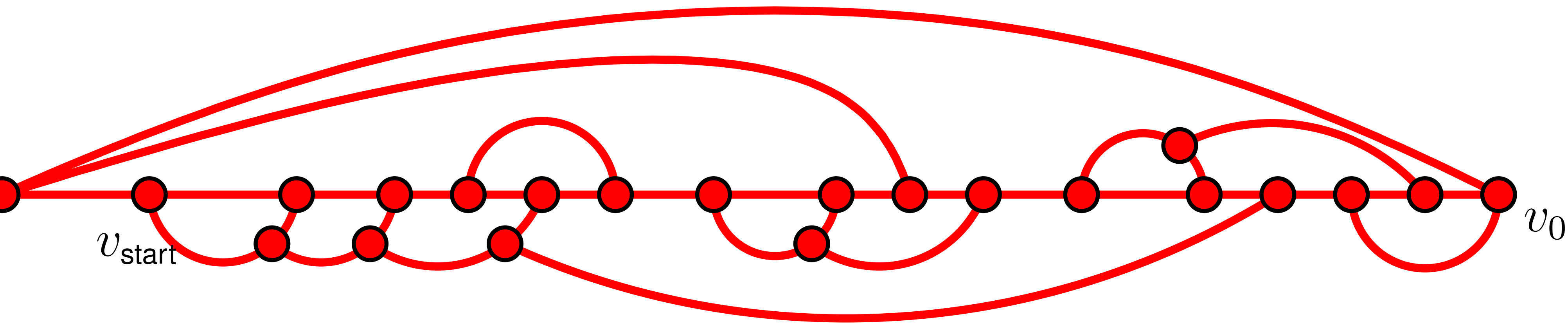
Lower bound Construction of a family of polytopes

Upper bound Induction on the number of 1- and 2-vertices “ahead”

For $n \leq 12$, we enumerated all the 3-connected cubic graphs with n faces (using `plantri`) ... BRINKMANN & MCKAY

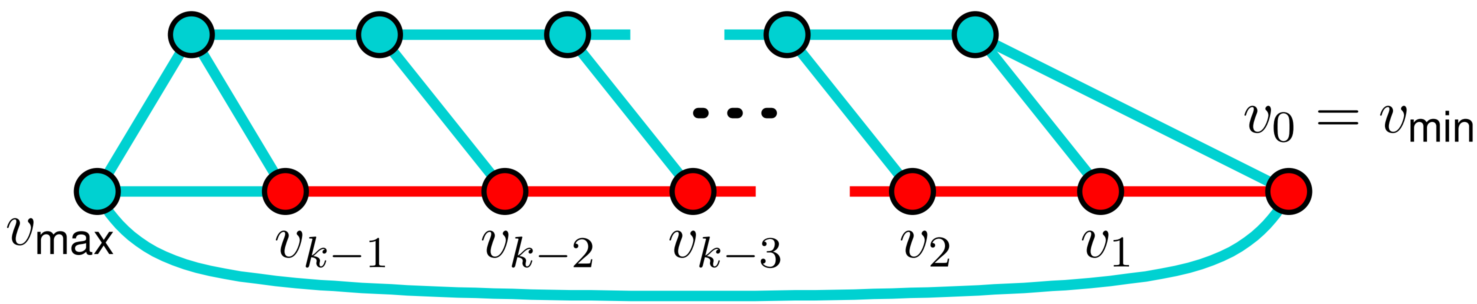
... and all the “abstract objective functions” on each of these ...

... to see what worst-case examples *look like*:

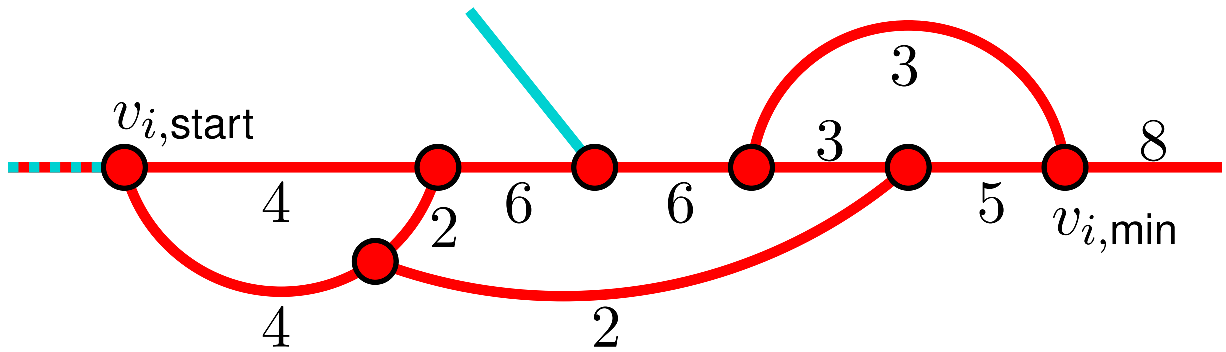


Random edge: lower bounds II

the backbone:



a configuration:

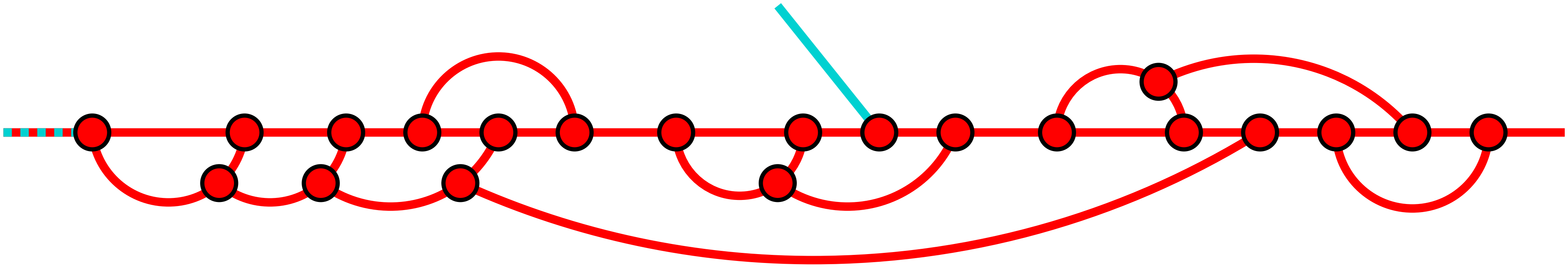


flow costs per configuration: $\frac{43}{8}$

facets per configuration: $3 + 1$

$$\Longrightarrow \quad \Lambda(\text{RE}) \geq \frac{43}{8 \cdot 4} = \frac{43}{32} \approx 1.3437$$

a worse configuration:



