Notes on Spherical Harmonics and Linear Representations of Lie Groups

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Chapter 1

Spherical Harmonics

1.1 Introduction, Spherical Harmonics on the Circle

Joseph Fourier (1768-1830) invented Fourier series in order to solve the heat equation [12]. Using Fourier series, every square-integrable periodic function, f, (of period 2π) can be expressed uniquely as the sum of a power series of the form

$$f(\theta) = a_0 + \sum_{k=1}^{\infty}(a_k \cos k\theta + b_k \cos k\theta),$$

where the *Fourier coefficents*, a_k, b_k, of f are given by the formulae

$$a_0 = \frac{1}{2\pi}\int_{\pi}^{\pi} f(\theta)\, d\theta, \quad a_k = \frac{1}{\pi}\int_{\pi}^{\pi} f(\theta)\cos k\theta\, d\theta, \quad b_k = \frac{1}{\pi}\int_{\pi}^{\pi} f(\theta)\sin k\theta\, d\theta,$$

for $k \geq 1$. The reader will find the above formulae in Fourier's famous book [12] in Chapter III, Section 233, page 256, essentially using the notation that we use nowdays.

This remarkable discovery has many theoretical and practical applications in physics, signal processing, engineering, *etc.* We can describe Fourier series in a more conceptual manner if we introduce the following inner product on square-integrable functions:

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(\theta)g(\theta)\, d\theta,$$

which we will also denote by

$$\langle f, g\rangle = \int_{S^1} f(\theta)g(\theta)\, d\theta,$$

where S^1 denotes the unit circle. After all, periodic functions of (period 2π) can be viewed as functions on the circle. With this inner product, the space $L^2(S^1)$ is a complete normed vector space, that is, a Hilbert space. Furthermore, if we define the subspaces, $\mathcal{H}_k(S^1)$, of $L^2(S^1)$, so that $\mathcal{H}_0(S^1)\,(=\mathbb{R})$ is the set of constant functions and $\mathcal{H}_k(S^1)$ is the two-dimensional space spanned by the functions $\cos k\theta$ and $\sin k\theta$, then it turns out that we have a Hilbert sum decomposition

$$L^2(S^1) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k(S^1)$$

into pairwise orthogonal subspaces, where $\bigcup_{k=0}^{\infty} \mathcal{H}_k(S^1)$ is dense in $L^2(S^1)$. The functions $\cos k\theta$ and $\sin k\theta$ are also orthogonal in $\mathcal{H}_k(S^1)$.

Now, it turns out that the spaces, $\mathcal{H}_k(S^1)$, arise naturally when we look for homogeneous solutions of the Laplace equation, $\Delta f = 0$, in $\mathbb{R}^2$. Roughly speaking, a homogeneous function in $\mathbb{R}^2$ is a function that can be expressed in polar coordinates, (r, θ), as

$$f(r, \theta) = r^k g(\theta).$$

Recall that the Laplacian on $\mathbb{R}^2$ expressed in cartesian coordinates, (x, y), is given by

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2},$$

where $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a function which is at least of class C^2. In polar coordinates, (r, θ), where $(x, y) = (r\cos\theta, r\sin\theta)$ and $r > 0$, the Laplacian is given by

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}.$$

If we restrict f to the unit circle, S^1, then the Laplacian on S^1 is given by

$$\Delta_{s^1} f = \frac{\partial^2 f}{\partial \theta^2}.$$

It turns out that *the space* $\mathcal{H}_k(S^1)$ *is the eigenspace of* Δ_{S^1} *for the eigenvalue* $-k^2$.

To show this, we consider another question, namely, *what are the harmonic functions on* $\mathbb{R}^2$, that is, the functions, f, that are solutions of the *Laplace equation*,

$$\Delta f = 0.$$

Our ancestors had the idea that the above equation can be solved by *separation of variables*. This means that we write $f(r, \theta) = F(r)g(\theta)$, where $F(r)$ and $g(\theta)$ are independent functions. To make things easier, let us assume that $F(r) = r^k$, for some integer $k \geq 0$, which means that we assume that f is a *homogeneous function of degree* k. Recall that a function, $f\colon \mathbb{R}^2 \to \mathbb{R}$, is *homogeneous of degree* k iff

$$f(tx, ty) = t^k f(x, y) \qquad \text{for all } t > 0.$$

Now, using the Laplacian in polar coordinates, we get

$$\begin{aligned}\Delta f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(r^k g(\theta))}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2(r^k g(\theta))}{\partial \theta^2} \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(kr^k g\right) + r^{k-2}\frac{\partial^2 g}{\partial\theta^2} \\ &= r^{k-2}k^2 g + r^{k-2}\frac{\partial^2 g}{\partial\theta^2} \\ &= r^{k-2}(k^2 g + \Delta_{S^1} g).\end{aligned}$$

Thus, we deduce that

$$\Delta f = 0 \quad \text{iff} \quad \Delta_{S^1} g = -k^2 g,$$

that is, g is an eigenfunction of Δ_{S^1} for the eigenvalue $-k^2$. But, the above equation is equivalent to the second-order differential equation

$$\frac{d^2 g}{d\theta^2} + k^2 g = 0,$$

whose general solution is given by

$$g(\theta) = a_n \cos k\theta + b_n \sin k\theta.$$

In summary, we found that the integers, $0, -1, -4, -9, \ldots, -k^2, \ldots$ are eigenvalues of Δ_{S^1} and that the functions $\cos k\theta$ and $\sin k\theta$ are eigenfunctions for the eigenvalue $-k^2$, with $k \geq 0$. So, it looks like the dimension of the eigenspace corresponding to the eigenvalue $-k^2$ is 1 when $k = 0$ and 2 when $k \geq 1$.

It can indeed be shown that Δ_{S^1} has no other eigenvalues and that the dimensions claimed for the eigenspaces are correct. Observe that if we go back to our homogeneous harmonic functions, $f(r, \theta) = r^k g(\theta)$, we see that this space is spanned by the functions

$$u_k = r^k \cos k\theta, \quad v_k = r^k \sin k\theta.$$

Now, $(x + iy)^k = r^k(\cos k\theta + i \sin k\theta)$, and since $\Re(x + iy)^k$ and $\Im(x + iy)^k$ are homogeneous polynomials, we see that u_k and v_k are homogeneous polynomials called *harmonic polynomials*. For example, here is a list of a basis for the harmonic polynomials (in two variables) of degree $k = 0, 1, 2, 3, 4$:

$k = 0$	1
$k = 1$	$x,\ y$
$k = 2$	$x^2 - y^2,\ xy$
$k = 3$	$x^3 - 3xy^2,\ 3x^2y - y^3$
$k = 4$	$x^4 - 6x^2y^2 + y^4,\ x^3y - xy^3.$

Therefore, the eigenfunctions of the Laplacian on S^1 are the restrictions of the harmonic polynomials on $\mathbb{R}^2$ to S^1 and we have a Hilbert sum decomposition, $L^2(S^1) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k(S^1)$. It turns out that this phenomenon generalizes to the sphere $S^n \subseteq \mathbb{R}^{n+1}$ for all $n \geq 1$.

Let us take a look at next case, $n = 2$.

1.2 Spherical Harmonics on the 2-Sphere

The material of section is very classical and can be found in many places, for example Andrews, Askey and Roy [1] (Chapter 9), Sansone [25] (Chapter III), Hochstadt [17] (Chapter 6) and Lebedev [21] (Chapter). We recommend the exposition in Lebedev [21] because we find it particularly clear and uncluttered. We have also borrowed heavily from some lecture notes by Hermann Gluck for a course he offered in 1997-1998.

Our goal is to find the homogeneous solutions of the Laplace equation, $\Delta f = 0$, in $\mathbb{R}^3$, and to show that they correspond to spaces, $\mathcal{H}_k(S^2)$, of eigenfunctions of the Laplacian, Δ_{S^2}, on the 2-sphere,

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

Then, the spaces $\mathcal{H}_k(S^2)$ will give us a Hilbert sum decomposition of the Hilbert space, $L^2(S^2)$, of square-integrable functions on S^2. This is the generalization of Fourier series to the 2-sphere and the functions in the spaces $\mathcal{H}_k(S^2)$ are called *spherical harmonics*.

The Laplacian in $\mathbb{R}^3$ is of course given by

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

If we use spherical coordinates

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta, \end{aligned}$$

in $\mathbb{R}^3$, where $0 \leq \theta < \pi$, $0 \leq \varphi < 2\pi$ and $r > 0$ (recall that φ is the so-called *azimuthal angle* in the xy-plane originating at the x-axis and θ is the so-called *polar angle* from the z-axis, angle defined in the plane obtained by rotating the xz-plane around the z-axis by the angle φ), then the Laplacian in spherical coordinates is given by

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\Delta_{S^2} f,$$

where

$$\Delta_{S^2} f = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2},$$

is the Laplacian on the sphere, S^2 (for example, see Lebedev [21], Chapter 8 or Section 1.3, where we derive this formula). Let us look for homogeneous harmonic functions, $f(r, \theta, \varphi) = r^k g(\theta, \varphi)$, on $\mathbb{R}^3$, that is, solutions of the Laplace equation

$$\Delta f = 0.$$

We get

$$\begin{aligned}\Delta f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial(r^k g)}{\partial r}\right) + \frac{1}{r^2}\Delta_{S^2}(r^k g)\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^{k+1}g\right) + r^{k-2}\Delta_{S^2}g\\ &= r^{k-2}k(k+1)g + r^{k-2}\Delta_{S^2}g\\ &= r^{k-2}(k(k+1)g + \Delta_{S^2}g).\end{aligned}$$

Therefore,

$$\Delta f = 0 \quad \text{iff} \quad \Delta_{S^2}g = -k(k+1)g,$$

that is, g is an eigenfunction of Δ_{S^2} for the eigenvalue $-k(k+1)$.

We can look for solutions of the above equation using the separation of variables method. If we let $g(\theta,\varphi) = \Theta(\theta)\Phi(\varphi)$, then we get the equation

$$\frac{\Phi}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{\Theta}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} = -k(k+1)\Theta\Phi,$$

that is, dividing by $\Theta\Phi$ and multiplying by $\sin^2\theta$,

$$\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + k(k+1)\sin^2\theta = -\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2}.$$

Since Θ and Φ are independent functions, the above is possible only if both sides are equal to a constant, say μ. This leads to two equations

$$\begin{aligned}&\frac{\partial^2\Phi}{\partial\varphi^2} + \mu\Phi = 0\\ &\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + k(k+1)\sin^2\theta - \mu = 0.\end{aligned}$$

However, we want Φ to be a periodic in φ since we are considering functions on the sphere, so μ be must of the form $\mu = m^2$, for some non-negative integer, m. Then, we know that the space of solutions of the equation

$$\frac{\partial^2\Phi}{\partial\varphi^2} + m^2\Phi = 0$$

is two-dimensional and is spanned by the two functions

$$\Phi(\varphi) = \cos m\varphi, \qquad \Phi(\varphi) = \sin m\varphi.$$

We still have to solve the equation

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + (k(k+1)\sin^2\theta - m^2)\Theta = 0,$$

which is equivalent to

$$\sin^2\theta\,\Theta'' + \sin\theta\cos\theta\,\Theta' + (k(k+1)\sin^2\theta - m^2)\Theta = 0.$$

a variant of Legendre's equation. For this, we use the change of variable, $\Theta(\theta) = y(\cos\theta)$ and, with $x = \cos\theta$, we get the second-order differential equation

$$(1-x^2)y'' - 2xy' + \left(k(k+1) - \frac{m^2}{1-x^2}\right)y = 0$$

sometimes called the *general Legendre equation*. The trick to solve this equation is to make the substitution

$$y(x) = (1-x^2)^{\frac{m}{2}}u(x),$$

see Lebedev [21], Chapter 7, Section 7.12. Then, we get

$$(1-x^2)u'' - 2(m+1)xu' + (k(k+1) - m(m+1))u = 0.$$

When $m = 0$, we get the *Legendre equation*:

$$(1-x^2)u'' - 2xu' + k(k+1)u = 0,$$

see Lebedev [21], Chapter 7, Section 7.3. This equation has two fundamental solution, $P_k(x)$ and $Q_k(x)$, called the *Legendre functions of the first and second kinds*. The $P_k(x)$ are actually polynomials and the $Q_k(x)$ are given by power series that diverge for $x = 1$, so we only keep the *Legendre polynomials*, $P_k(x)$. The Legendre polynomials can be defined in various ways. One definition is in terms of *Rodrigues' formula*:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n,$$

see Lebedev [21], Chapter 4, Section 4.2. In this version of the Legendre polynomials they are normalized so that $P_n(1) = 1$. There is also the following recurrence relation:

$$\begin{aligned}
P_0 &= 1\\
P_1 &= x\\
(n+1)P_{n+1} &= (2n+1)xP_n - nP_{n-1} \qquad n \geq 1,
\end{aligned}$$

see Lebedev [21], Chapter 4, Section 4.3. For example, the first six Legendre polynomials are:

$$\begin{aligned}
&1\\
&x\\
&\frac{1}{2}(3x^2-1)\\
&\frac{1}{2}(5x^3-3x)\\
&\frac{1}{8}(35x^4-30x^2+3)\\
&\frac{1}{8}(63x^5-70x^3+15x).
\end{aligned}$$

Let us now return to our differential equation

$$(1-x^2)u'' - 2(m+1)xu' + (k(k+1) - m(m+1))u = 0. \qquad (*)$$

Observe that if we differentiate with respect to x, we get the equation

$$(1-x^2)u''' - 2(m+2)xu'' + (k(k+1) - (m+1)(m+2))u' = 0.$$

This shows that if u is a solution of our equation $(*)$ for given k and m, then u' is a solution of the same equation for k and $m+1$. Thus, if $P_k(x)$ solves $(*)$ for given k and $m = 0$, then $P_k'(x)$ solves $(*)$ for the same k and $m = 1$, $P_k''(x)$ solves $(*)$ for the same k and $m = 2$, and in general, $d^m/dx^m(P_k(x))$ solves $(*)$ for k and m. Therefore, our original equation,

$$(1-x^2)y'' - 2xy' + \left(k(k+1) - \frac{m^2}{1-x^2}\right)y = 0 \qquad (\dagger)$$

has the solution

$$y(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m}(P_k(x)).$$

The function $y(x)$ is traditionally denoted $P_k^m(x)$ and called the *associated Legendre function*, see Lebedev [21], Chapter 7, Section 7.12. The index k is often called the *band index*. Obviously, $P_k^m(x) \equiv 0$ if $m > k$ and $P_k^0(x) = P_k(x)$, the Legendre polynomial of degree k. An associated Legendre function is not a polynomial in general and because of the factor $(1-x^2)^{\frac{m}{2}}$ it is only defined on the closed interval $[-1, 1]$.

Certain authors add the factor $(-1)^m$ in front of the expression for the associated Legendre function $P_k^m(x)$, as in Lebedev [21], Chapter 7, Section 7.12, see also footnote 29 on page 193. This seems to be common practice in the quantum mechanics literature where it is called the *Condon Shortley phase factor*.

The associated Legendre functions satisfy various recurrence relations that allows us to compute them. For example, for fixed $m \geq 0$, we have (see Lebedev [21], Chapter 7, Section 7.12) the recurrence

$$(k-m+1)P_{k+1}^m(x) = (2k+1)xP_k^m(x) - (k+m)P_{k-1}^m(x), \qquad k \geq 1$$

and for fixed $k \geq 2$ we have

$$P_k^{m+2}(x) = \frac{2(m+1)x}{(x^2-1)^{\frac{1}{2}}} P_k^{m+1}(x) + (k-m)(k+m+1)P_k^m(x), \qquad 0 \leq m \leq k-2$$

which can also be used to compute P_k^m starting from

$$\begin{aligned} P_k^0(x) &= P_k(x) \\ P_k^1(x) &= \frac{kx}{(x^2-1)^{\frac{1}{2}}} P_k(x) - \frac{k}{(x^2-1)^{\frac{1}{2}}} P_{k-1}(x). \end{aligned}$$

Observe that the recurrence relation for m fixed yield the following equation for $k = m$ (as $P_{m-1}^m = 0$):

$$P_{m+1}^m(x) = (2m+1)xP_m^m(x).$$

It it also easy to see that

$$P_m^m(x) = \frac{(2m)!}{2^m m!}(1-x^2)^{\frac{m}{2}}.$$

Observe that

$$\frac{(2m)!}{2^m m!} = (2m-1)(2m-3)\cdots 5\cdot 3\cdot 1,$$

an expression that is sometimes denoted $(2m-1)!!$ and called the *double factorial*.

Beware that some papers in computer graphics adopt the definition of associated Legendre functions with the scale factor $(-1)^m$ added so this factor is present in these papers, for example, Green [14].

