Surface Reconstruction from Triangular Meshes Using PPM

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CIS – Upenn

Joint work with

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DCT - UFMS

Dianna Xu

CS - Bryn Mawr

Problem Statement

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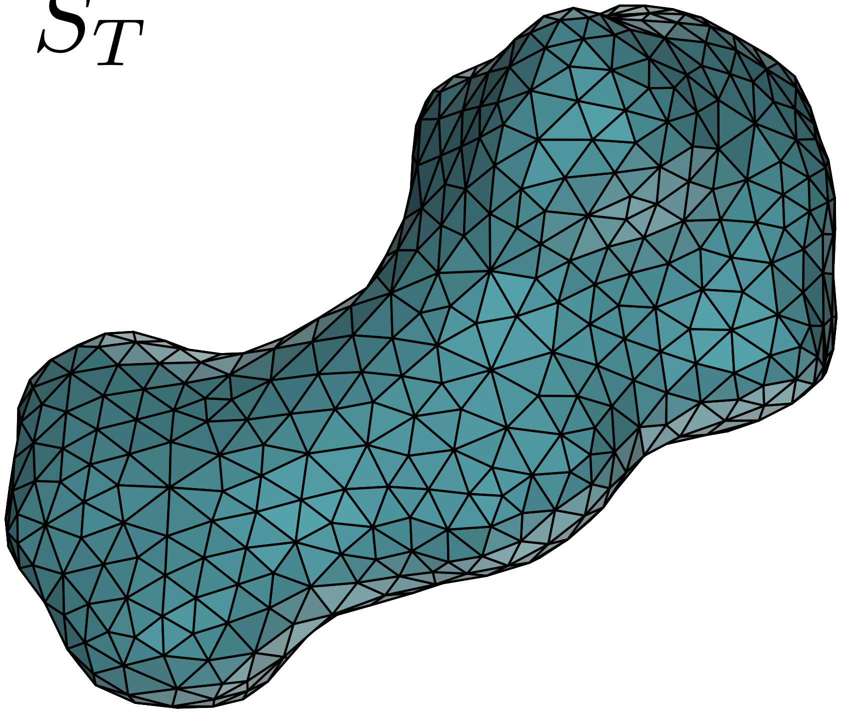
S_T



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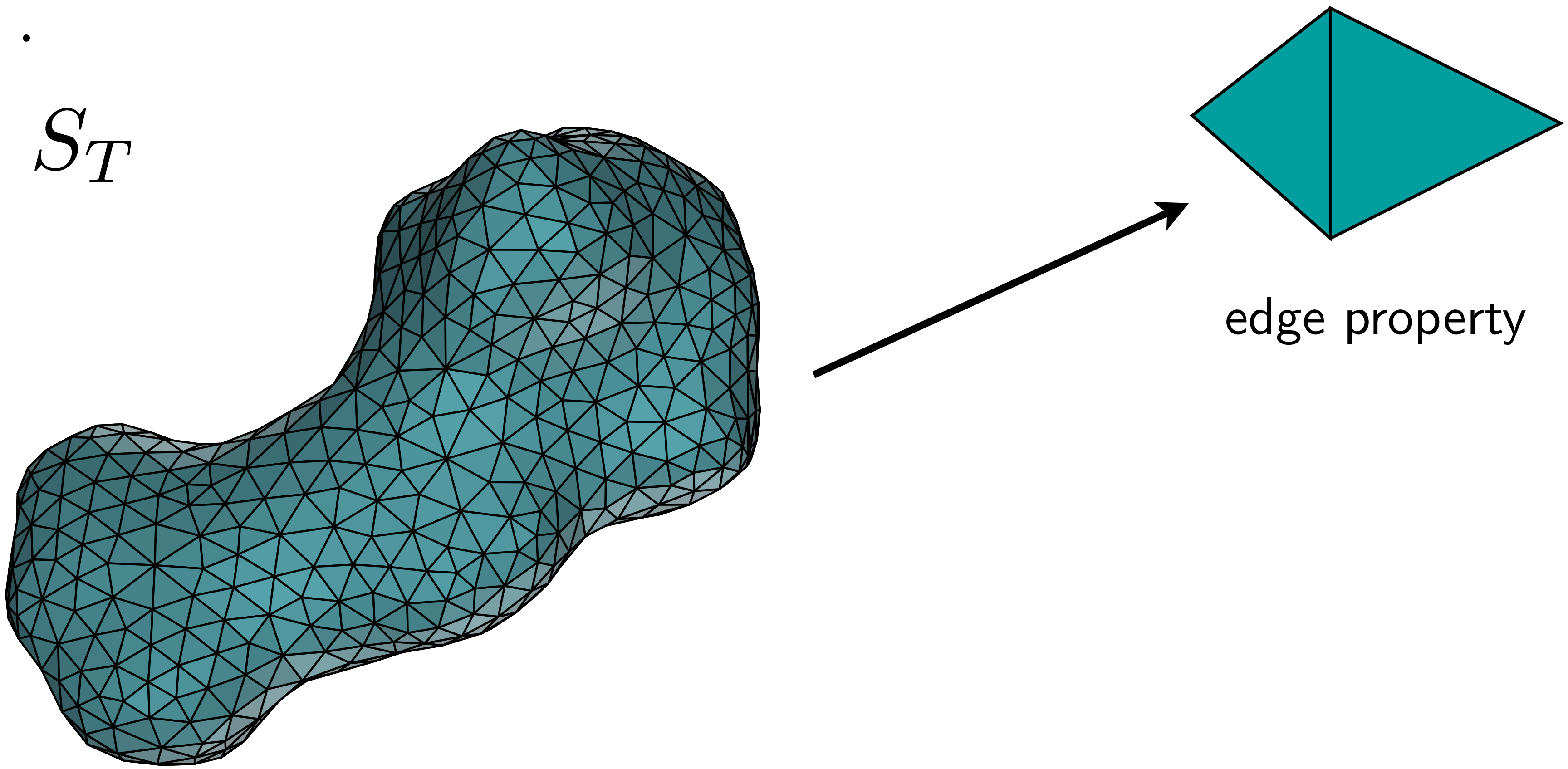

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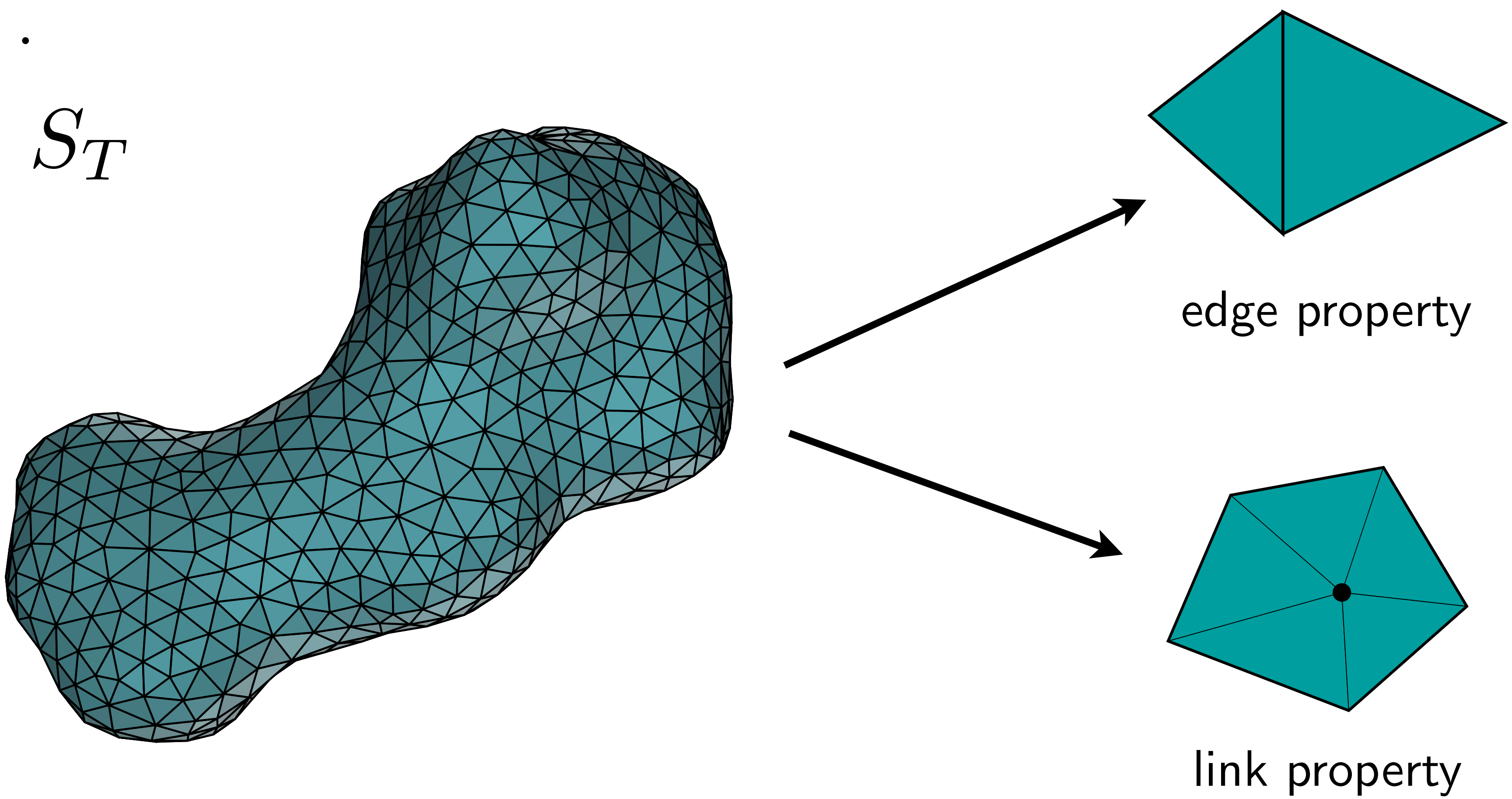

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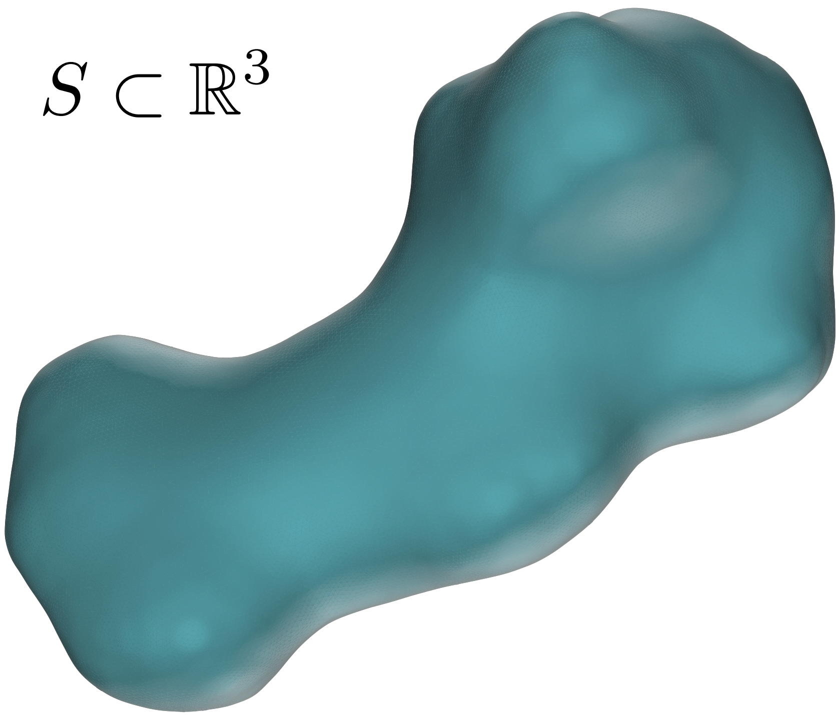

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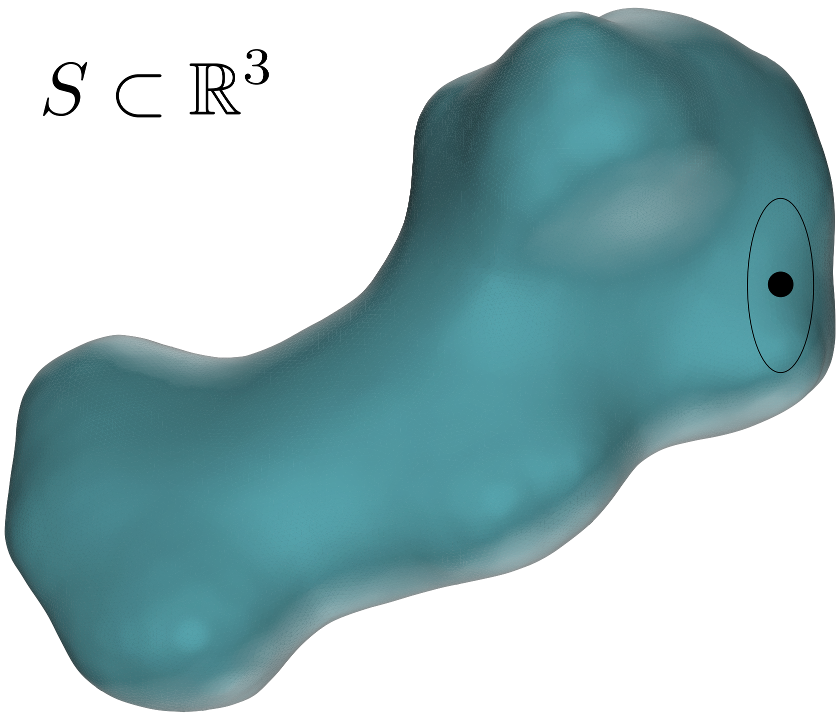

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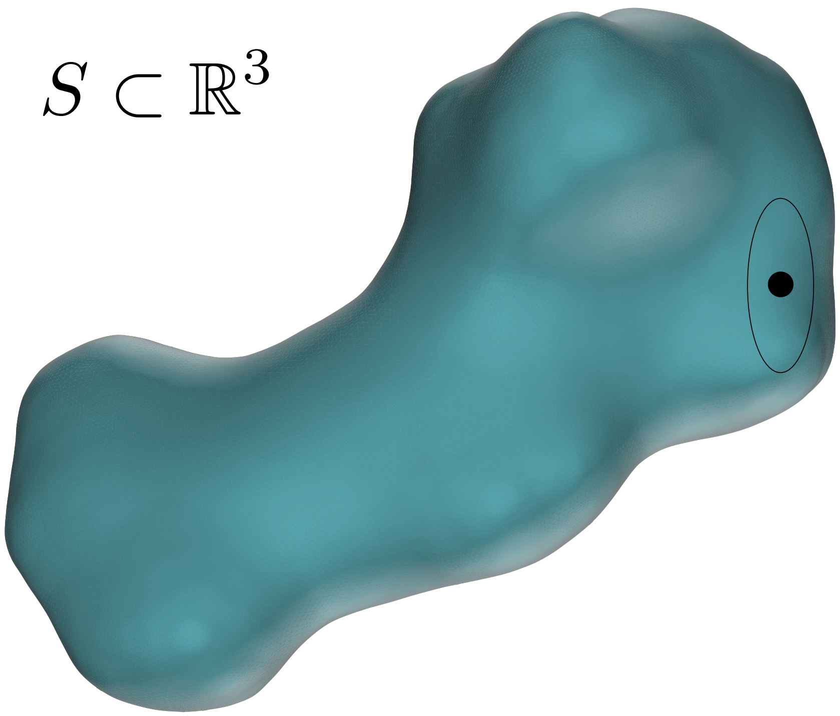

C^k diffeomorphism

$\longrightarrow$

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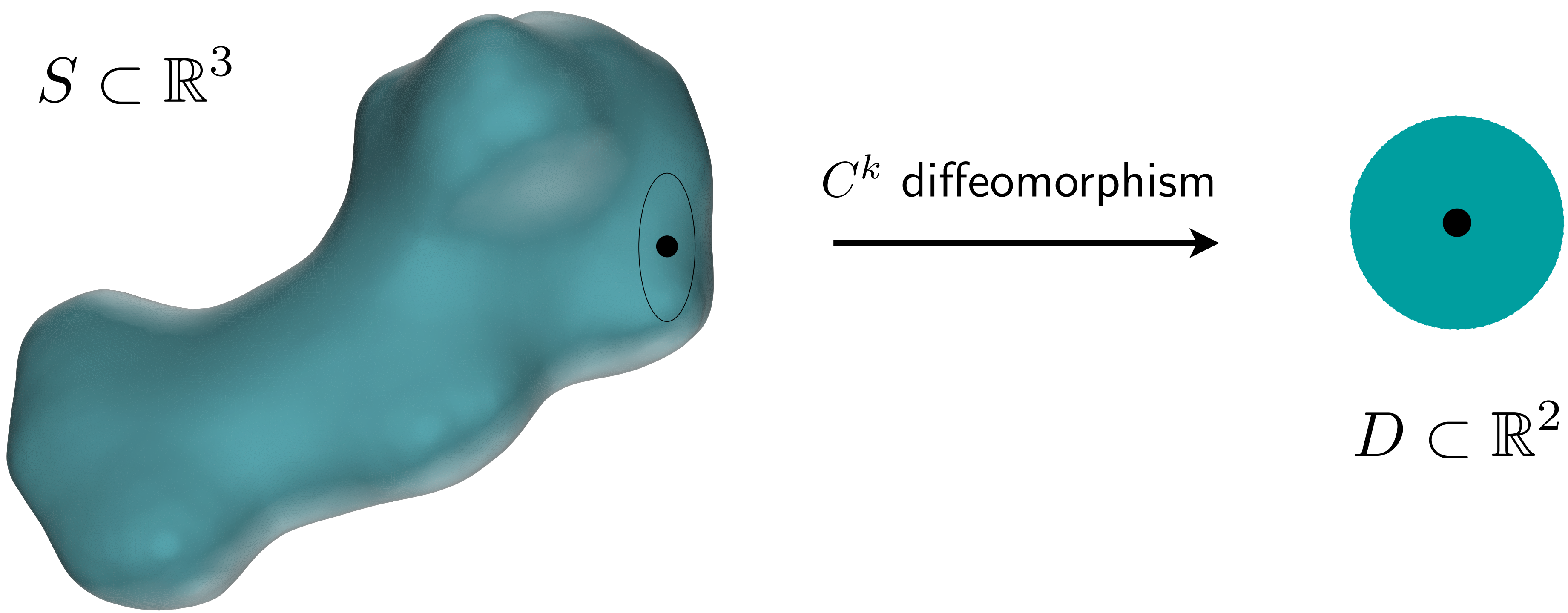

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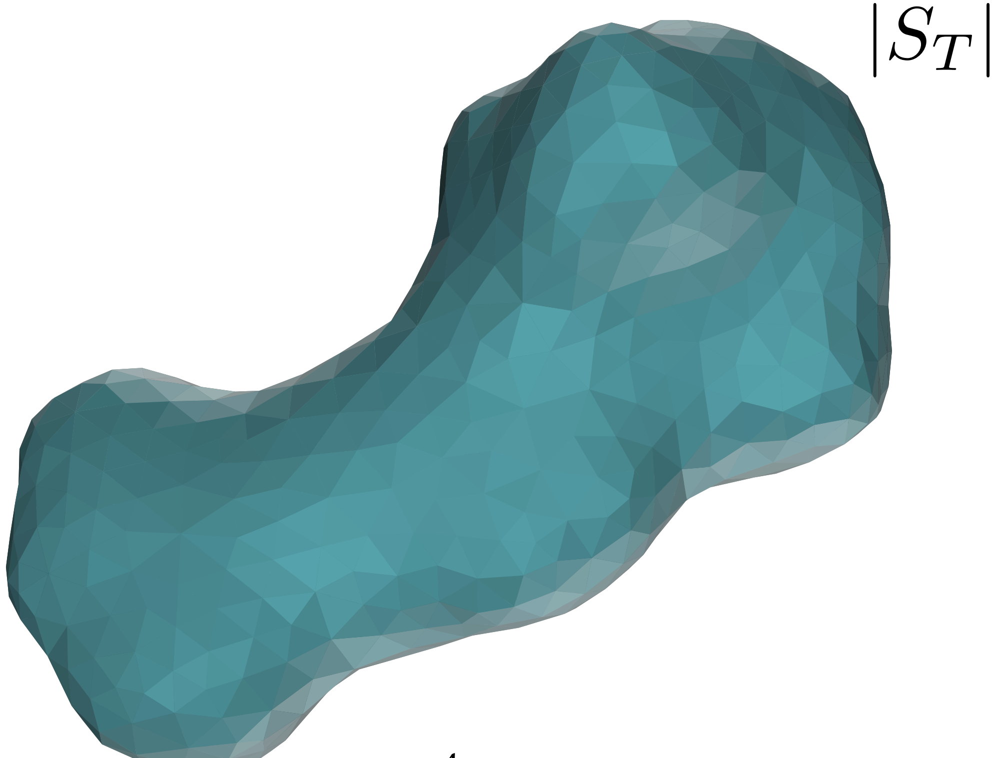
such that there exist a homeomorphism, $h : S \to |S_T|$, satisfying

$$\|h(p) - p\| \leq \epsilon\,,$$

for all $p \in S$.



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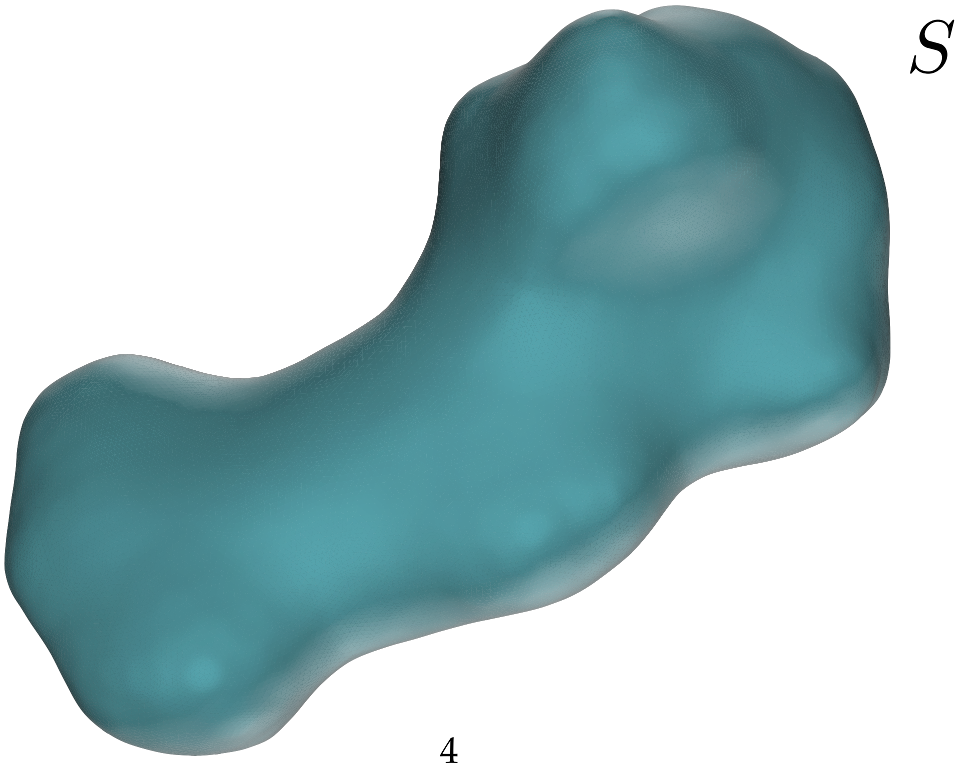

Related Work

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Reasonably well solved for $k = 1, 2$ but **not higher**.