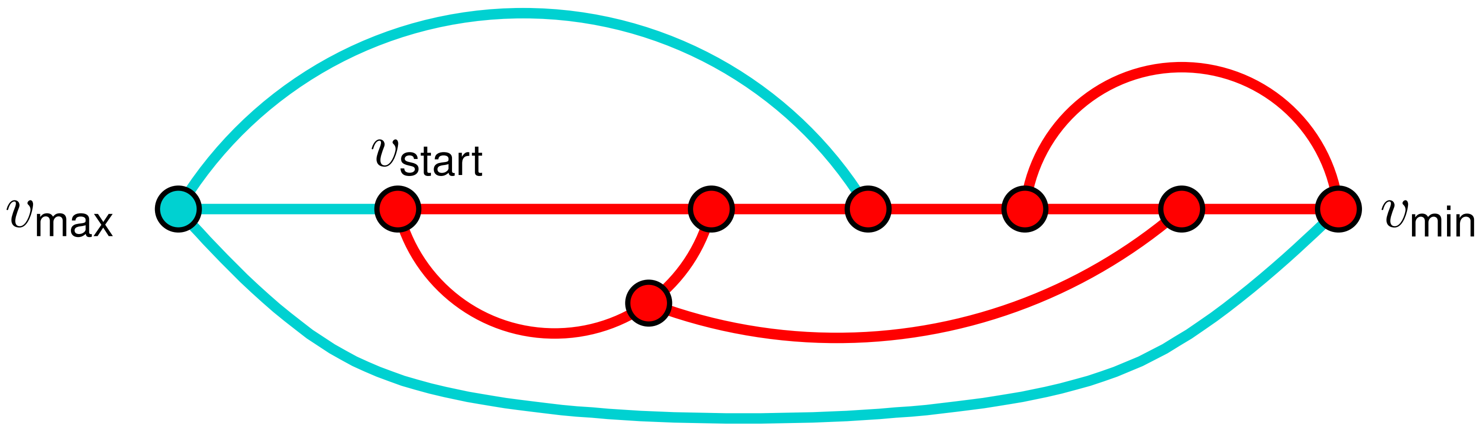
flow costs per configuration: $\frac{1897}{128}$

facets per configuration: $10 + 1$

$$\Longrightarrow \quad \Lambda(\mathrm{RE}) \ \geq \ \frac{1897}{1408} \ \approx \ 1.3473$$

Recursion formula for the expected number of pivot steps $\mathbf{E}(v)$ “starting from v”:

$$\mathbf{E}(v) \;=\; 1 + \frac{1}{|\,\delta^+(v)|} \sum_{u:(v,u)\in\delta^+(v)} \mathbf{E}(u),$$



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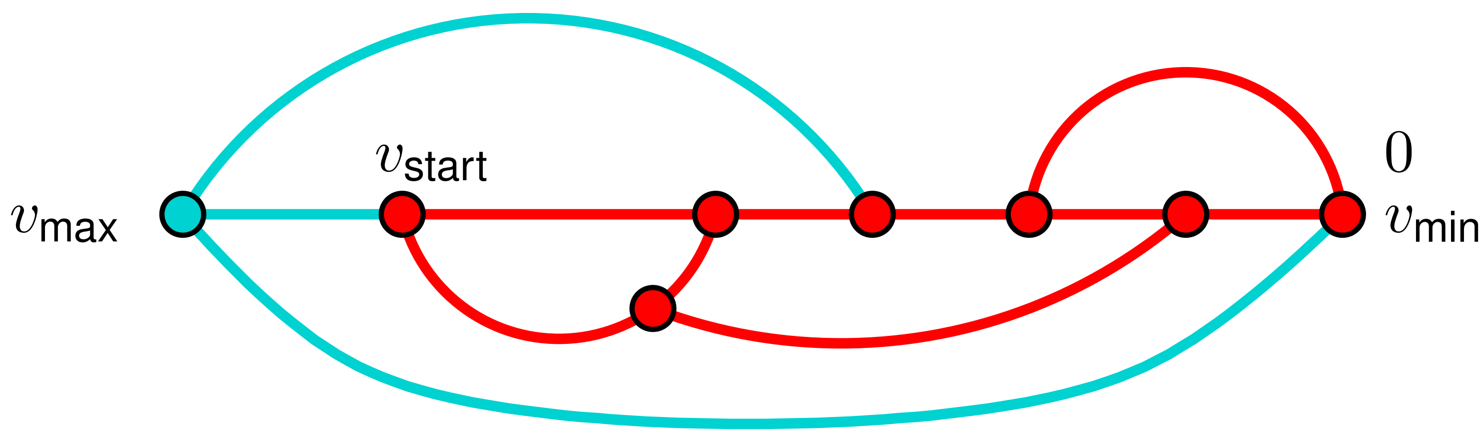

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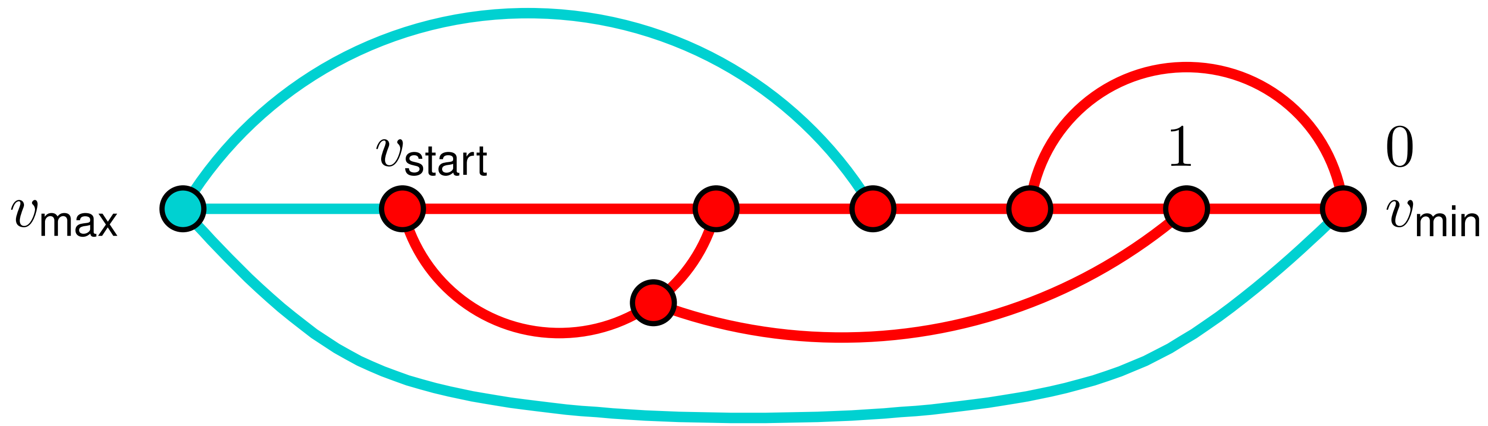