The equation above allows us to "lift" P_m^m to the higher band $m+1$. The computer graphics community (see Green [14]) uses the following three rules to compute $P_k^m(x)$ where $0 \leq m \leq k$:

(1) Compute

$$P_m^m(x) = \frac{(2m)!}{2^m m!}(1-x^2)^{\frac{m}{2}}.$$

If $m = k$, stop. Otherwise do step 2 once:

(2) Compute $P_{m+1}^m(x) = (2m+1)xP_m^m(x)$. If $k = m+1$, stop. Otherwise, iterate step 3:

(3) Starting from $i = m+1$, compute

$$(i-m+1)P_{i+1}^m(x) = (2i+1)xP_i^m(x) - (i+m)P_{i-1}^m(x)$$

until $i + 1 = k$.

If we recall that equation (†) was obtained from the equation

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + (k(k+1)\sin^2\theta - m^2)\Theta = 0$$

using the substitution $y(\cos\theta) = \Theta(\theta)$, we see that

$$\Theta(\theta) = P_k^m(\cos\theta)$$

is a solution of the above equation. Putting everything together, as $f(r,\theta,\varphi) = r^k\Theta(\theta)\Phi(\varphi)$, we proved that the homogeneous functions,

$$f(r,\theta,\varphi) = r^k\cos m\varphi\, P_k^m(\cos\theta), \qquad f(r,\theta,\varphi) = r^k\sin m\varphi\, P_k^m(\cos\theta),$$

are solutions of the Laplacian, Δ, in $\mathbb{R}^3$, and that the functions

$$\cos m\varphi\, P_k^m(\cos\theta), \qquad \sin m\varphi\, P_k^m(\cos\theta),$$

are eigenfunctions of the Laplacian, Δ_{S^2}, on the sphere for the eigenvalue $-k(k+1)$. For k fixed, as $0 \leq m \leq k$, we get $2k+1$ linearly independent functions.

The notation for the above functions varies quite a bit essentially because of the choice of normalization factors used in various fields (such as physics, seismology, geodesy, spectral analysis, magnetics, quantum mechanics *etc.*). We will adopt the notation y_l^m, where l is a nonnegative integer but m is allowed to be negative, with $-l \leq m \leq l$. Thus, we set

$$y_l^m(\theta,\varphi) = \begin{cases} N_l^0 P_l(\cos\theta) & \text{if } m = 0 \\ \sqrt{2} N_l^m \cos m\theta\, P_l^m(\cos\theta) & \text{if } m > 0 \\ \sqrt{2} N_l^m \sin(-m\theta)\, P_l^{-m}(\sin\theta) & \text{if } m < 0 \end{cases}$$

for $l = 0, 1, 2, \ldots$, and where the N_l^m are scaling factors. In physics and computer graphics, N_l^m is chosen to be

$$N_l^m = \sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}}.$$

The functions y_l^m are called the *real spherical harmonics of degree l and order m*. The index l is called the *band index*.

The functions, y_l^m, have some very nice properties but to explain these we need to recall the Hilbert space structure of the space, $L^2(S^2)$, of square-integrable functions on the sphere. Recall that we have an inner product on $L^2(S^2)$ given by

$$\langle f, g\rangle = \int_{S^2} f g\, \Omega_2 = \int_0^{2\pi}\int_0^{\pi} f(\theta,\varphi) g(\theta,\varphi) \sin\theta d\theta d\varphi,$$

where $f, g \in L^2(S^2)$ and where Ω_2 is the volume form on S^2 (induced by the metric on $\mathbb{R}^3$). With this inner product, $L^2(S^2)$ is a complete normed vector space using the norm, $\|f\| = \sqrt{\langle f, f\rangle}$, associated with this inner product, that is, $L^2(S^2)$ is a *Hilbert space*. Now, it can be shown that the Laplacian, Δ_{S^2}, on the sphere is a self-adjoint linear operator with respect to this inner product. As the functions, $y_{l_1}^{m_1}$ and $y_{l_2}^{m_2}$ with $l_1 \neq l_2$ are eigenfunctions corresponding to distinct eigenvalues ($-l_1(l_1+1)$ and $-l_2(l_2+1)$), they are orthogonal, that is,

$$\langle y_{l_1}^{m_1}, y_{l_2}^{m_2}\rangle = 0, \qquad \text{if} \quad l_1 \neq l_2.$$

It is also not hard to show that for a fixed l,

$$\langle y_l^{m_1}, y_l^{m_2}\rangle = \delta_{m_1,m_2},$$

that is, the functions y_l^m with $-l \leq m \leq l$ form an orthonormal system. It turns out that they form a basis of the eigenspace, $E_l = \mathcal{H}_l(S^2)$, of Δ_{S^2} associated with the eigenvalue $-l(l+1)$ and that Δ_{S^2} has no other eigenvalues. More is true. It turns out that $L^2(S^2)$ is the orthogonal Hilbert sum of the eigenspaces, $\mathcal{H}_l(S^2)$. This means that the $\mathcal{H}_l(S^2)$ are

(1) mutually orthogonal

(2) closed, and

(3) The space $L^2(S^2)$ is the Hilbert sum, $\bigoplus_{l=0}^{\infty} \mathcal{H}_l(S^2)$, which means that for every function, $f \in L^2(S^2)$, there is a unique sequence of spherical harmonics, $f_j \in \mathcal{H}_l(S^2)$, so that

$$f = \sum_{l=0}^{\infty} f_l,$$

that is, the sequence $\sum_{j=0}^{l} f_j$, converges to f (in the norm on $L^2(S^2)$). Observe that each f_l is a unique linear combination, $f_l = \sum_{m_l} a_{m_l\, l}\, y_l^{m_l}$.

Therefore, (3) give us a *Fourier decomposition on the sphere* generalizing the familiar Fourier decomposition on the circle. Furthermore, the *Fourier coefficients*, $a_{m_l l}$, can be computed using the fact that the y_l^m form an orthonormal Hilbert basis:

$$a_{m_l\, l} = \langle f, y_l^{m_l} \rangle.$$

We also have the corresponding homogeneous harmonic functions, $H_l^m(r, \theta, \varphi)$, on $\mathbb{R}^3$ given by

$$H_l^m(r, \theta, \varphi) = r^l y_l^m(\theta, \varphi).$$

If one starts computing explicity the H_l^m for small values of l and m, one finds that it is always possible to express these functions in terms of the cartesian coordinates x, y, z as *homogeneous polynomials*! This remarkable fact holds in general: The eigenfunctions of the Laplacian, Δ_{S^2}, and thus, the spherical harmonics, are the restrictions of homogeneous harmonic polynomials in $\mathbb{R}^3$. Here is a list of bases of the homogeneous harmonic polynomials of degree k in three variables up $k = 3$ (thanks to Herman Gluck):

$k = 0$	1
$k = 1$	$x,\ y,\ z$
$k = 2$	$x^2 - y^2,\ x^2 - z^2,\ xy,\ xz,\ yz$
$k = 3$	$x^3 - 3xy^2,\ 3x^2y - y^3,\ x^3 - 3xz^2,\ 3x^2z - z^3,$
	$y^3 - 3yx^2,\ 3y^2z - z^3,\ xyz.$

Subsequent sections will be devoted to a proof of the important facts stated earlier.

1.3 The Laplace-Beltrami Operator

In order to define rigorously the Laplacian on the sphere, $S^n \subseteq \mathbb{R}^{n+1}$, and establish its relationship with the Laplacian on $\mathbb{R}^{n+1}$, we review the definition of the Laplacian on a

Riemannian manifold, (M, g). Recall that a Riemannian metric, g, on a manifold, M, is a smooth family of inner products, $g = (g_p)$, where g_p is an inner product on the tangent space, T_pM, for every $p \in M$. The inner product, g_p, on T_pM, establishes a canonical duality between T_pM and T_p^*M, namely, we have the isomorphism, $\flat\colon T_pM \to T_p^*M$, defined such that for every $u \in T_pM$, the linear form, $u^\flat \in T_m^*M$, is given by

$$u^\flat(v) = g_p(u, v), \qquad v \in T_pM.$$

The inverse isomorphism, $\sharp\colon T_p^*M \to T_pM$, is defined such that for every $\omega \in T_p^*M$, the vector, $\omega^\sharp$, is the unique vector in T_pM so that

$$g_p(\omega^\sharp, v) = \omega(v), \qquad v \in T_pM.$$

The isomorphisms $\flat$ and $\sharp$ induce isomorphisms between vector fields, $X \in \mathfrak{X}(M)$, and one-forms, $\omega \in \mathcal{A}^1(M)$. In particular, for every smooth function, $f \in C^\infty(M)$, the vector field corresponding to the one-form, df, is the *gradient*, $\mathrm{grad}\, f$, of f. The gradient of f is uniquely determined by the condition

$$g_p((\mathrm{grad}\, f)_p, v) = df_p(v), \qquad v \in T_pM,\ p \in M.$$

If ∇_X is the covariant derivative associated with the Levi-Civita connection induced by the metric, g, then the *divergence* of a vector field, $X \in \mathfrak{X}(M)$, is the function, $\mathrm{div}\, X\colon M \to \mathbb{R}$, defined so that

$$(\mathrm{div}\, X)(p) = \mathrm{tr}(Y(p) \mapsto (\nabla_Y X)_p),$$

namely, for every p, $(\mathrm{div}\, X)(p)$ is the trace of the linear map, $Y(p) \mapsto (\nabla_Y X)_p$. Then, the *Laplace-Beltrami operator*, for short, *Laplacian*, is the linear operator, $\Delta\colon C^\infty(M) \to C^\infty(M)$, given by

$$\Delta f = \mathrm{div}\, \mathrm{grad}\, f.$$

For more details on the Laplace-Beltrami operator, we refer the reader to Gallot, Hulin and Lafontaine [13] (Chapter 4) or O'Neill [23] (Chapter 3), Postnikov [24] (Chapter 13), Helgason [16] (Chapter 2) or Warner [29] (Chapters 4 and 6).

All this being rather abstact, it is useful to know how $\mathrm{grad}\, f$, $\mathrm{div}\, X$ and Δf are expressed in a chart. If (U, φ) is a chart of M, where M is an orientable Riemannian manifold, with $p \in M$ and if, as usual,

$$\left(\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p \right)$$

denotes the basis of T_pM induced by φ, the local expression of the metric g at p is given by the $n \times n$ matrix, $(g_{ij})_p$, with

$$(g_{ij})_p = g_p\left(\left(\frac{\partial}{\partial x_i}\right)_p, \left(\frac{\partial}{\partial x_j}\right)_p \right).$$

The matrix $(g_{ij})_p$ is symmetric, positive definite and its inverse is denoted $(g^{ij})_p$. We also let $|g|_p = \det(g_{ij})_p$. For simplicity of notation we often omit the subscript p. Then, it can be shown that for every function, $f \in C^\infty(M)$, in local coordinates given by the chart (U, φ), we have

$$\operatorname{grad} f = \sum_{ij} g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i},$$

where, as usual

$$\frac{\partial f}{\partial x_j}(p) = \frac{\partial (f \circ \varphi^{-1})}{\partial u_j}(\varphi(p))$$

and $(u_1, \ldots, u_n)$ are the coordinate functions in $\mathbb{R}^n$. There are formulae for $\operatorname{div} X$ and Δf involving the Christoffel symbols but the following formulae will be more convenient for our purposes: For every vector field, $X \in \mathfrak{X}(M)$, expressed in local coordinates as

$$X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$$

we have

$$\operatorname{div} X = \frac{1}{\sqrt{|g|}} \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\sqrt{|g|}\, X_i \right)$$

and for every function, $f \in C^\infty(M)$, the Laplacian, Δf, is given by

$$\Delta f = \frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(\sqrt{|g|}\, g^{ij} \frac{\partial f}{\partial x_j} \right).$$

A derivation of the above formula can be found in Postnikov [24] (Chapter 13, Section 5).

One should check that for $M = \mathbb{R}^n$ with its standard coordinates, the Laplacian is given by the familiar formula

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}.$$

Remark: A different sign convention is also used in defining the divergence, namely,

$$\operatorname{div} X = -\frac{1}{\sqrt{|g|}} \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\sqrt{|g|}\, X_i \right).$$

With this convention, the Laplacian also has a negative sign. This has the advantage that the eigenvalues of the Laplacian are nonnegative.

As an application, let us derive the formula for the Laplacian in spherical coordinates,

$$\begin{aligned} x &= r \sin\theta \cos\varphi \\ y &= r \sin\theta \sin\varphi \\ z &= r \cos\theta. \end{aligned}$$

We have

$$\begin{aligned}
\frac{\partial}{\partial r} &= \sin\theta\cos\varphi\,\frac{\partial}{\partial x} + \sin\theta\sin\varphi\,\frac{\partial}{\partial y} + \cos\theta\,\frac{\partial}{\partial z} = \widehat{r} \\
\frac{\partial}{\partial \theta} &= r\left(\cos\theta\cos\varphi\,\frac{\partial}{\partial x} + \cos\theta\sin\varphi\,\frac{\partial}{\partial y} - \sin\theta\,\frac{\partial}{\partial z}\right) = r\widehat{\theta} \\
\frac{\partial}{\partial \varphi} &= r\left(-\sin\theta\sin\varphi\,\frac{\partial}{\partial x} + \sin\theta\cos\varphi\,\frac{\partial}{\partial y}\right) = r\widehat{\varphi}.
\end{aligned}$$

Observe that $\widehat{r}$, $\widehat{\theta}$ and $\widehat{\varphi}$ are pairwise orthogonal. Therefore, the matrix (g_{ij}) is given by

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

and $|g| = r^4\sin^2\theta$. The inverse of (g_{ij}) is

$$(g^{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix}.$$

We will let the reader finish the computation to verify that we get

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}.$$

Since (θ, φ) are coordinates on the sphere S^2 *via*

$$\begin{aligned}
x &= \sin\theta\cos\varphi \\
y &= \sin\theta\sin\varphi \\
z &= \cos\theta,
\end{aligned}$$

we see that in these coordinates, the metric, $(\widetilde{g}_{ij})$, on S^2 is given by the matrix

$$(\widetilde{g}_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix},$$

that $|\widetilde{g}| = \sin^2\theta$, and that the inverse of $(\widetilde{g}_{ij})$ is

$$(\widetilde{g}^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^{-2}\theta \end{pmatrix}.$$

It follows immediately that

$$\Delta_{S^2} f = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2},$$

so we have verified that

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\Delta_{S^2} f.$$

Let us now generalize the above formula to the Laplacian, Δ, on $\mathbb{R}^{n+1}$ and the Laplacian, Δ_{S^n}, on S^n, where

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$

Following Morimoto [22] (Chapter 2, Section 2), let us use "polar coordinates". The map from $\mathbb{R}_+ \times S^n$ to $\mathbb{R}^{n+1} - \{0\}$ given by

$$(r, \sigma) \mapsto r\sigma$$

is clearly a diffeomorphism. Thus, for any system of coordinates, $(u_1, \ldots, u_n)$, on S^n, the tuple $(u_1, \ldots, u_n, r)$ is a system of coordinates on $\mathbb{R}^{n+1} - \{0\}$ called *polar coordinates*. Let us establish the relationship between the Laplacian, Δ, on $\mathbb{R}^{n+1} - \{0\}$ in polar coordinates and the Laplacian, Δ_{S^n}, on S^n in local coordinates $(u_1, \ldots, u_n)$.