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Let us take a look at the most common approaches...

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The most popular is the parametric surface approach.

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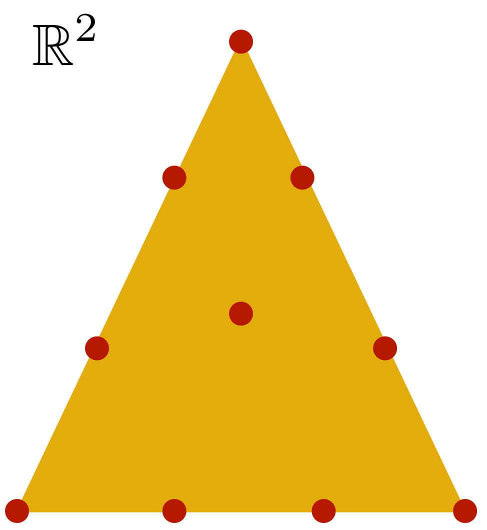
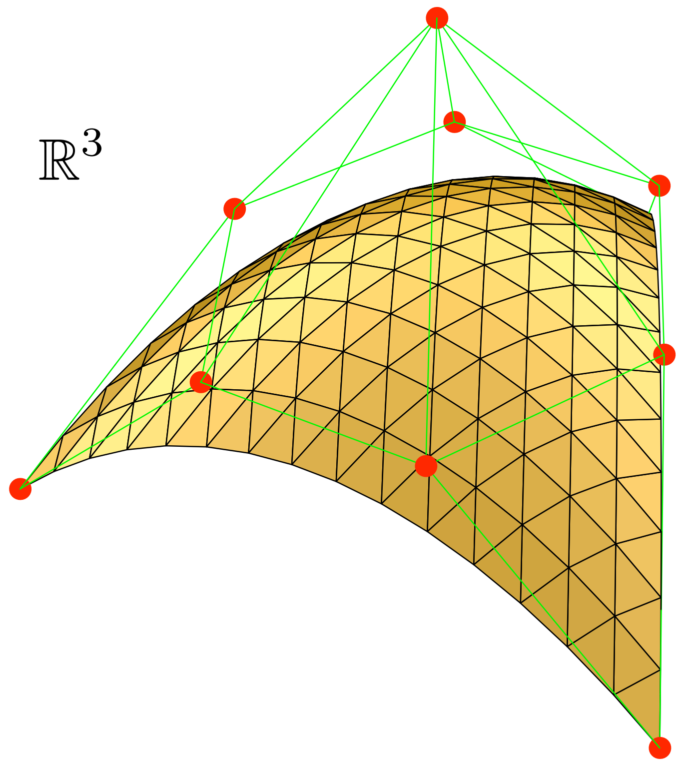
The idea is to assign a parametric surface patch with each triangle of S_T:





Related Work

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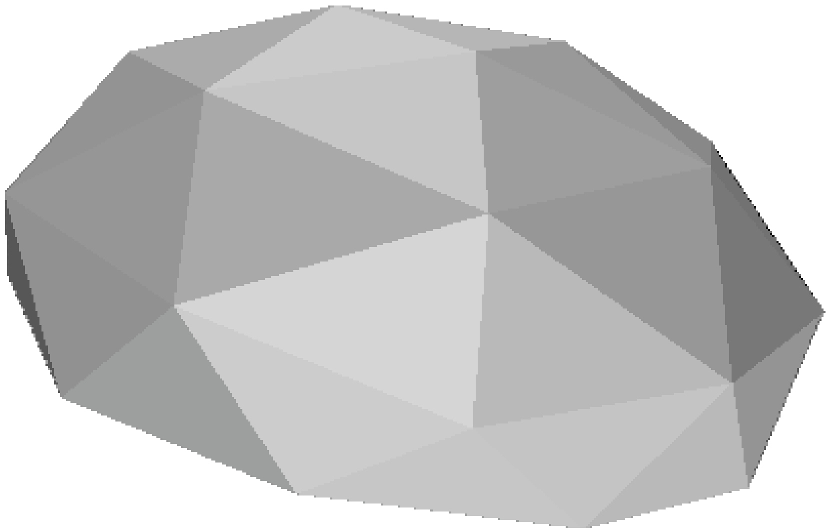
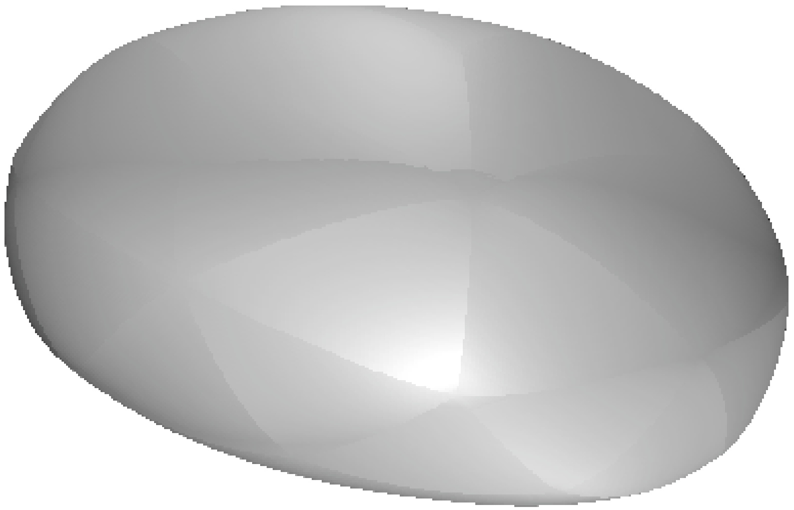
The patches are "stitched" together along their common vertices and edges.





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There are two main drawbacks with this approach:

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- To ensure continuity of order k, we need patches of order d, where d is a function of k and the value of d rapidly grows with k.

  Large values of d yield surfaces with poor visual quality. Also, the larger d is, the larger the number of control points, and the more difficult the placement of control points.

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- The placement of control points is constrained around vertices and along edges. So, local control of geometry is not very flexible.

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Some examples of C^k parametric approaches, for arbitrary k:

(Loop and DeRose, 1989), (Seidel, 1994), (Prautzsch, 1997), and (Reif, 1998).

Related Work

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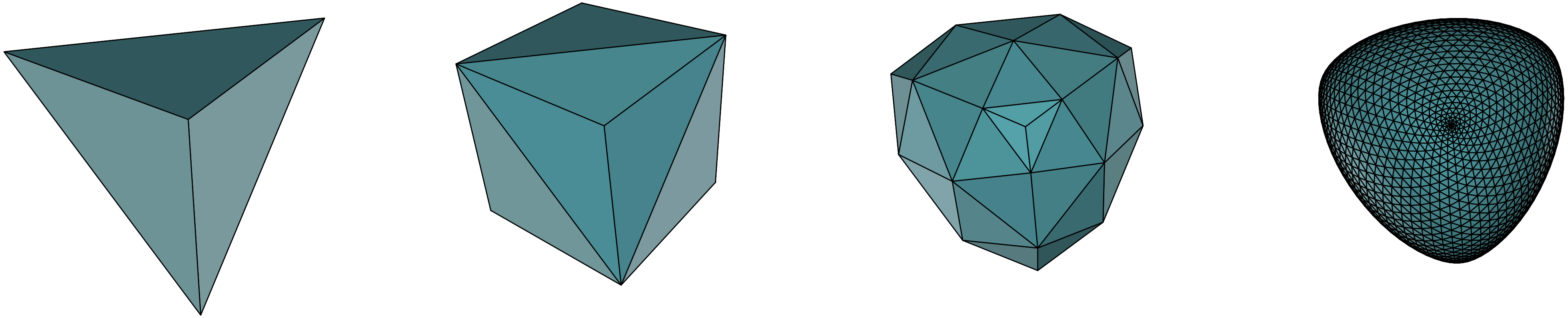
Another popular approach consists of using subdivision surfaces.



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For large values of k, the few existing schemes are rather complex.

See (Warren, 2002).

Related Work

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An often neglected approach, the **manifold-based** one, has the potential to easily produce C^k surfaces, for arbitrary k (including $k = \infty$).

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An often neglected approach, the **manifold-based** one, has the potential to easily produce C^k surfaces, for arbitrary k (including $k = \infty$).

The idea behind this approach is to build a surface from **open** parametric patches that overlap smoothly, as opposed to closed patches that stitch together along their common edges and vertices.

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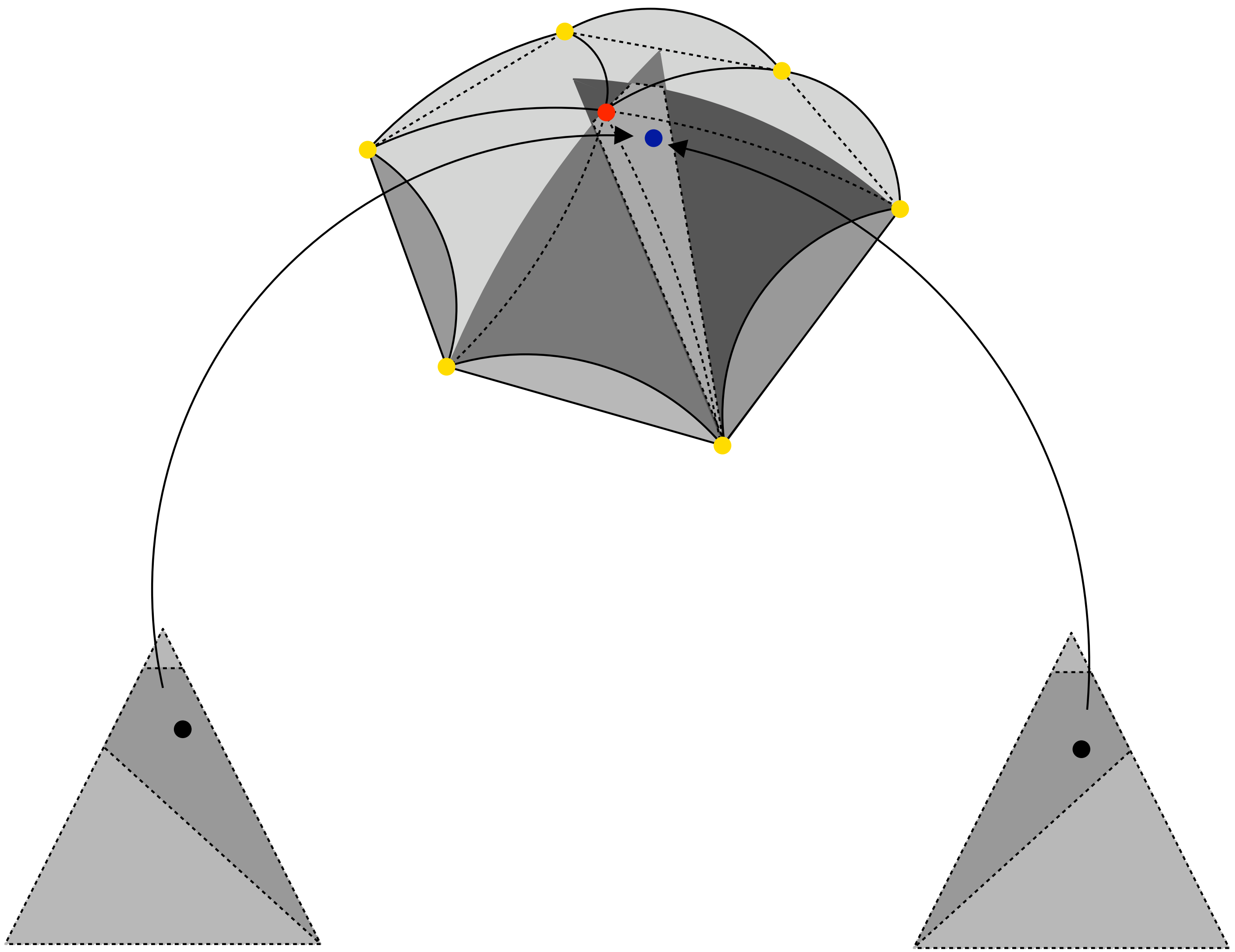
The manifold-based approach



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It does not yield a fully polynomial surface representation either.

Contributions

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We present a new manifold-based construction of C^k surfaces (including $k = \infty$).

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Our construction does not yield a fully polynomial surface, but it is guaranteed to produce an analytic representation of a truly C^k (including $k = \infty$) surface (i.e., with no singular points).

Gluing Data and PPS's

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Recall the definition of a manifold...

Gluing Data and PPS's

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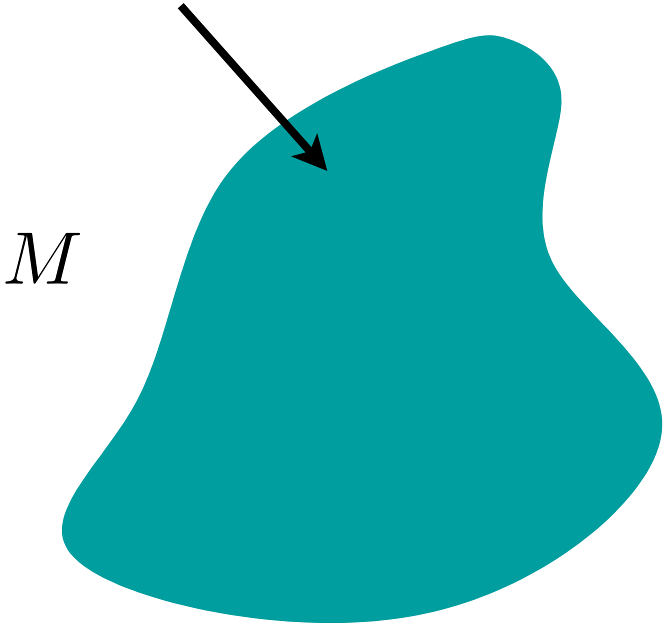
topological space



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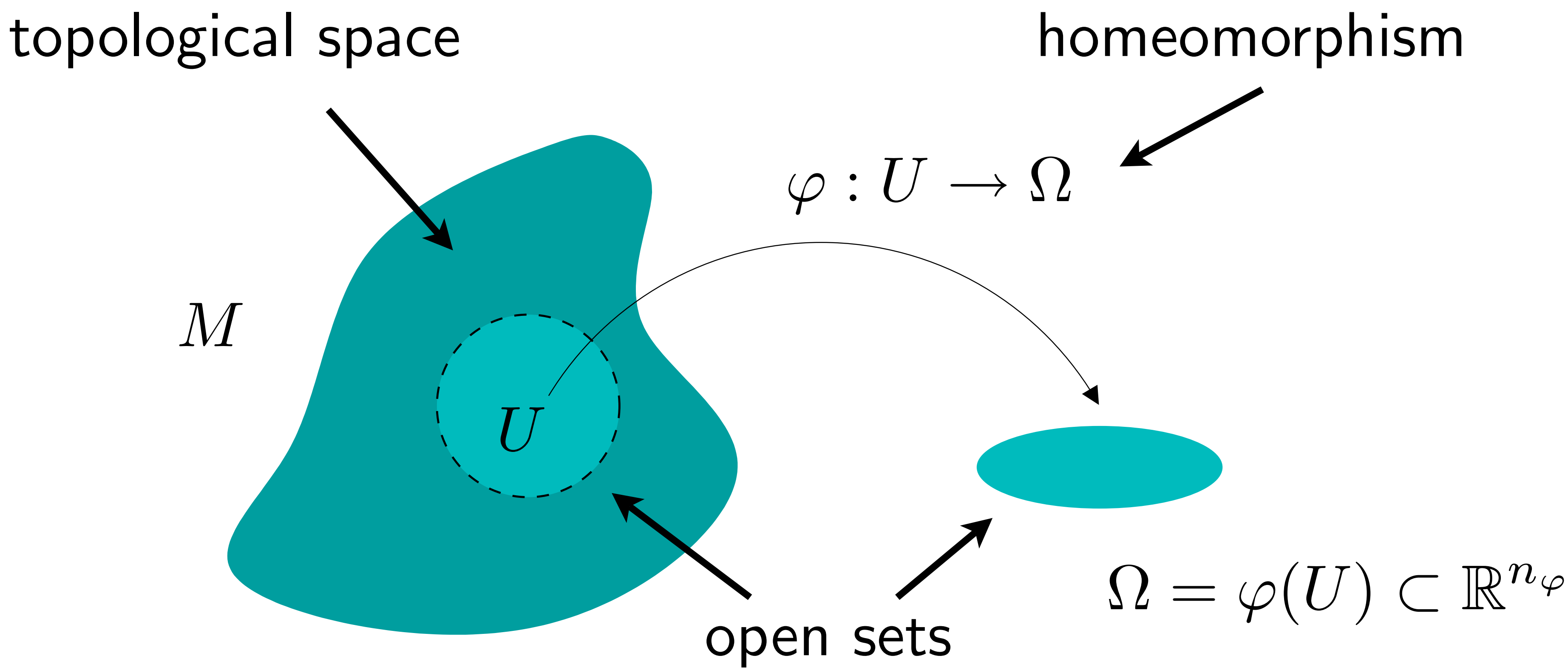

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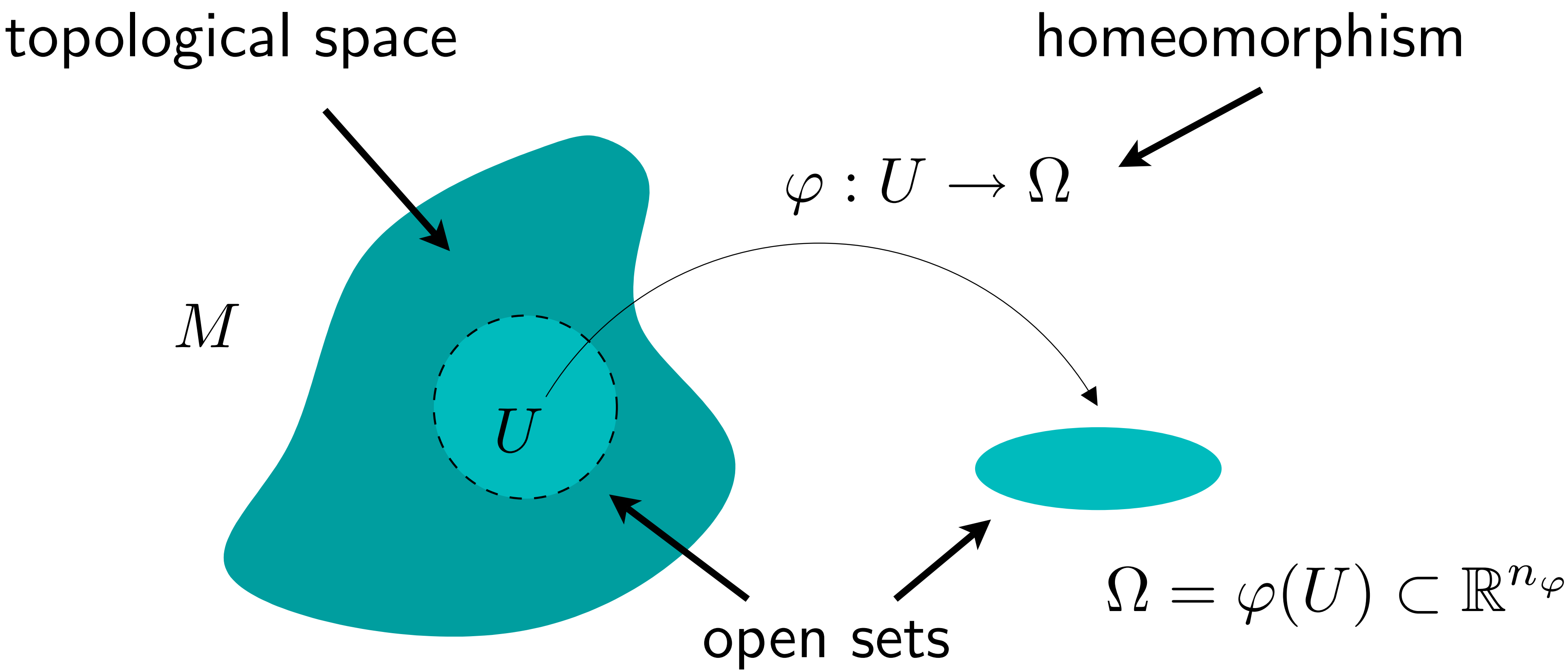

(U, φ) is called a **chart**.

