Chapter 8

Shellings, the Euler-Poincaré Formula for Polytopes, Dehn-Sommerville Equations, the Upper Bound Theorem

8.1 Shellings

The notion of shellability is motivated by the desire to give an inductive proof of the Euler-Poincaré formula in any dimension.

Historically, this formula was discovered by Euler for three dimensional polytopes in 1752 (but it was already known to Descartes around 1640).

If f_0, f_1 and f_2 denote the number of vertices, edges and triangles of the three dimensional polytope, P, (i.e., the number of i-faces of P for $i = 0, 1, 2$), then the *Euler formula* states that

$$f_0 - f_1 + f_2 = 2.$$

The proof of Euler's formula is not very difficult but one still has to exercise caution.

Euler's formula was generalized to arbitrary d-dimensional polytopes by Schläfli (1852) but the first correct proof was given by Poincaré (1893, 1899).

If f_i denotes the number of i-faces of the d-dimensional polytope, P, (with $f_{-1} = 1$ and $f_d = 1$), the *Euler-Poincaré formula* states that:

$$\sum_{i=0}^{d-1} (-1)^i f_i = 1 - (-1)^d,$$

which can also be written as

$$\sum_{i=0}^{d} (-1)^i f_i = 1,$$

by incorporating $f_d = 1$ in the first formula or as

$$\sum_{i=-1}^{d} (-1)^i f_i = 0,$$

by incorporating both $f_{-1} = 1$ and $f_d = 1$ in the first formula.

Earlier inductive "proofs" of the above formula were proposed, notably a proof by Schläfli in 1852, but it was later observed that all these proofs assume that the boundary of every polytope can be built up inductively in a nice way, what we call *shellability*.

Actually, counter-examples of shellability for various simplicial complexes suggested that polytopes were perhaps not shellable.

However, the fact that polytopes are shellable was finally proved in 1970 by Bruggesser and Mani [**?**] and soon after that (also in 1970) a striking application of shellability was made by McMullen [**?**] who gave the first proof of the so-called "upper bound theorem".

As shellability of polytopes is an important tool and as it yields one of the cleanest inductive proof of the Euler-Poincaré formula, we will sketch its proof in some details.

Definition 8.1.1 Let K be a pure polyhedral complex of dimension d. A *shelling of* K is a list, $F_1, \ldots, F_s$, of the cells (i.e., d-faces) of K such that either $d = 0$ (and thus, all F_i are points) or the following conditions hold:

(i) The boundary complex, $\mathcal{K}(\partial F_1)$, of the first cell, F_1, of K has a shelling.

(ii) For any j, $1 < j \leq s$, the intersection of the cell F_j with the previous cells is nonempty and is an initial segment of a shelling of the $(d-1)$-dimensional boundary complex of F_j, that is

$$F_j \cap \left(\bigcup_{i=1}^{j-1} F_i \right) = G_1 \cup G_2 \cup \cdots \cup G_r,$$

for some shelling $G_1, G_2, \ldots, G_r, \ldots, G_t$ of $\mathcal{K}(\partial F_j)$, with $1 \leq r \leq t$. As the intersection should be the initial segment of a shelling for the $(d-1)$-dimensional complex, ∂F_j, it has to be pure $(d-1)$-dimensional and connected for $d > 1$.

A polyhedral complex is *shellable* if it is pure and has a shelling.

Note that shellabiliy is only defined for pure complexes.

Here are some examples of shellable complexes:

(1) Every 0-dimensional complex, that is, evey set of points, is shellable, by definition.

(2) A 1-dimensional complex is a graph without loops and parallel edges. A 1-dimensional complex is shellable iff it is connected, which implies that it has no isolated vertices. Any ordering of the edges, $e_1, \ldots, e_s$, such that $\{e_1, \ldots, e_i\}$ induces a connected subgraph for every i will do. Such an ordering can be defined inductively, due to the connectivity of the graph.

(3) Every simplex is shellable. In fact, any ordering of its facets yields a shelling. This is easily shown by induction on the dimension, since the intersection of any two facets F_i and F_j is a facet of both F_i and F_j.

(4) The d-cubes are shellable. By induction on the dimension, it can be shown that every ordering of the $2d$ facets $F_1, \ldots, F_{2d}$ such that F_1 and F_{2d} are opposite (that is, $F_{2d} = -F_1$) yields a shelling.

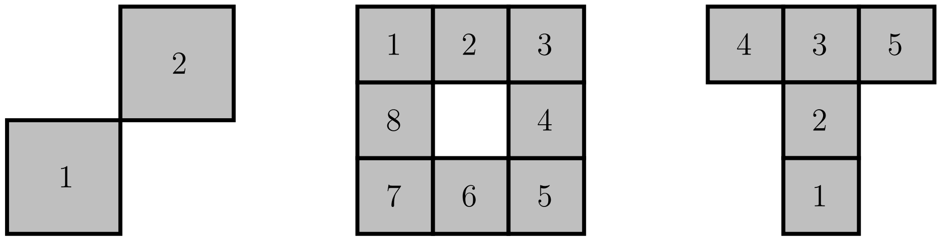

Figure 8.1: Non shellable and Shellable 2-complexes

However, already for 2-complexes, problems arise. For example, in Figure 8.1, the left and the middle 2-complexes are not shellable but the right complex is shellable.

The problem with the left complex is that cells 1 and 2 intersect at a vertex, which is not 1-dimensional, and in the middle complex, the intersection of cell 8 with its predecessors is not connected.

In contrast, the ordering of the right complex is a shelling.

However, observe that the reverse ordering is not a shelling because cell 4 has an empty intersection with cell 5!