Proposition 1.1 *If Δ is the Laplacian on $\mathbb{R}^{n+1} - \{0\}$ in polar coordinates $(u_1, \ldots, u_n, r)$ and Δ_{S^n} is the Laplacian on the sphere, S^n, in local coordinates $(u_1, \ldots, u_n)$, then*

$$\Delta f = \frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\Delta_{S^n} f.$$

Proof. Let us compute the $(n+1) \times (n+1)$ matrix, $G = (g_{ij})$, expressing the metric on $\mathbb{R}^{n+1}$ is polar coordinates and the $n \times n$ matrix, $\widetilde{G} = (\widetilde{g}_{ij})$, expressing the metric on S^n. Recall that that if $\sigma \in S^n$, then $\sigma \cdot \sigma = 1$ and so,

$$\frac{\partial\sigma}{\partial u_i} \cdot \sigma = 0,$$

as

$$\frac{\partial\sigma}{\partial u_i} \cdot \sigma = \frac{1}{2}\frac{\partial(\sigma \cdot \sigma)}{\partial u_i} = 0.$$

If $x = r\sigma$ with $\sigma \in S^n$, we have

$$\frac{\partial x}{\partial u_i} = r\frac{\partial\sigma}{\partial u_i}, \qquad 1 \leq i \leq n,$$

and

$$\frac{\partial x}{\partial r} = \sigma.$$

It follows that

$$\begin{aligned} g_{ij} &= \frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{\partial u_j} = r^2 \frac{\partial \sigma}{\partial u_i} \cdot \frac{\partial \sigma}{\partial u_j} = r^2 \widetilde{g}_{ij} \\ g_{in+1} &= \frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{\partial r} = r \frac{\partial \sigma}{\partial u_i} \cdot \sigma = 0 \\ g_{n+1n+1} &= \frac{\partial x}{\partial r} \cdot \frac{\partial x}{\partial r} = \sigma \cdot \sigma = 1. \end{aligned}$$

Consequently, we get

$$G = \begin{pmatrix} r^2 \widetilde{G} & 0 \\ 0 & 1 \end{pmatrix},$$

$|g| = r^{2n} |\widetilde{g}|$ and

$$G^{-1} = \begin{pmatrix} r^{-2} \widetilde{G}^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Using the above equations and

$$\Delta f = \frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(\sqrt{|g|}\, g^{ij} \, \frac{\partial f}{\partial x_j} \right),$$

we get

$$\begin{aligned} \Delta f &= \frac{1}{r^n \sqrt{|\widetilde{g}|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(r^n \sqrt{|\widetilde{g}|} \, \frac{1}{r^2} \widetilde{g}^{ij} \, \frac{\partial f}{\partial x_j} \right) + \frac{1}{r^n \sqrt{|\widetilde{g}|}} \frac{\partial}{\partial r} \left(r^n \sqrt{|\widetilde{g}|} \, \frac{\partial f}{\partial r} \right) \\ &= \frac{1}{r^2 \sqrt{|\widetilde{g}|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{|\widetilde{g}|}\, \widetilde{g}^{ij} \, \frac{\partial f}{\partial x_j} \right) + \frac{1}{r^n} \frac{\partial}{\partial r} \left(r^n \, \frac{\partial f}{\partial r} \right) \\ &= \frac{1}{r^2} \Delta_{S^n} f + \frac{1}{r^n} \frac{\partial}{\partial r} \left(r^n \, \frac{\partial f}{\partial r} \right), \end{aligned}$$

as claimed. □

It is also possible to express Δ_{S^n} in terms of $\Delta_{S^{n-1}}$. If $e_{n+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$, then we can view S^{n-1} as the intersection of S^n with the hyperplane, $x_{n+1} = 0$, that is, as the set

$$S^{n-1} = \{\sigma \in S^n \mid \sigma \cdot e_{n+1} = 0\}.$$

If $(u_1, \ldots, u_{n-1})$ are local coordinates on S^{n-1}, then $(u_1, \ldots, u_{n-1}, \theta)$ are local coordinates on S^n, by setting

$$\sigma = \sin\theta \, \widetilde{\sigma} + \cos\theta \, e_{n+1},$$

with $\widetilde{\sigma} \in S^{n-1}$ and $0 \leq \theta < \pi$. Using these local coordinate systems, it is a good exercise to find the relationship between Δ_{S^n} and $\Delta_{S^{n-1}}$, namely

$$\Delta_{S^n} f = \frac{1}{\sin^{n-1}\theta} \frac{\partial}{\partial\theta} \left(\sin^{n-1}\theta \, \frac{\partial f}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \Delta_{S_{n-1}} f.$$

A fundamental property of the divergence is known as *Green's Formula*. There are actually two Greens' Formulae but we will only need the version for an orientable manifold without boundary.

Theorem 1.2 *(Green's Formula) Let M be a compact, orientable, Riemannian manifold without boundary. Then, for every smooth vector field, $X \in \mathfrak{X}(M)$, we have*

$$\int_M (\operatorname{div} X)\, \Omega_M = 0,$$

where Ω_M is the volume form on M induced by the metric.

A proof of Theorem 1.2 can be found in Gallot, Hulin and Lafontaine [13] (Chapter 4, Proposition 4.9), see also Helgason [16] (Chapter 2, Section 2.4).

If M is a compact, orientable Riemannian manifold, then for any two smooth functions, $f, h \in C^\infty(M)$, we define $\langle f, h \rangle$ by

$$\langle f, h \rangle = \int_M fh\, \Omega_M.$$

Then, it is not hard to show that $\langle -, - \rangle$ is an inner product on $\mathcal{C}^\infty(M)$. One can also prove the following properties: For any two functions, $f, h \in C^\infty(M)$, and any vector field, $X \in \mathcal{C}^\infty(M)$, we have:

$$\begin{aligned} \operatorname{div}(fX) &= f \operatorname{div} X + X(f) = f \operatorname{div} X + g(\operatorname{grad} f, X) \\ \operatorname{grad} f\, (h) &= g(\operatorname{grad} f, \operatorname{grad} h) = \operatorname{grad} h\, (f). \end{aligned}$$

Using these identities, we obtain the following important result:

Proposition 1.3 *Let M be a compact, orientable, Riemannian manifold without boundary. The Laplacian on M is self-adjoint, that is, for any two functions, $f, h \in C^\infty(M)$, we have*

$$\langle \Delta f, h \rangle = \langle f, \Delta h \rangle$$

or equivalently

$$\int_M f \Delta h\, \Omega_M = \int_M h \Delta f\, \Omega_M.$$

Proof. By the two identities before Proposition 1.3,

$$f\Delta h = f \operatorname{div} \operatorname{grad} h = \operatorname{div}(f \operatorname{grad} h) - g(\operatorname{grad} f, \operatorname{grad} h)$$

and

$$h\Delta f = h \operatorname{div} \operatorname{grad} f = \operatorname{div}(h \operatorname{grad} f) - g(\operatorname{grad} h, \operatorname{grad} f),$$

so we get

$$f\Delta h - h\Delta f = \mathrm{div}(f\mathrm{grad}\, h - h\mathrm{grad}\, f).$$

By Green's Formula,

$$\int_M (f\Delta h - h\Delta f)\Omega_M = \int_M \mathrm{div}(f\mathrm{grad}\, h - h\mathrm{grad}\, f)\Omega_M = 0,$$

which proves that Δ is self-adjoint. □

The importance of Proposition 1.3 lies in the fact that as $\langle -, - \rangle$ is an inner product on $\mathcal{C}^\infty(M)$, the eigenspaces of Δ for distinct eigenvalues are pairwise orthogonal. We will make heavy use of this property in the next section on harmonic polynomials.

1.4 Harmonic Polynomials, Spherical Harmonics and $L^2(S^n)$

Harmonic homogeneous polynomials and their restrictions to S^n, where

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\},$$

turn out to play a crucial role in understanding the structure of the eigenspaces of the Laplacian on S^n (with $n \geq 1$). The results in this section appear in one form or another in Stein and Weiss [26] (Chapter 4), Morimoto [22] (Chapter 2), Helgason [16] (Introduction, Section 3), Dieudonné [6] (Chapter 7), Axler, Bourdon and Ramey [2] (Chapter 5) and Vilenkin [28] (Chapter IX). Some of these sources assume a fair amount of mathematical background and consequently, uninitiated readers will probably find the exposition rather condensed, especially Helgason. We tried hard to make our presentation more "user-friendly".

Definition 1.1 Let $\mathcal{P}_k(n+1)$ (resp. $\mathcal{P}_k^{\mathbb{C}}(n+1)$) denote the space of homogeneous polynomials of degree k in $n+1$ variables with real coefficients (resp. complex coefficients) and let $\mathcal{P}_k(S^n)$ (resp. $\mathcal{P}_k^{\mathbb{C}}(S^n)$) denote the restrictions of homogeneous polynomials in $\mathcal{P}_k(n+1)$ to S^n (resp. the restrictions of homogeneous polynomials in $\mathcal{P}_k^{\mathbb{C}}(n+1)$ to S^n). Let $\mathcal{H}_k(n+1)$ (resp. $\mathcal{H}_k^{\mathbb{C}}(n+1)$) denote the space of *(real) harmonic polynomials* (resp. *complex harmonic polynomials*), with

$$\mathcal{H}_k(n+1) = \{P \in \mathcal{P}_k(n+1) \mid \Delta P = 0\}$$

and

$$\mathcal{H}_k^{\mathbb{C}}(n+1) = \{P \in \mathcal{P}_k^{\mathbb{C}}(n+1) \mid \Delta P = 0\}.$$

Harmonic polynomials are sometimes called *solid harmonics*. Finally, Let $\mathcal{H}_k(S^n)$ (resp. $\mathcal{H}_k^{\mathbb{C}}(S^n)$) denote the space of *(real) spherical harmonics* (resp. *complex spherical harmonics*) be the set of restrictions of harmonic polynomials in $\mathcal{H}_k(n+1)$ to S^n (resp. restrictions of harmonic polynomials in $\mathcal{H}_k^{\mathbb{C}}(n+1)$ to S^n).

A function, $f\colon \mathbb{R}^n \to \mathbb{R}$ (resp. $f\colon \mathbb{R}^n \to \mathbb{C}$), is *homogeneous of degree* k iff

$$f(tx) = t^k f(x), \qquad \text{for all } x \in \mathbb{R}^n \text{ and } t > 0.$$

The restriction map, $\rho\colon \mathcal{H}_k(n+1) \to \mathcal{H}_k(S^n)$, is a surjective linear map. In fact, it is a bijection. Indeed, if $P \in \mathcal{H}_k(n+1)$, observe that

$$P(x) = \|x\|^k\, P\left(\frac{x}{\|x\|}\right), \qquad \text{with} \quad \frac{x}{\|x\|} \in S^n,$$

for all $x \neq 0$. Consequently, if $P \restriction S^n = Q \restriction S^n$, that is, $P(\sigma) = Q(\sigma)$ for all $\sigma \in S^n$, then

$$P(x) = \|x\|^k\, P\left(\frac{x}{\|x\|}\right) = \|x\|^k\, Q\left(\frac{x}{\|x\|}\right) = Q(x)$$

for all $x \neq 0$, which implies $P = Q$ (as P and Q are polynomials). Therefore, we have a linear isomorphism between $\mathcal{H}_k(n+1)$ and $\mathcal{H}_k(S^n)$ (and between $\mathcal{H}_k^{\mathbb{C}}(n+1)$ and $\mathcal{H}_k^{\mathbb{C}}(S^n)$).

It will be convenient to introduce some notation to deal with homogeneous polynomials. Given $n \geq 1$ variables, $x_1, \ldots, x_n$, and any n-tuple of nonnegative integers, $\alpha = (\alpha_1, \ldots, \alpha_n)$, let $|\alpha| = \alpha_1 + \cdots + \alpha_n$, let $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and let $\alpha! = \alpha_1! \cdots \alpha_n!$. Then, every homogeneous polynomial, P, of degree k in the variables $x_1, \ldots, x_n$ can be written uniquely as

$$P = \sum_{|\alpha|=k} c_\alpha x^\alpha,$$

with $c_\alpha \in \mathbb{R}$ or $c_\alpha \in \mathbb{C}$. It is well known that $\mathcal{P}_k(n)$ is a (real) vector space of dimension

$$d_k = \binom{n+k-1}{k}$$

and $\mathcal{P}_k^{\mathbb{C}}(n)$ is a complex vector space of the same dimension, d_k.

We can define an Hermitian inner product on $\mathcal{P}_k^{\mathbb{C}}(n)$ whose restriction to $\mathcal{P}_k(n)$ is an inner product by viewing a homogeneous polynomial as a differential operator as follows: For every $P = \sum_{|\alpha|=k} c_\alpha x^\alpha \in \mathcal{P}_k^{\mathbb{C}}(n)$, let

$$\partial(P) = \sum_{|\alpha|=k} c_\alpha \frac{\partial^k}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Then, for any two polynomials, $P, Q \in \mathcal{P}_k^{\mathbb{C}}(n)$, let

$$\langle P, Q \rangle = \partial(P)\overline{Q}.$$

A simple computation shows that

$$\left\langle \sum_{|\alpha|=k} a_\alpha x^\alpha, \sum_{|\alpha|=k} b_\alpha x^\alpha \right\rangle = \sum_{|\alpha|=k} \alpha!\, a_\alpha \overline{b}_\alpha.$$

Therefore, $\langle P, Q\rangle$ is indeed an inner product. Also observe that

$$\partial(x_1^2 + \cdots + x_n^2) = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} = \Delta.$$

Another useful property of our inner product is this:

$$\langle P, QR\rangle = \langle \partial(Q)P, R\rangle.$$

Indeed.

$$\begin{aligned}
\langle P, QR\rangle &= \langle QR, P\rangle \\
&= \partial(QR)\overline{P} \\
&= \partial(Q)(\partial(R)\overline{P}) \\
&= \partial(R)(\partial(Q)\overline{P}) \\
&= \langle R, \partial(Q)P\rangle \\
&= \langle \partial(Q)P, R\rangle.
\end{aligned}$$

In particular,

$$\langle (x_1^2 + \cdots + x_n^2)P, Q\rangle = \langle P, \partial(x_1^2 + \cdots + x_n^2)Q\rangle = \langle P, \Delta Q\rangle.$$

Let us write $\|x\|^2$ for $x_1^2 + \cdots + x_n^2$. Using our inner product, we can prove the following important theorem:

Theorem 1.4 *The map, $\Delta\colon \mathcal{P}_k(n) \to \mathcal{P}_{k-2}(n)$, is surjective for all $n, k \geq 2$ (and similarly for $\Delta\colon \mathcal{P}_k^{\mathbb{C}}(n) \to \mathcal{P}_{k-2}^{\mathbb{C}}(n)$). Furthermore, we have the following orthogonal direct sum decompositions:*

$$\mathcal{P}_k(n) = \mathcal{H}_k(n) \oplus \|x\|^2 \mathcal{H}_{k-2}(n) \oplus \cdots \oplus \|x\|^{2j} \mathcal{H}_{k-2j}(n) \oplus \cdots \oplus \|x\|^{2[k/2]} \mathcal{H}_{[k/2]}(n)$$

and

$$\mathcal{P}_k^{\mathbb{C}}(n) = \mathcal{H}_k^{\mathbb{C}}(n) \oplus \|x\|^2 \mathcal{H}_{k-2}^{\mathbb{C}}(n) \oplus \cdots \oplus \|x\|^{2j} \mathcal{H}_{k-2j}^{\mathbb{C}}(n) \oplus \cdots \oplus \|x\|^{2[k/2]} \mathcal{H}_{[k/2]}^{\mathbb{C}}(n),$$

with the understanding that only the first term occurs on the right-hand side when $k < 2$.

Proof. If the map $\Delta\colon \mathcal{P}_k^{\mathbb{C}}(n) \to \mathcal{P}_{k-2}^{\mathbb{C}}(n)$ is not surjective, then some nonzero polynomial, $Q \in \mathcal{P}_{k-2}^{\mathbb{C}}(n)$, is orthogonal to the image of Δ. In particular, Q must be orthogonal to ΔP with $P = \|x\|^2 Q \in \mathcal{P}_k^{\mathbb{C}}(n)$. So, using a fact established earlier,

$$0 = \langle Q, \Delta P\rangle = \langle \|x\|^2 Q, P\rangle = \langle P, P\rangle,$$

which implies that $P = \|x\|^2 Q = 0$ and thus, $Q = 0$, a contradiction. The same proof is valid in the real case.

We claim that we have an orthogonal direct sum decomposition,

$$\mathcal{P}_k^{\mathbb{C}}(n) = \mathcal{H}_k^{\mathbb{C}}(n) \oplus \|x\|^2 \mathcal{P}_{k-2}^{\mathbb{C}}(n),$$

and similarly in the real case, with the understanding that the second term is missing if $k < 2$. If $k = 0, 1$, then $\mathcal{P}_k^{\mathbb{C}}(n) = \mathcal{H}_k^{\mathbb{C}}(n)$ so this case is trivial. Assume $k \geq 2$. Since $\mathrm{Ker}\,\Delta = \mathcal{H}_k^{\mathbb{C}}(n)$ and Δ is surjective, $\dim(\mathcal{P}_k^{\mathbb{C}}(n)) = \dim(\mathcal{H}_k^{\mathbb{C}}(n)) + \dim(\mathcal{P}_{k-2}^{\mathbb{C}}(n))$, so it is sufficient to prove that $\mathcal{H}_k^{\mathbb{C}}(n)$ is orthogonal to $\|x\|^2 \mathcal{P}_{k-2}^{\mathbb{C}}(n)$. Now, if $H \in \mathcal{H}_k^{\mathbb{C}}(n)$ and $P = \|x\|^2 Q \in \|x\|^2 \mathcal{P}_{k-2}^{\mathbb{C}}(n)$, we have

$$\langle \|x\|^2 Q, H\rangle = \langle Q, \Delta H\rangle = 0,$$

so $\mathcal{H}_k^{\mathbb{C}}(n)$ and $\|x\|^2 \mathcal{P}_{k-2}^{\mathbb{C}}(n)$ are indeed orthogonal. Using induction, we immediately get the orthogonal direct sum decomposition

$$\mathcal{P}_k^{\mathbb{C}}(n) = \mathcal{H}_k^{\mathbb{C}}(n) \oplus \|x\|^2 \mathcal{H}_{k-2}^{\mathbb{C}}(n) \oplus \cdots \oplus \|x\|^{2j} \mathcal{H}_{k-2j}^{\mathbb{C}}(n) \oplus \cdots \oplus \|x\|^{2[k/2]} \mathcal{H}_{[k/2]}^{\mathbb{C}}(n)$$

and the corresponding real version. $\square$

Remark: Theorem 1.5 also holds for $n = 1$.