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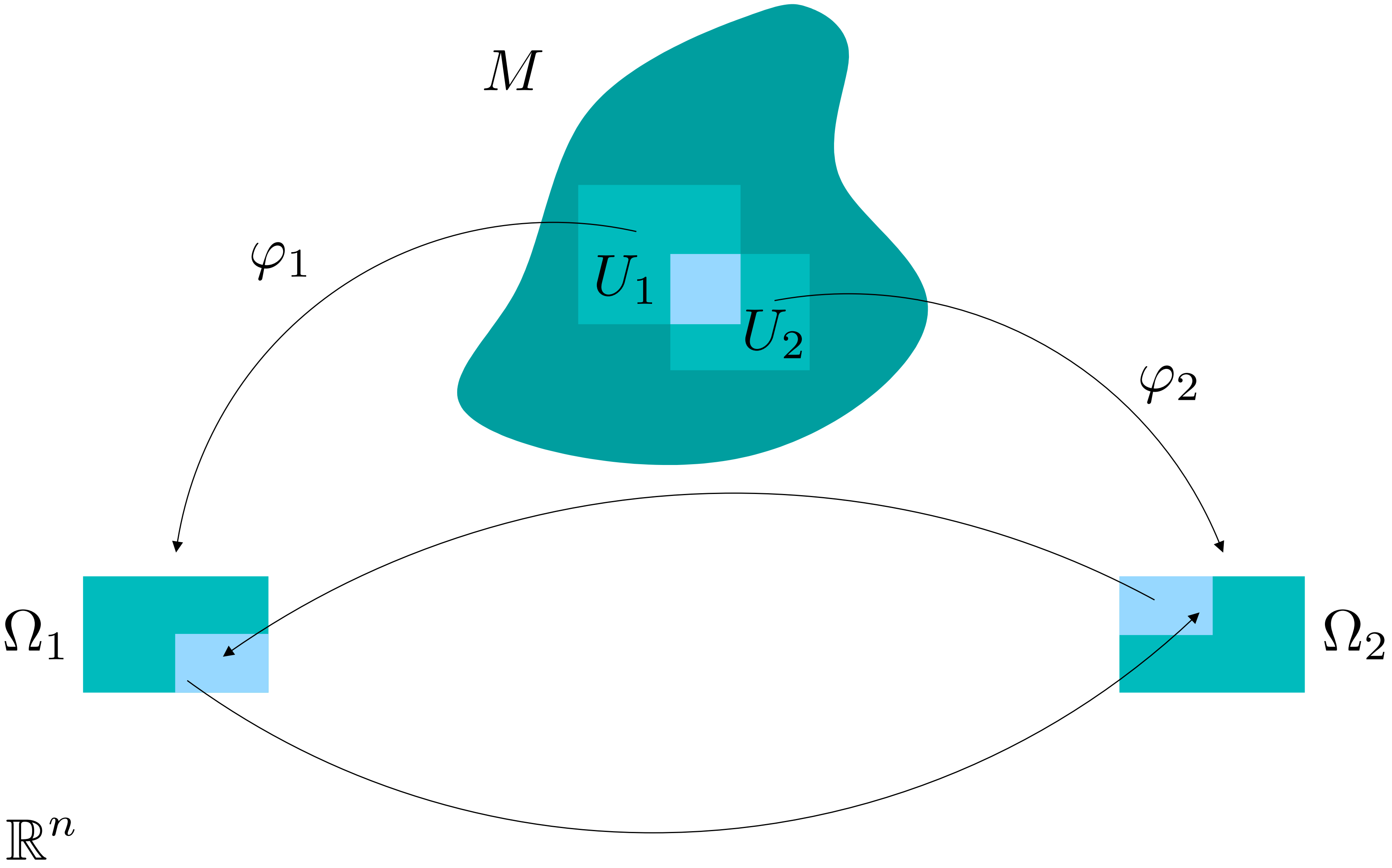

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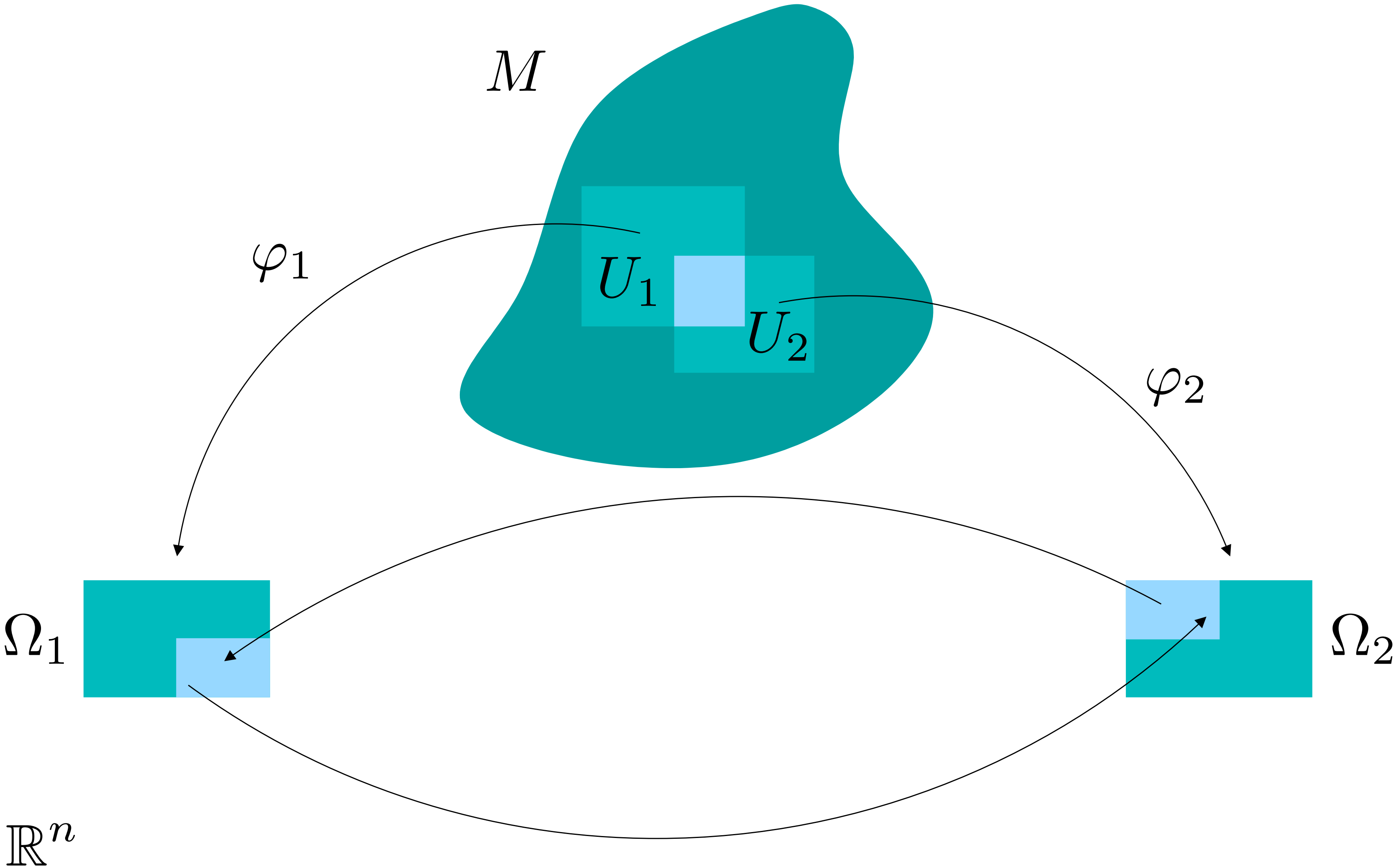

$$\varphi_{21} : \varphi_1(U_1 \cap U_2) \longrightarrow \varphi_2(U_1 \cap U_2)$$
$$\varphi_{12} : \varphi_2(U_1 \cap U_2) \longrightarrow \varphi_1(U_1 \cap U_2)$$

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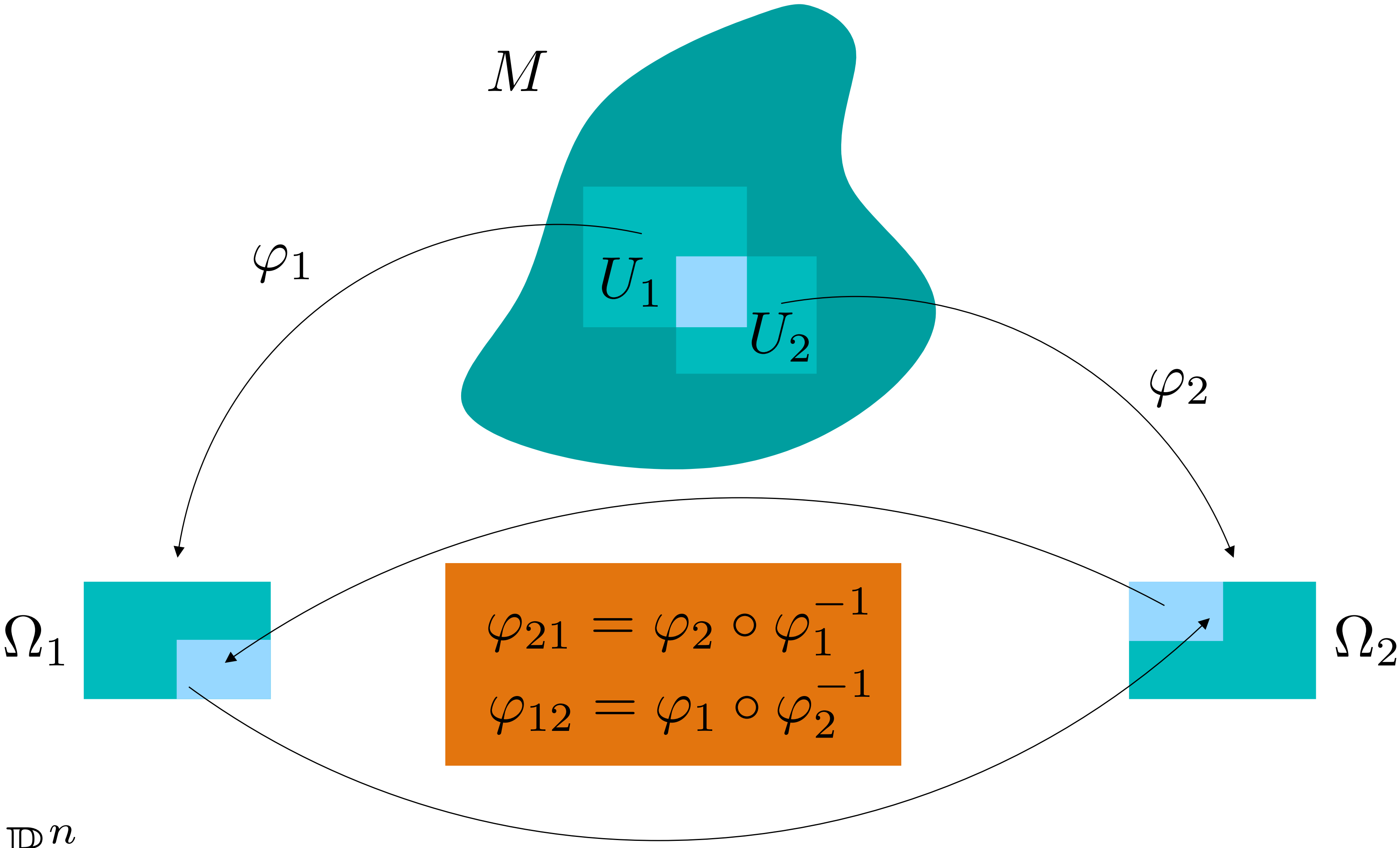

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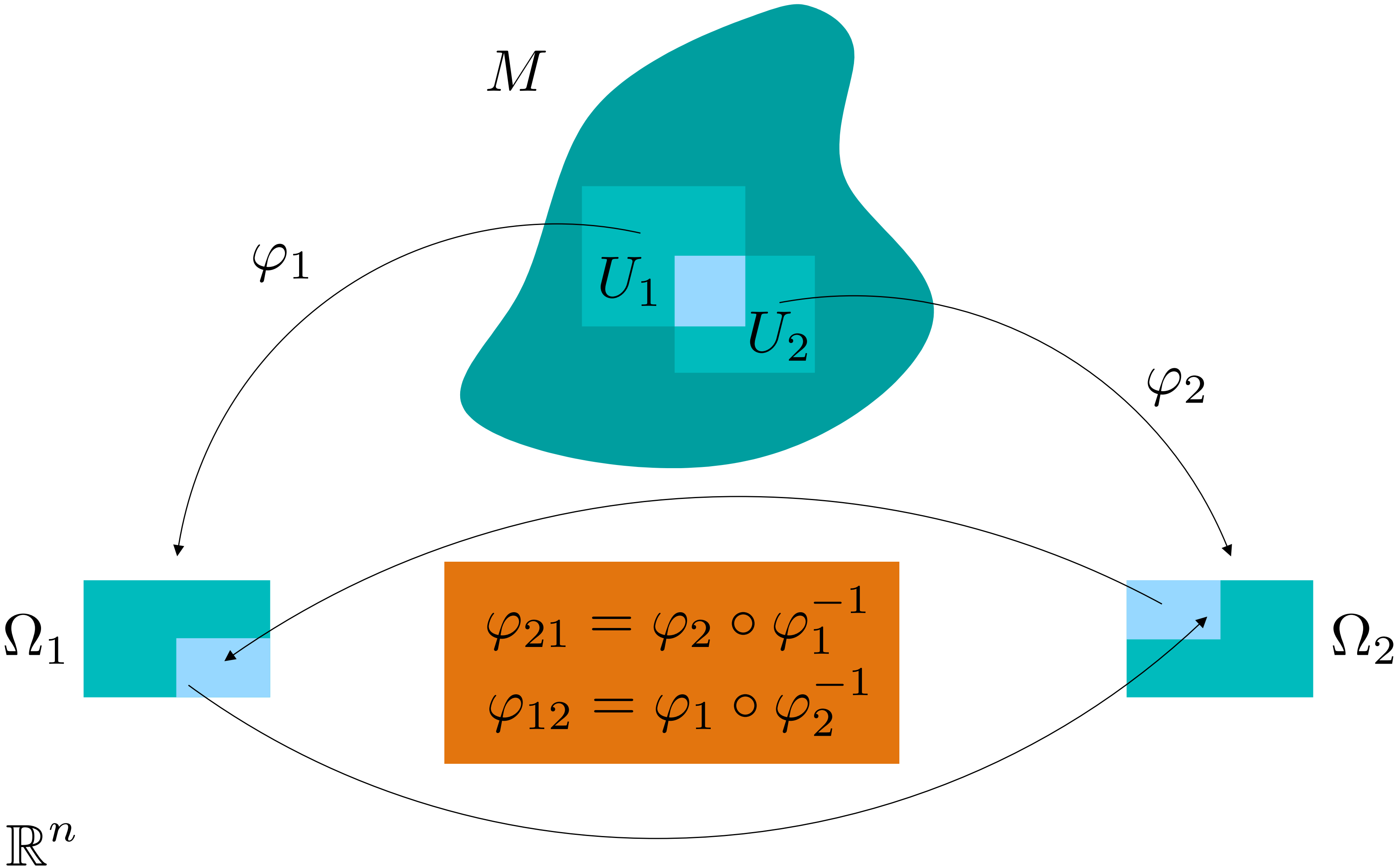

$$\varphi_{21} = \varphi_2 \circ \varphi_1^{-1}$$
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φ_{21} and φ_{12} are the **transition functions.**

Gluing Data and PPS's

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(2) $M = \displaystyle\bigcup_{i \in I} U_i$.

(3) Whenever $U_i \cap U_j \neq \emptyset$, the transition function φ_{ji} (resp. φ_{ij}) is a C^k diffeomorphism.

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Our plan is to define S constructively by **building a manifold**.

More specifically, we want to build a C^k 2-dimensional manifold in $\mathbb{R}^3$.

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A LITTLE PROBLEM:

Our definition of manifold is not constructive: it states what a manifold is by assuming it already exists! So, for our purposes, it is not useful!

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<u>THE KEY IDEA</u>:

The notion of a **set of gluing data**.

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A **set of gluing data** is a triple

$$\mathcal{G} = \big((\Omega_i)_{i \in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\varphi_{ji})_{(i,j) \in K \times K}\big)$$

satisfying the following properties, where I and K are countable sets, and I is non-empty:

Gluing Data and PPS's

Gluing Data and PPS's

(1) For every $i \in I$, the set Ω_i is a non-empty open subset of $\mathbb{R}^n$ called **parametrization domain**, for short, p-**domain**, and the Ω_i are pairwise disjoint (i.e., $\Omega_i \cap \Omega_j = \emptyset$ for all $i \neq j$).

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(2) For every pair $(i, j) \times I \times I$, the set Ω_{ij} is an open subset of Ω_i. Furthermore, $\Omega_{ii} = \Omega_i$ and $\Omega_{ji} \neq \emptyset$ if and only if $\Omega_{ij} \neq \emptyset$. Each non-empty Ω_{ij} (with $i \neq j$) is called **gluing domain**.

Gluing Data and PPS's

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(3) If we let

$$K = \{(i,j) \in I \times I \mid \Omega_{ij} \neq \emptyset\},$$

then $\varphi_{ji} \colon \Omega_{ij} \to \Omega_{ji}$ is a C^k bijection for every $(i,j) \in K$, called a **transition function** or **gluing function**.

Gluing Data

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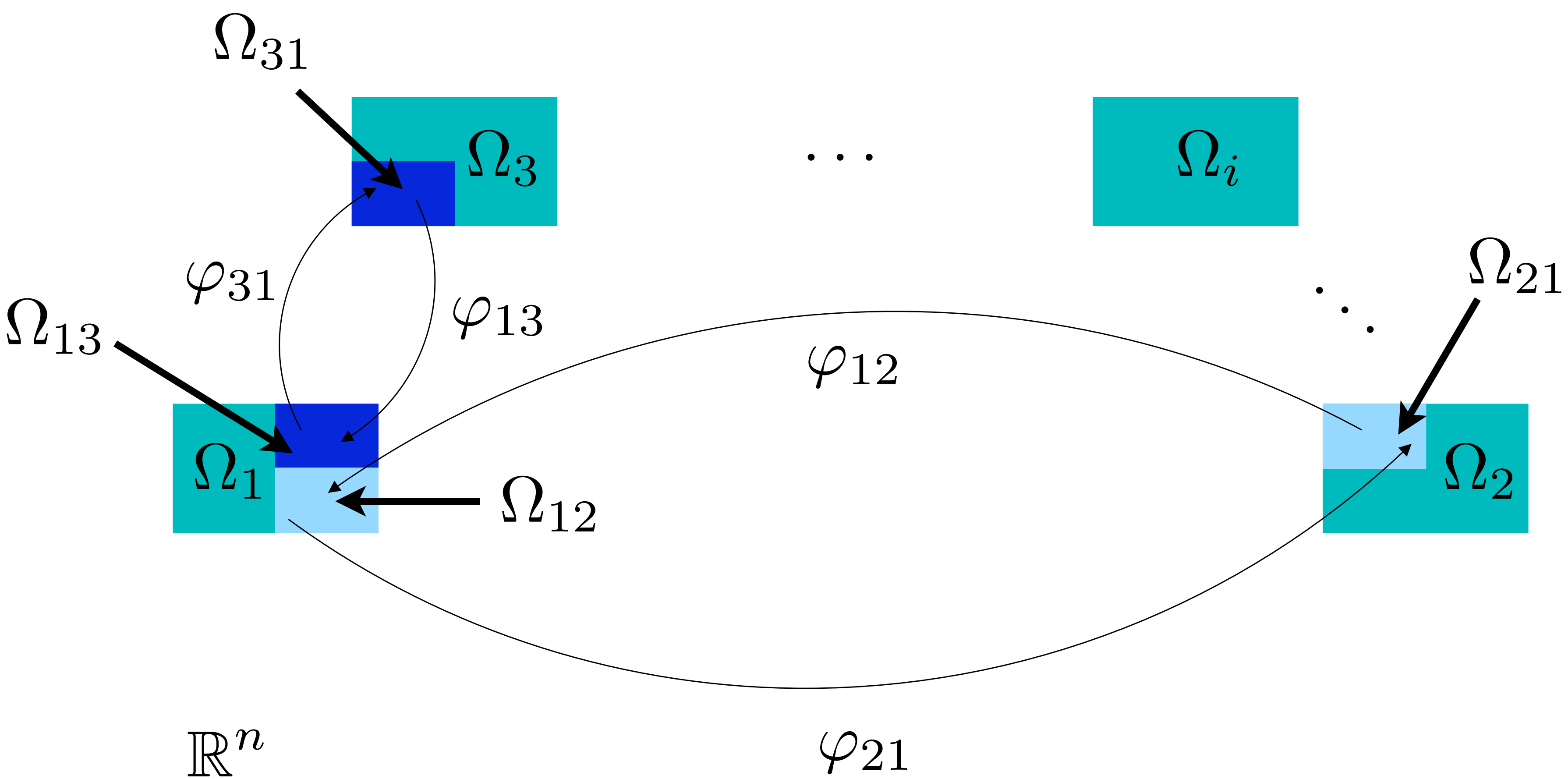

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(c) For all i, j, and k, if $\Omega_{ji} \cap \Omega_{jk} \neq \emptyset$ then $\varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk}) \subseteq \Omega_{ik}$ and $\varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x)$, for all $x \in \varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk})$.

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We form the quotient $M_{\mathcal{G}} = \Big(\coprod_i \Omega_i\Big) / \sim$

Gluing Data and PPS's

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Theorem 1. For every set of gluing data,

$$\mathcal{G} = \left((\Omega_i)_{i \in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\varphi_{ji})_{(i,j) \in K \times K} \right) ,$$

there is an n-dimensional C^k manifold, $M_{\mathcal{G}}$, whose transition functions are the $\varphi'_{ji}s$.

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there is an n-dimensional C^k manifold, $M_{\mathcal{G}}$, whose transition functions are the $\varphi'_{ji}s$.

A condition on the gluing data is needed to make sure that $M_{\mathcal{G}}$

is Hausdorff. Since it is quite technical, we will not show it here.

Gluing Data and PPS's

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Very nice, but...

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Very nice, but...

- our proof is not constructive;

- $M_{\mathcal{G}}$ is an "abstract entity", which may not be compact, orientable, etc.

So, the question that remains is **how** we can build a "concrete" manifold.

Gluing Data and PPS's

Gluing Data and PPS's

A **parametric** C^k **pseudo-manifold of dimension** n **in** $\mathbb{R}^m$ is a pair,

$$\mathcal{M} = (\mathcal{G}, (\theta_i)_{i \in I})$$

such that $\mathcal{G} = \big((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K}\big)$ is a set of gluing data, for some finite I, and each θ_i is a C^k function, $\theta_i : \Omega_i \rightarrow \mathbb{R}^m$, called a **parametrization** such that the following holds:

Gluing Data and PPS's

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(C) For all $(i, j) \in K$, we have $\theta_i = \theta_j \circ \varphi_{ji}$.