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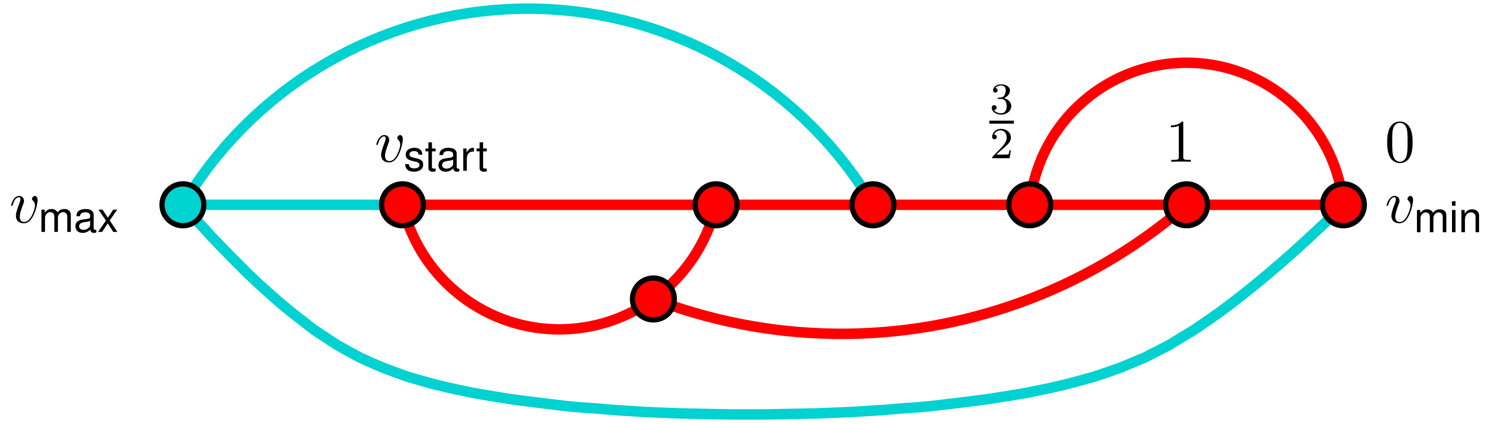

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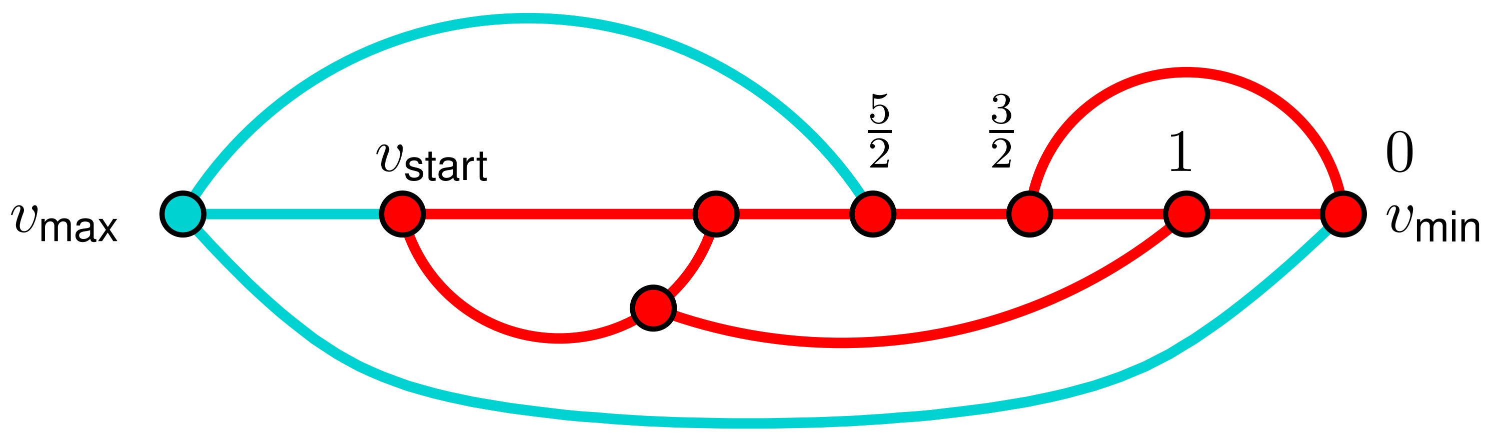

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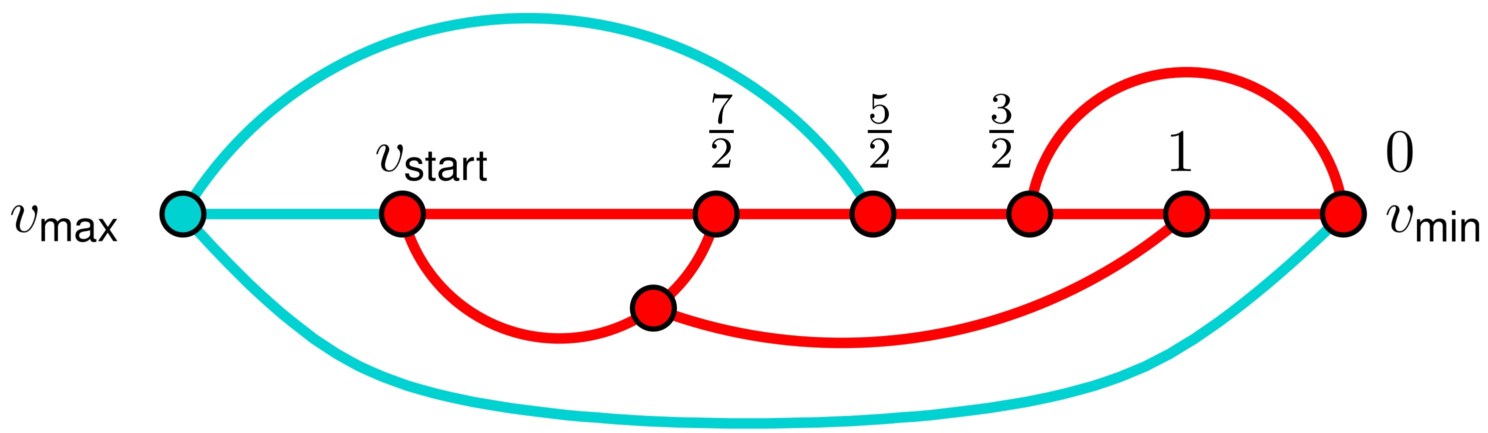

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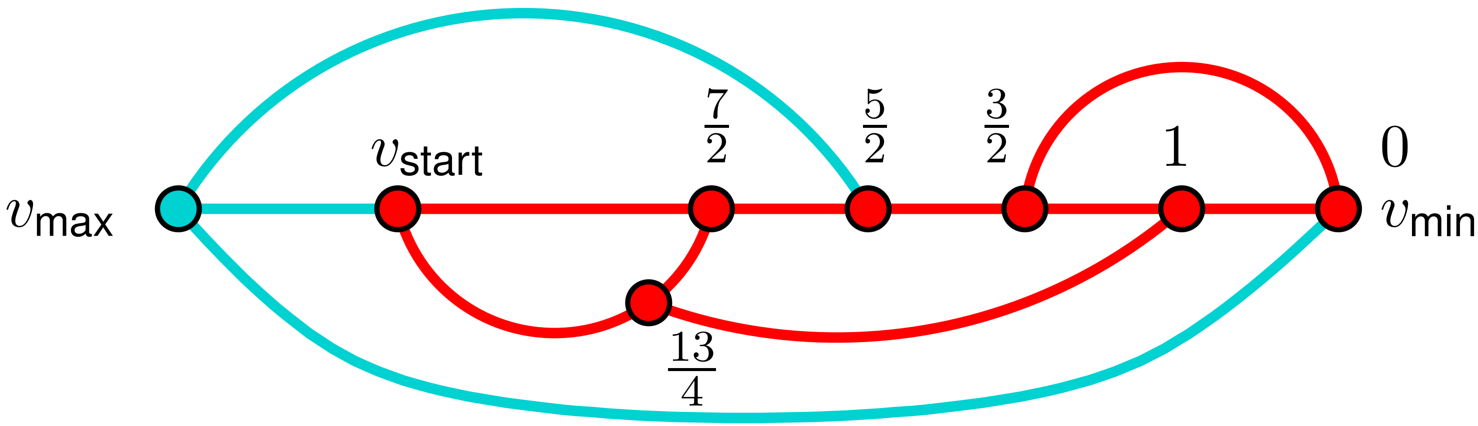

