Chapter 6

Polar Duality, Polyhedra and Polytopes

6.1 Polarity and Duality

In this section, we apply the intrinsic duality afforded by a Euclidean structure to the study of convex sets and, in particular, polytopes.

Let $E = \mathbb{E}^n$ be a Euclidean space of dimension n. Pick any origin, O, in $\mathbb{E}^n$ (we may assume $O = (0, \ldots, 0)$).

We know that the inner product on $E = \mathbb{E}^n$ induces a duality between E and its dual E^*, namely, $u \mapsto \varphi_u$, where φ_u is the linear form defined by $\varphi_u(v) = u \cdot v$, for all $v \in E$.

For geometric purposes, it is more convenient to recast this duality as a correspondence between points and hyperplanes, using the notion of polarity with respect to the unit sphere, $S^{n-1} = \{a \in \mathbb{E}^n \mid \|\mathbf{Oa}\| = 1\}$.

First, we need the following simple fact: For every hyperplane, H, not passing through O, there is a *unique* point, h, so that

$$H = \{a \in \mathbb{E}^n \mid \mathbf{Oh} \cdot \mathbf{Oa} = 1\}.$$

Using the above, we make the following definition:

Definition 6.1.1 Given any point, $a \neq O$, the *polar hyperplane of* a *(w.r.t.* S^{n-1}*)* or *dual of* a is the hyperplane, $a^\dagger$, given by

$$a^\dagger = \{b \in \mathbb{E}^n \mid \mathbf{Oa} \cdot \mathbf{Ob} = 1\}.$$

Given a hyperplane, H, not containing O, the *pole of* H *(w.r.t* S^{n-1}) or *dual of* H is the (unique) point, $H^\dagger$, so that

$$H = \{a \in \mathbb{E}^n \mid \mathbf{OH}^\dagger \cdot \mathbf{Oa} = 1\}.$$

We often abbreviate polar hyperplane to polar.

We immediately check that $a^{\dagger\dagger} = a$ and $H^{\dagger\dagger} = H$, so, we obtain a bijective correspondence between $\mathbb{E}^n - \{O\}$ and the set of hyperplanes not passing through O.

When a is outside the sphere S^{n-1}, there is a nice geometric interpetation for the polar hyperplane, $H = a^{\dagger}$. Indeed, in this case, since

$$H = a^{\dagger} = \{b \in \mathbb{E}^n \mid \mathbf{Oa} \cdot \mathbf{Ob} = 1\}$$

and $\|\mathbf{Oa}\| > 1$, the hyperplane H intersects S^{n-1} (along an $(n-2)$-dimensional sphere) and if b is any point on $H \cap S^{n-1}$, we claim that $\mathbf{Ob}$ and $\mathbf{ba}$ are orthogonal.

This means that $H \cap S^{n-1}$ is the set of points on S^{n-1} where the lines through a and tangent to S^{n-1} touch S^{n-1} (they form a cone tangent to S^{n-1} with apex a).

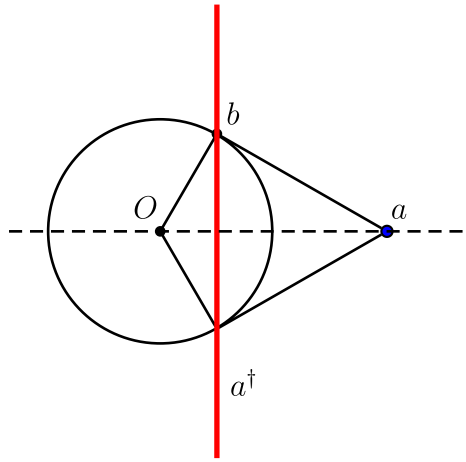

Figure 6.1: The polar, $a^\dagger$, of a point, a, outside the sphere S^{n-1}

Also, observe that for any point, $a \neq O$, and any hyperplane, H, not passing through O, if $a \in H$, then, $H^\dagger \in a^\dagger$, i.e, the pole, $H^\dagger$, of H belongs to the polar, $a^\dagger$, of a.