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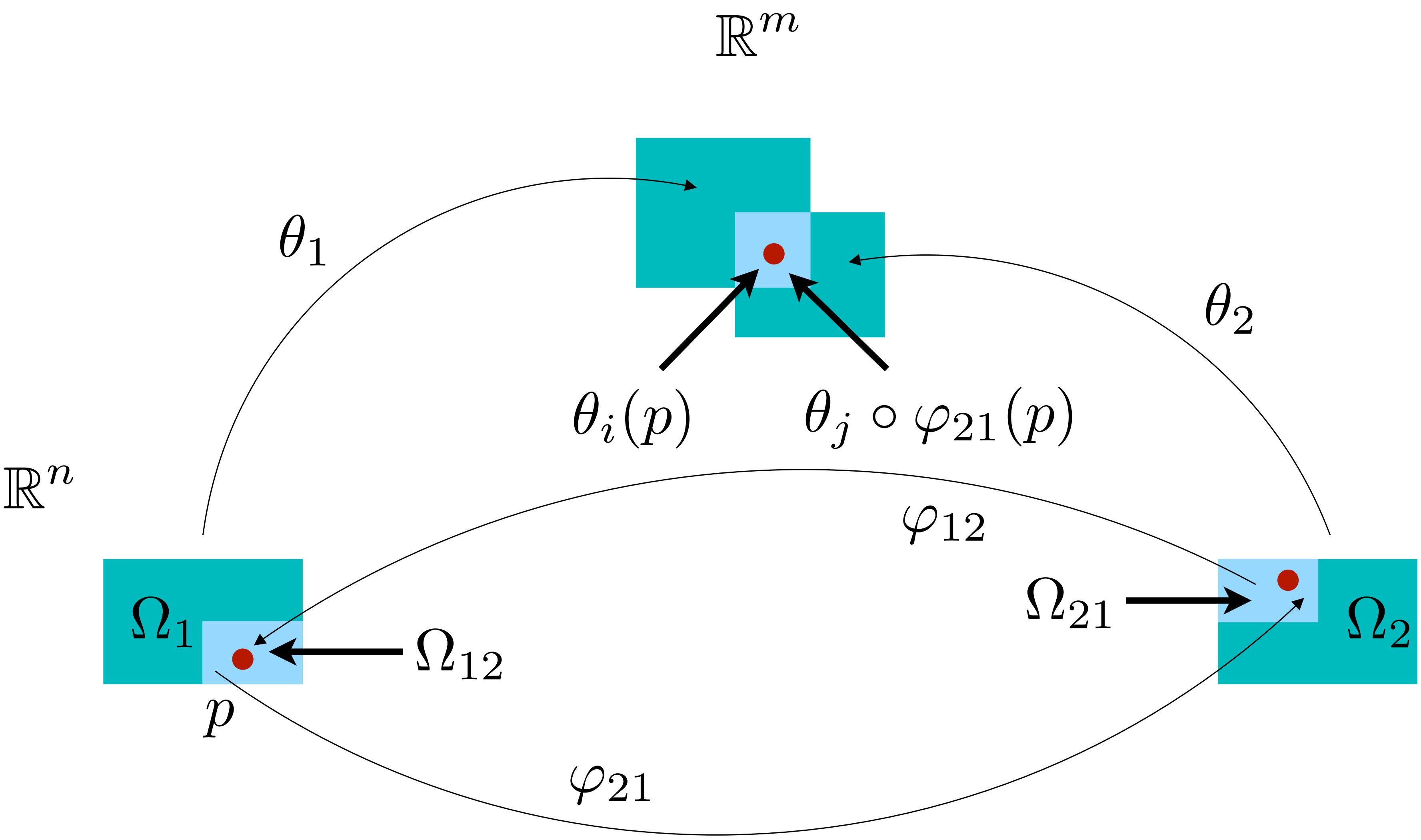

Gluing Data and PPS's

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of $\mathbb{R}^m$ is called the **image** of the parametric pseudo-manifold.

When $m = 3$ and $n = 2$, we say that $\mathcal{M}$ is a **parametric pseudo-surface**.

There is a (unique) surjective map: $\Theta \colon M_{\mathcal{G}} \longrightarrow M$.

Gluing Data and PPS's

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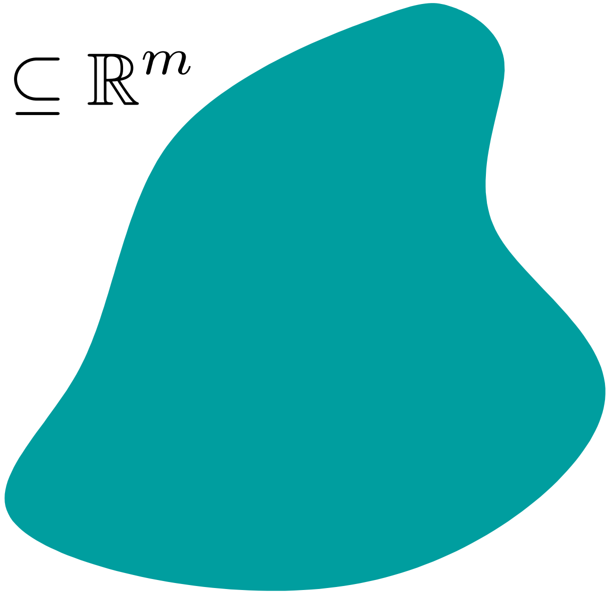
$M \subseteq \mathbb{R}^m$



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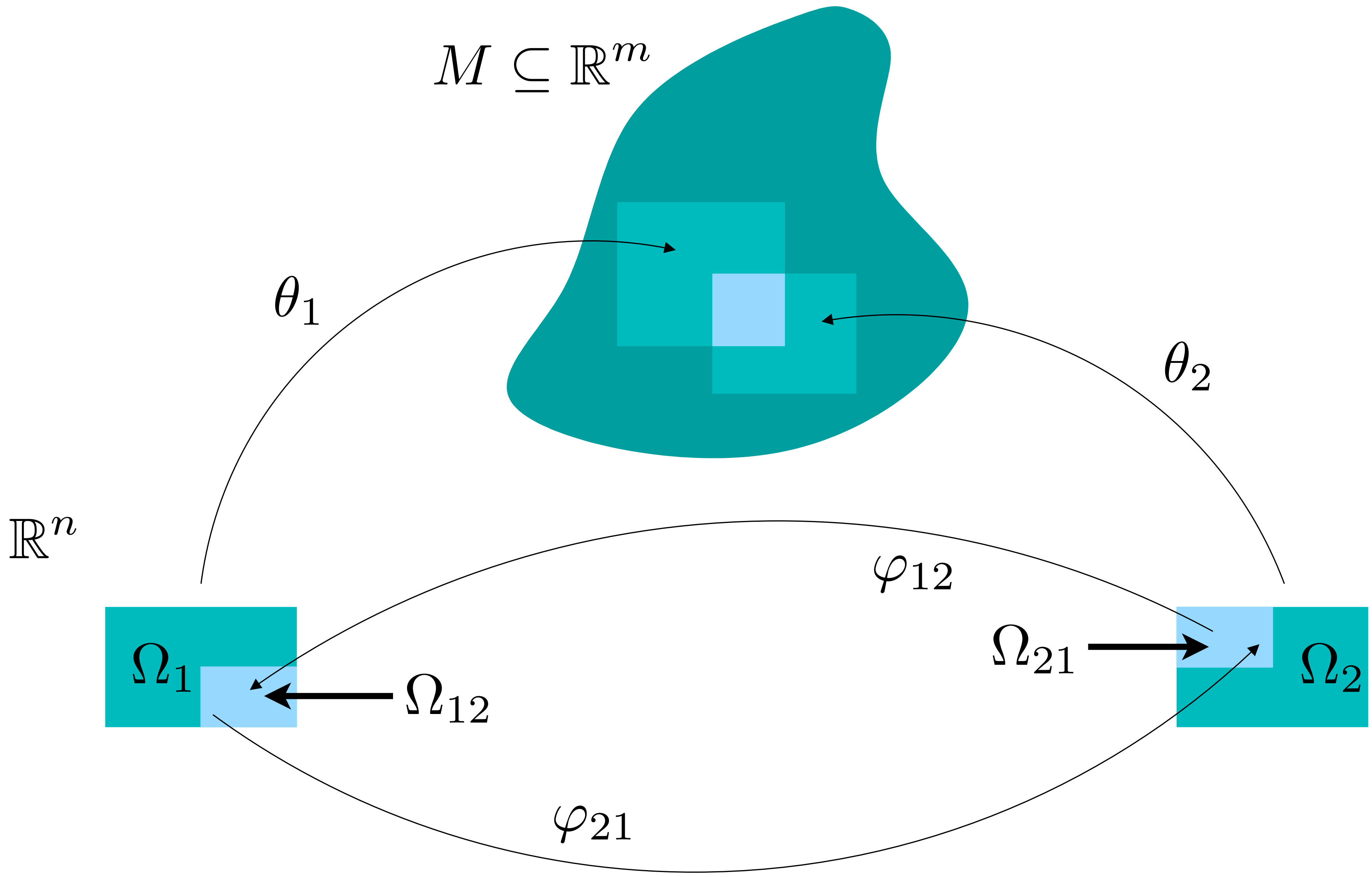

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(C') For all $(i, j) \in K$,

$$\theta_i(\Omega_i) \cap \theta_j(\Omega_j) = \theta_i(\Omega_{ij}) = \theta_j(\Omega_{ji}) \, .$$

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$$\theta_i(\Omega_i) \cap \theta_j(\Omega_j) = \theta_i(\Omega_{ij}) = \theta_j(\Omega_{ji})\,.$$

(C'') For all $(i, j) \notin K$,

$$\theta_i(\Omega_i) \cap \theta_j(\Omega_j) = \emptyset.$$

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Building PPS's

Now, let us go back to our original problem:

We want to define a surface, S, in $\mathbb{R}^3$ that approximates the underlying surface, $|S_T|$, of a given simplicial surface, S_T, in $\mathbb{R}^3$.

Building PPS's

Building PPS's

We solve this problem by defining a pseudo-surface, $\mathcal{M}$, so that S is the image, M, of $\mathcal{M}$.

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We use S_T to define the set of gluing data, $\mathcal{G}$, of $\mathcal{M}$.

We use $|S_T|$ to define the set of parametrizations, $(\theta_i)_{i \in I}$, of $\mathcal{M}$.

Building PPS's

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$\mathcal{G}$

Building PPS's

To define $\mathcal{M}$, we must

- define the p-domains, $(\Omega_i)_{i\in I}$,
- define the gluing domains, $(\Omega_{ij})_{(i,j)\in I\times I}$,
- define the transition functions, $(\varphi_{ij})_{(i,j)\in K\times K}$,
- and define the parametrizations, $(\theta_i)_{i\in I}$.

$\mathcal{G}$

$$\mathcal{M} = (\mathcal{G}, (\theta_i)_{i\in I})$$

Building PPS's

Building PPS's

p-Domains

Building PPS's

p-Domains

Let

$$I = \{(\sigma, v) \mid \sigma \text{ is a triangle of } S_T \text{ and } v \text{ is a vertex of } \sigma\}\,.$$

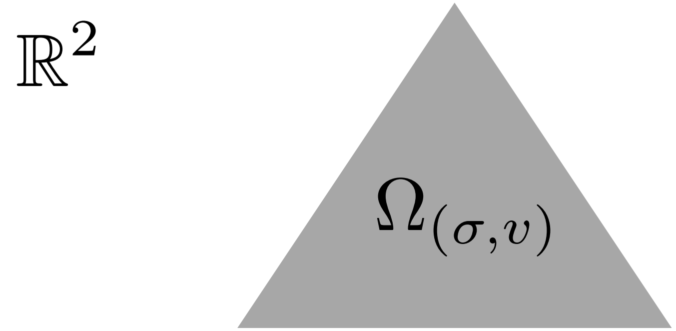
Building PPS's

p-Domains

Let

$$I = \{(\sigma, v) \mid \sigma \text{ is a triangle of } S_T \text{ and } v \text{ is a vertex of } \sigma\}.$$

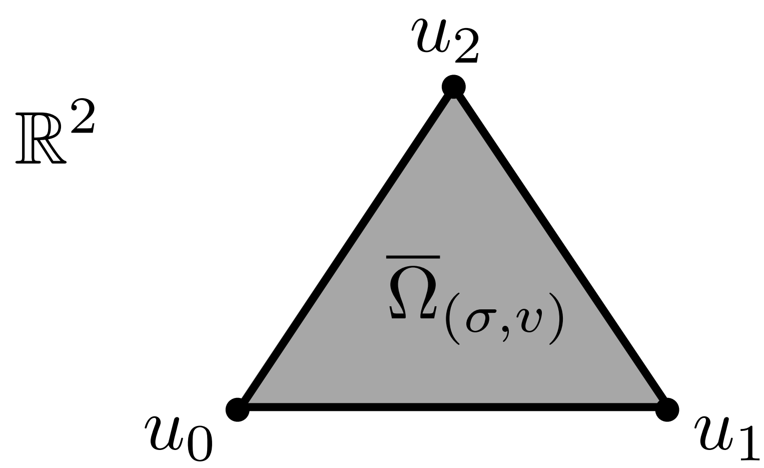
For each (σ, v) in I, we let $\Omega_{(\sigma,v)}$ be an *open* triangle in $\mathbb{R}^2$.



Building PPS’s

Building PPS's

We denote the (closed) triangle in $\mathbb{R}^2$ whose interior is $\Omega_{(\sigma,v)}$ by $\overline{\Omega}_{(\sigma,v)}$.



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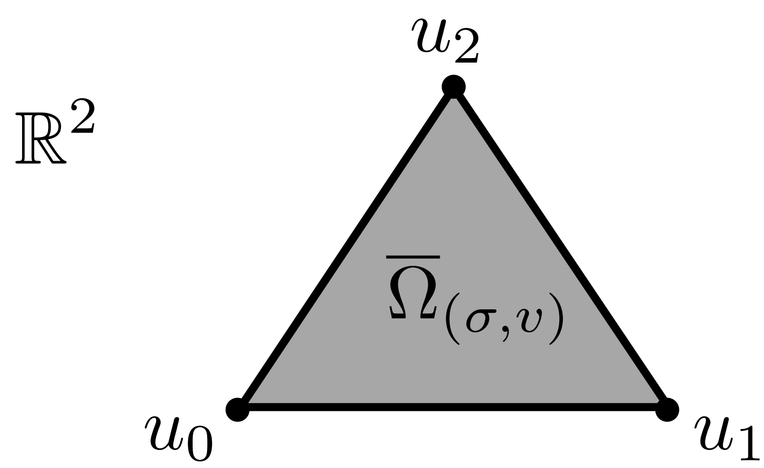

We fix an enumeration, $\langle u_0, u_1, u_2 \rangle$, of the vertices, u_0, u_1, and u_2 of $\overline{\Omega}_{(\sigma,v)}$. This enumeration will play an important role later on.

Building PPS's

Building PPS's

Gluing Domains

Building PPS's

Gluing Domains

Gluing domains are defined in terms of two abstractions, a P-polygon and its associated triangulation, and two maps, one of which is affine.

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A P-polygon is a regular n-gon inscribed in the unit circle centered at the origin.

Building PPS's

Building PPS's

For each vertex v of S_T, let m_v be the degree of v.

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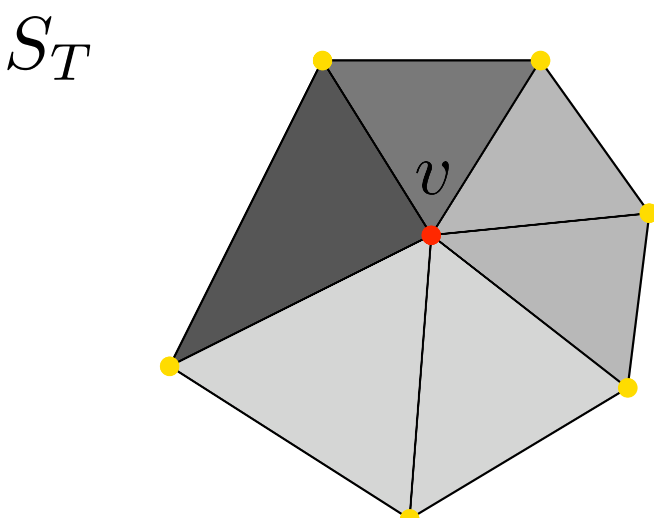
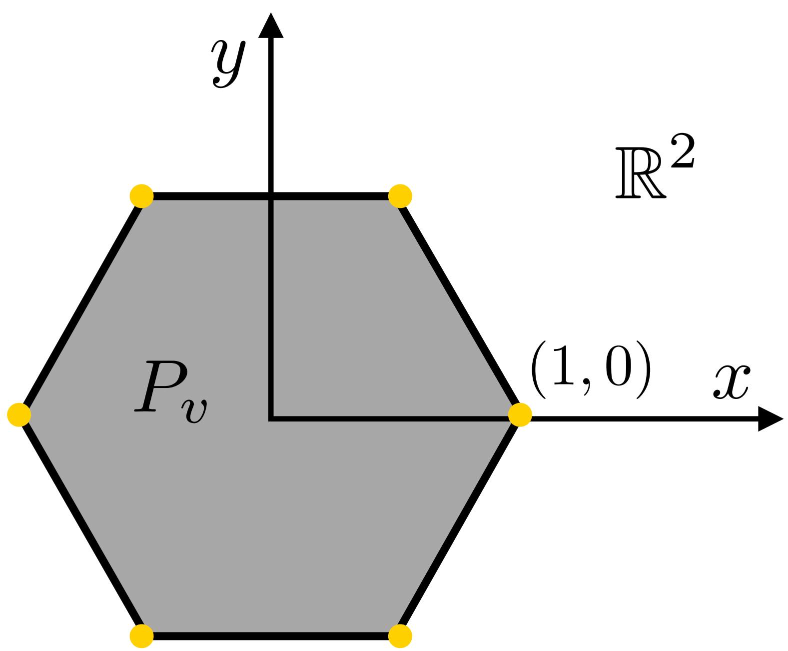
The **P-polygon, P_v, associated with** v is the regular m_v-gon inscribed in a unit circle centered at the origin and containing the vertex $(1,0)$.





Building PPS's

Building PPS's

We define a simplicial isomorphism between the vertices of the star, $st(v, S_T)$, of v in S_T and the vertices of T_v, as shown below:

Building PPS's

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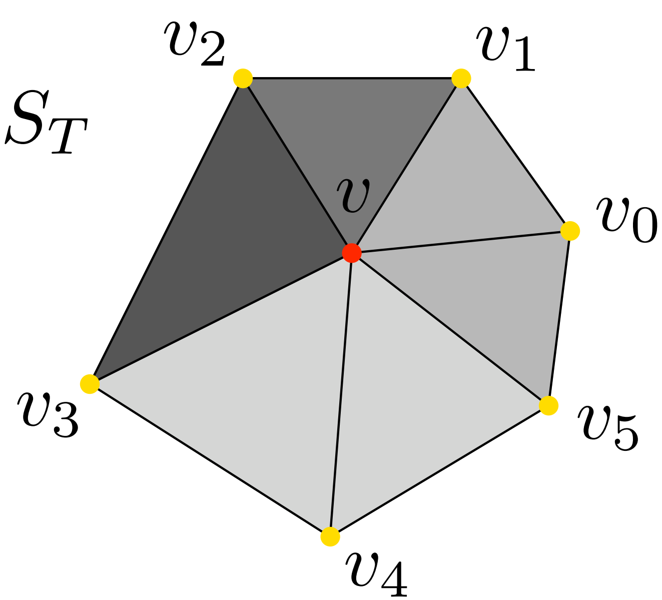
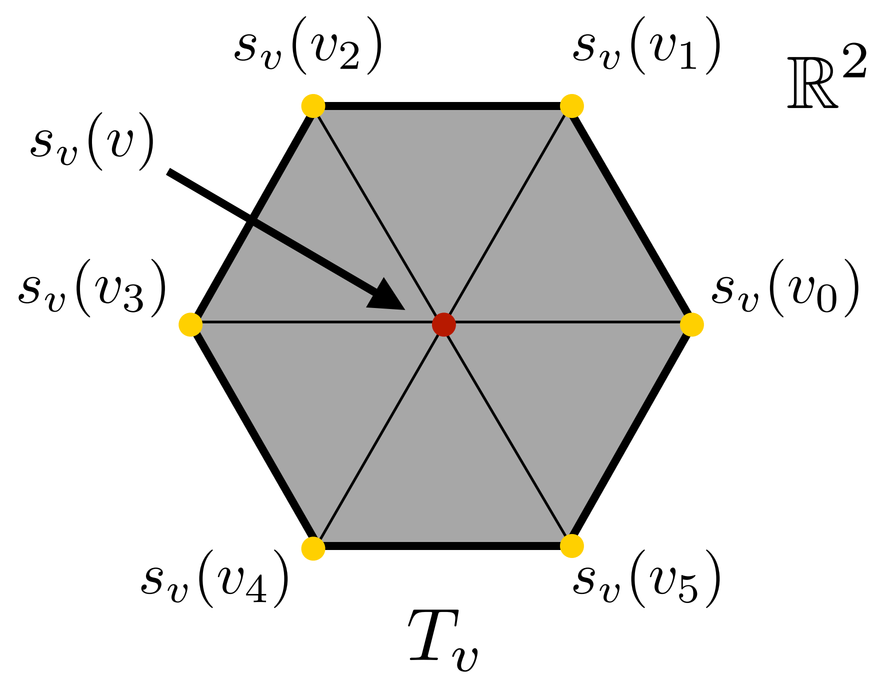
$$s_v : st(v, S_T)^{(0)} \to T_v^{(0)}$$





Building PPS's

Building PPS's

For each $i \in \{0, \ldots, m_v - 1\}$, we assign a point, $r_{i,v}$, with the triangle with vertices $s_v(v)$, $s_v(v_i)$, and $s_v(v_{i+1})$ of T_v (the index i is taken modulo m_v):