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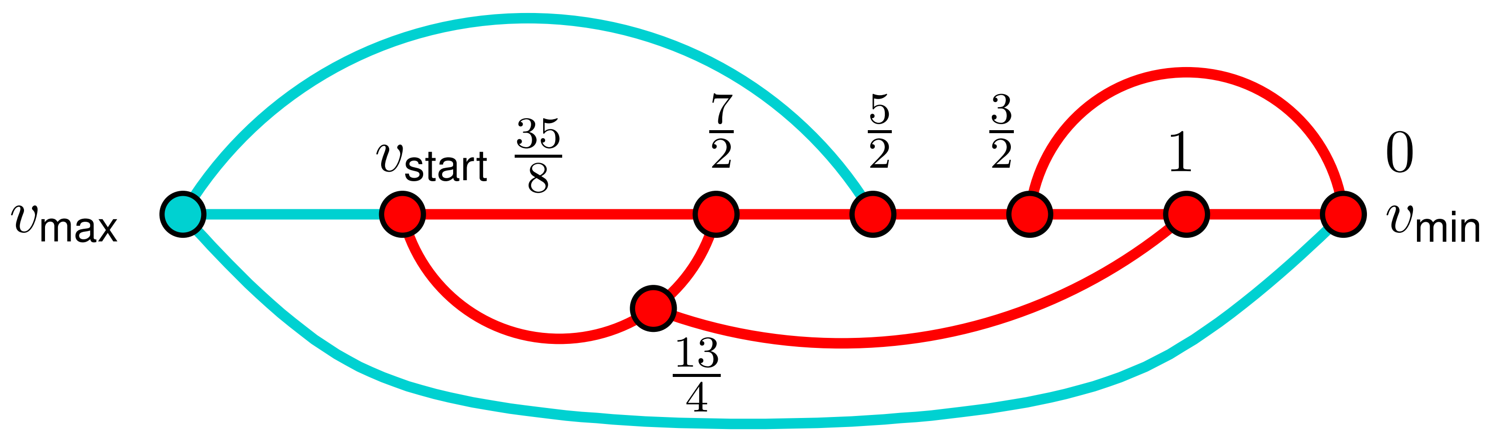
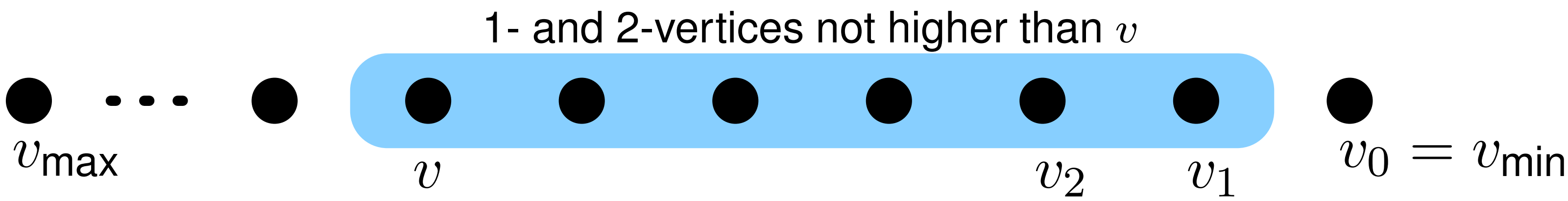


Random edge: upper bound II



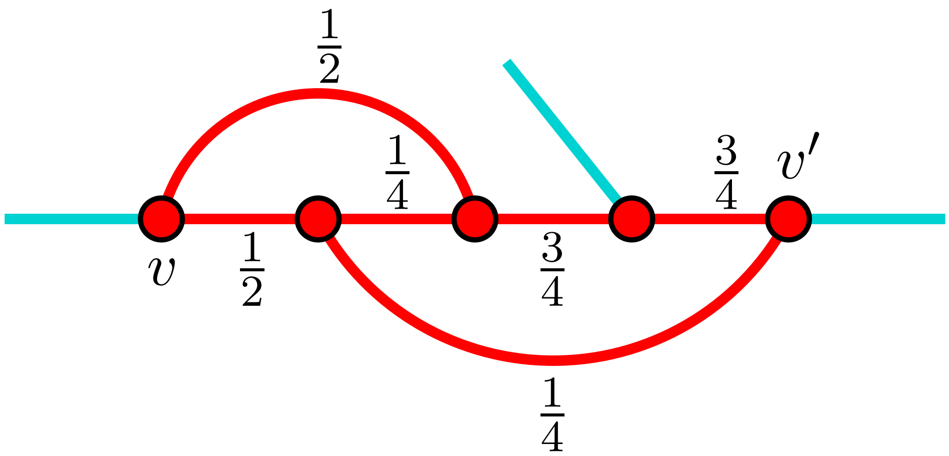
$$N_1(v) = \#\{\text{1-vertices not higher than } v\} \le n-3$$

$$N_2(v) = \#\{\text{2-vertices not higher than } v\} \le n-3$$

$$N(v) = N_1(v) + N_2(v) = \#\{\text{vertices below } v\} \le 2n-6$$

- Find feasible values for α and β such that $\mathbf{E}(v) \le \alpha N_1(v) + \beta N(v)$ holds and thus $\Lambda(\text{RE}) \le \alpha + 2\beta$.
- Induction on $N(v)$ rests on linear inequalities on α and β.
- Solve LP minimizing $\alpha + 2\beta$ on feasible region. The optimal solution is $\alpha = \frac{46}{87}$, $\beta = \frac{42}{87}$ and thus

$$\Lambda(\text{RE}) \le \frac{46}{87} + 2\,\frac{42}{87} \le 1.4943$$



$$E(v) = \tfrac{5}{2} \; + \; E(v')$$

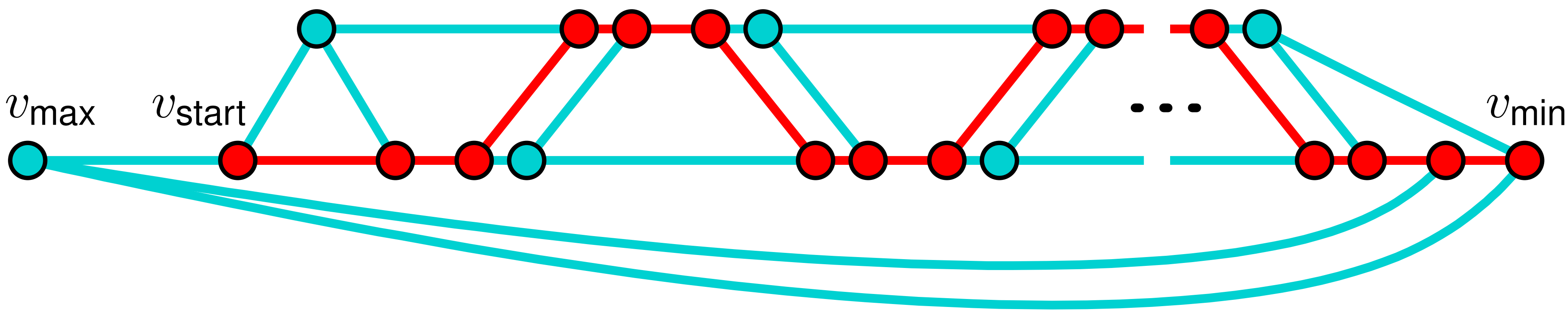
$$\le \tfrac{5}{2} \; + \; \alpha\, N_1(v') \; + \; \beta\, N(v') \overset{!}{\le} \alpha\, N_1(v) \; + \; \beta\, N(v)$$