If $a = (a_1, \ldots, a_n)$, the equation of the polar hyperplane, $a^\dagger$, is

$$a_1 X_1 + \cdots + a_n X_n = 1.$$

Now, we would like to extend this correspondence to subsets of $\mathbb{E}^n$, in particular, to convex sets.

Given a hyperplane, H, not containing O, we denote by H_- the closed half-space containing O.

Definition 6.1.2 Given any subset, A, of $\mathbb{E}^n$, the set

$$A^* = \{b \in \mathbb{E}^n \mid \mathbf{Oa}{\cdot}\mathbf{Ob} \leq 1, \quad \text{for all } a \in A\} = \bigcap_{\substack{a\in A\\ a\neq O}} (a^\dagger)_-,$$

is called the *polar dual* or *reciprocal* of A.

To simplify notation we write $a^\dagger_-$ for $(a^\dagger)_-$. Note that $\{O\}^* = \mathbb{E}^n$, so it is convenient to set $O^\dagger_- = \mathbb{E}^n$, even though $O^\dagger$ is undefined.

We use a different notation, $a^\dagger$ and $H^\dagger$, for polar hyperplanes and poles, as opposed to A^*, for polar duals, to avoid confusion. Indeed, $H^\dagger$ and H^*, where H is a hyperplane (resp. $a^\dagger$ and $\{a\}^*$, where a is a point) are very different things!

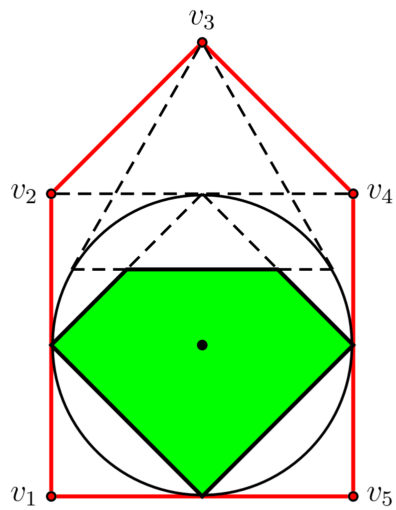

Figure 6.2: The polar dual of a polygon

In Figure 6.2, the polar dual of the polygon $(v_1, v_2, v_3, v_4, v_5)$ is the polygon shown in green.

This polygon is cut out by the half-planes determined by the polars of the vertices $(v_1, v_2, v_3, v_4, v_5)$ and containing the center of the circle.

By definition, A^* is convex even if A is not.

Furthermore, note that

(1) $A \subseteq A^{**}$.

(2) If $A \subseteq B$, then $B^* \subseteq A^*$.

(3) If A is convex and closed, then $A^* = (\partial A)^*$.

It follows immediately from (1) and (2) that $A^{***} = A^*$. Also, if $B^n(r)$ is the (closed) ball of radius $r > 0$ and center O, it is obvious by definition that $B^n(r)^* = B^n(1/r)$.

We would like to investigate the duality induced by the operation $A \mapsto A^*$.

Unfortunately, it is not always the case that $A^{**} = A$, but this is true when A is closed and convex, as shown in the following proposition:

Proposition 6.1.3 *Let A be any subset of $\mathbb{E}^n$ (with origin O).*

(i) If A is bounded, then $O \in \overset{\circ}{A^*}$; if $O \in \overset{\circ}{A}$, then A^* is bounded.

(ii) If A is a closed and convex subset containing O, then $A^{**} = A$.