Theorem 1.5 has some important corollaries. Since every polynomial in $n+1$ variables is the sum of homogeneous polynomials, we get:

Corollary 1.5 *The restriction to S^n of every polynomial (resp. complex polynomial) in $n+1 \geq 2$ variables is a sum of restrictions to S^n of harmonic polynomials (resp. complex harmonic polynomials).*

We can also derive a formula for the dimension of $\mathcal{H}_k(n)$ (and $\mathcal{H}_k^{\mathbb{C}}(n)$).

Corollary 1.6 *The dimension, $a_{k,n}$, of the space of harmonic polynomials, $\mathcal{H}_k(n)$, is given by the formula*

$$a_{k,n} = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}$$

if $n, k \geq 2$, with $a_{0,n} = 1$ and $a_{1,n} = n$, and similarly for $\mathcal{H}_k^{\mathbb{C}}(n)$. As $\mathcal{H}_k(n+1)$ is isomorphic to $\mathcal{H}_k(S^n)$ (and $\mathcal{H}_k^{\mathbb{C}}(n+1)$ is isomorphic to $\mathcal{H}_k^{\mathbb{C}}(S^n)$) we have

$$\dim(\mathcal{H}_k^{\mathbb{C}}(S^n)) = \dim(\mathcal{H}_k(S^n)) = a_{k,n+1} = \binom{n+k}{k} - \binom{n+k-2}{k-2}.$$

Proof. The cases $k = 0$ and $k = 1$ are trivial since in this case $\mathcal{H}_k(n) = \mathcal{P}_k(n)$. For $k \geq 2$, the result follows from the direct sum decomposition

$$\mathcal{P}_k(n) = \mathcal{H}_k(n) \oplus \|x\|^2 \mathcal{P}_{k-2}(n)$$

proved earlier. The proof is identical in the complex case. □

Observe that when $n = 2$, we get $a_{k,2} = 2$ for $k \geq 1$ and when $n = 3$, we get $a_{k,3} = 2k+1$ for all $k \geq 0$, which we already knew from Section 1.2. The formula even applies for $n = 1$ and yields $a_{k,1} = 0$ for $k \geq 2$.

Remark: It is easy to show that

$$a_{k,n+1} = \binom{n+k-1}{n-1} + \binom{n+k-2}{n-1}$$

for $k \geq 2$, see Morimoto [22] (Chapter 2, Theorem 2.4) or Dieudonné [6] (Chapter 7, formula 99), where a different proof technique is used.

Let $L^2(S^n)$ be the space of (real) square-integrable functions on the sphere, S^n. We have an inner product on $L^2(S^n)$ given by

$$\langle f, g\rangle = \int_{S^n} fg\,\Omega_n,$$

where $f, g \in L^2(S^n)$ and where Ω_n is the volume form on S^n (induced by the metric on $\mathbb{R}^{n+1}$). With this inner product, $L^2(S^n)$ is a complete normed vector space using the norm, $\|f\| = \|f\|_2 = \sqrt{\langle f, f\rangle}$, associated with this inner product, that is, $L^2(S^n)$ is a *Hilbert space*. In the case of complex-valued functions, we use the Hermitian inner product

$$\langle f, g\rangle = \int_{S^n} f\,\overline{g}\,\Omega_n$$

and we get the complex Hilbert space, $L^2_{\mathbb{C}}(S^n)$. We also denote by $C(S^n)$ the space of continuous (real) functions on S^n with the L^∞ norm, that is,

$$\|f\|_\infty = \sup\{|f(x)|\}_{x\in S^n}$$

and by $C_{\mathbb{C}}(S^n)$ the space of continuous complex-valued functions on S^n also with the L^∞ norm. Recall that $C(S^n)$ is dense in $L^2(S^n)$ (and $C_{\mathbb{C}}(S^n)$ is dense in $L^2_{\mathbb{C}}(S^n)$). The following proposition shows why the spherical harmonics play an important role:

Proposition 1.7 *The set of all finite linear combinations of elements in $\bigcup_{k=0}^\infty \mathcal{H}_k(S^n)$ (resp. $\bigcup_{k=0}^\infty \mathcal{H}_k^{\mathbb{C}}(S^n)$) is*

(i) dense in $C(S^n)$ (resp. in $C_{\mathbb{C}}(S^n)$) with respect to the L^∞-norm;

(ii) dense in $L^2(S^n)$ (resp. dense in $L^2_{\mathbb{C}}(S^n)$).

Proof. (i) As S^n is compact, by the Stone-Weierstrass approximation theorem (Lang [20], Chapter III, Corollary 1.3), if g is continuous on S^n, then it can be approximated uniformly by polynomials, P_j, restricted to S^n. By Corollary 1.5, the restriction of each P_j to S^n is a linear combination of elements in $\bigcup_{k=0}^{\infty} \mathcal{H}_k(S^n)$.

(ii) We use the fact that $C(S^n)$ is dense in $L^2(S^n)$. Given $f \in L^2(S^n)$, for every $\epsilon > 0$, we can choose a continuous function, g, so that $\|f - g\|_2 < \epsilon/2$. By (i), we can find a linear combination, h, of elements in $\bigcup_{k=0}^{\infty} \mathcal{H}_k(S^n)$ so that $\|g - h\|_\infty < \epsilon/(2\sqrt{\mathrm{vol}(S^n)})$, where $\mathrm{vol}(S^n)$ is the volume of S^n (really, area). Thus, we get

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2 < \epsilon/2 + \sqrt{\mathrm{vol}(S^n)}\, \|g - h\|_\infty < \epsilon/2 + \epsilon/2 = \epsilon,$$

which proves (ii). The proof in the complex case is identical. $\square$

We need one more proposition before showing that the spaces $\mathcal{H}_k(S^n)$ constitute an orthogonal Hilbert space decomposition of $L^2(S^n)$.

Proposition 1.8 *For every harmonic polynomial, $P \in \mathcal{H}_k(n+1)$ (resp. $P \in \mathcal{H}_k^{\mathbb{C}}(n+1)$), the restriction, $H \in \mathcal{H}_k(S^n)$ (resp. $H \in \mathcal{H}_k^{\mathbb{C}}(S^n)$), of P to S^n is an eigenfunction of Δ_{S^n} for the eigenvalue $-k(n+k-1)$.*

Proof. We have

$$P(r\sigma) = r^k H(\sigma), \qquad r > 0,\ \sigma \in S^n,$$

and by Proposition 1.1, for any $f \in C^\infty(\mathbb{R}^{n+1})$, we have

$$\Delta f = \frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n \frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\Delta_{S^n} f.$$

Consequently,

$$\begin{aligned}
\Delta P = \Delta(r^k H) &= \frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n \frac{\partial (r^k H)}{\partial r}\right) + \frac{1}{r^2}\Delta_{S^n}(r^k H) \\
&= \frac{1}{r^n}\frac{\partial}{\partial r}\left(k r^{n+k-1} H\right) + r^{k-2}\Delta_{S^n} H \\
&= \frac{1}{r^n} k(n+k-1) r^{n+k-2} H + r^{k-2}\Delta_{S^n} H \\
&= r^{k-2}(k(n+k-1)H + \Delta_{S^n} H).
\end{aligned}$$

Thus,

$$\Delta P = 0 \quad \text{iff} \quad \Delta_{S^n} H = -k(n+k-1)H,$$

as claimed. $\square$

From Proposition 1.8, we deduce that the space $\mathcal{H}_k(S^n)$ is a subspace of the eigenspace, E_k, of Δ_{S^n}, associated with the eigenvalue $-k(n+k-1)$ (and similarly for $\mathcal{H}_k^{\mathbb{C}}(S^n)$). Remarkably, $E_k = \mathcal{H}_k(S^n)$ but it will take more work to prove this.

What we can deduce immediately is that $\mathcal{H}_k(S^n)$ *and* $\mathcal{H}_l(S^n)$ *are pairwise orthogonal whenever* $k \neq l$. This is because, by Proposition 1.3, the Laplacian is self-adjoint and thus, any two eigenspaces, E_k and E_l are pairwise orthogonal whenever $k \neq l$ and as $\mathcal{H}_k(S^n) \subseteq E_k$ and $\mathcal{H}_l(S^n) \subseteq E_l$, our claim is indeed true. Furthermore, by Proposition 1.6, each $\mathcal{H}_k(S^n)$ is finite-dimensional and thus, closed. Finally, we know from Proposition 1.7 that $\bigcup_{k=0}^{\infty} \mathcal{H}_k(S^n)$ is dense in $L^2(S^n)$. But then, we can apply a standard result from Hilbert space theory (for example, see Lang [20], Chapter V, Proposition 1.9) to deduce the following important result:

Theorem 1.9 *The family of spaces,* $\mathcal{H}_k(S^n)$ *(resp.* $\mathcal{H}_k^{\mathbb{C}}(S^n)$*) yields a Hilbert space direct sum decomposition*

$$L^2(S^n) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k(S^n) \qquad (\textit{resp.} \quad L^2_{\mathbb{C}}(S^n) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k^{\mathbb{C}}(S^n)),$$

which means that the summands are closed, pairwise orthogonal, and that every $f \in L^2(S^n)$ *(resp.* $f \in L^2_{\mathbb{C}}2(S^n)$*) is the sum of a converging series*

$$f = \sum_{k=0}^{\infty} f_k,$$

in the L^2*-norm, where the* $f_k \in \mathcal{H}_k(S^n)$ *(resp.* $f_k \in \mathcal{H}_k^{\mathbb{C}}(S^n)$*) are uniquely determined functions. Furthermore, given any orthonormal basis,* $(Y_k^1, \ldots, Y_k^{a_{k,n+1}})$*, of* $\mathcal{H}_k(S^n)$*, we have*

$$f_k = \sum_{m_k=1}^{a_{k,n+1}} c_{k,m_k} Y_k^{m_k}, \qquad \textit{with} \quad c_{k,m_k} = \langle f, Y_k^{m_k} \rangle.$$

The coefficients c_{k,m_k} are “generalized” *Fourier coefficients* with respect to the Hilbert basis $\{Y_k^{m_k} \mid 1 \leq m_k \leq a_{k,n+1}, k \geq 0\}$. We can finally prove the main theorem of this section.

Theorem 1.10

(1) The eigenspaces (resp. complex eigenspaces) of the Laplacian, Δ_{S^n}*, on* S^n *are the spaces of spherical harmonics,*

$$E_k = \mathcal{H}_k(S^n) \qquad (\textit{resp.} \quad E_k = \mathcal{H}_k^{\mathbb{C}}(S^n))$$

and E_k *corresponds to the eigenvalue* $-k(n+k-1)$.

(2) We have the Hilbert space direct sum decompositions

$$L^2(S^n) = \bigoplus_{k=0}^{\infty} E_k \qquad (\textit{resp.} \quad L^2_{\mathbb{C}}(S^n) = \bigoplus_{k=0}^{\infty} E_k).$$

(3) *The complex polynomials of the form* $(c_1x_1 + \cdots + c_{n+1}x_{n+1})^k$*, with* $c_1^2 + \cdots + c_{n+1}^2 = 0$*, span the space* $\mathcal{H}_k^{\mathbb{C}}(n+1)$*, for* $k \geq 1$*.*

Proof. We follow essentially the proof in Helgason [16] (Introduction, Theorem 3.1). In (1) and (2) we only deal with the real case, the proof in the complex case being identical.

(1) We already know that the integers $-k(n+k-1)$ are eigenvalues of Δ_{S^n} and that $\mathcal{H}_k(S^n) \subseteq E_k$. We will prove that Δ_{S^n} has no other eigenvalues and no other eigenvectors using the Hilbert basis, $\{Y_k^{m_k} \mid 1 \leq m_k \leq a_{k,n+1},\ k \geq 0\}$, given by Theorem 1.9. Let λ be any eigenvalue of Δ_{S^n} and let $f \in L^2(S^n)$ be any eigenfunction associated with λ so that

$$\Delta f = \lambda\, f.$$

We have a unique series expansion

$$f = \sum_{k=0}^{\infty} \sum_{m_k=1}^{a_{k,n+1}} c_{k,m_k} Y_k^{m_k},$$

with $c_{k,m_k} = \langle f, Y_k^{m_k} \rangle$. Now, as Δ_{S^n} is self-adjoint and $\Delta Y_k^{m_k} = -k(n+k-1)Y_k^{m_k}$, the Fourier coefficients, d_{k,m_k}, of Δf are given by

$$d_{k,m_k} = \langle \Delta f, Y_k^{m_k} \rangle = \langle f, \Delta Y_k^{m_k} \rangle = -k(n+k-1)\langle f, Y_k^{m_k} \rangle = -k(n+k-1)c_{k,m_k}.$$

On the other hand, as $\Delta f = \lambda\, f$, the Fourier coefficients of Δf are given by

$$d_{k,m_k} = \lambda c_{k,m_k}.$$

By uniqueness of the Fourier expansion, we must have

$$\lambda c_{k,m_k} = -k(n+k-1)c_{k,m_k} \qquad \text{for all } k \geq 0.$$

If $\lambda = 0$, then $c_{k,m_k} = 0$ for all $k \geq 1$, so $f \in \mathcal{H}_0(S^n)$. If $\lambda \neq 0$, as $f \neq 0$, then there is a unique k so that

$$\lambda = -k(n+k-1)$$

and

$$c_{j,m_j} = 0 \qquad \text{for all } j \neq k,$$

which implies that $f \in \mathcal{H}_k(S^n)$. Therefore, the eigenvalues of Δ_{S^n} are exactly the integers $-k(n+k-1)$ and $E_k = \mathcal{H}_k(S^n)$, as claimed.

Since we just proved that $E_k = \mathcal{H}_k(S^n)$, (2) follows immediately from the Hilbert decomposition given by Theorem 1.9.

(3) If $H = (c_1x_1 + \cdots + c_{n+1}x_{n+1})^k$, with $c_1^2 + \cdots + c_{n+1}^2 = 0$, then for $k \leq 1$ is is obvious that $\Delta H = 0$ and for $k \geq 2$ we have

$$\Delta H = k(k-1)(c_1^2 + \cdots + c_{n+1}^2)(c_1x_1 + \cdots + c_{n+1}x_{n+1})^{k-2} = 0,$$

so $H \in \mathcal{H}_k^{\mathbb{C}}(n+1)$. A simple computation shows that for every $Q \in \mathcal{P}_k^{\mathbb{C}}(n+1)$, if $c = (c_1, \ldots, c_{n+1})$, then we have

$$\partial(Q)(c_1x_1 + \cdots + c_{n+1}x_{n+1})^m = m(m-1)\cdots(m-k+1)Q(c)(c_1x_1 + \cdots + c_{n+1}x_{n+1})^{m-k},$$

for all $m \geq k \geq 1$. Assume that $Q \in \mathcal{H}_k^{\mathbb{C}}(n+1)$ is orthogonal to all polynomials of the form $H = (c_1x_1 + \cdots + c_{n+1}x_{n+1})^k$, with $c_1^2 + \cdots + c_{n+1}^2 = 0$. Recall that

$$\langle P, \partial(Q)H\rangle = \langle QP, H\rangle$$

and apply this equation to $P = Q(c)$, H and Q. Since

$$\partial(Q)H = \partial(Q)(c_1x_1 + \cdots + c_{n+1}x_{n+1})^k = Q(c),$$

as Q is orthogonal to H, we get

$$\langle Q(c), Q(c)\rangle = \langle Q(c), \partial(Q)H\rangle = \langle Q\,Q(c), H\rangle = Q(c)\langle Q, H\rangle = 0,$$

which implies $Q(c) = 0$. Consequently, if $c_1^2 + \cdots + c_{n+1}^2 = 0$ then $Q(c) = 0$, that is, $Q(x_1, \ldots, x_{n+1})$ vanishes on the complex algebraic variety,

$$\{(x_1, \ldots, x_{n+1}) \in \mathbb{C}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 0\}.$$

By the Hilbert *Nullstellensatz*, some power, Q^m, belongs to the ideal, $(x_1^2 + \cdots + x_{n+1}^2)$, generated by $x_1^2 + \cdots + x_{n+1}^2$. Now, if $n \geq 2$, it is well-known that the polynomial $x_1^2 + \cdots + x_{n+1}^2$ is irreducible so the ideal $(x_1^2 + \cdots + x_{n+1}^2)$ is a prime ideal and thus, Q is divisible by $x_1^2 + \cdots + x_{n+1}^2$. However, we know from the proof of Theorem 1.4 that we have an orthogonal direct sum

$$\mathcal{P}_k^{\mathbb{C}}(n+1) = \mathcal{H}_k^{\mathbb{C}}(n+1) \oplus \|x\|^2\, \mathcal{P}_{k-2}^{\mathbb{C}}(n+1).$$

Since $Q \in \mathcal{H}_k^{\mathbb{C}}(n+1)$ and Q is divisible by $x_1^2 + \cdots + x_{n+1}^2$, we must have $Q = 0$. Therefore, if $n \geq 2$, we proved (3). However, when $n = 1$, we know from Section 1.1 that the complex harmonic homogeneous polynomials in two variables, $P(x, y)$, are spanned by the real and imaginary parts, U_k, V_k of the polynomial $(x + iy)^k = U_k + iV_k$. Since $(x - iy)^k = U_k - iV_k$ we see that

$$U_k = \frac{1}{2}\left((x+iy)^k + (x-iy)^k\right), \qquad U_k = \frac{1}{2i}\left((x+iy)^k - (x-iy)^k\right),$$

and as $1 + i^2 = 1 + (-i)^2 = 0$, the space $\mathcal{H}_k^{\mathbb{C}}(\mathbb{R}^2)$ is spanned by $(x+iy)^k$ and $(x-iy)^k$ (for $k \geq 1$), so (3) holds for $n = 1$ as well. $\square$

As an illustration of part (3) of Theorem 1.10, the polynomials $(x_1 + i\cos\theta x_2 + i\sin\theta x_3)^k$ are harmonic. Of course, the real and imaginary part of a complex harmonic polynomial $(c_1x_1 + \cdots + c_{n+1}x_{n+1})^k$ are real harmonic polynomials.