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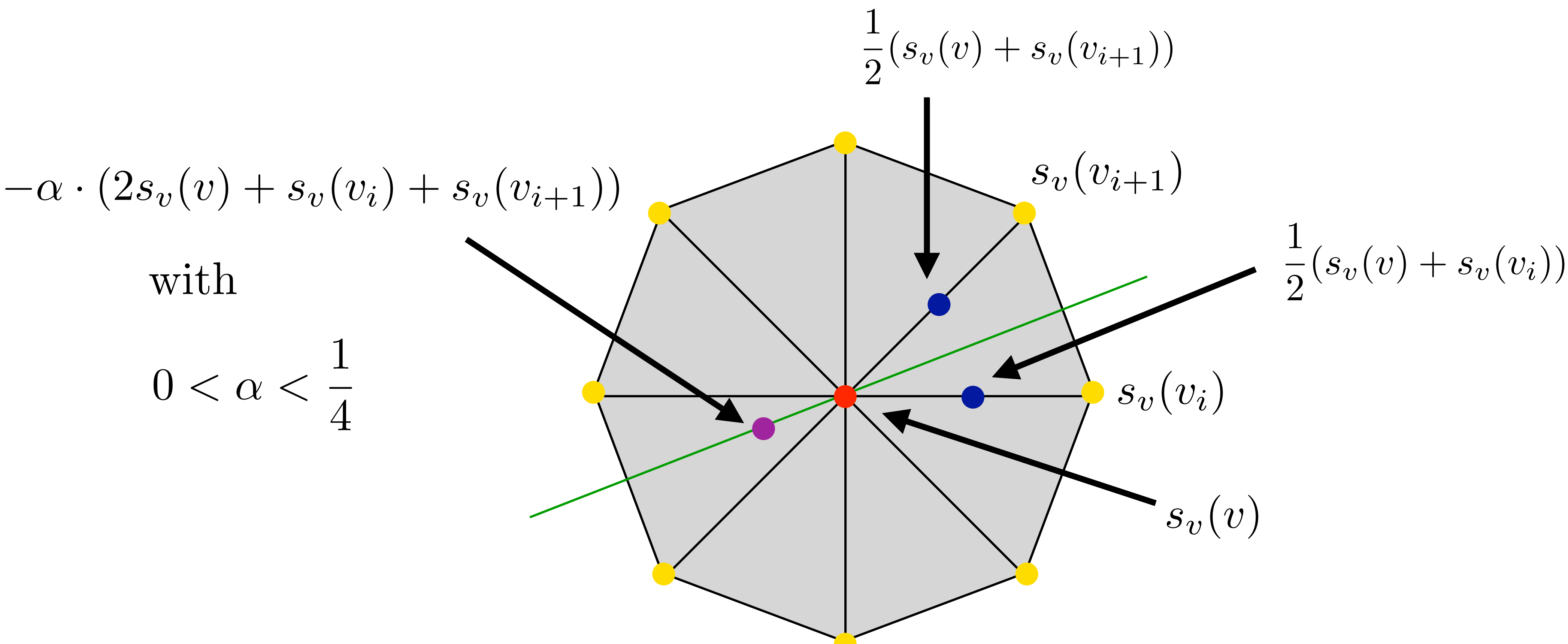
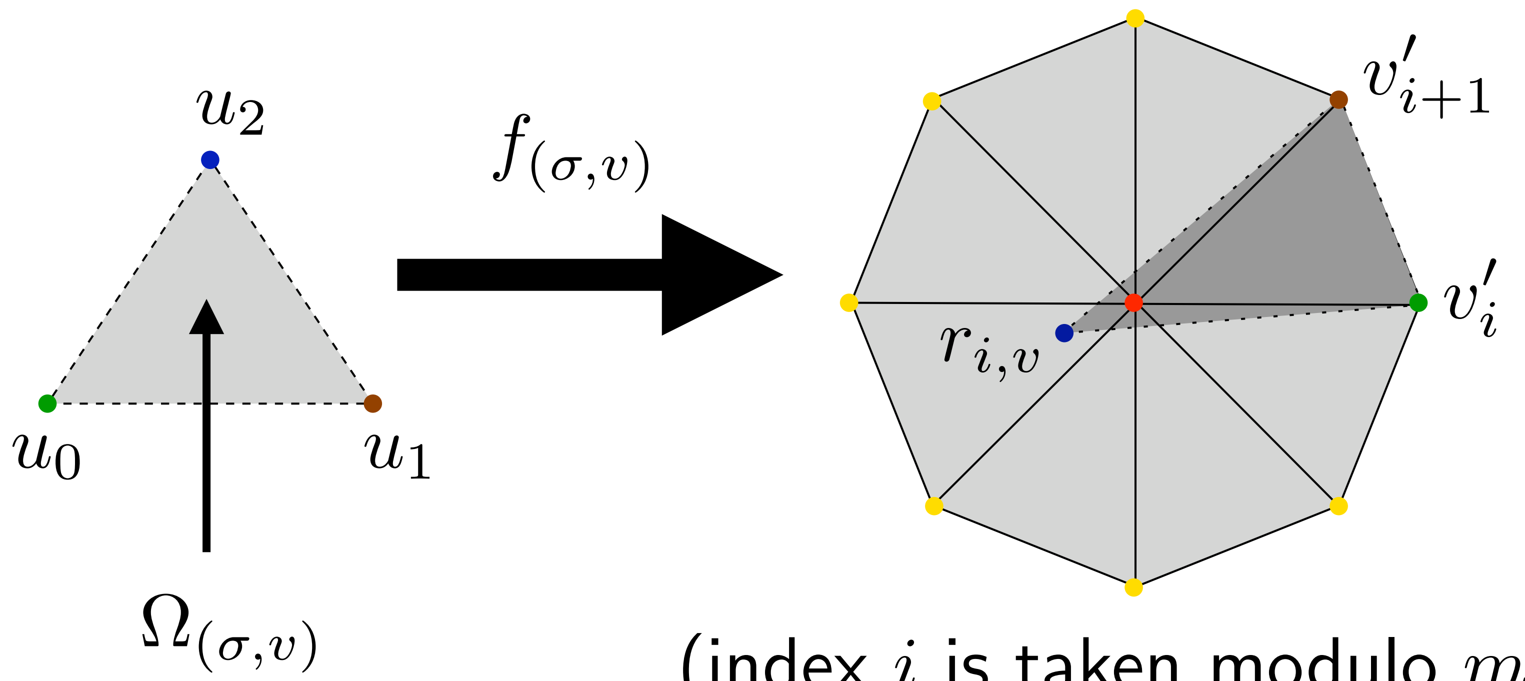


Building PPS's

Building PPS's

For each $(\sigma, v) \in I$, we let $f_{(\sigma,v)} : \mathbb{R}^2 \to \mathbb{R}^2$ denote the unique affine function that maps the vertices u_0, u_1, and u_2 of $\overline{\Omega}_{(\sigma,v)}$ to the vertices $s_v(v_i)$, $s_v(v_{i+1})$, and $r_{i,v}$, respectively:



(index i is taken modulo m_v)

Building PPS's

Building PPS's

For each edge $[u, w]$ of S_T, we define the function

$$g_{(u,w)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \,,$$

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Let

$$[u, x, w] \qquad \text{and} \qquad [u, w, y]$$

be the two triangles of S_T that share the edge $[u, w]$.

Building PPS's

Building PPS's

Then, the function $g_{(u,w)}$ takes the interior of the quadrilateral with vertices $s_u(u)$, $s_u(x)$, $s_u(w)$, and $s_u(y)$ onto the interior of the quadrilateral with vertices $s_w(u)$, $s_w(x)$, $s_w(w)$, and $s_w(y)$.

Building PPS's

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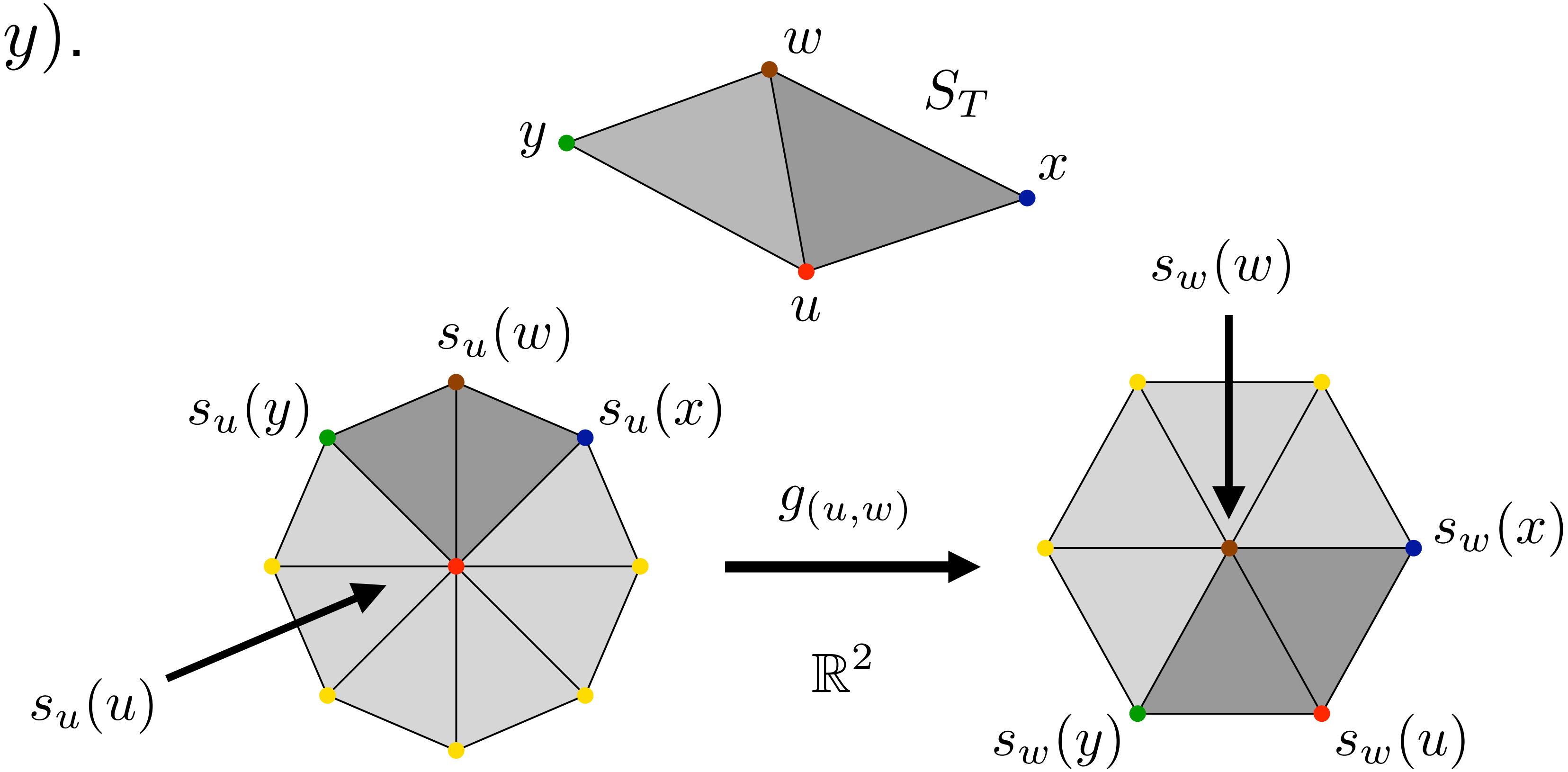

Building PPS's

Building PPS's

For every point p in the interior of the quadrilateral given by $s_u(u)$, $s_u(x)$, $s_u(w)$, and $s_u(y)$, we define the function $g_{(u,w)}(p)$ as

$$R^{-1}_{(w,u)} \circ H \circ R_{(u,w)}(p)\,,$$

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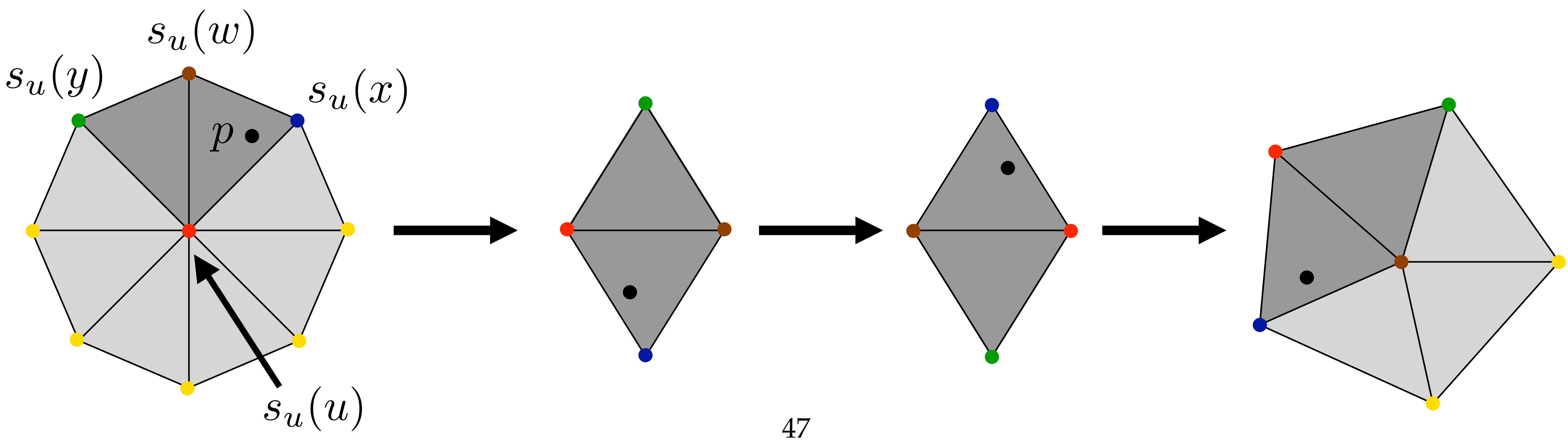

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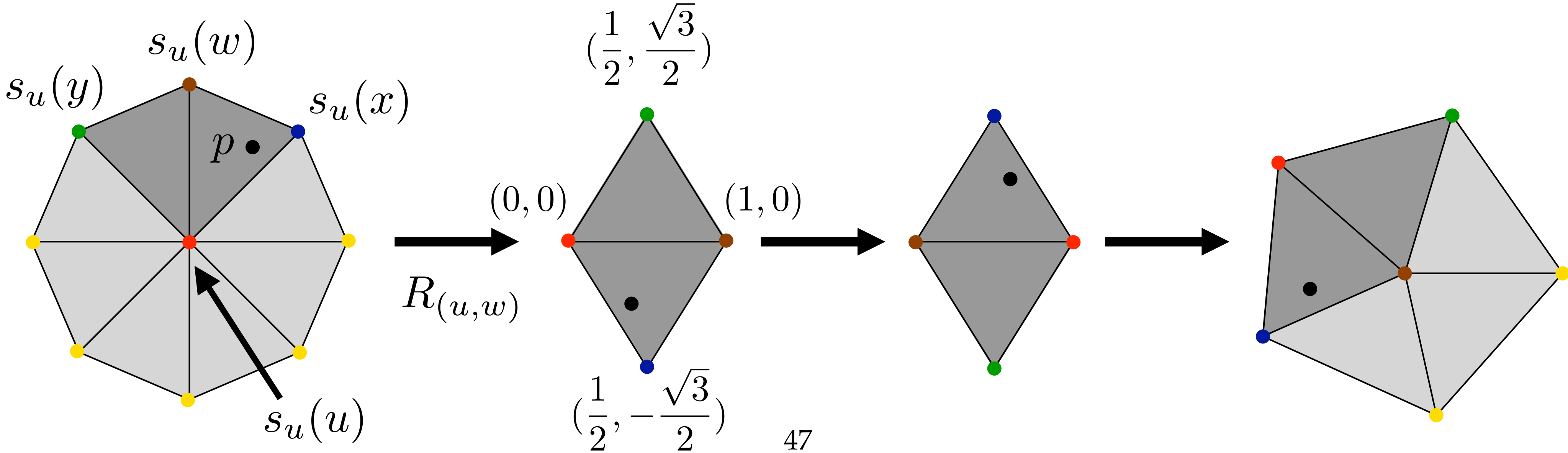

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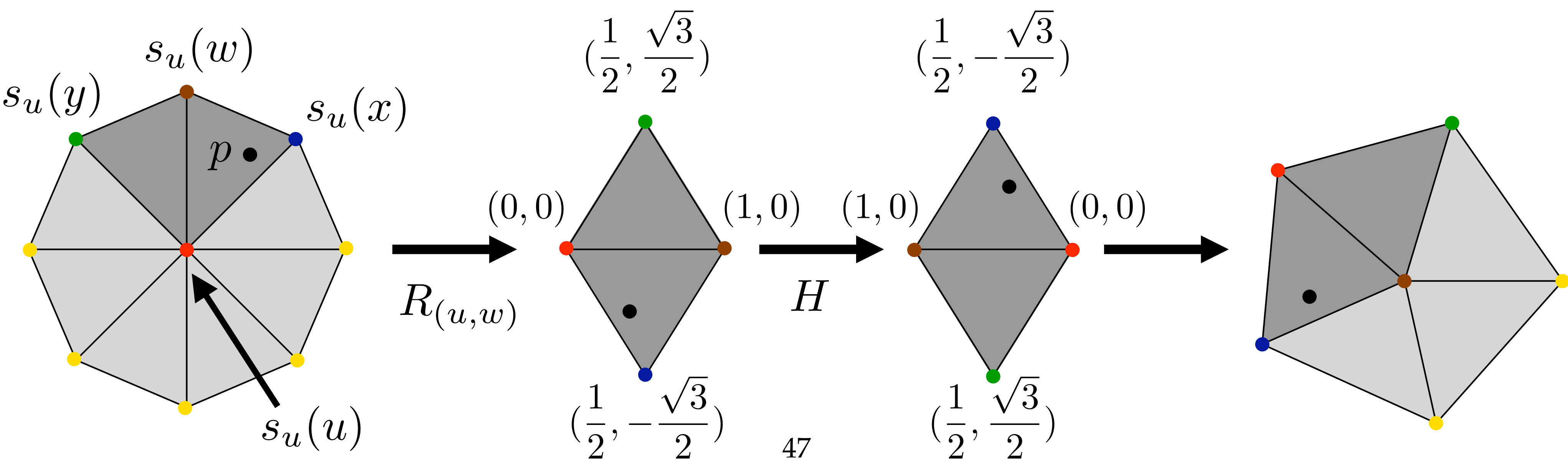

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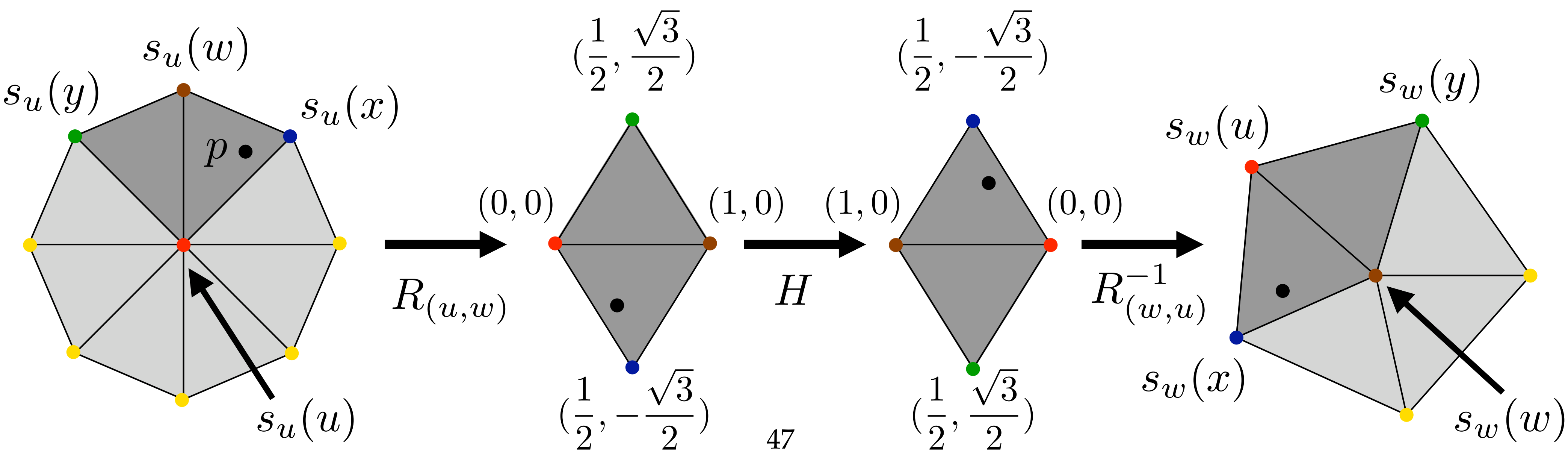

Building PPS's

Building PPS's

The function $R_{(u,w)}$ can be expressed by

$$\Pi^{-1} \circ (\mathsf{id} \times \rho_u) \circ \Pi \circ M_{\beta_i} \ ,$$

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Building PPS's

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- $\Pi(x, y) = (\sqrt{x^2 + y^2}, \theta)$,
- $\rho_u(\theta) = \theta \cdot \dfrac{m_u}{6}$.

Building PPS's

Building PPS's

For every point p outside the interior of the quadrilateral given by $s_u(u)$, $s_u(x)$, $s_u(w)$, and $s_u(y)$, the value $g_{(u,w)}(p)$ can be any point, $q \in \mathbb{R}^2$, outside the unit circle centered at the origin.