Note that

$$\begin{aligned} A^{**} &= \{c \in \mathbb{E}^n \mid \mathbf{Od} \cdot \mathbf{Oc} \leq 1 \quad \text{for all } d \in A^*\} \\ &= \{c \in \mathbb{E}^n \mid (\forall d \in \mathbb{E}^n)(\text{if} \quad \mathbf{Od} \cdot \mathbf{Oa} \leq 1 \\ &\qquad\qquad \text{for all } a \in A, \quad \text{then} \quad \mathbf{Od} \cdot \mathbf{Oc} \leq 1)\}. \end{aligned}$$

Remark: For an arbitrary subset, $A \subseteq \mathbb{E}^n$, it can be shown that $A^{**} = \overline{\mathrm{conv}(A \cup \{O\})}$, the topological closure of the convex hull of $A \cup \{O\}$.

Proposition 6.1.3 will play a key role in studying polytopes, but before doing this, we need one more proposition.

Proposition 6.1.4 *Let A be any closed convex subset of $\mathbb{E}^n$ such that $O \in \overset{\circ}{A}$. The polar hyperplanes of the points of the boundary of A constitute the set of supporting hyperplanes of A^*. Furthermore, for any $a \in \partial A$, the points of A^* where $H = a^\dagger$ is a supporting hyperplane of A^* are the poles of supporting hyperplanes of A at a.*

6.2 Polyhedra, $\mathcal{H}$-Polytopes and $\mathcal{V}$-Polytopes

There are two natural ways to define a convex polyhedron, A:

(1) As the convex hull of a finite set of points.

(2) As a subset of $\mathbb{E}^n$ cut out by a finite number of hyperplanes, more precisely, as the intersection of a finite number of (closed) half-spaces.

As stated, these two definitions are not equivalent because (1) implies that a polyhedron is bounded, whereas (2) allows unbounded subsets.

Now, if we require in (2) that the convex set A is bounded, it is quite clear for $n = 2$ that the two definitions (1) and (2) are equivalent; for $n = 3$, it is intuitively clear that definitions (1) and (2) are still equivalent, but proving this equivalence rigorously does not appear to be that easy.

What about the equivalence when $n \geq 4$?

It turns out that definitions (1) and (2) are equivalent for all n, but this is a nontrivial theorem and a rigorous proof does not come by so cheaply.

Fortunately, since we have Krein and Milman's theorem at our disposal and polar duality, we can give a rather short proof.

The hard direction of the equivalence consists in proving that definition (1) implies definition (2).

This is where the duality induced by polarity becomes handy, especially, the fact that $A^{**} = A$! (under the right hypotheses).

First, we give precise definitions (following Ziegler [**?**]).