Remarks:

1. Condition (i) in Definition 8.1.1 is redundant because, as we shall prove shortly, every polytope is shellable. However, if we want to use this definition for more general complexes, then condition (i) is necessary.
2. When K is a simplicial complex, condition (i) is of course redundant, as every simplex is shellable but condition (ii) can also be simplified to:

(ii') For any j, with $1 < j \leq s$, the intersection of F_j with the previous cells is nonempty and pure $(d-1)$-dimensional. This means that for every $i < j$ there is some $l < j$ such that $F_i \cap F_j \subseteq F_l \cap F_j$ and $F_l \cap F_j$ is a facet of F_j.

The following proposition yields an important piece of information about the local structure of shellable simplicial complexes:

Proposition 8.1.2 *Let K be a shellable simplicial complex and say $F_1, \ldots, F_s$ is a shelling for K. Then, for every vertex, v, the restriction of the above sequence to the link,* $\mathrm{Lk}(v)$, *and to the star,* $\mathrm{St}(v)$, *are shellings.*

Since the complex, $\mathcal{K}(P)$, associated with a polytope, P, has a single cell, namely P itself, note that by condition (i) in the definition of a shelling, $\mathcal{K}(P)$ is shellable iff the complex, $\mathcal{K}(\partial P)$, is shellable.

We will say simply say that "P is shellable" instead of "$\mathcal{K}(\partial P)$ is shellable".

Proposition 8.1.3 *Given any polytope, P, if $F_1, \ldots, F_s$ is a shelling of P, then the reverse sequence $F_s, \ldots, F_1$ is also a shelling of P.*

Proposition 8.1.3 generally fails for complexes that are not polytopes, see the right 2-complex in Figure 8.1.

We will now present the proof that every polytope is shellable, using a technique invented by Bruggesser and Mani (1970) known as *line shelling* [**?**].

We begin by explaining this idea in the 2-dimensional case, a convex polygon, since it is particularly simple.

Consider the 2-polytope, P, shown in Figure 8.2 (a polygon) whose faces are labeled F_1, F_2, F_3, F_4, F_5.

Pick any line, ℓ, intersecting the interior of P and intersecting the supporting lines of the facets of P (*i.e.*, the edges of P) in distinct points labeled z_1, z_2, z_3, z_4, z_5 (such a line can always be found, as will be shown shortly).

Orient the line, ℓ, (say, upward) and travel on ℓ starting from the point of P where ℓ leaves P, namely, z_1.

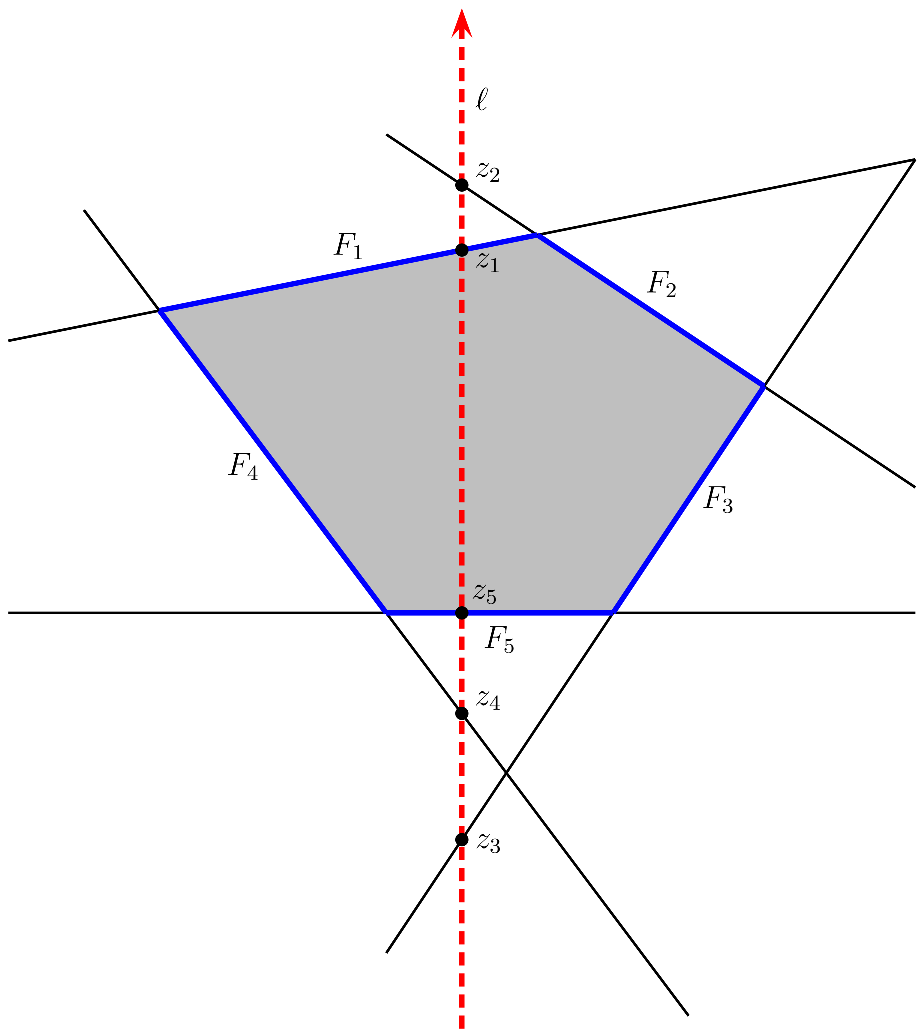

Figure 8.2: Shelling a polygon by travelling along a line

For a while we only see F_1 but then F_2 become visble when we cross z_2. We imagine that we travel very fast and when we reach "$+\infty$" in the upward direction on ℓ, we instantly come back on ℓ from below at "$-\infty$".