Building PPS's

Building PPS's

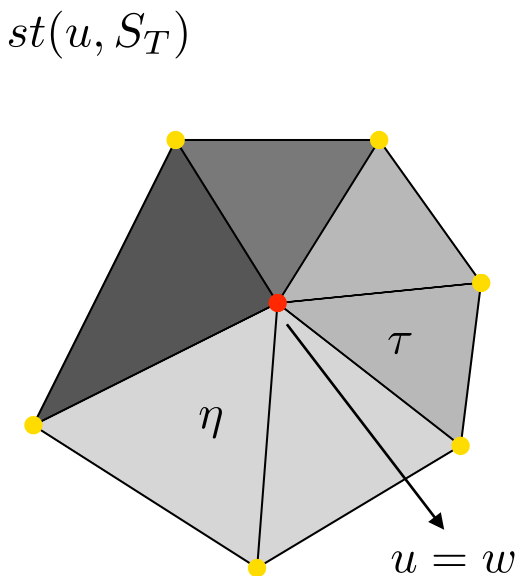
For any two $(\tau, u), (\eta, w) \in I$, we define $\Omega_{(\tau,u)(\eta,w)}$ as follows:

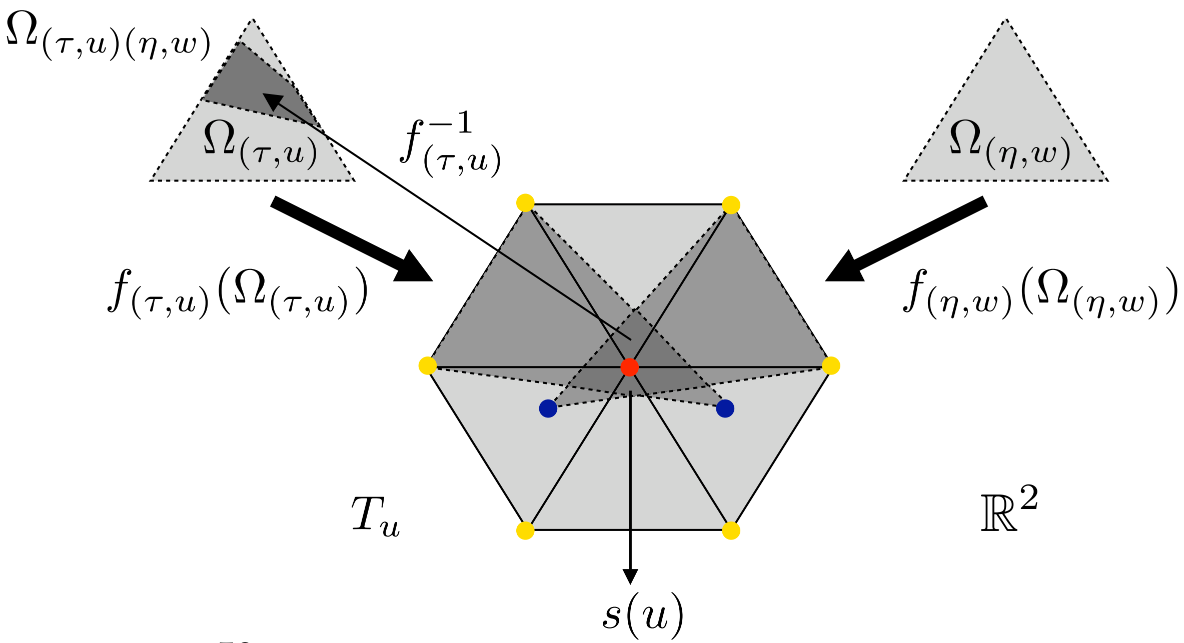
Building PPS's

For any two $(\tau, u), (\eta, w) \in I$, we define $\Omega_{(\tau,u)(\eta,w)}$ as follows:

(1) If $u = w$ then

$$\Omega_{(\tau,u),(\eta,w)} = f^{-1}_{(\tau,u)}\left(f_{(\tau,u)}(\Omega_{\tau,u}) \cap f_{(\eta,w)}(\Omega_{(\eta,w)})\right) .$$

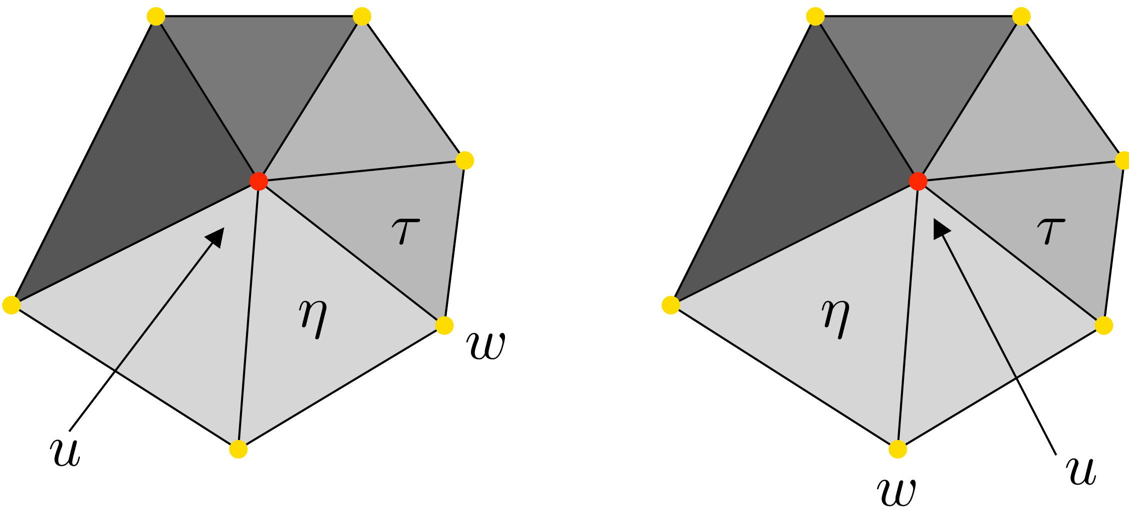




Building PPS's

Building PPS's

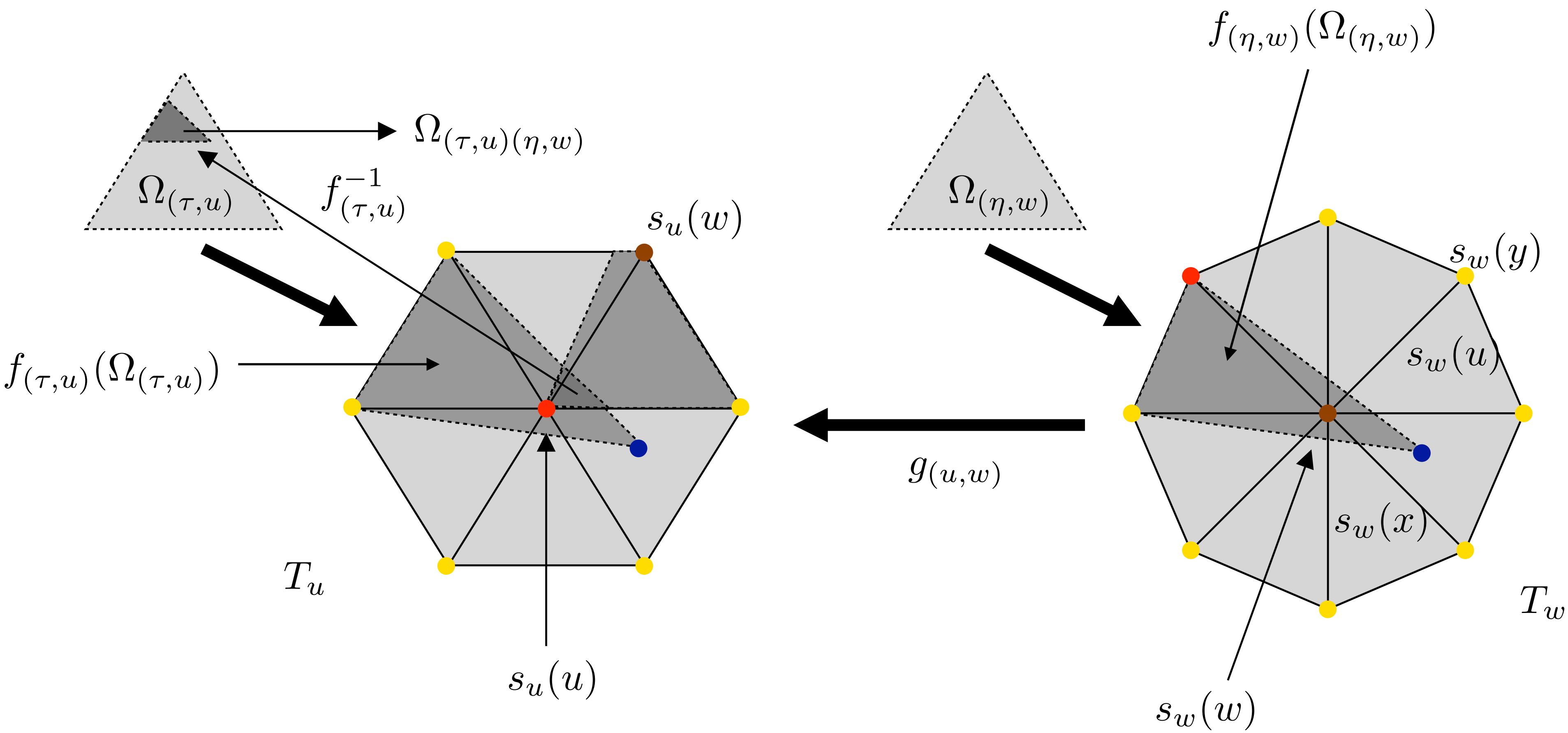
(2) If $u \neq w$ and w is a vertex of τ or u is a vertex of η then



Building PPS's

Building PPS's

$$\Omega_{(\tau,u),(\eta,w)} = f^{-1}_{(\tau,u)}\left(f_{(\tau,u)}(\Omega_{\tau,u}) \cap g_{(u,w)}(f_{(\eta,w)}(\Omega_{(\eta,w)}))\right) .$$



Building PPS’s

Building PPS's

(3) If $u \neq w$ and w is not a vertex of τ nor u is a vertex of η then

$$\Omega_{(\tau,u),(\eta,w)} = \emptyset\,.$$

Building PPS's

Building PPS's

We can show that the above definition of gluing domains satisfies condition (2) of the definition of sets of gluing data we saw before:

Building PPS's

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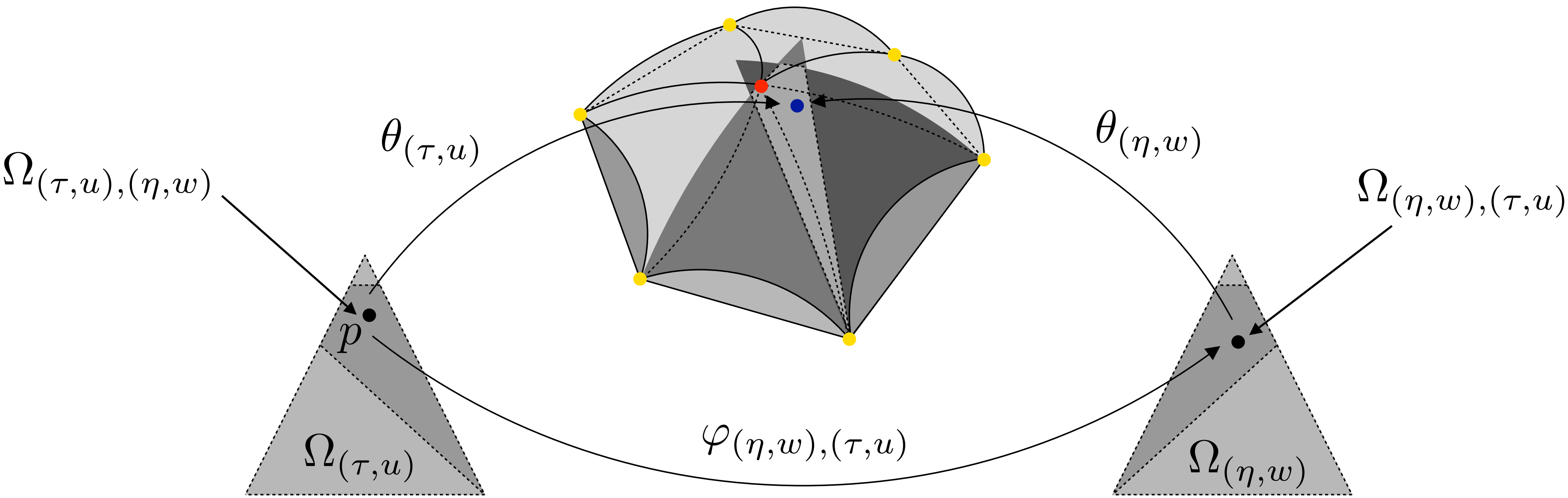
(2) For every pair $(i, j) \times I \times I$, the set Ω_{ij} is an open subset of Ω_i. Furthermore, $\Omega_{ii} = \Omega_i$ and $\Omega_{ji} \neq \emptyset$ if and only if $\Omega_{ij} \neq \emptyset$.

Building PPS's

Building PPS's

Parametrizations



Building PPS's

Building PPS's

For each $(\sigma, v) \in I$, we define the parametrization

$$\theta_{(\sigma,v)} : \Omega_{(\sigma,v)} \to \mathbb{R}^3 \,,$$

such that for each $p \in \Omega_{(\sigma,v)}$,

$$\theta_{(\sigma,v)}(p) = \sum_{(\tau,u)\in J(p)} \omega_{(\sigma,v)(\tau,u)} \cdot \psi_{\tau,u} \circ \varphi_{(\tau,u)(\sigma,v)}(p) \,,$$

where

Building PPS’s

Building PPS's

$$\omega_{(\sigma,v)(\tau,u)}(p) = \frac{\gamma_{\tau,u} \circ \psi_{(\tau,u)} \circ \varphi_{(\tau,u)(\sigma,v)}(p)}{\displaystyle\sum_{(\eta,w)\in J(p)} \gamma_{(\eta,w)} \circ \varphi_{(\eta,w)(\sigma,v)}(p)}$$

and

$$J(p) = \{(\eta, w) \in I \mid p \in \Omega_{(\sigma,v)(\eta,w)}\}\,.$$

Building PPS's

Building PPS's

The function

$$\psi_{(\tau,u)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

is a Bézier patch whose control points are defined on $\overline{\Omega}_{\tau,u}$.

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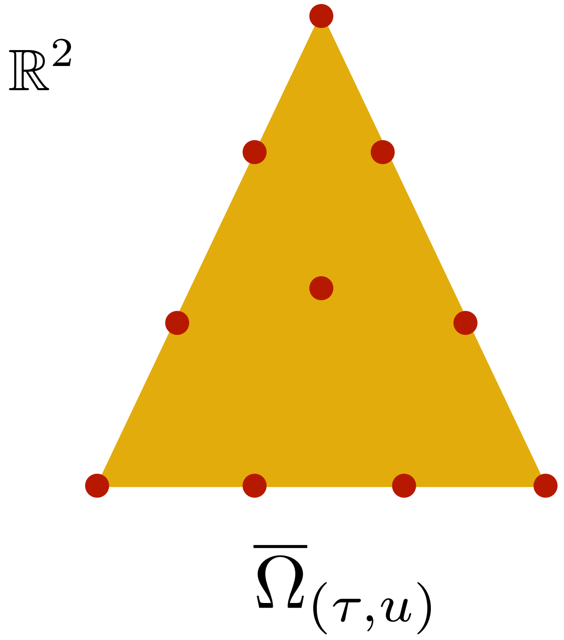
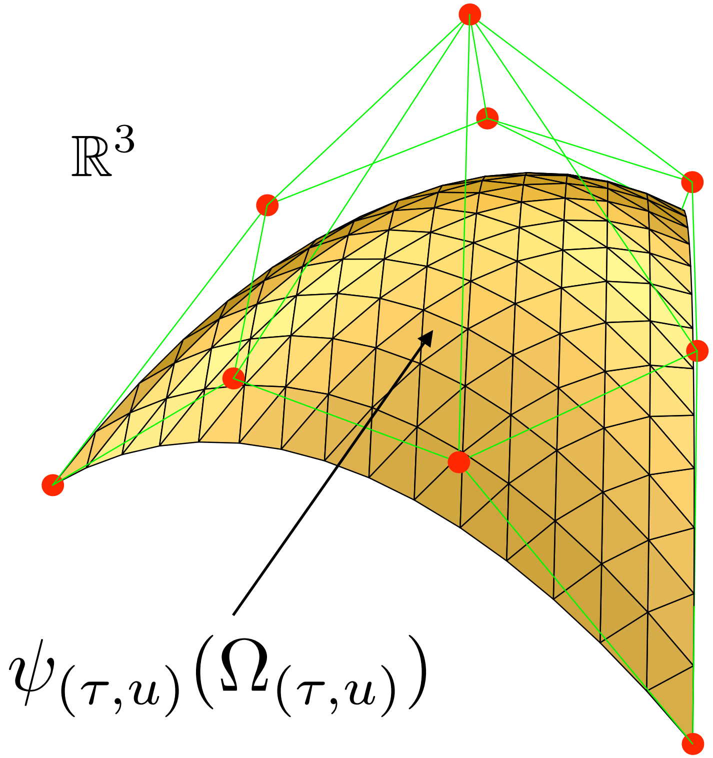




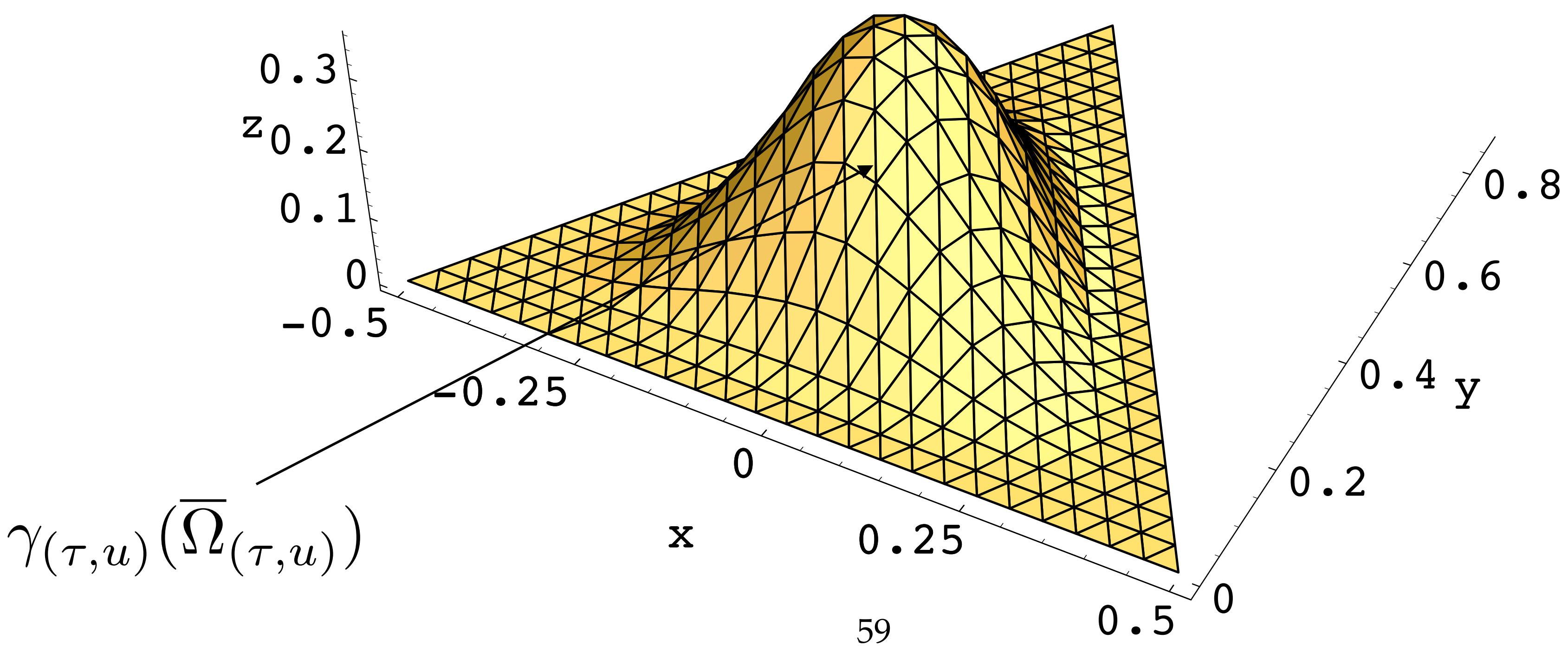
Building PPS's

Building PPS's

The function

$$\gamma_{(\tau,u)} : \mathbb{R}^2 \to \mathbb{R}$$

is a "hat" function defined as the product of three C^∞ curves:



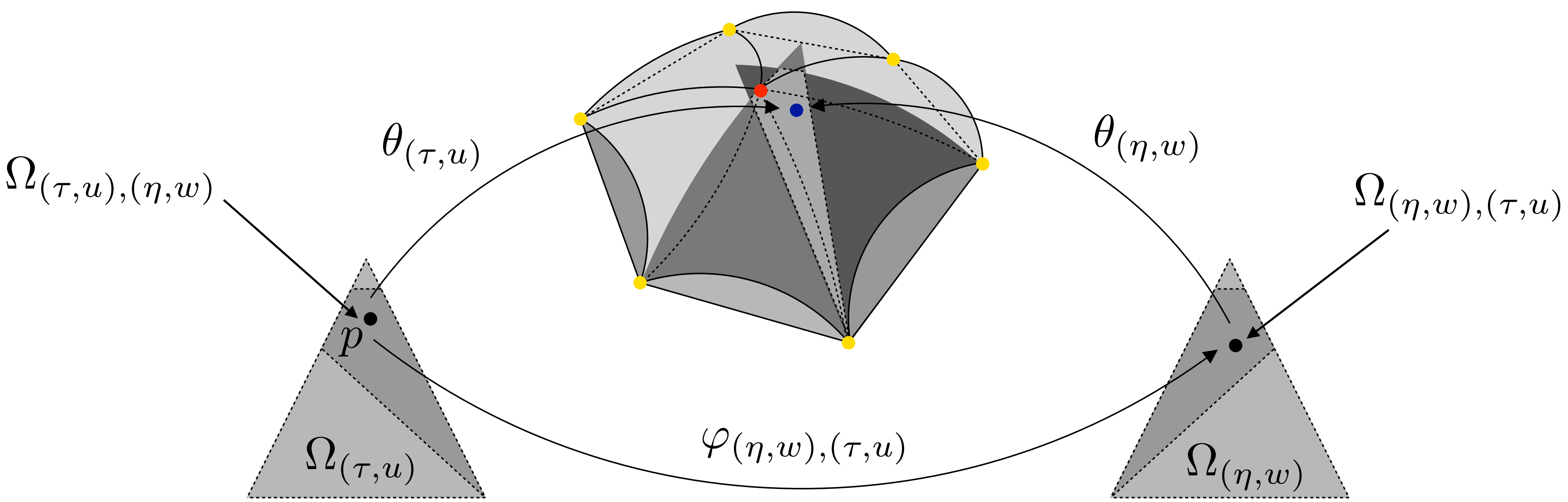
Building PPS's

Building PPS's

We can show that

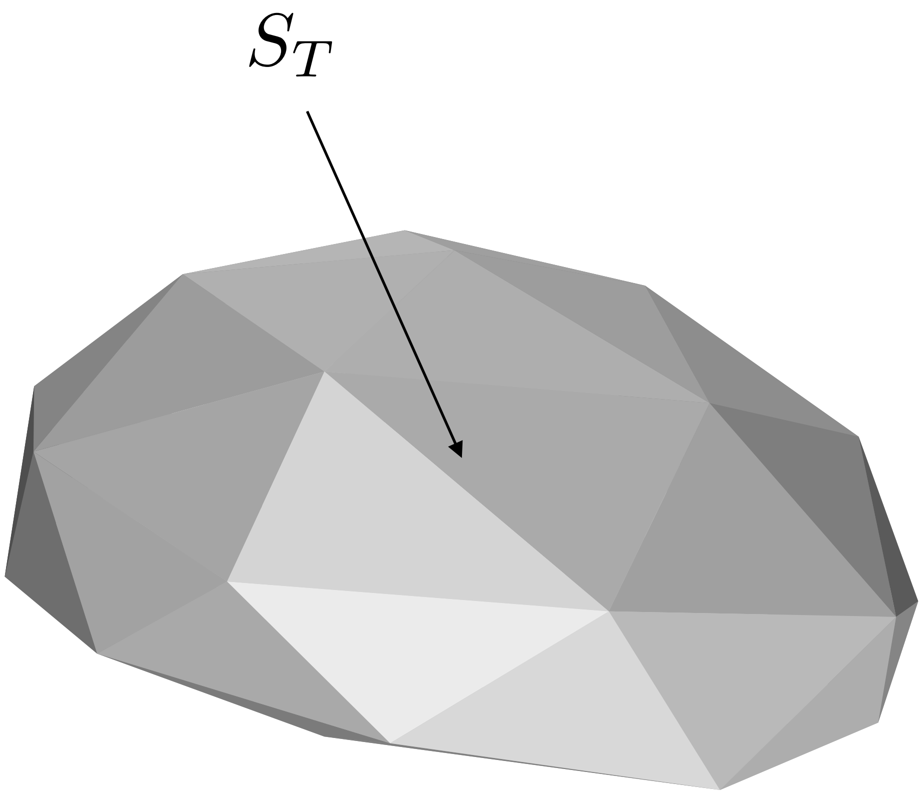
$$\theta_{(\tau,u)}(p) = \theta_{(\eta,w)}(\varphi_{(\eta,w)(\tau,u)}(p)) \, ,$$

for all $p \in \Omega_{(\tau,u)(\eta,w)}$ and for all $((\tau,u),(\eta,w)) \in K$.