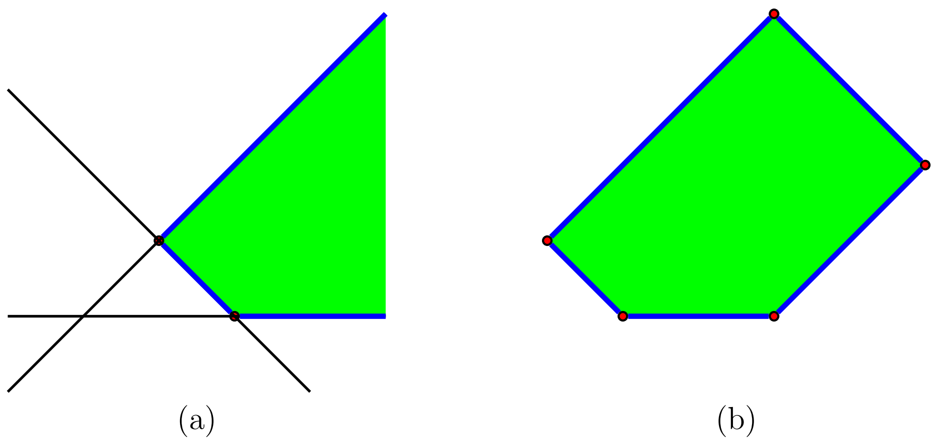

Figure 6.3: (a) An $\mathcal{H}$-polyhedron. (b) A $\mathcal{V}$-polytope

Definition 6.2.1 Let $\mathcal{E}$ be any affine Euclidean space of finite dimension, n.[1] An *$\mathcal{H}$-polyhedron* in $\mathcal{E}$, for short, a *polyhedron*, is any subset, $P = \bigcap_{i=1}^{p} C_i$, of $\mathcal{E}$ defined as the intersection of a finite number of closed half-spaces, C_i; an *$\mathcal{H}$-polytope* in $\mathcal{E}$ is a bounded polyhedron and a *$\mathcal{V}$-polytope* is the convex hull, $P = \mathrm{conv}(S)$, of a finite set of points, $S \subseteq \mathcal{E}$.

Examples of an $\mathcal{H}$-polyhedron and of a $\mathcal{V}$-polytope are shown in Figure 6.3.

[1]This means that the vector space, $\overrightarrow{\mathcal{E}}$, associated with $\mathcal{E}$ is a Euclidean space.

Obviously, polyhedra and polytopes are convex and closed (in $\mathcal{E}$). Since the notions of $\mathcal{H}$-polytope and $\mathcal{V}$-polytope are equivalent (see Theorem 6.3.1), we often use the simpler locution polytope.

Note that Definition 6.2.1 allows $\mathcal{H}$-polytopes and $\mathcal{V}$-polytopes to have an empty interior, which is sometimes an inconvenience.

This is not a problem. In fact, we can prove that we may always assume to $\mathcal{E} = \mathbb{E}^n$ and restrict ourselves to the affine hull of A (some copy of $\mathbb{E}^d$, for $d \leq n$, where $d = \dim(A)$, as in Definition 3.2.3).

Since the boundary of a closed half-space, C_i, is a hyperplane, H_i, and since hyperplanes are defined by affine forms, a closed half-space is defined by the locus of points satisfying a "linear" inequality of the form $a_i \cdot x \leq b_i$ or $a_i \cdot x \geq b_i$, for some vector $a_i \in \mathbb{R}^n$ and some $b_i \in \mathbb{R}$.

Since $a_i \cdot x \geq b_i$ is equivalent to $(-a_i) \cdot x \leq -b_i$, we may restrict our attention to inequalities with a $\leq$ sign.

Thus, if A is the $d \times p$ matrix whose i^{th} row is a_i, we see that the $\mathcal{H}$-polyhedron, P, is defined by the system of linear inequalities, $Ax \leq b$, where $b = (b_1, \ldots, b_p) \in \mathbb{R}^p$.

We write

$$P = P(A, b), \quad \textit{with} \quad P(A, b) = \{x \in \mathbb{R}^n \mid Ax \leq b\}.$$

An equation, $a_i \cdot x = b_i$, may be handled as the conjunction of the two inequalities $a_i \cdot x \leq b_i$ and $(-a_i) \cdot x \leq -b_i$.

Also, if $0 \in P$, observe that we must have $b_i \geq 0$ for $i = 1, \ldots, p$. In this case, every inequality for which $b_i > 0$ can be normalized by dividing both sides by b_i, so we may assume that $b_i = 1$ or $b_i = 0$.

Remark: Some authors call "convex" polyhedra and "convex" polytopes what we have simply called polyhedra and polytopes.

Since Definition 6.2.1 implies that these objects are convex and since we are not going to consider non-convex polyhedra in this chapter, we stick to the simpler terminology.

One should consult Ziegler [**?**], Berger [**?**], Grunbaum [**?**] and especially Cromwell [**?**], for pictures of polyhedra and polytopes.

Even better, take a look at the web sites listed in the web page for CIS610!