At this point, we only see the face of P corresponding to the lowest supporting line of faces of P, i.e., the line corresponding to the smallest z_i, in our case, z_3.

Our trip stops when we reach z_5, the intersection of F_5 and ℓ. During the second phase of our trip, we saw F_3, F_4 and F_5 and the entire trip yields the sequence F_1, F_2, F_3, F_4, F_5, which is easily seen to be a shelling of P.

This is the crux of the Bruggesser-Mani method for shelling a polytope: We travel along a suitably chosen line and record the order in which the faces become visible during this trip. This is why such shellings are called *line shellings*.

In order to prove that polytopes are shellable we need the notion of points and lines in "general position".

Recall from the equivalence of $\mathcal{V}$-polytopes and $\mathcal{H}$-polytopes that a polytope, P, in $\mathbb{E}^d$ with nonempty interior is cut out by t irredundant hyperplanes, H_i, and by picking the origin in the interior of P the equations of the H_i may be assumed to be of the form

$$a_i \cdot z = 1$$

where a_i and a_j are not proportional for all $i \neq j$, so that

$$P = \{z \in \mathbb{E}^d \mid a_i \cdot z \leq 1,\ 1 \leq i \leq t\}.$$

Definition 8.1.4 Let P be any polytope in $\mathbb{E}^d$ with nonempty interior and assume that P is cut out by the irredudant hyperplanes, H_i, of equations $a_i \cdot z = 1$, for $i = 1, \ldots, t$. A point, $c \in \mathbb{E}^d$, is said to be in *general position* w.r.t. P is c does not belong to any of the H_i, that is, if $a_i \cdot c \neq 1$ for $i = 1, \ldots, t$. A line, ℓ, is said to be in *general position* w.r.t. P if ℓ is not parallel to any of the H_i and if ℓ intersects the H_i in distinct points.

The following proposition showing the existence of lines in general position w.r.t. a polytope illustrates a very useful technique, the "perturbation method".

Proposition 8.1.5 *Let P be any polytope in $\mathbb{E}^d$ with nonempty interior. For any two points, x and y in $\mathbb{E}^d$, with x outside of P; y in the interior of P; and x in general position w.r.t. P, for $\lambda \in \mathbb{R}$ small enough, the line, ℓ_λ, through x and y_λ with*

$$y_\lambda = y + (\lambda, \lambda^2, \ldots, \lambda^d),$$

intersects P in its interior and is in general position w.r.t. P.

It should be noted that the perturbation method involving $\Lambda = (\lambda, \lambda^2, \ldots, \lambda^d)$ is quite flexible.

For example, by adapting the proof of Proposition 8.1.5 we can prove that for any two distinct facets, F_i and F_j of P, there is a line in general position w.r.t. P intersecting F_i and F_j. Start with x outside P and very close to F_i and y in the interior of P and very close to F_j.

Given any point, x, strictly outside a polytope, P, we say that a facet, F, of P is *visible from* x iff for every $y \in F$ the line through x and y intersects F only in y (equivalently, x and the interior of P are strictly separared by the supporting hyperplane of F).

We now prove the following fundamental theorem due to Bruggesser and Mani [**?**] (1970):

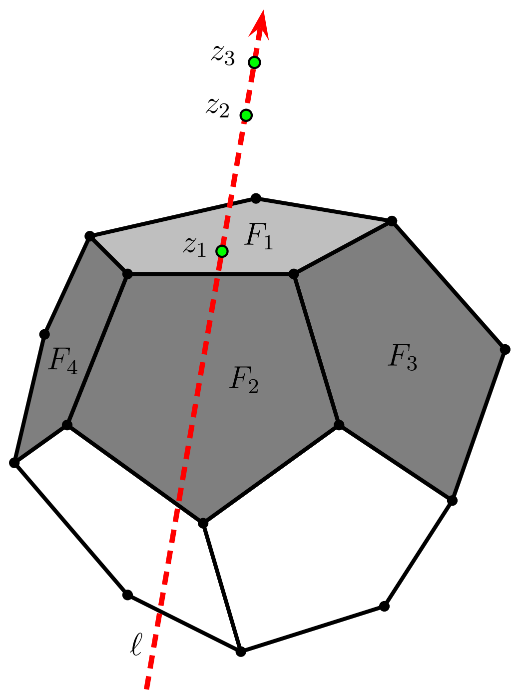

Figure 8.3: Shelling a polytope by travelling along a line, ℓ

Theorem 8.1.6 *(Existence of Line Shellings for Polytopes) Let P be any polytope in $\mathbb{E}^d$ of dimension d. For every point, x, outside P and in general position w.r.t. P, there is a shelling of P in which the facets of P that are visible from x come first.*

Remark: The trip along the line ℓ is often described as a *rocket flight* starting from the surface of P viewed as a little planet (for instance, this is the description given by Ziegler [**?**] (Chapter 8)).

Observe that if we reverse the direction of ℓ, we obtain the reversal of the original line shelling. Thus, the reversal of a line shelling is not only a shelling but a line shelling as well.

We can easily prove the following corollary:

Corollary 8.1.7 *Given any polytope, P, the following facts hold:*

(1) For any two facets F and F', there is a shelling of P in which F comes first and F' comes last.

(2) For any vertex, v, of P, there is a shelling of P in which the facets containing v form an initial segment of the shelling.

Remark: A *plane triangulation*, K, is a pure two-dimensional complex in the plane such that $|K|$ is homeomorphic to a closed disk.