$$\iff \quad \tfrac{5}{2} \le \big(N_1(v) - N_1(v')\big)\;\alpha \; + \; \big(N(v) - N(v')\big)\;\beta$$

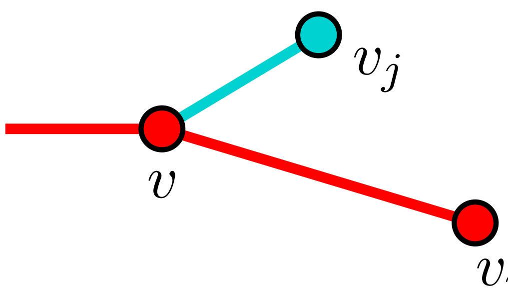
We have $N_1(v) - N_1(v') = 2$ and $N(v) - N(v') = 3$, thus

this case is o.k. if $\quad \frac{5}{2} \le 2\,\alpha + 3\,\beta$.

Move to the neighbor with the smallest objective function value.



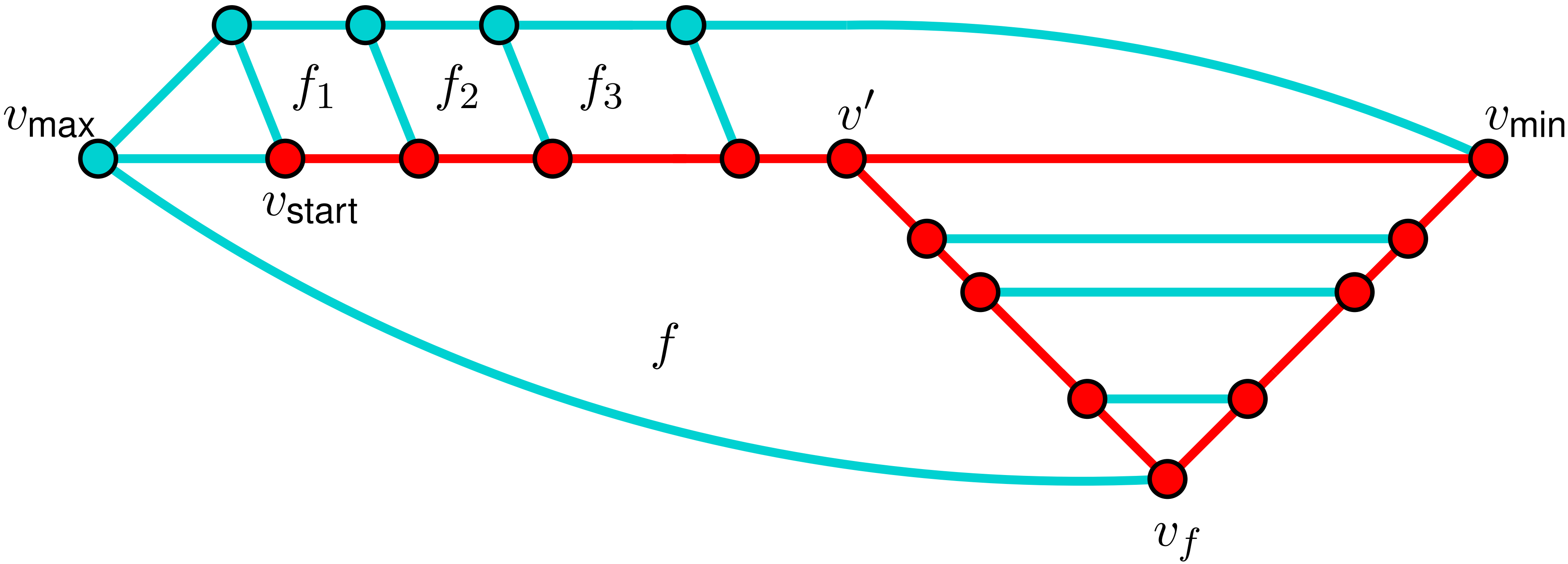
- Example using all $(n-3)$ 1-vertices and half of the $(n-3)$ 2-vertices
- For every 2-vertex, one vertex is skipped



THEOREM

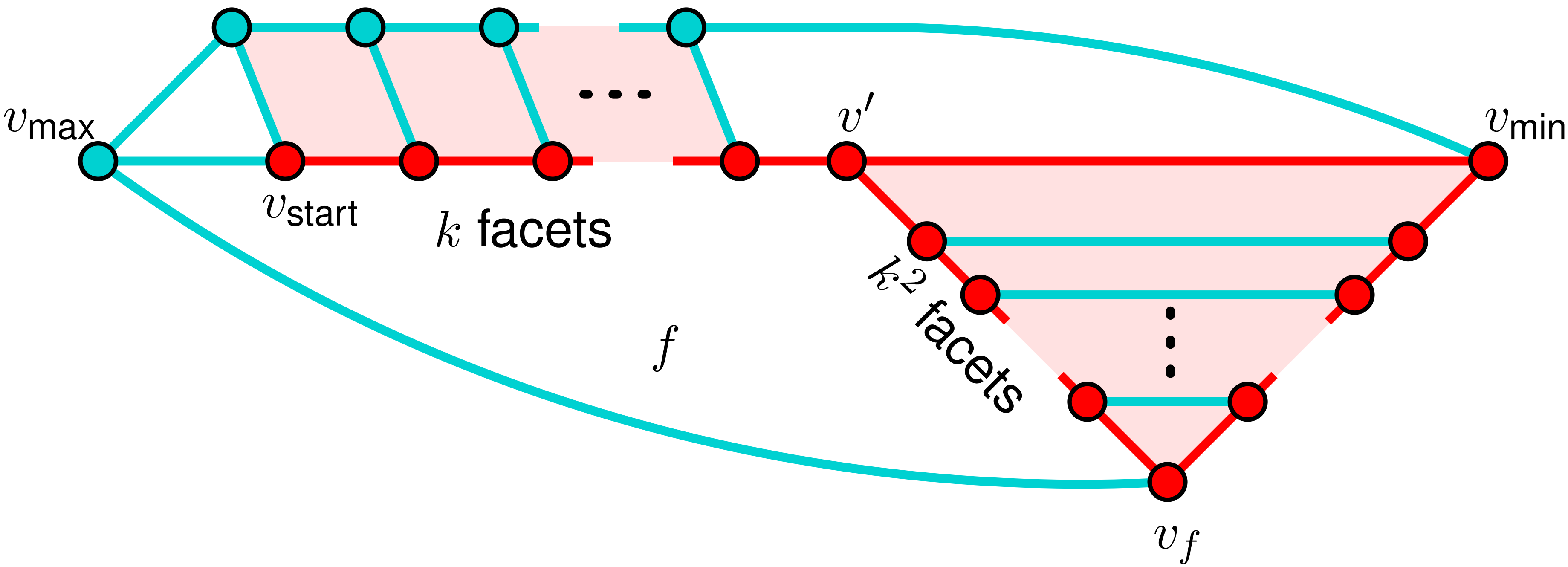
$$\Lambda(\text{GD}) \;=\; 1.5$$

Choose a facet that contains v uniformly at random, and solve by applying (RF) recursively.