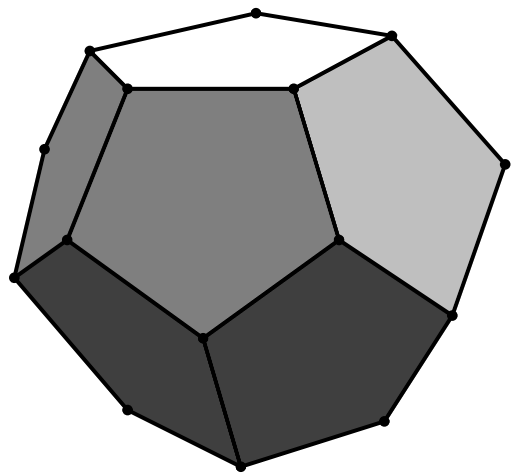

Figure 6.4: Example of a polytope (a dodecahedron)

Figure 6.4 shows the picture a polytope whose faces are all pentagons. This polytope is called a *dodecahedron*. The dodecahedron has 12 faces, 30 edges and 20 vertices.

Obviously, an n-simplex is a $\mathcal{V}$-polytope. The *standard n-cube* is the set

$$\{(x_1, \ldots, x_n) \in \mathbb{E}^n \mid |x_i| \leq 1, \quad 1 \leq i \leq n\}.$$

The standard cube is a $\mathcal{V}$-polytope. The *standard n-cross-polytope* (or *n-co-cube*) is the set

$$\{(x_1, \ldots, x_n) \in \mathbb{E}^n \mid \sum_{i=1}^{n} |x_i| \leq 1\}.$$

It is also a $\mathcal{V}$-polytope.

What happens if we take the dual of a $\mathcal{V}$-polytope (resp. an $\mathcal{H}$-polytope)? The following proposition, although very simple, is an important step in answering the above question.

Proposition 6.2.2 *Let $S = \{a_i\}_{i=1}^{p}$ be a finite set of points in $\mathbb{E}^n$ and let $A = \mathrm{conv}(S)$ be its convex hull. If $S \neq \{O\}$, then, the dual, A^*, of A w.r.t. the center O is an $\mathcal{H}$-polyhedron; furthermore, if $O \in \overset{\circ}{A}$, then A^* is an $\mathcal{H}$-polytope, i.e., the dual of a $\mathcal{V}$-polytope with nonempty interior is an $\mathcal{H}$-polytope. If $A = S = \{O\}$, then $A^* = \mathbb{E}^d$.*

Thus, the dual of the convex hull of a finite set of points, $\{a_1, \ldots, a_p\}$, is the intersection of the half-spaces containing O determined by the polar hyperplanes of the points a_i. (Recall that $(a_i)^{\dagger}_{-} = \mathbb{E}^n$ if $a_i = O$.)

It is convenient to restate Proposition 6.2.2 using matrices.

First, observe that the proof of Proposition 6.2.2 shows that

$$\mathrm{conv}(\{a_1, \ldots, a_p\})^* = \mathrm{conv}(\{a_1, \ldots, a_p\} \cup \{O\})^*.$$

Therefore, we may assume that not all $a_i = 0$ $(1 \leq i \leq p)$. If we pick O as an origin, then every point a_j can be identified with a vector in $\mathbb{E}^n$ and O corresponds to the zero vector, 0.

Observe that any set of p points, $a_j \in \mathbb{E}^n$, corresponds to the $n \times p$ matrix, A, whose j^{th} column is a_j.

Then, the equation of the the polar hyperplane, $a_j^\dagger$, of any a_j ($\neq 0$) is $a_j \cdot x = 1$, that is

$$a_j^\top x = 1.$$

Consequently, the system of inequalities defining $\mathrm{conv}(\{a_1, \ldots, a_p\})^*$ can be written in matrix form as

$$\mathrm{conv}(\{a_1, \ldots, a_p\})^* = \{x \in \mathbb{R}^n \mid A^\top x \leq \mathbf{1}\},$$

where $\mathbf{1}$ denotes the vector of $\mathbb{R}^p$ with all coordinates equal to 1. We write
$P(A^\top, \mathbf{1}) = \{x \in \mathbb{R}^n \mid A^\top x \leq \mathbf{1}\}$.