Edelsbrunner proves that every plane triangulation has a shelling and from this, that $\chi(K) = 1$, where $\chi(K) = f_0 - f_1 + f_2$ is the Euler-Poincaré characteristic of K, where f_0 is the number of vertices, f_1 is the number of edges and f_2 is the number of triangles in K (see Edelsbrunner [**?**], Chapter 3).

This result is an immediate consequence of Corollary 8.1.7 if one knows about the stereographic projection map, which will be discussed in the next Chapter.

We now have all the tools needed to prove the famous Euler-Poincaré Formula for Polytopes.

8.2 The Euler-Poincaré Formula for Polytopes

We begin by defining a very important topological concept, the Euler-Poincaré characteristic of a complex.

Definition 8.2.1 Let K be a d-dimensional complex. For every i, with $0 \leq i \leq d$, we let f_i denote the number of i-faces of K and we let

$$\mathbf{f}(K) = (f_0, \cdots, f_d) \in \mathbb{N}^{d+1}$$

be the *f-vector* associated with K (if necessary we write $f_i(K)$ instead of f_i). The *Euler-Poincaré characteristic*, $\chi(K)$, of K is defined by

$$\chi(K) = f_0 - f_1 + f_2 + \cdots + (-1)^d f_d = \sum_{i=0}^{d} (-1)^i f_i.$$

Given any d-dimensional polytope, P, the *f-vector* associated with P is the f-vector associated with $\mathcal{K}(P)$, that is,

$$\mathbf{f}(P) = (f_0, \cdots, f_d) \in \mathbb{N}^{d+1},$$

where f_i, is the number of i-faces of P (= the number of i-faces of $\mathcal{K}(P)$ and thus, $f_d = 1$), and the *Euler-Poincaré characteristic*, $\chi(P)$, of P is defined by

$$\chi(P) = f_0 - f_1 + f_2 + \cdots + (-1)^d f_d = \sum_{i=0}^{d} (-1)^i f_i.$$

Moreover, the *f-vector* associated with the boundary, ∂P, of P is the f-vector associated with $\mathcal{K}(\partial P)$, that is,

$$\mathbf{f}(\partial P) = (f_0, \cdots, f_{d-1}) \in \mathbb{N}^d$$

where f_i, is the number of i-faces of ∂P (with $0 \leq i \leq d-1$), and the *Euler-Poincaré characteristic*, $\chi(\partial P)$, of ∂P is defined by

$$\chi(\partial P) = f_0 - f_1 + f_2 + \cdots + (-1)^{d-1} f_{d-1} = \sum_{i=0}^{d-1} (-1)^i f_i.$$

Observe that $\chi(P) = \chi(\partial P) + (-1)^d$, since $f_d = 1$.

Remark: It is convenient to set $f_{-1} = 1$. Then, some authors, including Ziegler [**?**] (Chapter 8), define the *reduced Euler-Poincaré characteristic*, $\chi'(K)$, of a complex (or a polytope), K, as

$$\begin{aligned}\chi'(K) &= -f_{-1} + f_0 - f_1 + f_2 + \cdots + (-1)^d f_d \\ &= \sum_{i=-1}^{d} (-1)^i f_i = -1 + \chi(K),\end{aligned}$$

i.e., they incorporate $f_{-1} = 1$ into the formula.

A crucial observation for proving the Euler-Poincaré formula is that the Euler-Poincaré characteristic is additive.

This means that if K_1 and K_2 are any two complexes such that $K_1 \cup K_2$ is also a complex, which implies that $K_1 \cap K_2$ is also a complex (because we must have $F_1 \cap F_2 \in K_1 \cap K_2$ for every face F_1 of K_1 and every face F_2 of K_2), then

$$\chi(K_1 \cup K_2) = \chi(K_1) + \chi(K_2) - \chi(K_1 \cap K_2).$$

This follows immediately because for any two sets A and B

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

To prove our next theorem we will use complete induction on $\mathbb{N} \times \mathbb{N}$ ordered by the lexicographic ordering.

Recall that the lexicographic ordering on $\mathbb{N} \times \mathbb{N}$ is defined as follows:

$$(m, n) < (m', n') \quad \text{iff} \quad \begin{cases} m = m' \quad \textit{and} \quad n < n' \\ \text{or} \\ m < m'. \end{cases}$$

Theorem 8.2.2 *(Euler-Poincaré Formula) For every polytope, P, we have*

$$\chi(P) = \sum_{i=0}^{d} (-1)^i f_i = 1 \qquad (d \geq 0),$$

and so,

$$\chi(\partial P) = \sum_{i=0}^{d-1} (-1)^i f_i = 1 - (-1)^d \qquad (d \geq 1).$$

Proof. We prove the following statement: For every d-dimensional polytope, P, if $d = 0$ then

$$\chi(P) = 1,$$

else if $d \geq 1$ then for every shelling $F_1, \ldots, F_{f_{d-1}}$, of P, for every j, with $1 \leq j \leq f_{d-1}$, we have

$$\chi(F_1 \cup \cdots \cup F_j) = \begin{cases} 1 & \text{if } 1 \leq j < f_{d-1} \\ 1 - (-1)^d & \text{if } j = f_{d-1}. \end{cases}$$

We proceed by complete induction on $(d, j) \geq (0, 1)$. $\square$

Remark: Other combinatorial proofs of the Euler-Poincaré formula are given in Grünbaum [**?**] (Chapter 8), Boissonnat and Yvinec [**?**] (Chapter 7) and Ewald [**?**] (Chapter 3).