THEOREM

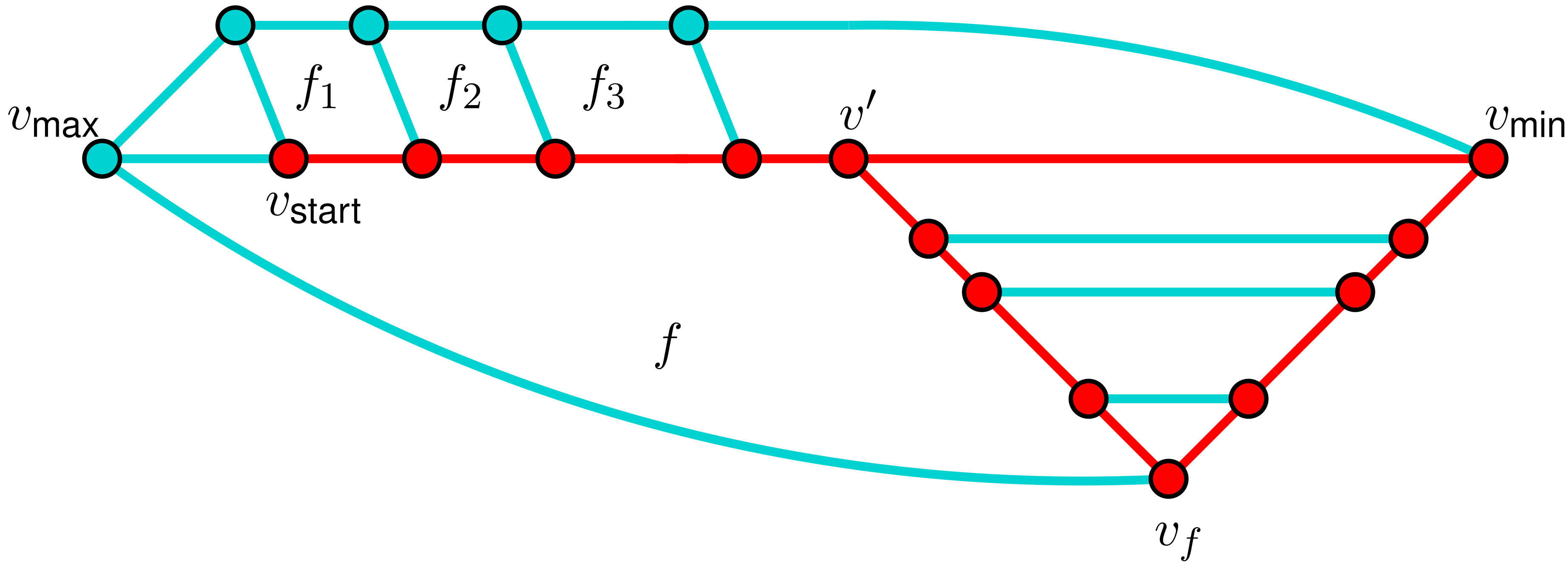
$$\Lambda(\text{RF}) = 2$$



$$\mathbf{E}(v_{\text{start}}) \geq \left(1 - \left(\tfrac{1}{2}\right)^k\right)(2k^2 + k + 1) \quad \text{and} \quad n = k^2 + k + 4$$

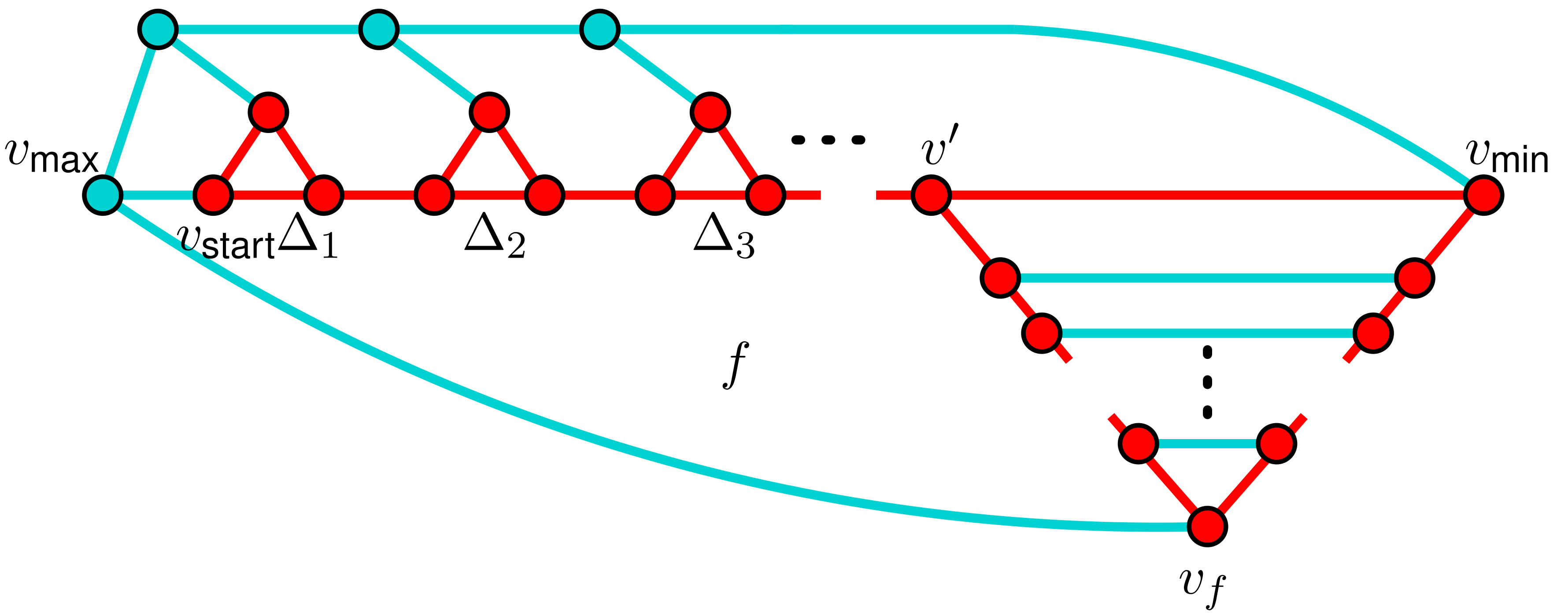
$$\Longrightarrow \Lambda(\text{RF}) \geq 2$$

At a 1-vertex, follow the outgoing edge.
Otherwise choose a facet that contains v uniformly at random, and solve by applying (RF-B) recursively.