In the next section, we try to show how spherical harmonics fit into the broader framework of linear respresentations of (Lie) groups.

1.5 Spherical Functions and Linear Representations of Lie Groups; A Glimpse

In this section, we indicate briefly how Theorem 1.10 (except part (3)) can be viewed as a special case of a famous theorem known as the *Peter-Weyl Theorem* about unitary representations of compact Lie groups. First, we review the notion of a linear representation of a group. A good and easy-going introduction to representations of Lie groups can be found in Hall [15]. We begin with finite-dimensional representations.

Definition 1.2 Given a Lie group, G, and a vector space, V, of dimension n, a *linear representation* of G of *dimension* (or *degree* n) is a group homomorphism, $U\colon G \to \mathbf{GL}(V)$, such that the map, $g \mapsto U(g)(u)$, is continuous for every $u \in V$ and where $\mathbf{GL}(V)$ denotes the group of invertible linear maps from V to itself. The space, V, called the *representation space* may be a real or a complex vector space. If V has a Hermitian (resp Euclidean) inner product, $\langle -, - \rangle$, we say that $U\colon G \to \mathbf{GL}(V)$ is a *unitary representation* iff

$$\langle U(g)(u), U(g)(v) \rangle = \langle u, v \rangle, \qquad \text{for all } g \in G \text{ and all } u, v \in V.$$

Thus, a linear representation of G is a map, $U\colon G \to \mathbf{GL}(V)$, satisfying the properties:

$$\begin{aligned} U(gh) &= U(g)U(h) \\ U(g^{-1}) &= U(g)^{-1} \\ U(1) &= I. \end{aligned}$$

For simplicity of language, we usually abbreviate *linear representation* as *representation*. The representation space, V, is also called a *G-module* since the representation, $U\colon G \to \mathbf{GL}(V)$, is equivalent to the left action, $\cdot\colon G \times V \to V$, with $g \cdot v = U(g)(v)$. The representation such that $U(g) = I$ for all $g \in G$ is called the *trivial representation*.

As an example, we describe a class of representations of $\mathbf{SL}(2, \mathbb{C})$, the group of complex matrices with determinant $+1$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1.$$

Recall that $\mathcal{P}_k^{\mathbb{C}}(2)$ denotes the vector space of complex homogeneous polynomials of degree k in two variables, (z_1, z_2). For every matrix, $A \in \mathbf{SL}(2, \mathbb{C})$, with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for every homogeneous polynomial, $Q \in \mathcal{P}_k^{\mathbb{C}}(2)$, we define $U_k(A)(Q(z_1, z_2))$ by

$$U_k(A)(Q(z_1, z_2)) = Q(dz_1 - bz_2, -cz_1 + az_2).$$

If we think of the homogeneous polynomial, $Q(z_1, z_2)$, as a function, $Q\binom{z_1}{z_2}$, of the vector $\binom{z_1}{z_2}$, then

$$U_k(A)\left(Q\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) = QA^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = Q\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

The expression above makes it clear that

$$U_k(AB) = U_k(A)U_k(B)$$

for any two matrices, $A, B \in \mathbf{SL}(2, \mathbb{C})$, so U_k is indeed a representation of $\mathbf{SL}(2, \mathbb{C})$ into $\mathcal{P}_k^{\mathbb{C}}(2)$. It can be shown that the representations, U_k, are irreducible and that every representation of $\mathbf{SL}(2, \mathbb{C})$ is equivalent to one of the U_k's (see Bröcker and tom Dieck [4], Chapter 2, Section 5). The representations, U_k, are also representations of $\mathbf{SU}(2)$. Again, they are irreducible representations of $\mathbf{SU}(2)$ and they constitute all them (up to equivalence). The reader should consult Hall [15] for more examples of representations of Lie groups.

One might wonder why we considered $\mathbf{SL}(2, \mathbb{C})$ rather than $\mathbf{SL}(2, \mathbb{R})$. This is because it can be shown that $\mathbf{SL}(2, \mathbb{R})$ has *no* nontrivial unitary (finite-dimensional) representations! For more on representations of $\mathbf{SL}(2, \mathbb{R})$, see Dieudonné [6] (Chapter 14).

Given any basis, $(e_1, \ldots, e_n)$, of V, each $U(g)$ is represented by an $n \times n$ matrix, $U(g) = (U_{ij}(g))$. We may think of the scalar functions, $g \mapsto U_{ij}(g)$, as *special functions* on G. As explained in Dieudonné [6] (see also Vilenkin [28]), essentially all special functions (Legendre polynomials, ultraspherical polynomials, Bessel functions, *etc.*) arise in this way by choosing some suitable G and V. There is a natural and useful notion of equivalence of representations:

Definition 1.3 Given any two representations, $U_1 \colon G \to \mathbf{GL}(V_1)$ and $U_2 \colon G \to \mathbf{GL}(V_2)$, a *$G$-map* (or *morphism of representations*), $\varphi \colon U_1 \to U_2$, is a linear map, $\varphi \colon V_1 \to V_2$, so that the following diagram commutes for every $g \in G$:

$$\begin{array}{ccc} V_1 & \xrightarrow{U_1(g)} & V_1 \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ V_2 & \xrightarrow{U_2(g)} & V_2. \end{array}$$

The space of all G-maps between two representations as above is denoted $\mathrm{Hom}_G(U_1, U_2)$. Two representations $U_1 \colon G \to \mathbf{GL}(V_1)$ and $U_2 \colon G \to \mathbf{GL}(V_2)$ are *equivalent* iff $\varphi \colon V_1 \to V_2$ is an invertible linear map (which implies that $\dim V_1 = \dim V_2$). In terms of matrices, the representations $U_1 \colon G \to \mathbf{GL}(V_1)$ and $U_2 \colon G \to \mathbf{GL}(V_2)$ are equivalent iff there is some invertible $n \times n$ matrix, P, so that

$$U_2(g) = PU_1(g)P^{-1}, \qquad g \in G.$$

If $W \subseteq V$ is a subspace of V, then in some cases, a representation $U\colon G \to \mathbf{GL}(V)$ yields a representation $U\colon G \to \mathbf{GL}(W)$. This is interesting because under certain conditions on G every representation may be decomposed into a "sum" of so-called irreducible representations and thus, the study of all representations of G boils down to the study of irreducible representations of G (for instance, see Knapp [19] (Chapter 4, Corollary 4.7) or Bröcker and tom Dieck [4] (Chapter 2, Proposition 1.9).

Definition 1.4 Let $U\colon G \to \mathbf{GL}(V)$ be a representation of G. If $W \subseteq V$ is a subspace of V, then we say that W is *invariant* (or *stable*) under U iff $U(g)(w) \in W$, for all $g \in G$ and all $w \in W$. If W is invariant under U, then we have a homomorphism, $U\colon G \to \mathbf{GL}(W)$, called a *subrepresentation* of G. A representation, $U\colon G \to \mathbf{GL}(V)$, with $V \neq (0)$ is *irreducible* iff it only has the two subrepresentations, $U\colon G \to \mathbf{GL}(W)$, corresponding to $W = (0)$ or $W = V$.

An easy but crucial lemma about irreducible representations is "Schur's Lemma".

Lemma 1.11 *(Schur's Lemma) Let $U_1\colon G \to \mathbf{GL}(V)$ and $U_2\colon G \to \mathbf{GL}(W)$ be any two real or complex representations of a group, G. If U_1 and U_2 are irreducible, then the following properties hold:*

(i) Every G-map, $\varphi\colon U_1 \to U_2$, is either the zero map or an isomorphism.

(ii) If U_1 is a complex representation, then every G-map, $\varphi\colon U_1 \to U_1$, is of the form, $\varphi = \lambda\mathrm{id}$, for some $\lambda \in \mathbb{C}$.

Proof. (i) Observe that the kernel, $\mathrm{Ker}\ \varphi \subseteq V$, of φ is invariant under U_1. Thus, $U_1\colon G \to \mathbf{GL}(\mathrm{Ker}\ \varphi)$ is a subrepresentation of U_1 and as U_1 is irreducible, either $\mathrm{Ker}\ \varphi = (0)$ or $\mathrm{Ker}\ \varphi = V$. In the second case, $\varphi = 0$. If $\mathrm{Ker}\ \varphi = (0)$, then φ is injective. However, it is clear that $\varphi(V) \subseteq W$ is invariant under U_2 and as $\varphi(V) \neq (0)$ (as $V \neq (0)$ since U_1 is irreducible) and U_2 is irreducible, we must have $\varphi(V) = W$, that is, φ is an isomorphism.

(ii) Since V is a complex vector space, the linear map, φ, has some eigenvalue, $\lambda \in \mathbb{C}$. Let $E_\lambda \subseteq V$ be the eigenspace associated with λ. It is easy to check that E_λ is invariant under U_1, so $U_1\colon G \to \mathbf{GL}(E_\lambda)$ is a subrepresentation of U_1 and as U_1 is irreducible and $E_\lambda \neq (0)$, we must have $E_\lambda = V$. $\square$

An interesting corollary of Schur's Lemma is that every complex irreducible representtaion of a commutative group is one-dimensional.

Let us now restrict our attention to compact Lie groups. If G is a compact Lie group, then it is known that it has a left and right-invariant volume form, ω_G, so we can define the integral of a (real or complex) continuous function, f, defined on G as

$$\int_G f = \int_G f\,\omega_G,$$

also denoted $\int_G f\, d\mu_G$ or simply $\int_G f(t)\, dt$, with ω_G normalized so that $\int_G \omega_G = 1$. (See Knapp [19], Chapter 8 or Warner [29], 4 and 6.) Because G is compact, the *Haar measure*, μ_G, induced by ω_G is both left and right-invariant (G is a *unimodular group*) and our integral has the following invariance properties:

$$\int_G f(t)\, dt = \int_G f(st)\, dt = \int_G f(tu)\, dt = \int_G f(t^{-1})\, dt,$$

for all $s, u \in G$.

Since G is a compact Lie group, we can use an "averaging trick" to show that every (finite-dimensional) representation is equivalent to a unitary representation (see Bröcker and tom Dieck [4] (Chapter 2, Theorem 1.7) or Knapp [19] (Chapter 4, Proposition 4.6).

If we define the Hermitian inner product,

$$\langle f, g\rangle = \int_G f\, \overline{g}\, \omega_G,$$

then, with this inner product, the space of square-integrable functions, $L^2_{\mathbb{C}}(G)$, is a *Hilbert space*. We can also define the *convolution*, $f * g$, of two functions, $f, g \in L^2_{\mathbb{C}}(G)$, by

$$(f * g)(x) = \int_G f(xt^{-1})g(t)dt = \int_G f(t)g(t^{-1}x)dt$$

In general, $f * g \neq g * f$ unless G is commutative. With the convolution product, $L^2_{\mathbb{C}}(G)$ becomes an associative algebra (non-commutative in general).

This leads us to consider unitary representations of G into the infinite-dimensional vector space, $L^2_{\mathbb{C}}(G)$. The definition is the same as in Definition 1.2, except that $\mathbf{GL}(L^2_{\mathbb{C}}(G))$ is the group of automorphisms (unitary operators), $\mathrm{Aut}(L^2_{\mathbb{C}}(G))$, of the Hilbert space, $L^2_{\mathbb{C}}(G)$ and

$$\langle U(g)(u), U(g)(v)\rangle = \langle u, v\rangle$$

with respect to the inner product on $L^2_{\mathbb{C}}(G)$. Also, in the definition of an irreducible representation, $U\colon G \to V$, we require that the only *closed* subrepresentations, $U\colon G \to W$, of the representation, $U\colon G \to V$, correspond to $W = (0)$ or $W = V$.

The *Peter Weyl Theorem* gives a decomposition of $L^2_{\mathbb{C}}(G)$ as a Hilbert sum of spaces that correspond to irreducible unitary representations of G. We present a version of the Peter Weyl Theorem found in Dieudonné [6] (Chapters 3-8) and Dieudonné [7] (Chapter XXI, Sections 1-4), which contains complete proofs. Other versions can be found in Bröcker and tom Dieck [4] (Chapter 3), Knapp [19] (Chapter 4) or Duistermaat and Kolk [10] (Chapter 4). A good preparation of these fairly advanced books is Deitmar [5].

Theorem 1.12 *(Peter-Weyl (1927)) Given a compact Lie group, G, there is a decomposition of $L^2_{\mathbb{C}}(G)$ as a Hilbert sum,*

$$L^2_{\mathbb{C}}(G) = \bigoplus_{\rho} \mathfrak{a}_\rho,$$

of countably many two-sided ideals, $\mathfrak{a}_\rho$, where each $\mathfrak{a}_\rho$ is isomorphic to a finite-dimensional algebra of $n_\rho \times n_\rho$ complex matrices. More precisely, there is a basis of $\mathfrak{a}_\rho$ consisting of smooth pairwise orthogonal functions, $m_{ij}^{(\rho)}$, satisfying various properties, including

$$\langle m_{ij}^{(\rho)}, m_{ij}^{(\rho)} \rangle = n_\rho,$$

and if we form the matrix, $M_\rho(g) = (\frac{1}{n_\rho} m_{ij}^{(\rho)}(g))$, then the map, $g \mapsto M_\rho(g)$ is an **irreducible unitary representation** *of G in the vector space $\mathbb{C}^{n_\rho}$. Furthermore, every irreducible representation of G is equivalent to some M_ρ, so the set of indices, ρ, corresponds to the set of equivalence classes of irreducible unitary representations of G. The function, u_ρ, given by*

$$u_\rho(g) = \sum_{j=1}^{n_\rho} m_{jj}^{(\rho)}(g) = n_\rho \mathrm{tr}(M_\rho(g))$$

is the unit of the algebra $\mathfrak{a}_\rho$ and the orthogonal projection of $L^2_{\mathbb{C}}(G)$ onto $\mathfrak{a}_\rho$ is the map

$$f \mapsto u_\rho * f,$$

that is, convolution with u_ρ.

Remark: The function, $\chi_\rho = \frac{1}{n_\rho} u_\rho = \mathrm{tr}(M_\rho)$, is the *character* of G associated with the representation of G into M_ρ. The functions, χ_ρ, form an orthogonal system. Beware that they are *not* homomorphims of G into $\mathbb{C}$ unless G is commutative. The characters of G are the key to the definition of the Fourier transform on a (compact) group, G.

A complete proof of Theorem 1.12 is given in Dieudonné [7], Chapter XXI, Section 2, but see also Sections 3 and 4.

There is more to the Peter Weyl Theorem: It gives a description of all unitary representations of G into a separable Hilbert space (see Dieudonné [7], Chapter XXI, Section 4). If $V \colon G \to \mathrm{Aut}(E)$ is such a representation, then for every ρ as above, the map

$$x \mapsto V(u_\rho)(x) = \int_G (V(s)(x)) u_\rho(s)\, ds$$

is an orthogonal projection of E onto a closed subspace, E_ρ. Then, E is the Hilbert sum, $E = \bigoplus_\rho E_\rho$, of those E_ρ such that $E_\rho \neq (0)$ and each such E_ρ is itself a (countable) Hilbert sum of closed spaces invariant under V. The subrepresentations of V corresponding to these subspaces of E_ρ are all equivalent to $M_{\overline{\rho}} = \overline{M_\rho}$ and hence, irreducible. This is why *all* (unitary) representations of G are equivalent to a representation into some M_ρ.

An interesting special case is the case of the so-called regular representation of G in $L^2_{\mathbb{C}}(G)$ itself. The (left) *regular representation*, $\mathbf{R}$, of G in $L^2_{\mathbb{C}}(G)$ is defined by

$$(\mathbf{R}_s(f))(t) = \lambda_s(f)(t) = f(s^{-1}t), \qquad f \in L^2_{\mathbb{C}}(G),\ s, t \in G.$$

It turns out that we also get the same Hilbert sum,

$$L^2_{\mathbb{C}}(G) = \bigoplus_{\rho} \mathfrak{a}_\rho,$$

but this time, the $\mathfrak{a}_\rho$ generally do not correspond to irreducible subrepresentations. However, $\mathfrak{a}_\rho$ splits into n_ρ left ideals, $\mathfrak{b}_j^{(\rho)}$, where $\mathfrak{b}_j^{(\rho)}$ corresponds to the jth columm of M_ρ and all the subrepresentations of G in $\mathfrak{b}_j^{(\rho)}$ are equivalent to $M_{\overline{\rho}}$ and thus, are irreducible (see Dieudonné [6], Chapter 3).