Coxeter gives a proof very close to Poincaré's own proof using notions of homology theory [**?**] (Chapter IX).

We feel that the proof based on shellings is the most direct and one of the most elegant.

Incidently, the above proof of the Euler-Poincaré formula is very close to Schläfli proof from 1852 but Schläfli did not have shellings at his disposal so his "proof" had a gap. The Bruggesser-Mani proof that polytopes are shellable fills this gap!

8.3 Dehn-Sommerville Equations for Simplicial Polytopes and h-Vectors

If a d-polytope, P, has the property that its faces are all simplices, then it is called a *simplicial polytope*.

It is easily shown that a polytope is simplicial iff its facets are simplices, in which case, every facet has d vertices.

The polar dual of a simplicial polytope is called a *simple polytope*. We see immediately that every vertex of a simple polytope belongs to d facets.

For simplicial (and simple) polytopes it turns out that other remarkable equations besides the Euler-Poincaré formula hold among the number of i-faces.

These equations were discovered by Dehn for $d = 4, 5$ (1905) and by Sommerville in the general case (1927).

Although it is possible (and not difficult) to prove the Dehn-Sommerville equations by "double counting", as in Grünbaum [**?**] (Chapter 9) or Boissonnat and Yvinec (Chapter 7, but beware, these are the dual formulae for simple polytopes), it turns out that instead of using the f-vector associated with a polytope it is preferable to use what's known as the h-vector because for simplicial polytopes the h-numbers have a natural interpretation in terms of shellings.

Furthermore, the statement of the Dehn-Sommerville equations in terms of h-vectors is transparent:

$$h_i = h_{d-i},$$

and the proof is very simple in terms of shellings.

In the rest of this section, we restrict our attention to simplicial complexes.

In order to motivate h-vectors, we begin by examining more closely the structure of the new faces that are created during a shelling when the cell F_j is added to the partial shelling $F_1, \ldots, F_{j-1}$.

If K is a simplicial polytope and V is the set of vertices of K, then every i-face of K can be identified with an $(i+1)$-subset of V (that is, a subset of V of cardinality $i+1$).

Definition 8.3.1 For any shelling, $F_1, \ldots, F_s$, of a simplicial complex, K, of dimension d, for every j, with $1 \leq j \leq s$, the *restriction*, R_j, of the facet, F_j, is the set of "obligatory" vertices

$$R_j = \{v \in F_j \mid F_j - \{v\} \subseteq F_i, \text{ for some } i,\ 1 \leq i < j\}.$$

The crucial property of the R_j is that the new faces, G, added at step j (when F_j is added to the shelling) are precisely the faces in the set

$$I_j = \{G \subseteq V \mid R_j \subseteq G \subseteq F_j\}.$$

But then, we obtain a partition, $\{I_1, \ldots, I_s\}$, of the set of faces of the simplicial complex (other that K itself). Note that the empty face is allowed.

Now, if we define

$$h_i = |\{j \mid |R_j| = i,\ 1 \leq j \leq s\}|,$$

for $i = 0, \ldots, d$, then it turns out that we can recover the f_k in terms of the h_i as follows:

$$f_{k-1} = \sum_{j=1}^{s} \binom{d - |R_j|}{k - |R_j|} = \sum_{i=0}^{k} h_i \binom{d-i}{k-i},$$

with $1 \leq k \leq d$.

But more is true: The above equations are invertible and the h_k can be expressed in terms of the f_i as follows:

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1},$$

with $0 \leq k \leq d$ (remember, $f_{-1} = 1$).

Let us explain all this in more detail. Consider the example of a connected graph shown in Figure 8.4:

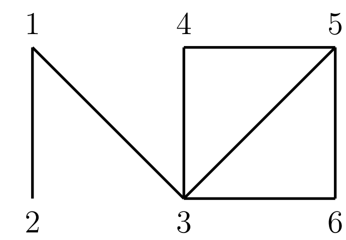

Figure 8.4: A connected 1-dimensional complex, G

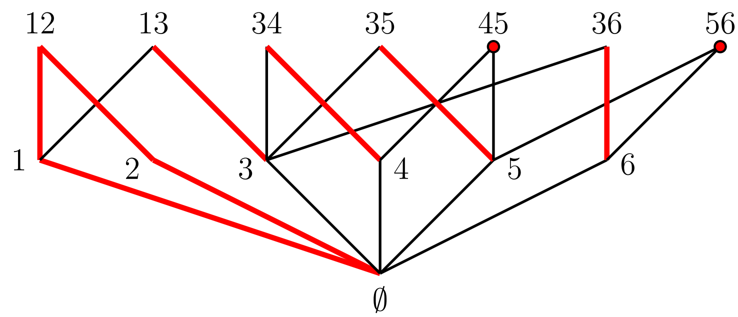

Figure 8.5: the partition associated with a shelling of G

A shelling order of its 7 edges is given by the sequence

$$12,\ 13,\ 34,\ 35,\ 45,\ 36,\ 56.$$

The partial order of the faces of G together with the blocks of the partition $\{I_1, \ldots, I_7\}$ associated with the seven edges of G are shown in Figure 8.5, with the blocks I_j shown in red:

The "minimal" new faces (corresponding to the R_j's) added at every stage of the shelling are

$$\emptyset,\ 3,\ 4,\ 5,\ 45,\ 6,\ 56.$$

Again, if h_i is the number of blocks, I_j, such that the corresponding restriction set, R_j, has size i, that is,

$$h_i = |\{j \mid |R_j| = i,\ 1 \leq j \leq s\}|,$$

for $i = 0, \ldots, d$, where the simplicial polytope, K, has dimension $d-1$, we define the *h-vector* associated with K as

$$\mathbf{h}(K) = (h_0, \ldots, h_d).$$

Then, in the above example, as $R_1 = \{\emptyset\}$, $R_2 = \{3\}$, $R_3 = \{4\}$, $R_4 = \{5\}$, $R_5 = \{4,5\}$, $R_6 = \{6\}$ and $R_7 = \{5,6\}$, we get $h_0 = 1$, $h_1 = 4$ and $h_2 = 2$, that is,

$$\mathbf{h}(G) = (1, 4, 2).$$

Now, let us show that if K is a shellable simplicial complex, then the f-vector can be recovered from the h-vector.