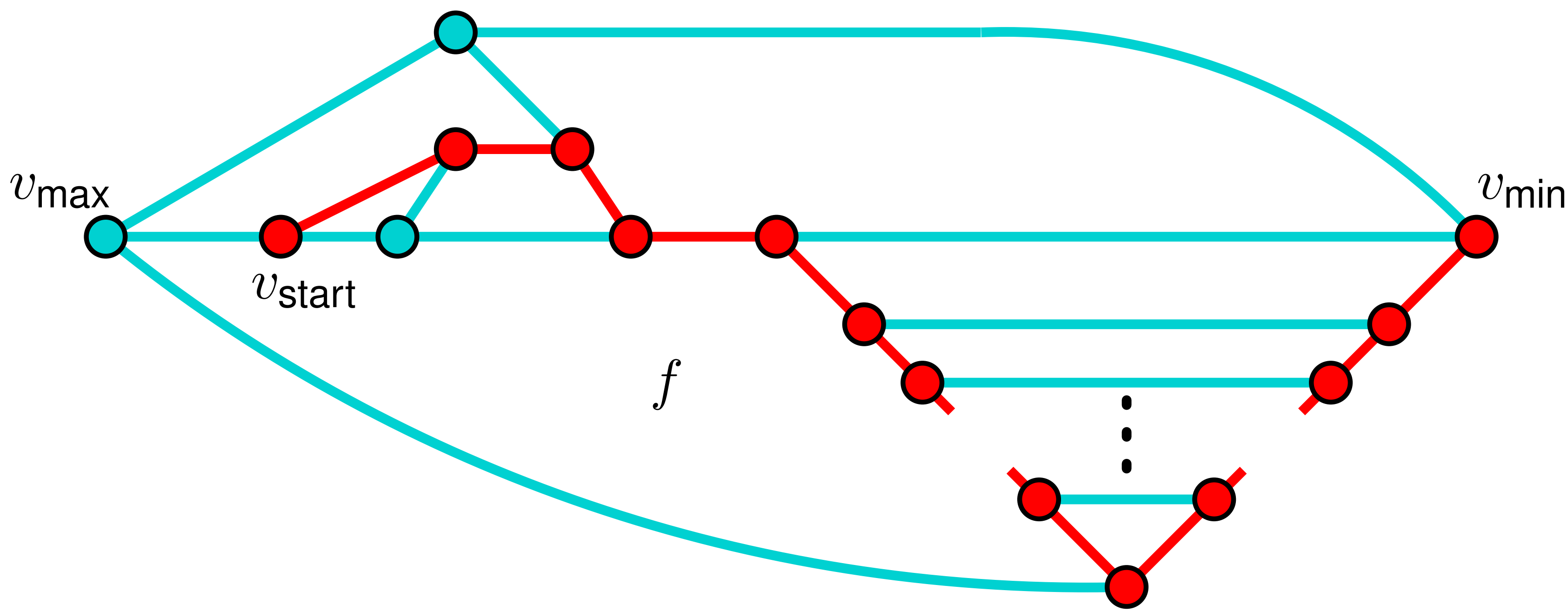
- follows path of 1-vertices deterministically
- cut off the first k 1-vertices

At a 1-vertex, follow the outgoing edge.
Otherwise choose a facet that contains v uniformly at random, and solve by applying (RF-B) recursively.

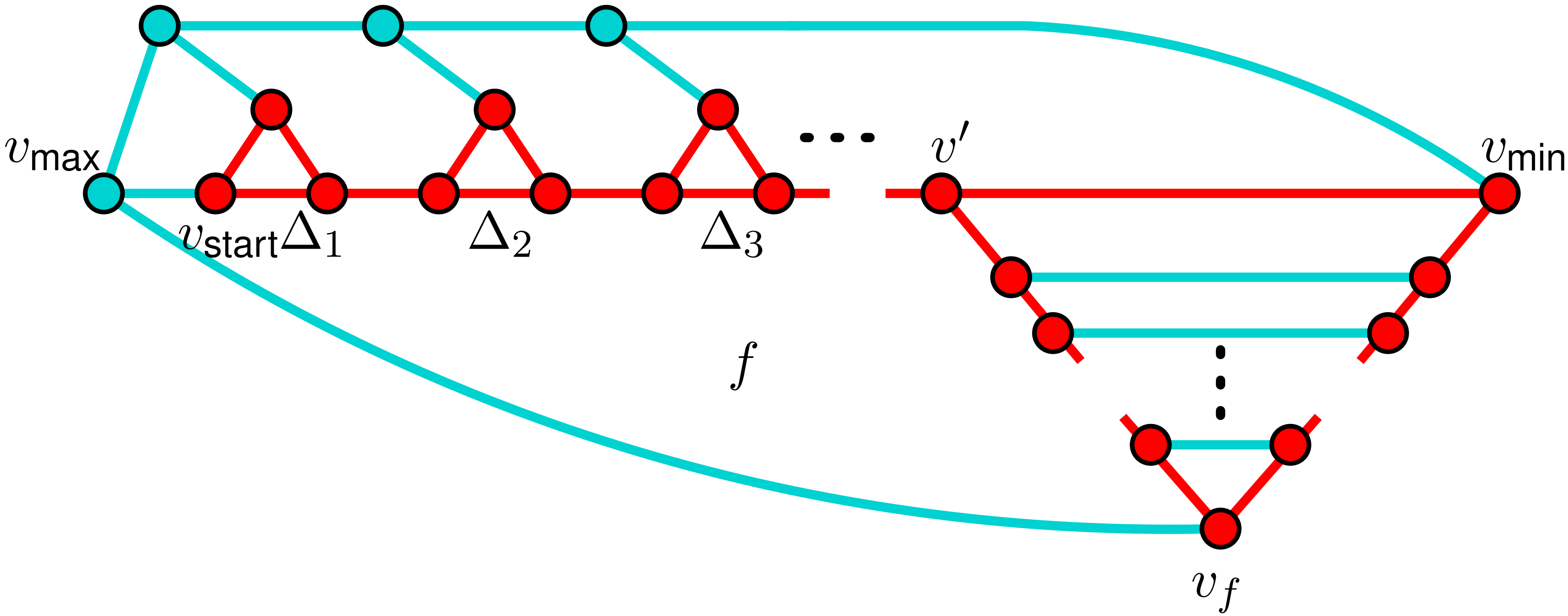


Choose an edge that leaves a facet which was entered least often in previous moves. (A tie-breaking rule is needed.)