Finally, assume that besides the compact Lie group, G, we also have a closed subgroup, K, or G. Then, we know that G/K is a manifold called a *homogeneous space* and G acts on M on the left. For example, if $G = \mathbf{SO}(n+1)$ and $K = \mathbf{SO}(n)$, then $S^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$ (for instance, see Warner [29], Chapter 3). The subspace of $L^2_{\mathbb{C}}(G)$ consisting of the functions $f \in L^2_{\mathbb{C}}(G)$ that are right-invariant under the action of K, that is, such that

$$f(su) = f(s) \qquad \text{for all } s \in G \text{ and all } u \in K$$

form a closed subspace of $L^2_{\mathbb{C}}(G)$ denoted $L^2_{\mathbb{C}}(G/K)$. For example, if $G = \mathbf{SO}(n+1)$ and $K = \mathbf{SO}(n)$, then $L^2_{\mathbb{C}}(G/K) = L^2_{\mathbb{C}}(S^n)$.

It turns out that $L^2_{\mathbb{C}}(G/K)$ is invariant under the regular representation, $\mathbf{R}$, of G in $L^2_{\mathbb{C}}(G)$, so we get a subrepresentation (of the regular representation) of G in $L^2_{\mathbb{C}}(G/K)$. Again, the Peter-Weyl gives us a Hilbert sum decomposition of $L^2_{\mathbb{C}}(G/K)$ of the form

$$L^2_{\mathbb{C}}(G/K) = \bigoplus_{\rho} L_\rho = L^2_{\mathbb{C}}(G/K) \cap \mathfrak{a}_\rho,$$

for the same ρ's as before. However, these subrepresentations of $\mathbf{R}$ in L_ρ are not necessarily irreducible. What happens is that there is some d_ρ with $0 \leq d_\rho \leq n_\rho$ so that if $d_\rho \geq 1$, then L_σ is the direct sum of the first d_ρ columns of M_ρ (see Dieudonné [6], Chapter 6 and Dieudonné [8], Chapter XXII, Sections 4-5).

We can also consider the subspace of $L^2_{\mathbb{C}}(G)$ consisting of the functions, $f \in L^2_{\mathbb{C}}(G)$, that are left-invariant under the action of K, that is, such that

$$f(ts) = f(s) \qquad \text{for all } s \in G \text{ and all } t \in K.$$

This is a closed subspace of $L^2_{\mathbb{C}}(G)$ denoted $L^2_{\mathbb{C}}(K\backslash G)$. Then, we get a Hilbert sum decomposition of $L^2_{\mathbb{C}}(K\backslash G)$ of the form

$$L^2_{\mathbb{C}}(K\backslash G) = \bigoplus_{\rho} L'_\rho = L^2_{\mathbb{C}}(K\backslash G) \cap \mathfrak{a}_\rho,$$

and for the same d_ρ as before, L'_σ is the direct sum of the first d_ρ rows of M_ρ. We can also consider

$$\begin{aligned} L^2_{\mathbb{C}}(K\backslash G/K) &= L^2_{\mathbb{C}}(G/K) \cap L^2_{\mathbb{C}}(K\backslash G) \\ &= \{f \in L^2_{\mathbb{C}}(G) \mid f(tsu) = f(s)\} \qquad \text{for all } s \in G \text{ and all } t, u \in K. \end{aligned}$$

From our previous discussion, we see that we have a Hilbert sum decomposition

$$L^2_{\mathbb{C}}(K\backslash G/K) = \bigoplus_{\rho} L_\rho \cap L'_\rho$$

and each $L_\rho \cap L'_\rho$ for which $d_\rho \geq 1$ is a matrix algebra of dimension d_ρ^2. As a consequence, *the algebra $L^2_{\mathbb{C}}(K\backslash G/K)$ is commutative iff $d_\rho \leq 1$ for all ρ.*

If the algebra $L^2_{\mathbb{C}}(K\backslash G/K)$ is commutative (for the convolution product), we say that (G, K) is a *Gelfand pair* (see Dieudonné [6], Chapter 8 and Dieudonné [8], Chapter XXII, Sections 6-7). In this case, the L_ρ in the Hilbert sum decomposition of $L^2_{\mathbb{C}}(G/K)$ are nontrivial of dimension n_ρ iff $d_\rho = 1$ and the subrepresentation, $\mathbf{U}$, (of the regular representation) of G into L_ρ is irreducible and equivalent to $M_{\overline{\rho}}$. The space L_ρ is generated by the functions, $m^{(\rho)}_{1,1}, \ldots, m^{(\rho)}_{n_\rho,1}$, but the function

$$\omega_\rho(s) = \frac{1}{n_\rho}\, m^{(\rho)}_{1,1}(s)$$

plays a special role. This function called a *zonal spherical function* has some interesting properties. First, $\omega_\rho(e) = 1$ (where e is the identity element of the group, G) and

$$\omega_\rho(ust) = \omega_\rho(s) \qquad \text{for all } s \in G \text{ and all } u, t \in K.$$

In addition, ω_ρ is of positive type. A function, $f\colon G \to \mathbb{C}$, is of *positive type* iff

$$\sum_{j,k=1}^{n} f(s_j^{-1} s_k) z_j \overline{z}_k \geq 0,$$

for every finite set, $\{s_1, \ldots, s_n\}$, of elements of G and every finite tuple, $(z_1, \ldots, z_n) \in \mathbb{C}^n$. Because the subrepresentation of G into L_ρ is irreducible, the function ω_ρ generates L_ρ under left translation. This means the following: If we recall that for any function, f, on G,

$$\lambda_s(f)(t) = f(s^{-1}t), \qquad s, t \in G,$$

then, L_ρ is generated by the functions $\lambda_s(\omega_\rho)$, as s varies in G. The function ω_ρ also satisfies the following property:

$$\omega_\rho(s) = \langle \mathbf{U}(s)(\omega_\rho), \omega_\rho \rangle.$$

The set of zonal spherical functions on G/K is denoted $S(G/K)$. It is a countable set.

The notion of Gelfand pair also applies to locally-compact unimodular groups that are not necessary compact but we will not discuss this notion here. Curious readers may consult Dieudonné [6] (Chapters 8 and 9) and Dieudonné [8] (Chapter XXII, Sections 6-9).

It turns out that $G = \mathbf{SO}(n+1)$ and $K = \mathbf{SO}(n)$ form a Gelfand pair (see Dieudonné [6], Chapters 7-8 and Dieudonné [9], Chapter XXIII, Section 38). In this particular case,

$\rho = k$ is any nonnegative integer and $L_\rho = E_k$, the eigenspace of the Laplacian on S^n corresponding to the eigenvalue $-k(n+k-1)$. Therefore, the regular representation of $\mathbf{SO}(n)$ into $E_k = \mathcal{H}_k^{\mathbb{C}}(S^n)$ is irreducible. This can be proved more directly, for example, see Helgason [16] (Introduction, Theorem 3.1) or Bröcker and tom Dieck [4] (Chapter 2, Proposition 5.10).

The zonal spherical harmonics, ω_k, can be expressed in terms of the *ultraspherical polynomials* (also called *Gegenbauer polynomials*), $P_k^{(n-1)/2}$ (up to a constant factor), see Stein and Weiss [26] (Chapter 4), Morimoto [22] (Chapter 2) and Dieudonné [6] (Chapter 7). For $n = 2$, $P_k^{\frac{1}{2}}$ is just the ordinary Legendre polynomial (up to a constant factor). We will say more about the zonal spherical harmonics and the ultraspherical polynomials in the next two sections.

The material in this section belongs to the overlapping areas of *representation theory* and *noncommutative harmonic analysis*. These are deep and vast areas. Besides the references cited earlier, for noncommutative harmonic analysis, the reader may consult Folland [11] or Taylor [27], but they may find the pace rather rapid. Another great survey on both topics is Kirillov [18], although it is not geared for the beginner.

1.6 Reproducing Kernel, Zonal Spherical Functions and Gegenbauer Polynomials

We now return to S^n and its spherical harmonics. The previous section suggested that zonal spherical functions play a special role. In this section, we describe the zonal spherical functions on S^n and show that they essentially come from certain polynomials generalizing the Legendre polyomials known as the *Gegenbauer Polynomials*. Most proof will be omitted. We refer the reader to Stein and Weiss [26] (Chapter 4) and Morimoto [22] (Chapter 2) for a complete exposition with proofs.

Recall that the space of spherical harmonics, $\mathcal{H}_k^{\mathbb{C}}(S^n)$, is the image of the space of homogeneous harmonic poynomials, $\mathcal{P}_k^{\mathbb{C}}(n+1)$, under the restriction map. It is a finite-dimensional space of dimension

$$a_{k,n+1} = \binom{n+k}{k} - \binom{n+k-2}{k-2},$$

if $n \geq 1$ and $k \geq 2$, with $a_{0,n+1} = 1$ and $a_{1,n+1} = n+1$. Let $(Y_k^1, \ldots, Y_k^{a_{k,n+1}})$ be any orthonormal basis of $\mathcal{H}_k^{\mathbb{C}}(S^n)$ and define $F_k(\sigma, \tau)$ by

$$F_k(\sigma, \tau) = \sum_{i=1}^{a_{k,n+1}} Y_k^i(\sigma)\overline{Y_k^i(\tau)}, \qquad \sigma, \tau \in S^n.$$

The following proposition is easy to prove (see Morimoto [22], Chapter 2, Lemma 1.19 and Lemma 2.20):

Proposition 1.13 *The function F_k is independent of the choice of orthonormal basis. Furthermore, for every rotation, $R \in \mathbf{SO}(n+1)$, we have*

$$F_k(R\sigma, R\tau) = F_k(\sigma, \tau), \qquad \sigma, \tau \in S^n.$$

Clearly, F_k is a symmetric function. Since we can pick an orthonormal basis of real orthogonal functions for $\mathcal{H}_k^{\mathbb{C}}(S^n)$ (pick a basis of $\mathcal{H}_k(S^n)$), Proposition 1.13 shows that F_k is a real-valued function.

The function F_k satisfies the following property which justifies its name, the *reproducing kernel* for $\mathcal{H}_k^{\mathbb{C}}(S^n)$:

Proposition 1.14 *For every spherical harmonic, $H \in \mathcal{H}_j^{\mathbb{C}}(S^n)$, we have*

$$\int_{S^n} H(\tau) F_k(\sigma, \tau)\, d\tau = \delta_{j\,k} H(\sigma), \qquad \sigma, \tau \in S^n,$$

for all $j, k \geq 0$.

Proof. When $j \neq k$, since $\mathcal{H}_k^{\mathbb{C}}(S^n)$ and $\mathcal{H}_j^{\mathbb{C}}(S^n)$ are orthogonal and since $F_k(\sigma, \tau) = \sum_{i=1}^{a_{k,n+1}} Y_k^i(\sigma)\overline{Y_k^i(\tau)}$, it is clear that the integral in Proposition 1.14 vanishes. When $j = k$, we have

$$\begin{aligned}
\int_{S^n} H(\tau) F_k(\sigma, \tau)\, d\tau &= \int_{S^n} H(\tau) \sum_{i=1}^{a_{k,n+1}} Y_k^i(\sigma)\overline{Y_k^i(\tau)}\, d\tau \\
&= \sum_{i=1}^{a_{k,n+1}} Y_k^i(\sigma) \int_{S^n} H(\tau)\overline{Y_k^i(\tau)}\, d\tau \\
&= \sum_{i=1}^{a_{k,n+1}} Y_k^i(\sigma)\, \langle H, Y_k^i \rangle \\
&= H(\sigma),
\end{aligned}$$

since $(Y_k^1, \ldots, Y_k^{a_{k,n+1}})$ is an orthonormal basis. □

In Stein and Weiss [26] (Chapter 4), the function $F_k(\sigma, \tau)$ is denoted by $Z_\sigma^{(k)}(\tau)$ and it is called the *zonal harmonic of degree k with pole σ*.

The value, $F_k(\sigma, \tau)$, of the function F_k depends only on $\sigma \cdot \tau$ as stated in the following proposition which is proved in Morimoto [22] (Chapter 2, Lemma 2.23):

Proposition 1.15 *For all $\sigma, \tau, \sigma', \tau' \in S^n$, if $\sigma \cdot \tau = \sigma' \cdot \tau'$, then $F_k(\sigma, \tau) = F_k(\sigma', \tau')$. Consequently, there is some function, $\varphi \colon \mathbb{R} \to \mathbb{R}$, such that $F_k(\omega, \tau) = \varphi(\omega \cdot \tau)$.*

We are now ready to define zonal functions. Remarkably, the function φ in Proposition 1.15 comes from a real polynomial. We need the following proposition which is of independent interest:

Proposition 1.16 *If P is any (complex) polynomial in n variables such that*

$$P(R(x)) = P(x) \qquad \text{for all rotations, } R \in \mathbf{SO}(n), \text{ and all } x \in \mathbb{R}^n,$$

then P is of the form

$$P(x) = \sum_{j=0}^{m} c_j (x_1^2 + \cdots + x_n^2)^j,$$

for some $c_1, \ldots, c_m \in \mathbb{C}$.

Proof. Write P as the sum of its homogeneous pieces, $P = \sum_{l=0}^{k} Q_l$, where Q_l is homogeneous of degree l. Then, for every $\epsilon > 0$ and every rotation, R, we have

$$\sum_{l=0}^{k} \epsilon^l Q_l(x) = P(\epsilon x) = P(R(\epsilon x)) = P(\epsilon R(x)) = \sum_{l=0}^{k} \epsilon^l Q_l(R(x)),$$

which implies that

$$Q_l(R(x)) = Q_l(x), \qquad l = 0, \ldots, k.$$

If we let $F_l(x) = \|x\|^{-l} Q_l(x)$, then F_l is a homogeneous function of degree 0 and F_l is invariant under all rotations. This is only possible if F_l is a constant function, thus $F_l(x) = a_l$ for all $x \in \mathbb{R}^n$. But then, $Q_l(x) = a_l \|x\|^l$. Since Q_l is a polynomial, l must be even whenever $a_l \neq 0$. It follows that

$$P(x) = \sum_{j=0}^{m} c_j \|x\|^{2j}$$

with $c_j = a_{2j}$ for $k = 0, \ldots, m$ and where m is the largest integer $\leq k/2$. $\square$

Proposition 1.16 implies that if a polynomial function on the sphere, S^n, in particular, a spherical harmonic, is invariant under all rotations, then it is a constant. If we relax this condition to invariance under all rotations leaving some given point, $\tau \in S^n$, invariant, then we obtain zonal harmonics.

The following theorem from Morimoto [22] (Chapter 2, Theorem 2.24) gives the relationship between zonal harmonics and the Gegenbauer polynomials:

Theorem 1.17 *Fix any $\tau \in S^n$. For every constant, $c \in \mathbb{C}$, there is a unique homogeneous harmonic polynomial, $Z_k^\tau \in \mathcal{H}_k^{\mathbb{C}}(n+1)$, satisfying the following conditions:*

(1) $Z_k^\tau(\tau) = c$;

(2) For every rotation, $R \in \mathbf{SO}(n+1)$, if $R\tau = \tau$, then $Z_k^\tau(R(x)) = Z_k^\tau(x)$, for all $x \in \mathbb{R}^{n+1}$.

Furthermore, we have

$$Z_k^\tau(x) = c \|x\|^{\frac{k}{2}} P_{k,n}\left(\frac{x}{\|x\|} \cdot \tau\right),$$

for some polynomial, $P_{k,n}(t)$, of degree k.

Remark: The proof given in Morimoto [22] is essentially the same as the proof of Theorem 2.12 in Stein and Weiss [26] (Chapter 4) but Morimoto makes an implicit use of Proposition 1.16 above. Also, Morimoto states Theorem 1.17 only for $c = 1$ but the proof goes through for any $c \in \mathbb{C}$, including $c = 0$, and we will need this extra generality in the proof of the Funk-Hecke formula.

Proof. Let $e_{n+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ and for any $\tau \in S^n$, let R_τ be some rotation such that $R_\tau(e_{n+1}) = \tau$. Assume $Z \in \mathcal{H}_k^{\mathbb{C}}(n+1)$ satisfies conditions (1) and (2) and let Z' be given by $Z'(x) = Z(R_\tau(x))$. As $R_\tau(e_{n+1}) = \tau$, we have $Z'(e_{n+1}) = Z(\tau) = c$. Furthermore, for any rotation, S, such that $S(e_{n+1}) = e_{n+1}$, observe that

$$R_\tau \circ S \circ R_\tau^{-1}(\tau) = R_\tau \circ S(e_{n+1}) = R_\tau(e_{n+1}) = \tau,$$

and so, as Z satisfies property (2) for the rotation $R_\tau \circ S \circ R_\tau^{-1}$, we get

$$Z'(S(x)) = Z(R_\tau \circ S(x)) = Z(R_\tau \circ S \circ R_\tau^{-1} \circ R_\tau(x)) = Z(R_\tau(x)) = Z'(x),$$

which proves that Z' is a harmonic polynomial satisfying properties (1) and (2) with respect to e_{n+1}. Therefore, we may assume that $\tau = e_{n+1}$.