Indeed, if $|R_j| = i$, then each $(k-1)$-face in the block I_j must use all i nodes in R_j, so that there are only $d-i$ nodes available and, among those, $k-i$ must be chosen. Therefore,

$$f_{k-1} = \sum_{j=1}^{s} \binom{d-|R_j|}{k-|R_j|}$$

and, by definition of h_i, we get

$$\begin{aligned} f_{k-1} &= \sum_{i=0}^{k} h_i \binom{d-i}{k-i} \\ &= h_k + \binom{d-k+1}{1} h_{k-1} + \cdots + \binom{d}{k} h_0, \end{aligned}$$

where $1 \leq k \leq d$.

Moreover, the formulae are invertible, that is, the h_i can be expressed in terms of the f_k. For this, form the two polynomials

$$f(x) = \sum_{i=0}^{d} f_{i-1}x^{d-i} = f_{d-1} + f_{d-2}x + \cdots + f_0 x^{d-1} + f_{-1}x^d$$

with $f_{-1} = 1$ and

$$h(x) = \sum_{i=0}^{d} h_i x^{d-i} = h_d + h_{d-1}x + \cdots + h_1 x^{d-1} + h_0 x^d.$$

Then, it is easy to see that

$$f(x) = \sum_{i=0}^{d} h_i(x+1)^{d-i} = h(x+1).$$

Consequently, $h(x) = f(x-1)$ and by comparing the coefficients of x^{d-k} on both sides of the above equation, we get

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}.$$

In particular, $h_0 = 1$, $h_1 = f_0 - d$, and

$$h_d = f_{d-1} - f_{d-2} + f_{d-3} + \cdots + (-1)^{d-1} f_0 + (-1)^d.$$

It is also easy to check that

$$h_0 + h_1 + \cdots + h_d = f_{d-1}.$$

Now, we just showed that if K is shellable, then its f-vector and its h-vector are related as above.

But even if K is not shellable, the above suggests defining the h-vector from the f-vector as above. Thus, we make the definition:

Definition 8.3.2 For any $(d-1)$-dimensional simplicial complex, K, the *h-vector* associated with K is the vector

$$\mathbf{h}(K) = (h_0, \ldots, h_d) \in \mathbb{Z}^{d+1},$$

given by

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}.$$

Note that if K is shellable, then the interpretation of h_i as the number of cells, F_j, such that the corresponding restriction set, R_j, has size i shows that $h_i \geq 0$.

However, for an arbitrary simplicial complex, some of the h_i can be strictly negative. Such an example is given in Ziegler [**?**] (Section 8.3).

We summarize below most of what we just showed:

Proposition 8.3.3 *Let K be a $(d-1)$-dimensional pure simplicial complex. If K is shellable, then its h-vector is nonnegative and h_i counts the number of cells in a shelling whose restriction set has size i. Moreover, the h_i do not depend on the particular shelling of K.*

We are now ready to prove the Dehn-Sommerville equations.

For $d = 3$, these are easily obtained by double counting. Indeed, for a simplicial polytope, every edge belongs to two facets and every facet has three edges. It follows that

$$2f_1 = 3f_2.$$

Together with Euler's formula

$$f_0 - f_1 + f_2 = 2,$$

we see that

$$f_1 = 3f_0 - 6 \quad \text{and} \quad f_2 = 2f_0 - 4,$$

namely, that the number of vertices of a simplicial 3-polytope determines its number of edges and faces, these being linear functions of the number of vertices.

For arbitrary dimension d, we have

Theorem 8.3.4 *(Dehn-Sommerville Equations) If K is any simplicial d-polytope, then the components of the h-vector satisfy*

$$h_k = h_{d-k} \qquad k = 0, 1, \ldots, d.$$

Equivalently

$$f_{k-1} = \sum_{i=k}^{d} (-1)^{d-i} \binom{i}{k} f_{i-1} \qquad k = 0, \ldots, d.$$

Furthermore, the equation $h_0 = h_d$ is equivalent to the Euler-Poincaré formula.

Clearly, the Dehn-Sommerville equations, $h_k = h_{d-k}$, are linearly independent for
$0 \leq k < \lfloor \frac{d+1}{2} \rfloor$.

For example, for $d = 3$, we have the two independent equations

$$h_0 = h_3,\ h_1 = h_2,$$

and for $d = 4$, we also have two independent equations

$$h_0 = h_4,\ h_1 = h_3,$$

since $h_2 = h_2$ is trivial.

When $d = 3$, we know that $h_1 = h_2$ is equivalent to $2f_1 = 3f_2$ and when $d = 4$, if one unravels $h_1 = h_3$ in terms of the f_i' one finds

$$2f_2 = 4f_3,$$

that is $f_2 = 2f_3$.

More generally, it is easy to check that

$$2f_{d-2} = df_{d-1}$$

for all d. For $d = 5$, we find three independent equations

$$h_0 = h_5,\ h_1 = h_4,\ h_2 = h_3,$$

and so on.