Experimental Results

Experimental Results



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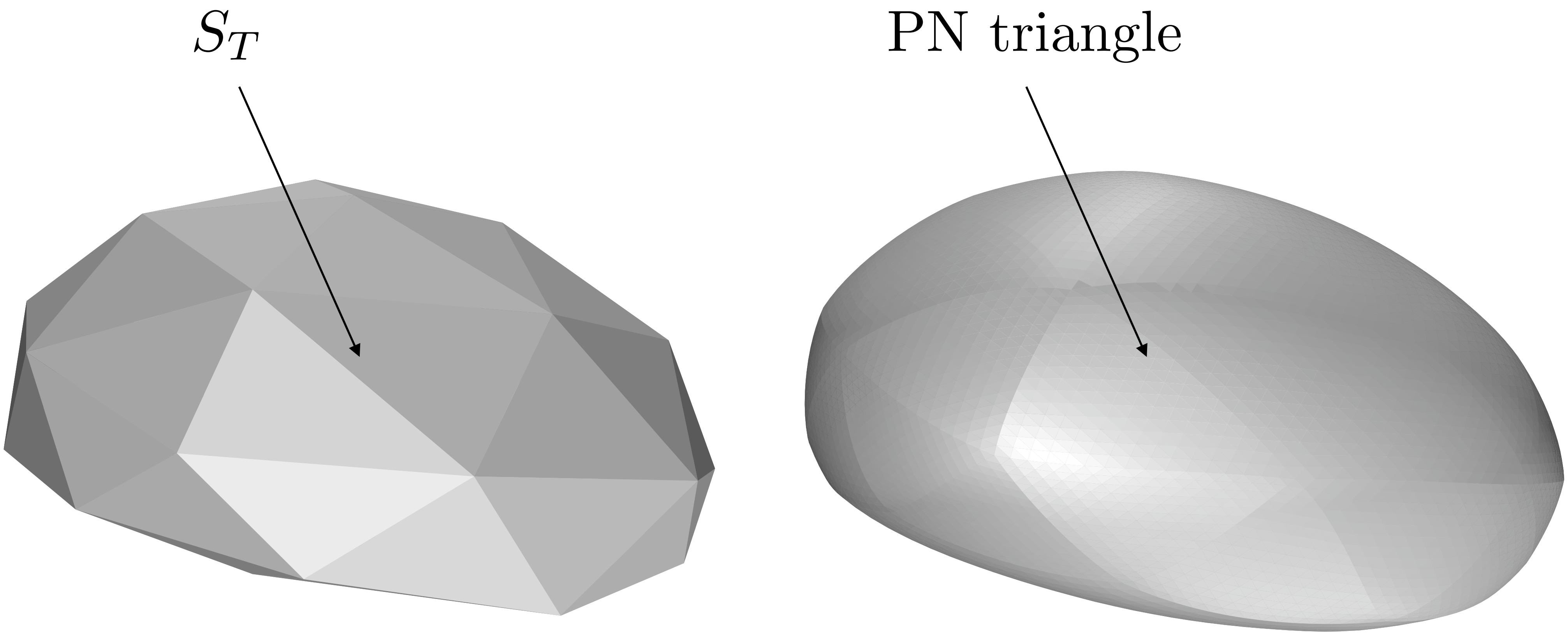
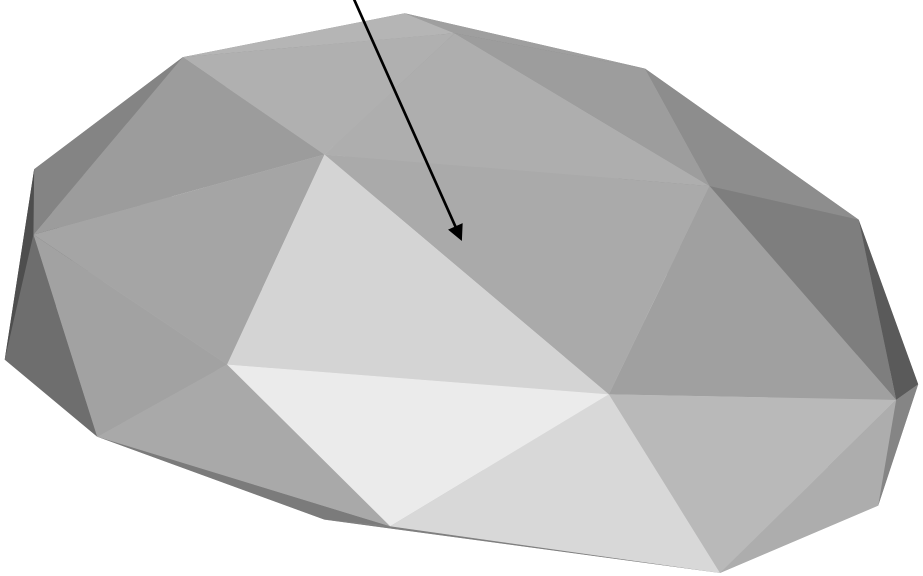
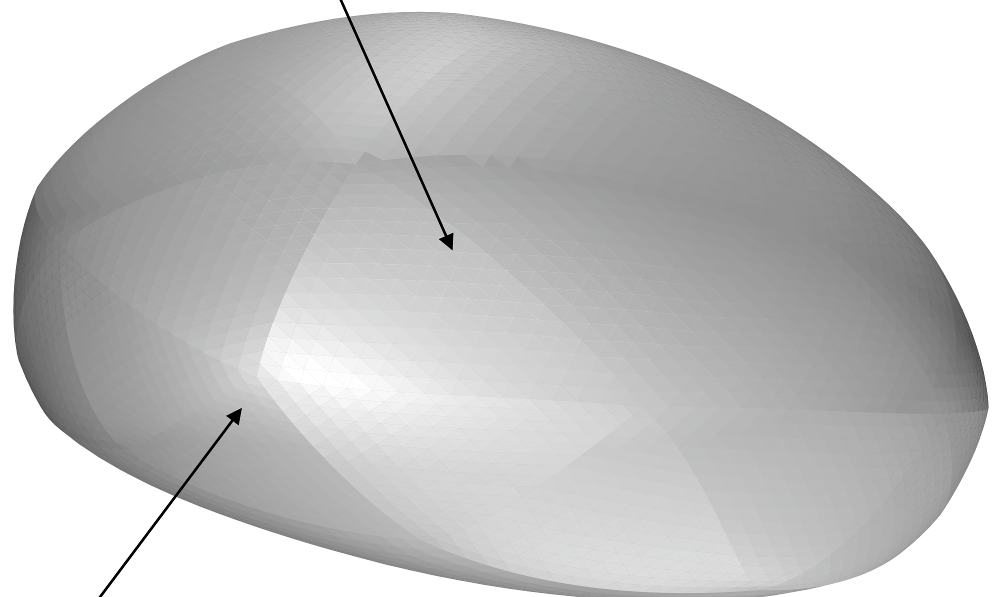


Experimental Results

S_T



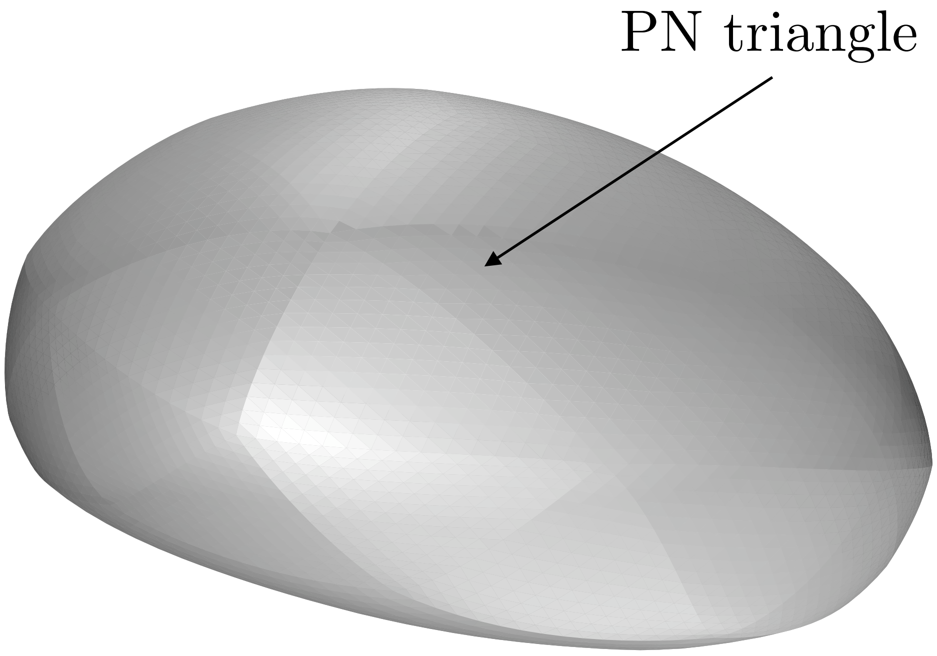
PN triangle



Shading reveals lack of smoothness

Experimental Results

Experimental Results



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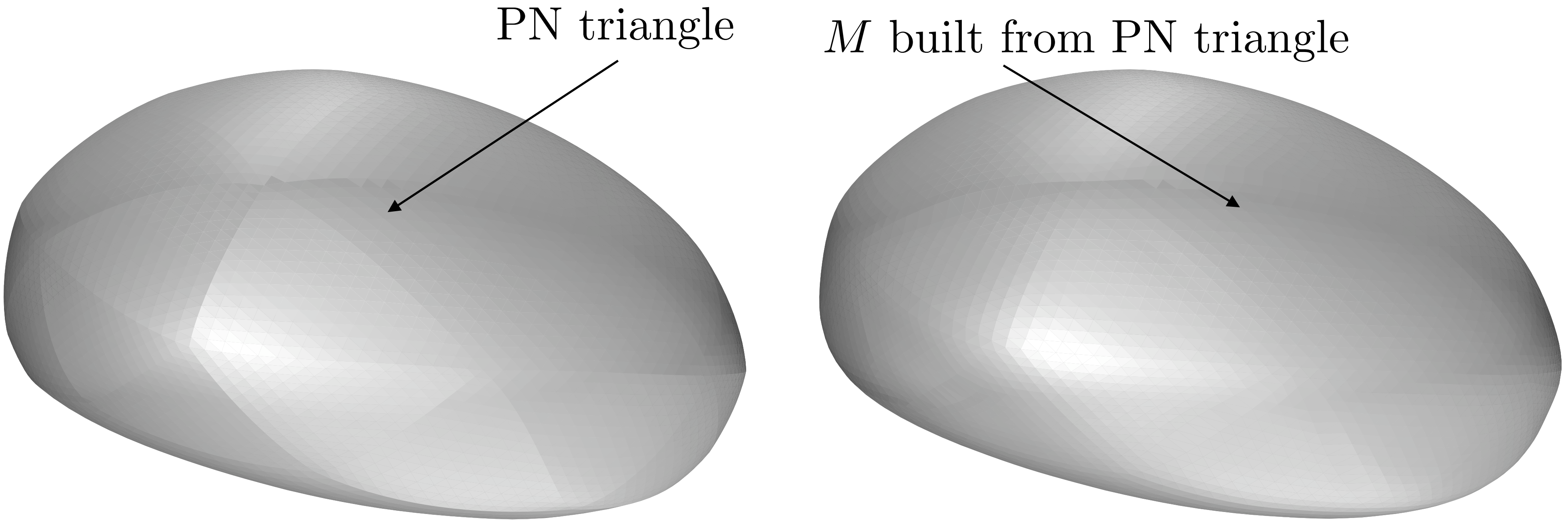

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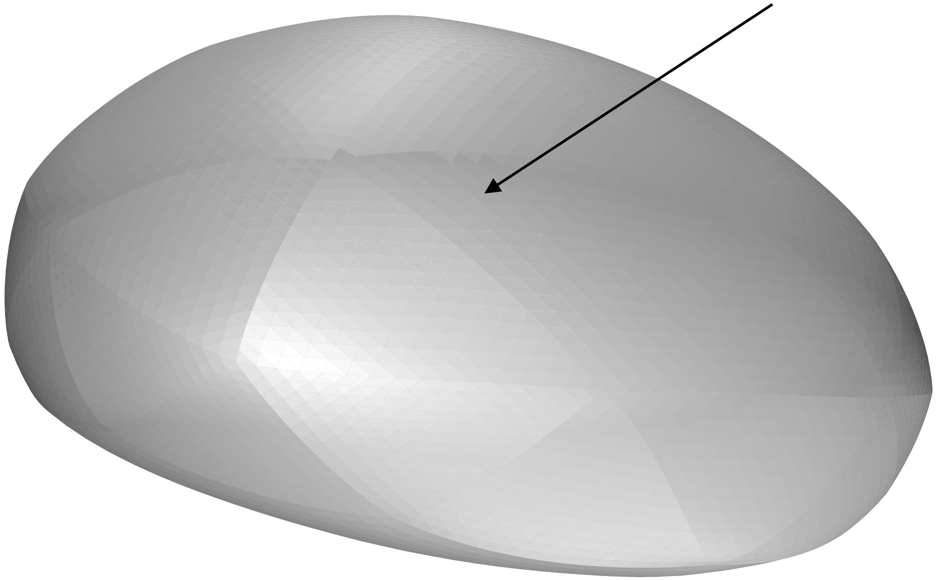
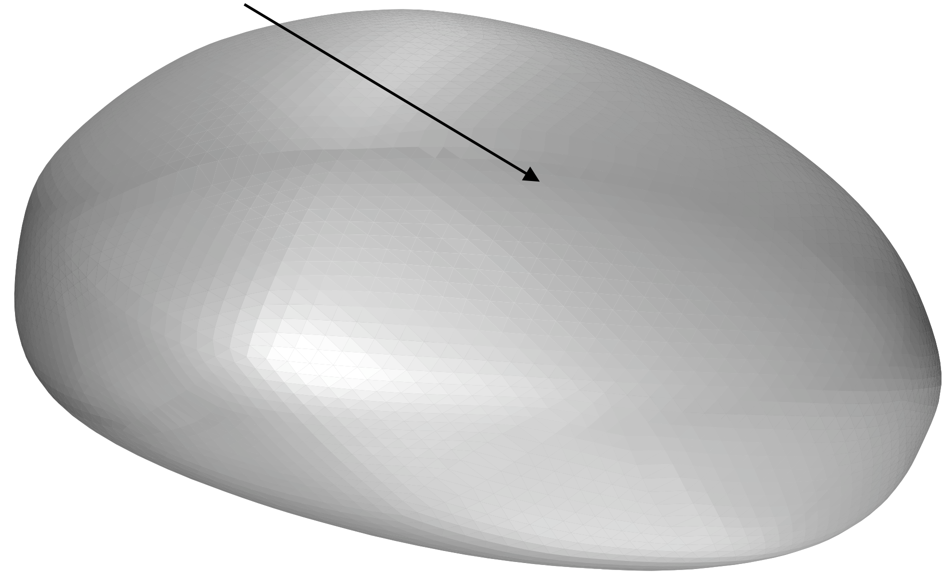
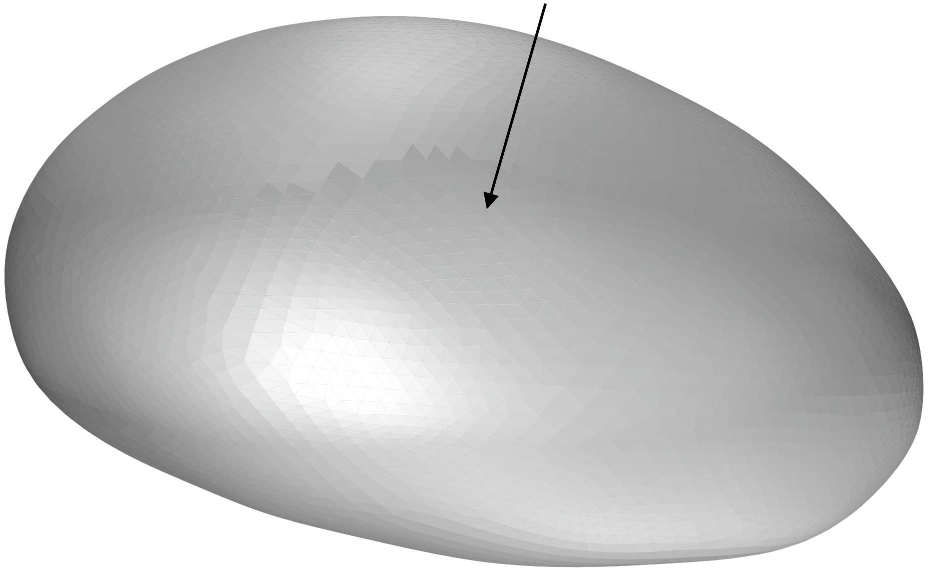
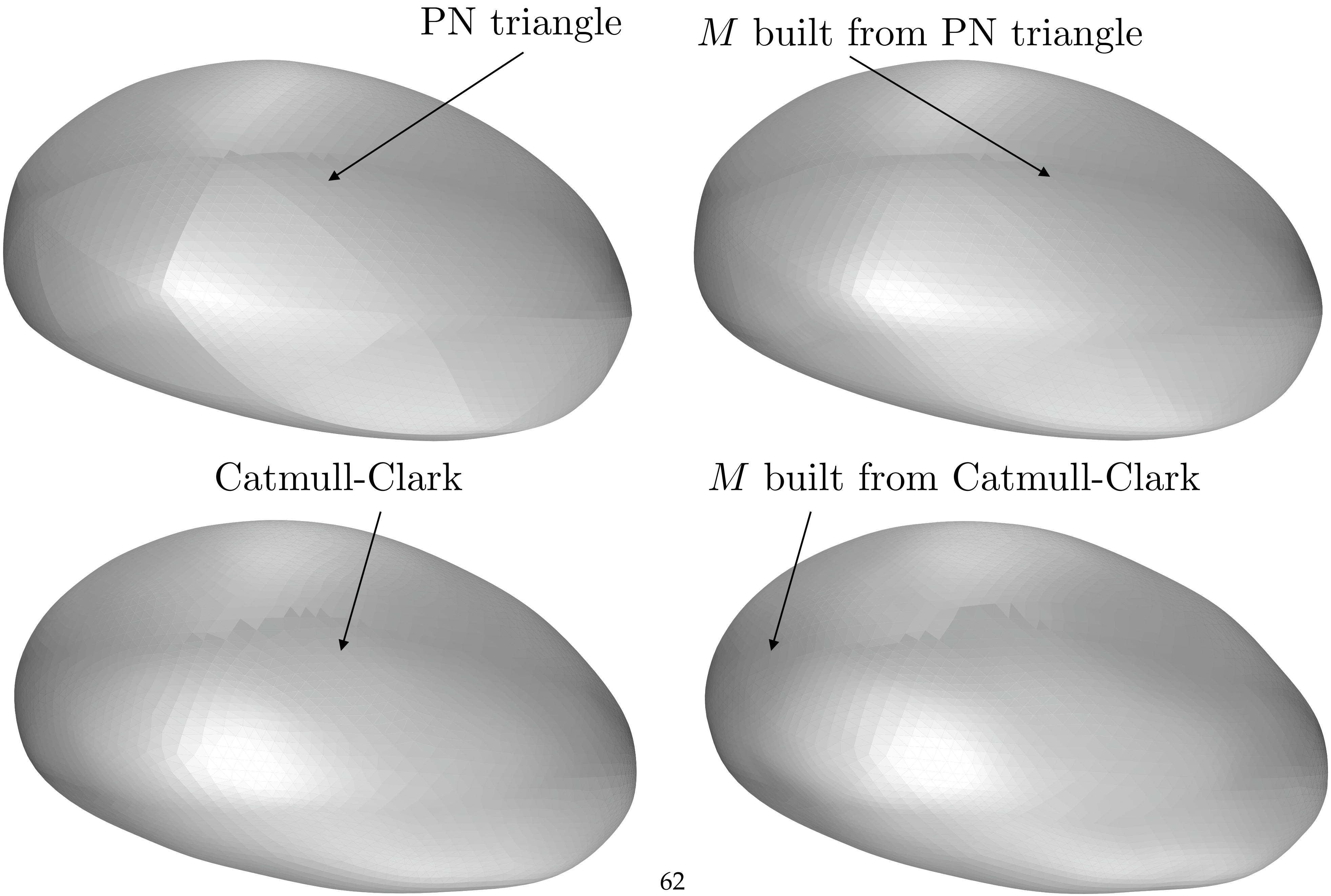