Write

$$Z(x) = \sum_{j=0}^{k} x_{n+1}^{k-j} P_j(x_1, \ldots, x_n),$$

where $P_j(x_1, \ldots, x_n)$ is a homogeneous polynomial of degree j. Since Z is invariant under every rotation, R, fixing e_{n+1} and since the monomials x_{n+1}^{k-j} are clearly invariant under such a rotation, we deduce that every $P_j(x_1, \ldots, x_n)$ is invariant under all rotations of $\mathbb{R}^n$ (clearly, there is a one-two-one correspondence between the rotations of $\mathbb{R}^{n+1}$ fixing e_{n+1} and the rotations of $\mathbb{R}^n$). By Proposition 1.16, we conclude that

$$P_j(x_1, \ldots, x_n) = c_j(x_1^2 + \cdots + x_n^2)^{\frac{j}{2}},$$

which implies that $P_j = 0$ is j is odd. Thus, we can write

$$Z(x) = \sum_{i=0}^{[k/2]} c_i x_{n+1}^{k-2i}(x_1^2 + \cdots + x_n^2)^i$$

where $[k/2]$ is the greatest integer, m, such that $2m \leq k$. If $k < 2$, then $Z = c_0$, so $c_0 = c$ and Z is uniquely determined. If $k \geq 2$, we know that Z is a harmonic polynomial so we assert that $\Delta Z = 0$. A simple computation shows that

$$\Delta(x_1^2 + \cdots + x_n^2)^i = 2i(n + 2i - 2)(x_1^2 + \cdots + x_n^2)^{i-1}$$

and

$$\begin{aligned}
\Delta x_{n+1}^{k-2i}(x_1^2 + \cdots + x_n^2)^i &= (k-2i)(k-2i-1)x_{n+1}^{k-2i-2}(x_1^2 + \cdots + x_n^2)^i \\
&\quad + x_{n+1}^{k-2i}\Delta(x_1^2 + \cdots + x_n^2)^i \\
&= (k-2i)(k-2i-1)x_{n+1}^{k-2i-2}(x_1^2 + \cdots + x_n^2)^i \\
&\quad + 2i(n+2i-2)x_{n+1}^{k-2i}(x_1^2 + \cdots + x_n^2)^{i-1},
\end{aligned}$$

so we get

$$\Delta Z = \sum_{i=0}^{[k/2]-1} \left((k-2i)(k-2i-1)c_i + 2(i+1)(n+2i)c_{i+1}\right) x_{n+1}^{k-2i-2}(x_1^2 + \cdots + x_n^2)^i.$$

Then, $\Delta Z = 0$ yields the relations

$$2(i+1)(n+2i)c_{i+1} = -(k-2i)(k-2i-1)c_i, \qquad i = 0, \ldots, [k/2]-1,$$

which shows that Z is uniquely determined up to the constant c_0. Since we are requiring $Z(e_{n+1}) = c$, we get $c_0 = c$ and Z is uniquely determined. Now, on S^n, we have $x_1^2 + \cdots + x_{n+1}^2 = 1$, so if we let $t = x_{n+1}$, for $c_0 = 1$, we get a polynomial in one variable,

$$P_{k,n}(t) = \sum_{i=0}^{[k/2]} c_i t^{k-2i}(1-t^2)^i.$$

Thus, we proved that when $Z(e_{n+1}) = c$, we have

$$Z(x) = c\, \|x\|^{\frac{k}{2}}\; P_{k,n}\left(\frac{x_{n+1}}{\|x\|}\right) = c\, \|x\|^{\frac{k}{2}}\; P_{k,n}\left(\frac{x}{\|x\|} \cdot e_{n+1}\right).$$

When $Z(\tau) = c$, we write $Z = Z' \circ R_\tau^{-1}$ with $Z' = Z \circ R_\tau$ and where R_τ is a rotation such that $R_\tau(e_{n+1}) = \tau$. Then, as $Z'(e_{n+1}) = c$, using the formula above for Z', we have

$$\begin{aligned} Z(x) = Z'(R_\tau^{-1}(x)) &= c\left\|R_\tau^{-1}(x)\right\|^{\frac{k}{2}}\; P_{k,n}\left(\frac{R_\tau^{-1}(x)}{\|R_\tau^{-1}(x)\|} \cdot e_{n+1}\right) \\ &= c\, \|x\|^{\frac{k}{2}}\; P_{k,n}\left(\frac{x}{\|x\|} \cdot R_\tau(e_{n+1})\right) \\ &= c\, \|x\|^{\frac{k}{2}}\; P_{k,n}\left(\frac{x}{\|x\|} \cdot \tau\right), \end{aligned}$$

since R_τ is an isometry. □

The function, Z_k^τ, is called a *zonal function* and its restriction to S^n is a *zonal spherical function*. The polynomial, $P_{k,n}$, is called the *Gegenbauer polynomial* of degree k and dimension $n+1$ or *ultraspherical polynomial*. By definition, $P_{k,n}(1) = 1$.

The proof of Theorem 1.17 shows that for k even, say $k = 2m$, the polynomial $P_{2m,n}$ is of the form

$$P_{2m,n} = \sum_{j=0}^{m} c_{m-j} t^{2j}(1-t^2)^{m-j}$$

and for k odd, say $k = 2m+1$, the polynomial $P_{2m+1,n}$ is of the form

$$P_{2m+1,n} = \sum_{j=0}^{m} c_{m-j} t^{2j+1}(1-t^2)^{m-j}.$$

Consequently, $P_{k,n}(-t) = (-1)^k P_{k,n}(t)$, for all $k \geq 0$. The proof also shows that the "natural basis" for these polynomials consists of the polynomials, $t^i(1-t^2)^{\frac{k-i}{2}}$, with $k-i$ even. Indeed, with this basis, there are simple recurrence equations for computing the coefficients of $P_{k,n}$.

Remark: Morimoto [22] calls the polynomials, $P_{k,n}$, "Legendre polynomials". For $n = 2$, they are indeed the Legendre polynomials. Stein and Weiss denotes our (and Morimoto's) $P_{k,n}$ by $P_k^{\frac{n-1}{2}}$ (up to a constant factor) and Dieudonné [6] (Chapter 7) by $G_{k,n+1}$.

When $n = 2$, using the notation of Section 1.2, the zonal functions on S^2 are the spherical harmonics, y_l^0, for which $m = 0$, that is (up to a constant factor),

$$y_l^0(\theta, \varphi) = \sqrt{\frac{(2l+1)}{4\pi}}\, P_l(\cos\theta),$$

where P_l is the Legendre polynomial of degree l. For example, for $l = 2$, $P_l(t) = \frac{1}{2}(3t^2 - 1)$.

If we put $Z(r^k\sigma) = r^k F_k(\sigma, \tau)$ for a fixed τ, then by the definition of $F_k(\sigma, \tau)$ it is clear that Z is a homogeneous harmonic polynomial. The value $F_k(\tau, \tau)$ does not depend of τ because by transitivity of the action of $\mathbf{SO}(n+1)$ on S^n, for any other $\sigma \in S^n$, there is some rotation, R, so that $R\tau = \sigma$ and by Proposition 1.13, we have $F_k(\sigma, \sigma) = F_k(R\tau, R\tau) = F_k(\tau, \tau)$. To compute $F_k(\tau, \tau)$, since

$$F_k(\tau, \tau) = \sum_{i=1}^{a_{k,n+1}} \left\| Y_k^i(\tau) \right\|^2,$$

and since $(Y_k^1, \ldots, Y_k^{a_{k,n+1}})$ is an orthonormal basis of $\mathcal{H}_k^{\mathbb{C}}(S^n)$, observe that

$$\begin{aligned} a_{k,n+1} &= \sum_{i=1}^{a_{k,n+1}} \int_{S^n} \left\| Y_k^i(\tau) \right\|^2 \, d\tau && (1.1) \\ &= \int_{S^n} \left(\sum_{i=1}^{a_{k,n+1}} \left\| Y_k^i(\tau) \right\|^2 \right) d\tau && (1.2) \\ &= \int_{S^n} F_k(\tau, \tau)\, d\tau = F_k(\tau, \tau)\, \mathrm{vol}(S^n). && (1.3) \end{aligned}$$

Therefore,

$$F_k(\tau, \tau) = \frac{a_{k,n+1}}{\mathrm{vol}(S^n)}.$$

Beware that Morimoto [22] uses the normalized measure on S^n, so the factor involving $\mathrm{vol}(S^n)$ does not appear.

Remark: Recall that

$$\mathrm{vol}(S^{2d}) = \frac{2^{d+1}\pi^d}{1 \cdot 3 \cdots (2d-1)} \quad \text{if} \quad d \geq 1 \quad \text{and} \quad \mathrm{vol}(S^{2d+1}) = \frac{2\pi^{d+1}}{d!} \quad \text{if} \quad d \geq 0.$$

Now, if $R\tau = \tau$, then Proposition 1.13 shows that

$$Z(R(r^k\sigma)) = r^k F_k(R\sigma, \tau) = r^k F_k(R\sigma, R\tau) = r^k F_k(\sigma, \tau) = Z(r^k\sigma).$$

Therefore, the function Z_k^τ satisfies conditions (1) and (2) of Theorem 1.17 with $c = \frac{a_{k,n+1}}{\mathrm{vol}(S^n)}$ and by uniqueness, we get

$$F_k(\sigma, \tau) = \frac{a_{k,n+1}}{\mathrm{vol}(S^n)} P_{k,n}(\sigma \cdot \tau).$$

Consequently, we have obtained the so-called *addition formula*:

Proposition 1.18 *(Addition Formula) If $(Y_k^1, \ldots, Y_k^{a_{k,n+1}})$ is any orthonormal basis of $\mathcal{H}_k^{\mathbb{C}}(S^n)$, then*

$$P_{k,n}(\sigma \cdot \tau) = \frac{\mathrm{vol}(S^n)}{a_{k,n+1}} \sum_{i=1}^{a_{k,n+1}} Y_k^i(\sigma)\overline{Y_k^i(\tau)}.$$

Again, beware that Morimoto [22] does not have the factor $\mathrm{vol}(S^n)$.

For $n = 1$, we can write $\sigma = (\cos\theta, \sin\theta)$ and $\tau = (\cos\varphi, \sin\varphi)$ and it is easy to see that the addition formula reduces to

$$P_{k,1}(\cos(\theta - \varphi)) = \cos k\theta \cos k\varphi + \sin k\theta \sin k\varphi = \cos k(\theta - \varphi),$$

the standard addition formula for trigonometric functions.

Theorem 1.18 implies that $P_{k,n}$ has real coefficients. Furthermore Proposition 1.14 is reformulated as

$$\frac{a_{k,n+1}}{\mathrm{vol}(S^n)} \int_{S^n} P_{k,n}(\sigma \cdot \tau) H(\tau)\, d\tau = H(\sigma), \tag{rk}$$

showing that the Gengenbauer polynomials are reproducing kernels. A neat application of this formula is a formula for obtaining the kth spherical harmonic component of a function, $f \in L^2_{\mathbb{C}}(S^n)$.

Proposition 1.19 *For every function, $f \in L^2_{\mathbb{C}}\mathbb{C}(S^n)$, if $f = \sum_{k=0}^\infty f_k$ is the unique decomposition of f over the Hilbert sum $\bigoplus_{k=0}^\infty \mathcal{H}_k^{\mathbb{C}}(S^k)$, then f_k is given by*

$$f_k(\sigma) = \frac{a_{k,n+1}}{\mathrm{vol}(S^n)} \int_{S^n} f(\tau) P_{k,n}(\sigma \cdot \tau)\, d\tau,$$

for all $\sigma \in S^n$.

Proof. If we recall that $\mathcal{H}_k^{\mathbb{C}}(S^k)$ and $\mathcal{H}_j^{\mathbb{C}}(S^k)$ are orthogonal for all $j \neq k$, using the formula (rk), we have

$$\begin{aligned}
\frac{a_{k,n+1}}{\mathrm{vol}(S^n)} \int_{S^n} f(\tau) P_{k,n}(\sigma \cdot \tau)\, d\tau &= \frac{a_{k,n+1}}{\mathrm{vol}(S^n)} \int_{S^n} \sum_{j=0}^{\infty} f_j(\tau) P_{k,n}(\sigma \cdot \tau)\, d\tau \\
&= \frac{a_{k,n+1}}{\mathrm{vol}(S^n)} \sum_{j=0}^{\infty} \int_{S^n} f_j(\tau) P_{k,n}(\sigma \cdot \tau)\, d\tau \\
&= \frac{a_{k,n+1}}{\mathrm{vol}(S^n)} \int_{S^n} f_k(\tau) P_{k,n}(\sigma \cdot \tau)\, d\tau \\
&= f_k(\sigma),
\end{aligned}$$

as claimed. □

We know from the previous section that the kth zonal function generates $\mathcal{H}_k^{\mathbb{C}}(S^n)$. Here is an explicit way to prove this fact.

Proposition 1.20 *If $H_1, \ldots, H_m \in \mathcal{H}_k^{\mathbb{C}}(S^n)$ are linearly independent, then there are m points, $\sigma_1, \ldots, \sigma_m$, on S^n, so that the $m \times m$ matrix, $(H_j(\sigma_i))$, is invertible.*

Proof. We proceed by induction on m. The case $m = 1$ is trivial. For the induction step, we may assume that we found m points, $\sigma_1, \ldots, \sigma_m$, on S^n, so that the $m \times m$ matrix, $(H_j(\sigma_i))$, is invertible. Consider the function

$$\sigma \mapsto \begin{vmatrix} H_1(\sigma) & \ldots & H_m(\sigma) & H_{m+1}(\sigma) \\ H_1(\sigma_1) & \ldots & H_m(\sigma_1) & H_{m+1}(\sigma_1) \\ \vdots & \ddots & \vdots & \vdots \\ H_1(\sigma_m) & \ldots & H_m(\sigma_m) & H_{m+1}(\sigma_m). \end{vmatrix}$$

Since $H_1, \ldots, H_{m+1}$ are linearly independent, the above function does not vanish for all σ. Therefore, we can find σ_{m+1} so that the $(m+1) \times (m+1)$ matrix, $(H_j(\sigma_i))$, is invertible. □

We say that $a_{k,n+1}$ points, $\sigma_1, \ldots, \sigma_{a_{k,n+1}}$ on S^n form a *fundamental system* iff the $(m+1) \times (m+1)$ matrix, $(P_{n,k}(\sigma_i \cdot \sigma_j))$, is invertible.

Theorem 1.21 *The following properties hold:*

(i) There is a fundamental system, $\sigma_1, \ldots, \sigma_{a_{k,n+1}}$, for every $k \geq 1$.

(ii) Every spherical harmonic, $H \in \mathcal{H}_k^{\mathbb{C}}(S^n)$, can be written as

$$H(\sigma) = \sum_{j=1}^{a_{k,n+1}} c_j\, P_{k,n}(\sigma_j \cdot \sigma),$$

for some unique $c_j \in \mathbb{C}$.

Proof. (i) By the addition formula,

$$P_{k,n}(\sigma_i \cdot \sigma_j) = \frac{\mathrm{vol}(S^n)}{a_{k,n+1}} \sum_{l=1}^{a_{k,n+1}} Y_k^l(\sigma_i)\overline{Y_k^l(\sigma_j)}$$

for any orthonormal basis, $(Y_k^1, \ldots, Y_k^{a_{k,n+1}})$. It follows that the matrix $(P_{k,n}(\sigma_i \cdot \sigma_j))$ can be written as

$$(P_{k,n}(\sigma_i \cdot \sigma_j)) = \frac{\mathrm{vol}(S^n)}{a_{k,n+1}} YY^*,$$

where $Y = (Y_k^l(\sigma_i))$, and by Proposition 1.18, we can find $\sigma_1, \ldots, \sigma_{a_{k,n+1}} \in S^n$ so that Y and thus also Y^* are invertible and so, $(P_{n,k}(\sigma_i \cdot \sigma_j))$ is invertible.