It can be shown that for general d-polytopes, the Euler-Poincaré formula is the only equation satisfied by all h-vectors and for simplicial d-polytopes, the $\lfloor \frac{d+1}{2} \rfloor$ Dehn-Sommerville equations, $h_k = h_{d-k}$, are the only equations satisfied by all h-vectors (see Grünbaum [**?**], Chapter 9).

As we saw for 3-dimensional simplicial polytopes, the number of vertices, $n = f_0$, determines the number of edges and the number of faces, and these are linear in f_0.

For $d \geq 4$, this is no longer true and the number of facets is no longer linear in n but in fact quadratic.

It is then natural to ask which d-polytopes with a prescribed number of vertices have the maximum number of k-faces.

This question which remained an open problem for some twenty years was eventually settled by McMullen in 1970 [**?**].

8.4 The Upper Bound Theorem and Cyclic Polytopes

Given a d-polytope with n vertices, what is an upper bound on the number of its i-faces?

This question is not only important from a theoretical point of view but also from a computational point of view because of its implications for algorithms in combinatorial optimization and in computational geometry.

The answer to the above problem is that there is a class of polytopes called *cyclic polytopes* such that the cyclic d-polytope, $C_d(n)$, has the maximum number of i-faces among all d-polytopes with n vertices.

This result stated by Motzkin in 1957 became known as the *upper bound conjecture* until it was proved by McMullen in 1970, using shellings [**?**] (just after Bruggesser and Mani's proof that polytopes are shellable). It is now known as the *upper bound theorem*.

Another proof of the upper bound theorem was given later by Alon and Kalai [**?**] (1985). A version of this proof can also be found in Ewald [**?**] (Chapter 3).

First, consider the cases $d = 2$ and $d = 3$. When $d = 2$, our polytope is a polygon in which case $n = f_0 = f_1$. Thus, this case is trivial.

For $d = 3$, we claim that $2f_1 \geq 3f_2$. Indeed, every edge belongs to exactly two faces so if we add up the number of sides for all faces, we get $2f_1$. Since every face has at least three sides, we get $2f_1 \geq 3f_2$. Then, using Euler's relation, it is easy to show that

$$f_1 \leq 6n - 3 \quad f_2 \leq 2n - 4$$

and we know that equality is achieved for simplicial polytopes.

Let us now consider the general case. The rational curve, $c\colon \mathbb{R} \to \mathbb{R}^d$, given parametrically by

$$c(t) = (t, t^2, \ldots, t^d)$$

is at the heart of the story.

This curve if often called the *moment curve* or *rational normal curve* of degree d. For $d = 3$, it is known as the *twisted cubic*. Here is the definition of the cyclic polytope, $C_d(n)$.

Definition 8.4.1 For any sequence, $t_1 < \ldots < t_n$, of distinct real number, $t_i \in \mathbb{R}$, with $n > d$, the convex hull,

$$C_d(n) = \mathrm{conv}(c(t_1), \ldots, c(t_n))$$

of the n points, $c(t_1), \ldots, c(t_n)$, on the moment curve of degree d is called a *cyclic polytope*.

The first interesting fact about the cyclic polytope is that it is simplicial.

Proposition 8.4.2 *Every $d+1$ of the points $c(t_1), \ldots,$ $c(t_n)$ are affinely independent. Consequently, $C_d(n)$ is a simplicial polytope and the $c(t_i)$ are vertices.*

Proposition 8.4.3 *For any k with $2 \leq 2k \leq d$, every subset of k vertices of $C_d(n)$ is a $(k-1)$-face of $C_d(n)$. Hence*

$$f_k(C_d(n)) = \binom{n}{k+1} \qquad \text{if} \quad 0 \leq k < \left\lfloor \frac{d}{2} \right\rfloor.$$

Observe that Proposition 8.4.3 shows that any subset of $\lfloor \frac{d}{2} \rfloor$ vertices of $C_d(n)$ forms a face of $C_d(n)$.

When a d-polytope has this property it is called a *neighborly polytope*. Therefore, cyclic polytopes are neighborly.

Proposition 8.4.3 also shows a phenomenon that only manifests itself in dimension at least 4: For $d \geq 4$, the polytope $C_d(n)$ has n pairwise adjacent vertices. For $n >> d$, this is counter-intuitive.

Finally, the combinatorial structure of cyclic polytopes is completely determined as follows:

Proposition 8.4.4 *(Gale evenness condition, Gale (1963)). Let n and d be integers with $2 \leq d < n$. For any sequence $t_1 < t_2 < \cdots < t_n$, consider the cyclic polytope*

$$C_d(n) = \mathrm{conv}(c(t_1), \ldots, c(t_n)).$$

A subset, $S \subseteq \{1, \ldots, n\}$ with $|S| = d$ determines a facet of $C_d(n)$ iff for all $i < j$ not in S, then the number of $k \in S$ between i and j is even:

$$|\{k \in S \mid i < k < j\}| \equiv 0 \pmod 2 \quad \textit{for} \quad i, j \notin S$$

In particular, Proposition 8.4.4 shows that the combinatorial structure of $C_d(n)$ does not depend on the specific choice of the sequence $t_1 < \ldots < t_n$. This justifies our notation $C_d(n)$.

Here is the celebrated upper bound theorem first proved by McMullen [**?**].

Theorem 8.4.5 *(Upper Bound Theorem, McMullen (1970)) Let P be any d-polytope with n vertices. Then, for every k, with $1 \leq k \leq d$, the polytope P has at most as many $(k-1)$-faces as the cyclic polytope, $C_d(n)$, that is*

$$f_{k-1}(P) \leq f_{k-1}(C_d(n)).$$

Moreover, equality for some k with $\lfloor \frac{d}{2} \rfloor \leq k \leq d$ implies that P is neighborly.