Experimental Results



Experimental Results

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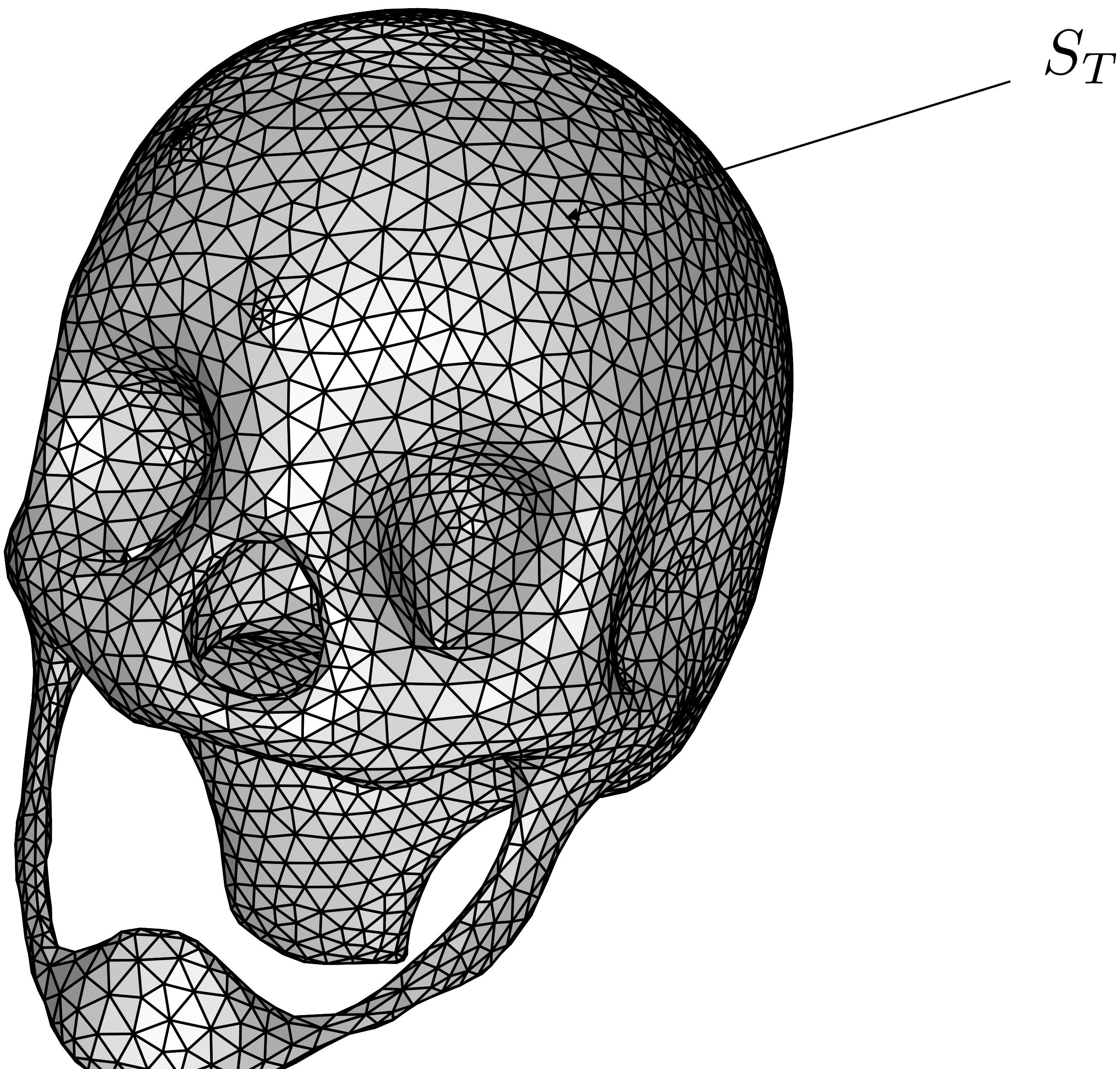

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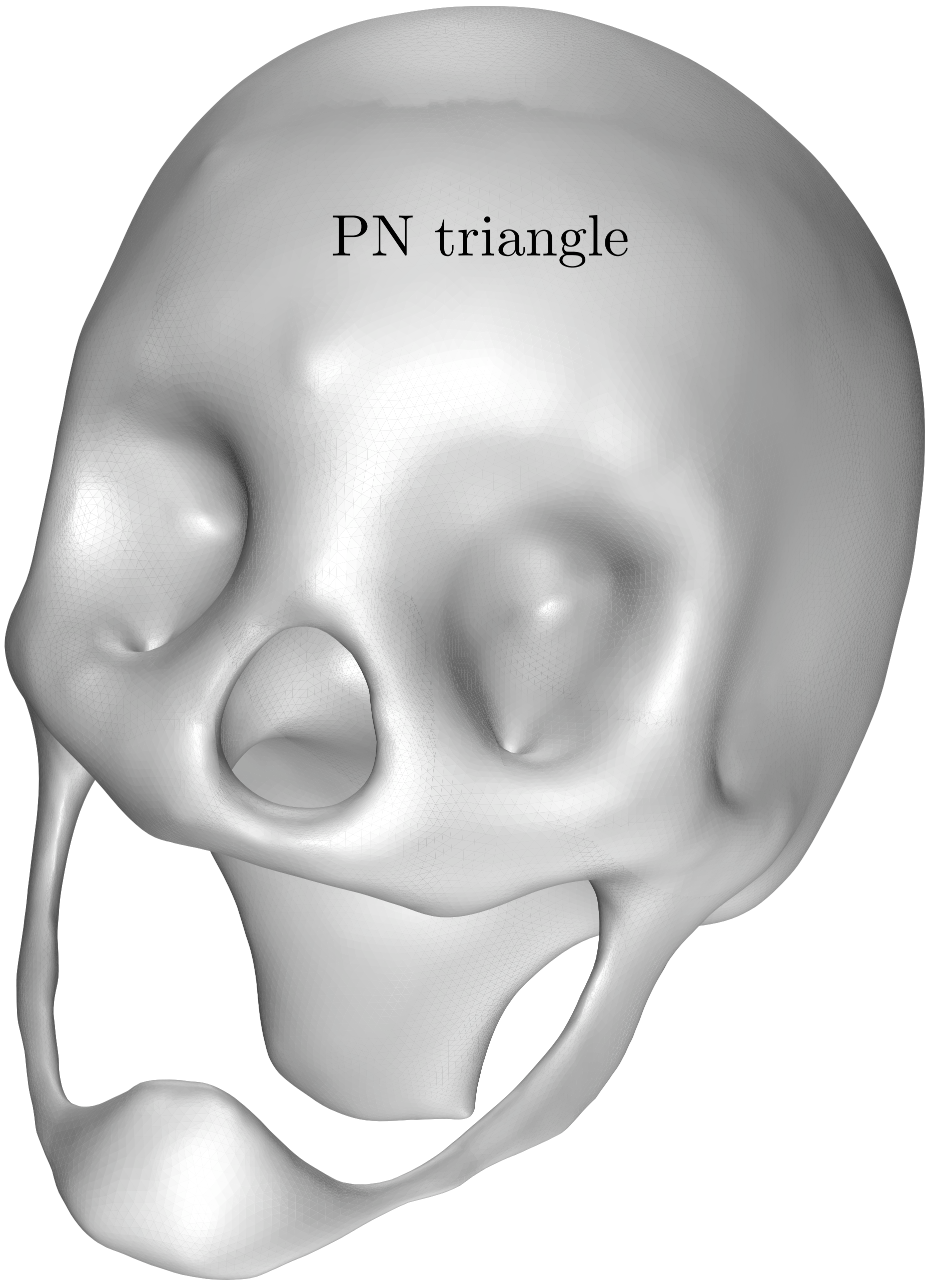

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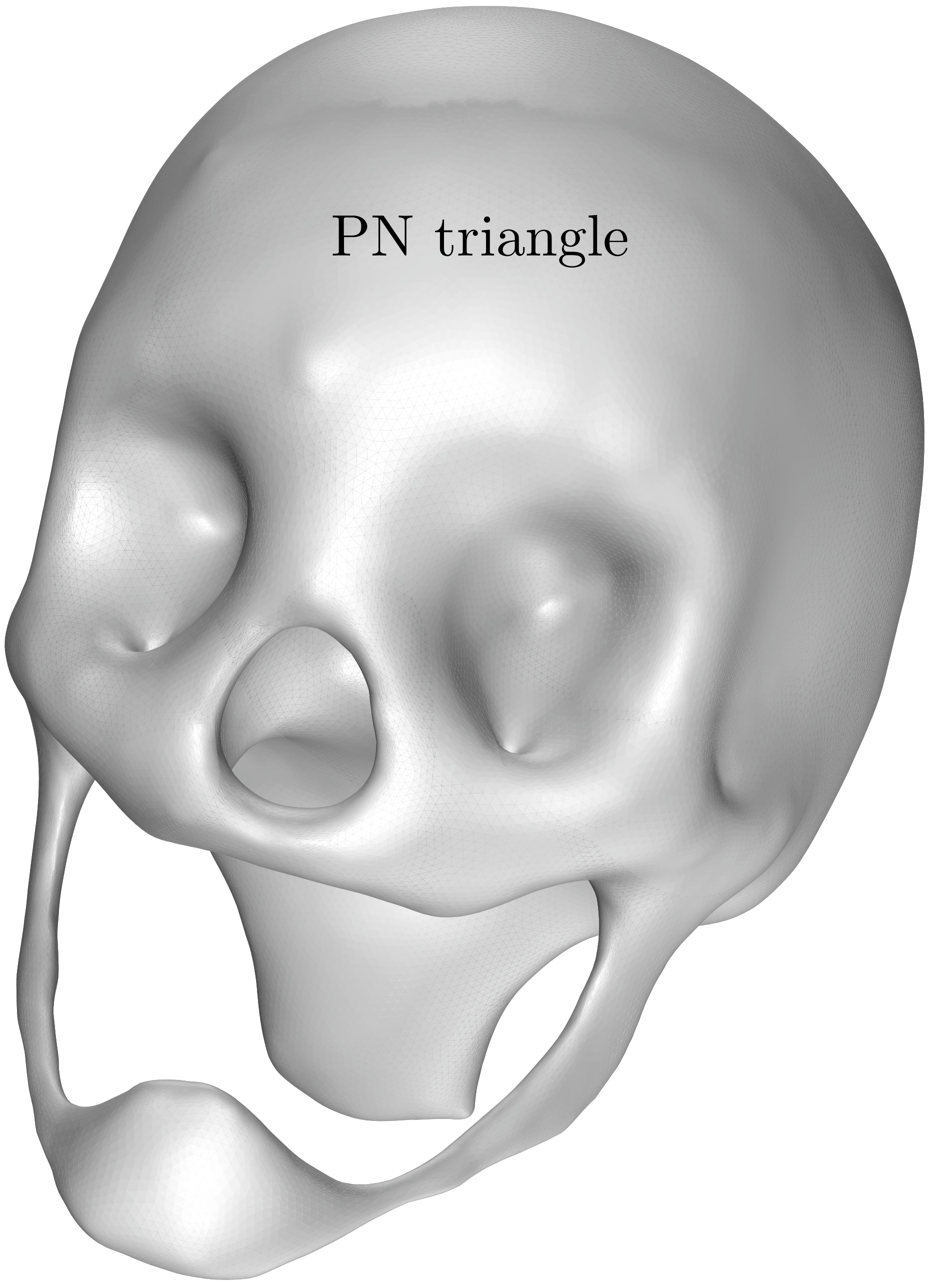
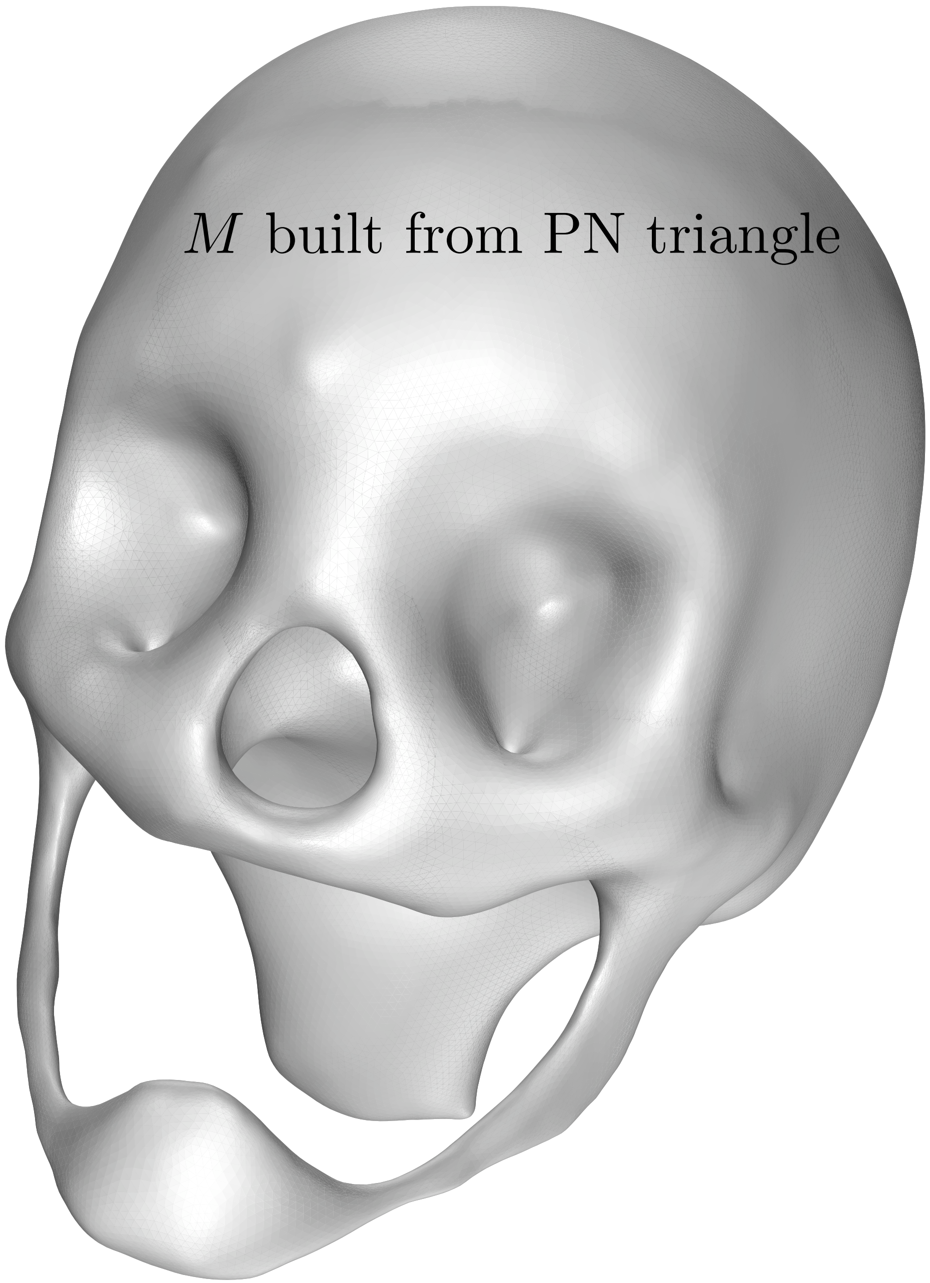
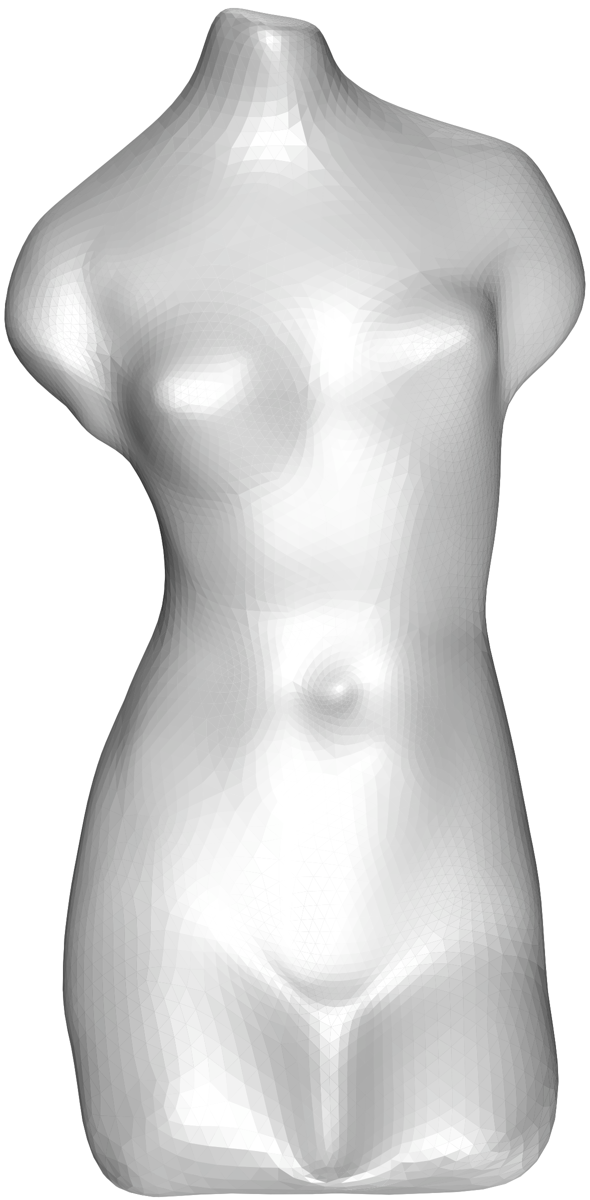




Experimental Results

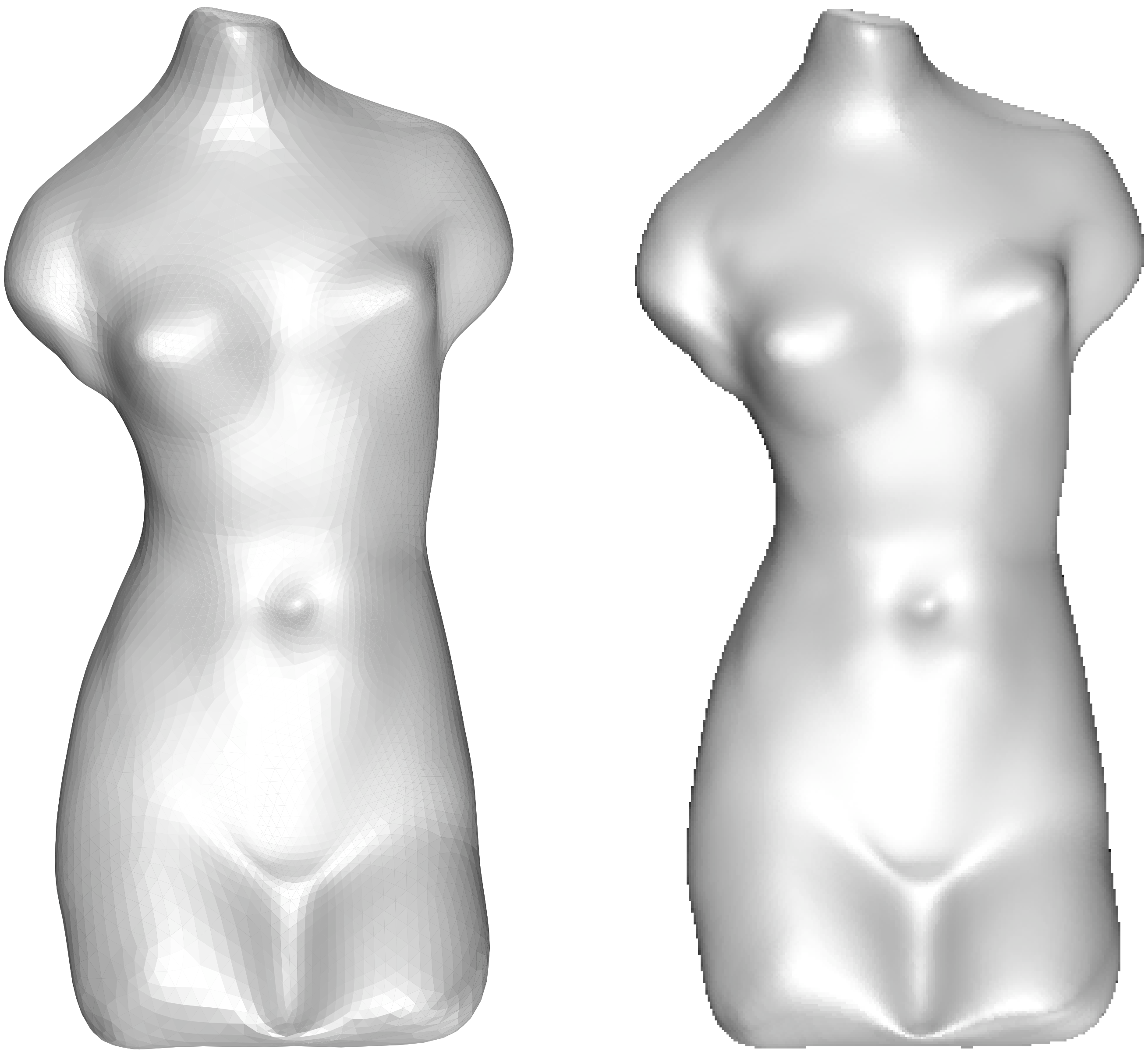
Experimental Results

S_T



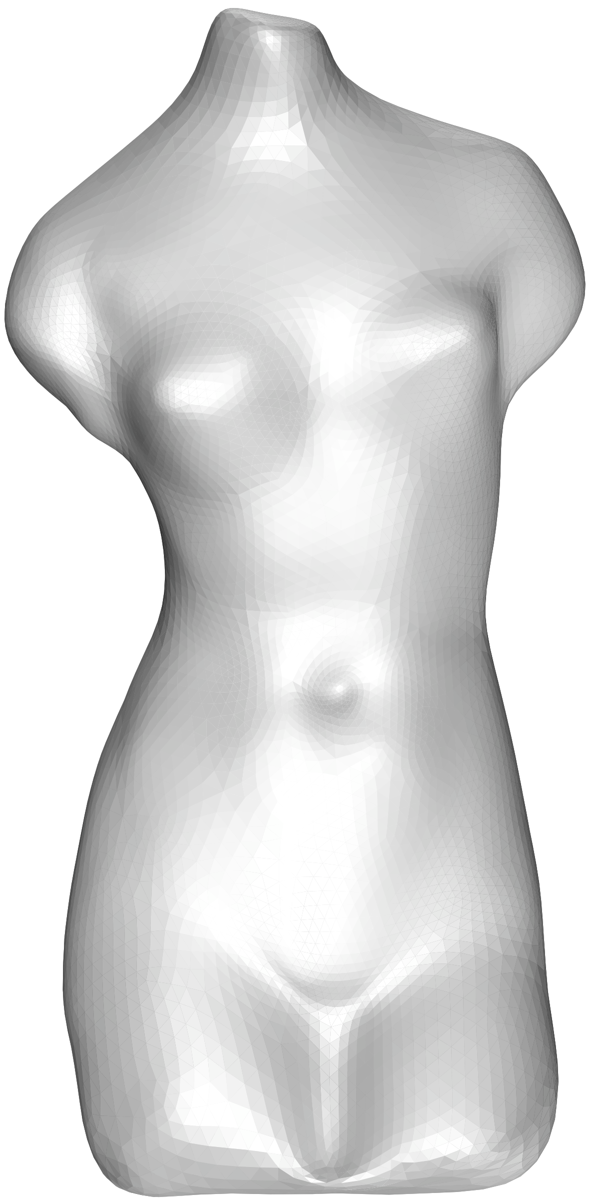
Experimental Results

S_T Catmull-Clark



Experimental Results

S_T



Catmull-Clark

M from Catmull-Clark