(ii) Again, by the addition formula,

$$P_{k,n}(\sigma \cdot \sigma_j) = \frac{\mathrm{vol}(S^n)}{a_{k,n+1}} \sum_{i=1}^{a_{k,n+1}} Y_k^i(\sigma)\overline{Y_k^i(\sigma_j)}.$$

However, as $(Y_k^1, \ldots, Y_k^{a_{k,n+1}})$ is an orthonormal basis, (i) proved that the matrix Y^* is invertible so the $Y_k^i(\sigma)$ can be expressed uniquely in terms of the $P_{k,n}(\sigma \cdot \sigma_j)$, as claimed. $\square$

A neat geometric characterization of the zonal spherical functions is given in Stein and Weiss [26]. For this, we need to define the notion of a parallel on S^n. A *parallel of S^n orthogonal to a point $\tau \in S^n$* is the intersection of S^n with any (affine) hyperplane orthogonal to the line through the center of S^n and τ. Clearly, any rotation, R, leaving τ fixed leaves every parallel orthogonal to τ globally invariant and for any two points, σ_1 and σ_2, on such a parallel there is a rotation leaving τ fixed that maps σ_1 to σ_2. Consequently, the zonal function, Z_k^τ, defined by τ is constant on the parallels orthogonal to τ. In fact, this property characterizes zonal harmonics, up to a constant.

The theorem below is proved in Stein and Weiss [26] (Chapter 4, Theorem 2.12). The proof uses Proposition 1.16 and it is very similar to the proof of Theorem 1.17 so, to save space, it is omitted.

Theorem 1.22 *Fix any point, $\tau \in S^n$. A spherical harmonic, $Y \in \mathcal{H}_k^{\mathbb{C}}(S^n)$, is constant on parallels orthogonal to τ iff $Y = cZ_k^\tau$, for some constant, $c \in \mathbb{C}$.*

In the next section, we show how the Gegenbauer polynomials can actually be computed.

1.7 More on the Gegenbauer Polynomials

The Gegenbauer polynomials are characterized by a formula generalizing the Rodrigues formula defining the Legendre polynomials (see Section 1.2). The expression

$$\left(k + \frac{n-2}{2}\right)\left(k - 1 + \frac{n-2}{2}\right) \cdots \left(1 + \frac{n-2}{2}\right)$$

can be expressed in terms of the Γ function as

$$\frac{\Gamma\left(k+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}.$$

Recall that the Γ function is a generalization of factorial that satisfies the equation

$$\Gamma(z+1) = z\Gamma(z).$$

For $z = x + iy$ with $x > 0$, $\Gamma(z)$ is given by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt,$$

where the integral converges absolutely. If n is an integer $n \geq 0$, then $\Gamma(n+1) = n!$.

It is proved in Morimoto [22] (Chapter 2, Theorem 2.35) that

Proposition 1.23 *The Gegenbauer polynomial, $P_{k,n}$, is given by Rodrigues' formula:*

$$P_{k,n}(t) = \frac{(-1)^k}{2^k}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(k+\frac{n}{2}\right)}\frac{1}{(1-t^2)^{\frac{n-2}{2}}}\frac{d^k}{dt^k}(1-t^2)^{k+\frac{n-2}{2}},$$

with $n \geq 2$.

The Gegenbauer polynomials satisfy the following orthogonality properties with respect to the kernel $(1-t^2)^{\frac{n-2}{2}}$ (see Morimoto [22] (Chapter 2, Theorem 2.34):

Proposition 1.24 *The Gegenbauer polynomial, $P_{k,n}$, have the following properties:*

$$\begin{aligned}\int_{-1}^{-1}(P_{k,n}(t))^2(1-t^2)^{\frac{n-2}{2}}\,dt &= \frac{\mathrm{vol}(S^n)}{a_{k,n+1}\mathrm{vol}(S^{n-1})}\\ \int_{-1}^{-1}P_{k,n}(t)P_{l,n}(t)(1-t^2)^{\frac{n-2}{2}}\,dt &= 0, \qquad k \neq l.\end{aligned}$$

The Gegenbauer polynomials satisfy a second-order differential equation generalizing the Legendre equation from Section 1.2.

Proposition 1.25 *The Gegenbauer polynomial, $P_{k,n}$, are solutions of the differential equation*

$$(1-t^2)P''_{k,n}(t) - ntP'_{k,n}(t) + k(k+n-1)P_{k,n}(t) = 0.$$

Proof. For a fixed τ, the function H given by $H(\sigma) = P_{k,n}(\sigma \cdot \tau) = P_{k,n}(\cos\theta)$, belongs to $\mathcal{H}_k^{\mathbb{C}}(S^n)$, so

$$\Delta_{S^n} H = -k(k+n-1)H.$$

Recall from Section 1.3 that

$$\Delta_{S^n} f = \frac{1}{\sin^{n-1}\theta}\frac{\partial}{\partial\theta}\left(\sin^{n-1}\theta\,\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\Delta_{S_{n-1}} f,$$

in the local coordinates where

$$\sigma = \sin\theta\,\widetilde{\sigma} + \cos\theta\, e_{n+1},$$

with $\widetilde{\sigma} \in S^{n-1}$ and $0 \leq \theta < \pi$. If we make the change of variable $t = \cos\theta$, then it is easy to see that the above formula becomes

$$\Delta_{S^n} f = (1-t^2)\frac{\partial^2 f}{\partial t^2} - nt\frac{\partial f}{\partial t} + \frac{1}{1-t^2}\Delta_{S^{n-1}}$$

(see Morimoto [22], Chapter 2, Theorem 2.9.) But, H being zonal, it only depends on θ, that is, on t, so $\Delta_{S^{n-1}} H = 0$ and thus,

$$-k(k+n-1)P_{k,n}(t) = \Delta_{S^n} P_{k,n}(t) = (1-t^2)\frac{\partial^2 P_{k,n}}{\partial t^2} - nt\frac{\partial P_{k,n}}{\partial t},$$

which yields our equation. □

Note that for $n = 2$, the differential equation of Proposition 1.25 is the Legendre equation from Section 1.2.

The Gegenbauer poynomials also appear as coefficients in some simple generating functions. The following proposition is proved in Morimoto [22] (Chapter 2, Theorem 2.53 and Theorem 2.55):

Proposition 1.26 *For all r and t such that $-1 < r < 1$ and $-1 \leq t \leq 1$, for all $n \geq 1$, we have the following generating formula:*

$$\sum_{k=0}^{\infty} a_{k,n+1}\, r^k P_{k,n}(t) = \frac{1-r^2}{(1-2rt+r^2)^{\frac{n+1}{2}}}.$$

Furthermore, for all r and t such that $0 \leq r < 1$ and $-1 \leq t \leq 1$, if $n = 1$, then

$$\sum_{k=1}^{\infty} \frac{r^k}{k}\, P_{k,1}(t) = -\frac{1}{2}\log(1-2rt+r^2)$$

and if $n \geq 2$, then

$$\sum_{k=0}^{\infty} \frac{n-1}{2k+n-1}\, a_{k,n+1}\, r^k P_{k,n}(t) = \frac{1}{(1-2rt+r^2)^{\frac{n-1}{2}}}.$$

In Stein and Weiss [26] (Chapter 4, Section 2), the polynomials, $P_k^\lambda(t)$, where $\lambda > 0$ are defined using the following generating formula:

$$\sum_{k=0}^{\infty} r^k P_k^\lambda(t) = \frac{1}{(1 - 2rt + r^2)^\lambda}.$$

Each polynomial, $P_k^\lambda(t)$, has degree k and is called an *ultraspherical polynomial of degree k associated with λ*. In view of Proposition 1.26, we see that

$$P_k^{\frac{n-1}{2}}(t) = \frac{n-1}{2k+n-1}\, a_{k,n+1}\, P_{k,n}(t),$$

as we mentionned ealier. There is also an integral formula for the Gegenbauer polynomials known as *Laplace representation*, see Morimoto [22] (Chapter 2, Theorem 2.52).

1.8 The Funk-Hecke Formula

The Funk-Hecke Formula (also known as Hecke-Funk Formula) basically allows one to perform a sort of convolution of a "kernel function" with a spherical function in a convenient way. Given a measurable function, K, on $[-1, 1]$ such that the integral

$$\int_{-1}^{1} |K(t)|(1 - t^2)^{\frac{n-2}{2}}\, dt$$

makes sense, given a function $f \in L^2_{\mathbb{C}}(S^n)$, we can view the expression

$$K \star f(\sigma) = \int_{S^n} K(\sigma \cdot \tau) f(\tau)\, d\tau$$

as a sort of *convolution* of K and f. Actually, the use of the term convolution is really unfortunate because in a "true" convolution, $f * g$, either the argument of f or the argument of g should be multiplied by the inverse of the variable of integration, which means that the integration should really be taking place over the group $\mathbf{SO}(n+1)$. We will come back to this point later. For the time being, let us call the expression $K \star f$ defined above a *pseudo-convolution*. Now, if f is expressed in terms of spherical harmonics as

$$f = \sum_{k=0}^{\infty} \sum_{m_k=1}^{a_{k,n+1}} c_{k,m_k} Y_k^{m_k},$$

then the Funk-Hecke Formula states that

$$K \star Y_k^{m_k}(\sigma) = \alpha_k Y_k^{m_k}(\sigma),$$

for some fixed constant, α_k, and so

$$K \star f = \sum_{k=0}^{\infty} \sum_{m_k=1}^{a_{k,n+1}} \alpha_k c_{k,m_k} Y_k^{m_k}.$$

Thus, if the constants, α_k are known, then it is "cheap" to compute the pseudo-convolution $K \star f$.

This method was used in a ground-breaking paper by Basri and Jacobs [3] to compute the reflectance function, r, from the lighting function, ℓ, as a pseudo-convolution $K \star \ell$ (over S^2) with the *Lambertian kernel*, K, given by

$$K(\sigma \cdot \tau) = \max(\sigma \cdot \tau, 0).$$

Below, we give a proof of the Funk-Hecke formula due to Morimoto [22] (Chapter 2, Theorem 2.39) but see also Andrews, Askey and Roy [1] (Chapter 9). This formula was first published by Funk in 1916 and then by Hecke in 1918.

Theorem 1.27 *(Funk-Hecke Formula) Given any measurable function, K, on $[-1,1]$ such that the integral*

$$\int_{-1}^{1} |K(t)|(1-t^2)^{\frac{n-2}{2}}\, dt$$

makes sense, for every function, $H \in \mathcal{H}_k^{\mathbb{C}}(S^n)$, we have

$$\int_{S^n} K(\sigma \cdot \xi) H(\xi)\, d\xi = \left(\mathrm{vol}(S_{n-1}) \int_{-1}^{1} K(t) P_{k,n}(t)(1-t^2)^{\frac{n-2}{2}}\, dt\right) H(\sigma).$$

Observe that when $n = 2$, the term $(1-t^2)^{\frac{n-2}{2}}$ is missing and we are simply requiring that $\int_{-1}^{1} |K(t)|\, dt$ makes sense.

Proof. We first prove the formula in the case where H is a zonal harmonic and then use the fact that the $P_{k,n}$'s are reproducing kernels (formula (rk)).

For all $\sigma, \tau \in S^n$ define H by

$$H(\sigma) = P_{k,n}(\sigma \cdot \tau)$$

and F by

$$F(\sigma, \tau) = \int_{S^n} K(\sigma \cdot \xi) P_{k,n}(\xi \cdot \tau)\, d\xi.$$

Since the volume form on the sphere is invariant under orientation-preserving isometries, for every $R \in \mathbf{SO}(n+1)$, we have

$$F(R\sigma, R\tau) = F(\sigma, \tau).$$

On the other hand, for σ fixed, it is not hard to see that as a function in τ, the function $F(\sigma, -)$ is a spherical harmonic, because $P_{k,n}$ satisfies a differential equation that implies that $\Delta_{S^2} F(\sigma, -) = -k(k+n-1)F(\sigma, -)$. Now, for every rotation, R, that fixes σ,

$$F(\sigma, \tau) = F(R\sigma, R\tau) = F(\sigma, R\tau),$$

which means that $F(\sigma, -)$ satisfies condition (2) of Theorem 1.17. By Theorem 1.17, we get

$$F(\sigma, \tau) = F(\sigma, \sigma) P_{k,n}(\sigma \cdot \tau).$$

If we use local coordinates on S^n where

$$\sigma = \sqrt{1-t^2}\,\widetilde{\sigma} + t\, e_{n+1},$$

with $\widetilde{\sigma} \in S^{n-1}$ and $-1 \le t \le 1$, it is not hard to show that the volume form on S^n is given by

$$d\sigma_{S^n} = (1-t^2)^{\frac{n-2}{2}}\, dt d\sigma_{S^{n-1}}.$$

Using this, we have

$$F(\sigma, \sigma) = \int_{S^n} K(\sigma \cdot \xi) P_{k,n}(\xi \cdot \sigma)\, d\xi = \mathrm{vol}(S^{n-1}) \int_{-1}^{1} K(t) P_{k,n}(t) (1-t^2)^{\frac{n-2}{2}}\, dt,$$

and thus,

$$F(\sigma, \tau) = \left(\mathrm{vol}(S^{n-1}) \int_{-1}^{1} K(t) P_{k,n}(t) (1-t^2)^{\frac{n-2}{2}}\, dt \right) P_{k,n}(\sigma \cdot \tau),$$

which is the Funk-Hecke formula when $H(\sigma) = P_{k,n}(\sigma \cdot \tau)$.

Let us now consider any function, $H \in \mathcal{H}_k^{\mathbb{C}}(S^n)$. Recall that by the reproducing kernel property (rk), we have

$$\frac{a_{k,n+1}}{\mathrm{vol}(S^n)} \int_{S^n} P_{k,n}(\xi \cdot \tau) H(\tau)\, d\tau = H(\xi).$$

Then, we can compute $\int_{S^n} K(\sigma \cdot \xi) H(\xi)\, d\xi$ using Fubini's Theorem and the Funk-Hecke formula in the special case where $H(\sigma) = P_{k,n}(\sigma \cdot \tau)$, as follows:

$$\begin{aligned}
&\int_{S^n} K(\sigma \cdot \xi) H(\xi)\, d\xi \\
&= \int_{S^n} K(\sigma \cdot \xi) \left(\frac{a_{k,n+1}}{\mathrm{vol}(S^n)} \int_{S^n} P_{k,n}(\xi \cdot \tau) H(\tau)\, d\tau \right) d\xi \\
&= \frac{a_{k,n+1}}{\mathrm{vol}(S^n)} \int_{S^n} H(\tau) \left(\int_{S^n} K(\sigma \cdot \xi) P_{k,n}(\xi \cdot \tau)\, d\xi \right) d\tau \\
&= \frac{a_{k,n+1}}{\mathrm{vol}(S^n)} \int_{S^n} H(\tau) \left(\left(\mathrm{vol}(S^{n-1}) \int_{-1}^{1} K(t) P_{k,n}(t)(1-t^2)^{\frac{n-2}{2}}\, dt \right) P_{k,n}(\sigma \cdot \tau) \right) d\tau \\
&= \left(\mathrm{vol}(S^{n-1}) \int_{-1}^{1} K(t) P_{k,n}(t)(1-t^2)^{\frac{n-2}{2}}\, dt \right) \left(\frac{a_{k,n+1}}{\mathrm{vol}(S^n)} \int_{S^n} P_{k,n}(\sigma \cdot \tau) H(\tau)\, d\tau \right) \\
&= \left(\mathrm{vol}(S^{n-1}) \int_{-1}^{1} K(t) P_{k,n}(t)(1-t^2)^{\frac{n-2}{2}}\, dt \right) H(\sigma),
\end{aligned}$$

which proves the Funk-Hecke formula in general. □

The Funk-Hecke formula can be used to derive an "addition theorem" for the ultraspherical polynomials (Gegenbauer polynomials). We omit this topic and we refer the interested reader to Andrews, Askey and Roy [1] (Chapter 9, Section 9.8).

Remark: Oddly, in their computation of $K \star \ell$, Basri and Jacobs [3] first expand K in terms of spherical harmonics as

$$K = \sum_{n=0}^{\infty} k_n Y_n^0,$$

and then use the Funk-Hecke formula to compute $K \star Y_n^m$ and they get (see page 222)

$$K \star Y_n^m = \alpha_n Y_n^m, \quad \text{with} \quad \alpha_n = \sqrt{\frac{4\pi}{2n+1}}\, k_n.$$

However, there is no need to expand K as the Funk-Hecke formula yields directly

$$K \star Y_n^m(\sigma) = \int_{S^2} K(\sigma \cdot \xi) Y_n^m(\xi)\, d\xi = \left(\int_{-1}^{1} K(t) P_n(t)\, dt \right) Y_n^m(\sigma),$$

where $P_n(t)$ is the standard Legendre polynomial of degree n since we are in the case of S^2. By the definition of K $(K(t) = \max(t, 0))$ and since $\mathrm{vol}(S^1) = 2\pi$, we get

$$K \star Y_n^m = \left(2\pi \int_0^1 t P_n(t)\, dt \right) Y_n^m,$$

which is equivalent to Basri and Jacobs' formula (14) since their α_n on page 222 is given by

$$\alpha_n = \sqrt{\frac{4\pi}{2n+1}}\, k_n,$$

but from page 230,

$$k_n = \sqrt{(2n+1)\pi} \int_0^1 t P_n(t)\, dt.$$

What remains to be done is to compute $\int_0^1 t P_n(t)\, dt$, which is done by using the Rodrigues Formula and integrating by parts (see Appendix A of Basri and Jacobs [3]).

1.9 Convolution on G/K, for a Gelfand Pair (G, K)

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