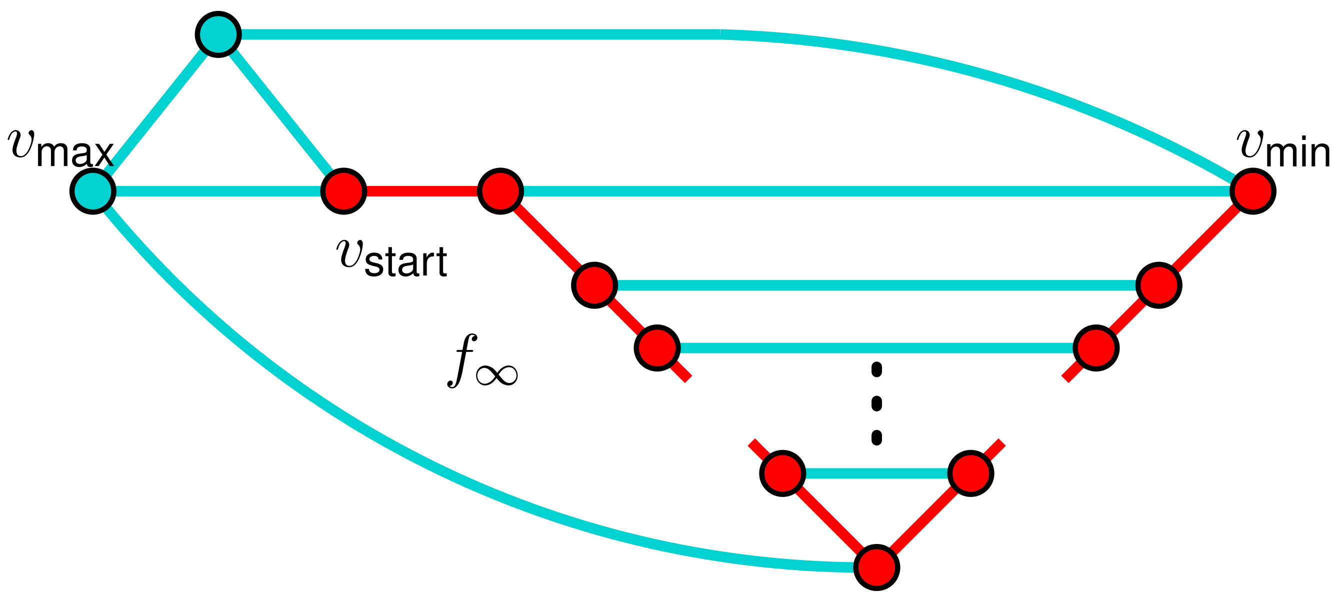
- Greatest decrease as tie-breaking rule



- Random edge as tie-breaking rule



Choose the edge that leaves the facet with the smallest number.

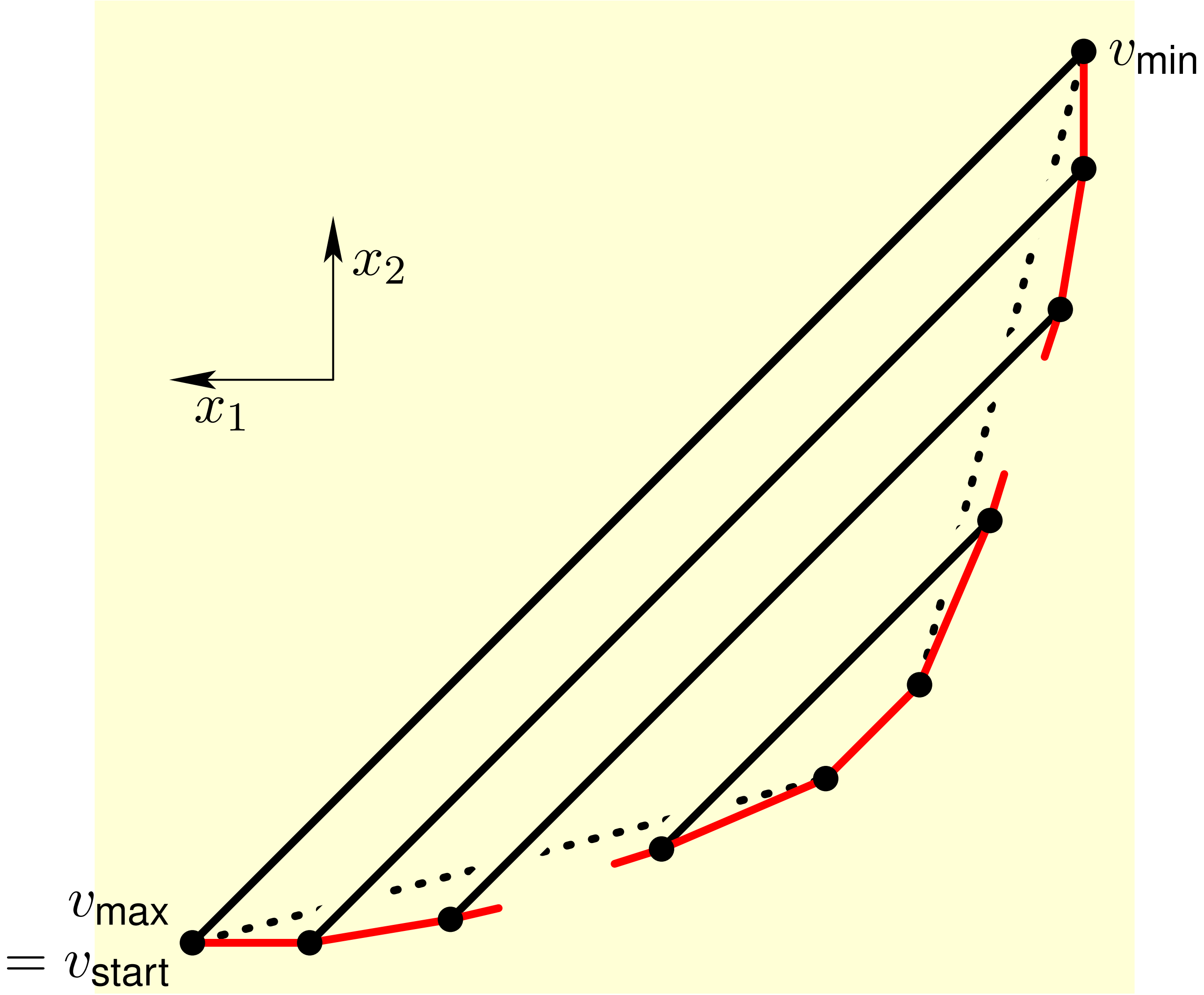


The original rule proposed by Dantzig:

> Select the edge according to the best "reduced cost coefficient" in the simplex tableau.

Observation: DANTZIG's rule isn't better than BLAND's rule.

Move along steepest decreasing edge vw, with $\frac{\langle c, w-v \rangle}{\|w-v\| \, \|c\|}$ minimal



- Best general upper bound: $O\left(\frac{f_0}{\sqrt{d}}\right)$ KAIBEL 2004
- Klee–Minty cubes: $\Theta(d^2)$ GÄRTNER, HENK, Z. 1998
  PEMANTLE & BALOGH 2004
- Linear assignment: polynomial TOVEY 1986
- $n = d + 1$: $\Theta(\log d)$
  $n = d + 2$: $\Theta(\log^2 d)$ GÄRTNER ET AL. 2001
- d-cube AOF's: $\Omega\left(c^{\sqrt[3]{d}}\right)$ MATOUŠEK & SZABO 2004

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▷ d-cube AOF's:	$\Omega(c^{\sqrt[3]{d}})$	MATOUŠEK & SZABO 2004
▷ **Conjecture**: On polytopes	$O(nd)$	**???**