The first step in the proof of Theorem 8.4.5 is to prove that among all d-polytopes with a given number, n, of vertices, the maximum number of i-faces is achieved by simplicial d-polytopes.

Proposition 8.4.6 *Given any d-polytope, P, with n-vertices, it is possible to form a simplicial polytope, P', by perturbing the vertices of P such that P' also has n vertices and*

$$f_{k-1}(P) \leq f_{k-1}(P') \qquad \textit{for} \quad 1 \leq k \leq d.$$

Furthermore, equality for $k > \lfloor \frac{d}{2} \rfloor$ can occur only if P is simplicial.

Proposition 8.4.6 allows us to restict our attention to simplicial polytopes. Now, it is obvious that

$$f_{k-1} \leq \binom{n}{k}$$

for any polytope P (simplicial or not) and we also know that equality holds if $k \leq \lfloor \frac{d}{2} \rfloor$ for neighborly polytopes such as the cyclic polytopes. For $k > \lfloor \frac{d}{2} \rfloor$, it turns out that equality can only be achieved for simplices.

However, for a *simplicial* polytope, the Dehn-Sommerville equations $h_k = h_{d-k}$ together with the equations giving f_k in terms of the h_i's show that $f_0, f_1, \ldots, f_{\lfloor \frac{d}{2} \rfloor}$ already determine the whole f-vector.

Thus, it is possible to express the f_{k-1} in terms of $h_0, h_1, \ldots, h_{\lfloor \frac{d}{2} \rfloor}$ for $k \geq \lfloor \frac{d}{2} \rfloor$. It turns out that we get

$$f_{k-1} = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor}{}^{*} \left(\binom{d-i}{k-i} + \binom{i}{k-d+i} \right) h_i,$$

where the meaning of the superscript $*$ is that when d is even we only take half of the last term for $i = \frac{d}{2}$ and when d is odd we take the whole last term for $i = \frac{d-1}{2}$ (for details, see Ziegler [**?**], Chapter 8).

As a consequence if we can show that the neighborly polytopes maximize not only f_{k-1} but also h_{k-1} when $k \leq \lfloor \frac{d}{2} \rfloor$, the upper bound theorem will be proved.

Indeed, McMullen proved the following theorem which is "more than enough" to yield the desired result ([**?**]):

Theorem 8.4.7 *(McMullen (1970)) For every simplicial d-polytope with $f_0 = n$ vertices, we have*

$$h_k(P) \leq \binom{n-d-1+k}{k} \qquad \textit{for} \quad 0 \leq k \leq d.$$

Furthermore, equality holds for all l and all k with $0 \leq k \leq l$ iff $l \leq \lfloor \frac{d}{2} \rfloor$ and P is l-neighborly. (a polytope is l-neighborly iff any subset of l or less vertices determine a face of P.)

Since cyclic d-polytopes are neighborly (which means that they are $\lfloor \frac{d}{2} \rfloor$-neighborly), Theorem 8.4.5 follows from Proposition 8.4.6, and Theorem 8.4.7.

Corollary 8.4.8 *For every simplicial neighborly d-polytope with n vertices, we have*

$$f_{k-1} = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor^{*}} \left(\binom{d-i}{k-i} + \binom{i}{k-d+i} \right) \binom{n-d-1+i}{i},$$

for $1 \leq k \leq d$. This gives the maximum number of $(k-1)$-faces for any d-polytope with n-vertices, for all k with $1 \leq k \leq d$. In particular, the number of facets of the cyclic polytope, $C_d(n)$, is

$$f_{d-1} = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor^{*}} 2 \binom{n-d-1+i}{i}$$

and, more explicitly,

$$f_{d-1} = \binom{n - \lfloor \frac{d+1}{2} \rfloor}{n-d} + \binom{n - \lfloor \frac{d+2}{2} \rfloor}{n-d}.$$

Corollary 8.4.8 implies that the number of facets of any d-polytope is $O(n^{\lfloor \frac{d}{2} \rfloor})$.

An unfortunate consequence of this upper bound is that the complexity of any convex hull algorithms for n points in $\mathbb{E}^d$ is $O(n^{\lfloor \frac{d}{2} \rfloor})$.

The $O(n^{\lfloor \frac{d}{2} \rfloor})$ upper bound can be obtained more directly using a pretty argument using shellings due to R. Seidel [**?**].

Remark: There is also a *lower bound theorem* due to Barnette (1971, 1973) which gives a lower bound on the f-vectors all d-polytopes with n vertices.

In this case, there is an analog of the cyclic polytopes called *stacked polytopes*.

These polytopes, $P_d(n)$, are simplicial polytopes obtained from a simplex by building shallow pyramids over the facets of the simplex. Then, it turns out that if $d \geq 2$, then

$$f_k \geq \begin{cases} \binom{d}{k} n - \binom{d+1}{k+1} k & \text{if } 0 \leq k \leq d-2 \\ (d-1)n - (d+1)(d-2) & \text{if } k = d-1. \end{cases}$$

There has been a lot of progress on the combinatorics of f-vectors and h-vectors since 1971, especially by R. Stanley, G. Kalai and L. Billera and K. Lee, among others. We recommend two excellent surveys:

1. Bayer and Lee [**?**] summarizes progress in this area up to 1993.

2. Billera and Björner [**?**] is a more advanced survey which reports on results up to 1997.

In fact, many of the chapters in Goodman and O'Rourke [**?**] should be of interest to the reader.