Proposition 6.2.3 *Given any set of p points, $\{a_1, \ldots, a_p\}$, in $\mathbb{R}^n$ with $\{a_1, \ldots, a_p\} \neq \{0\}$, if A is the $n \times p$ matrix whose j^{th} column is a_j, then*

$$\mathrm{conv}(\{a_1, \ldots, a_p\})^* = P(A^\top, \mathbf{1}),$$

with $P(A^\top, \mathbf{1}) = \{x \in \mathbb{R}^n \mid A^\top x \leq \mathbf{1}\}$.

Conversely, given any $p \times n$ matrix, A, not equal to the zero matrix, we have

$$P(A, \mathbf{1})^* = \mathrm{conv}(\{a_1, \ldots, a_p\} \cup \{0\}),$$

where $a_i \in \mathbb{R}^n$ is the i^{th} row of A or, equivalently,

$$P(A, \mathbf{1})^* = \{x \in \mathbb{R}^n \mid x = A^\top t,\ t \in \mathbb{R}^p,\ t \geq 0,\ \mathbb{I}t = 1\},$$

where $\mathbb{I}$ is the row vector of length p whose coordinates are all equal to 1.

Using the above, the reader should check that the dual of a simplex is a simplex and that the dual of an n-cube is an n-cross polytope.

We will see shortly that if A is an $\mathcal{H}$-polytope and if $O \in \overset{\circ}{A}$, then A^* is also an $\mathcal{H}$-polytope.

For this, we will prove first that an $\mathcal{H}$-polytope is a $\mathcal{V}$-polytope. This requires taking a closer look at polyhedra.

Note that some of the hyperplanes cutting out a polyhedron may be redundant.

If $A = \bigcap_{i=1}^{t} C_i$ is a polyhedron (where each closed half-space, C_i, is associated with a hyperplane, H_i, so that $\partial C_i = H_i$), we say that $\bigcap_{i=1}^{t} C_i$ *is an irredundant decomposition of* A if A cannot be expressed as
$A = \bigcap_{i=1}^{m} C_i'$ with $m < t$ (for some closed half-spaces, C_i').

Proposition 6.2.4 *Let A be a polyhedron with nonempty interior and assume that $A = \bigcap_{i=1}^{t} C_i$ is an irredundant decomposition of A. Then,*

(i) Up to order, the C_i's are uniquely determined by A.

(ii) If $H_i = \partial C_i$ is the boundary of C_i, then $H_i \cap A$ is a polyhedron with nonempty interior in H_i, denoted $\mathrm{Facet}_i\, A$*, and called a facet of A.*

(iii) We have $\partial A = \bigcup_{i=1}^{t} \mathrm{Facet}_i\, A$, where the union is irredundant, i.e., $\mathrm{Facet}_i\, A$ *is not a subset of* $\mathrm{Facet}_j\, A$*, for all $i \neq j$.*

As a consequence, if A is a polyhedron, then so are its facets and the same holds for $\mathcal{H}$-polytopes.

If A is an $\mathcal{H}$-polytope and H is a hyperplane with $H \cap \overset{\circ}{A} \neq \emptyset$, then $H \cap A$ is an $\mathcal{H}$-polytope whose facets are of the form $H \cap F$, where F is a facet of A.

We can use induction and define k-faces, for $0 \leq k \leq n-1$.

Definition 6.2.5 Let $A \subseteq \mathbb{E}^n$ be a polyhedron with nonempty interior. We define a *k-face of A* to be a facet of a $(k+1)$-face of A, for $k = 0, \ldots, n-2$, where an *$(n-1)$-face* is just a facet of A. The 1-faces are called *edges*. Two k-faces are *adjacent* if their intersection is a $(k-1)$-face.

The polyhedron A itself is also called a *face* (of itself) or *n-face* and the k-faces of A with $k \leq n-1$ are called *proper faces* of A.

If $A = \bigcap_{i=1}^{t} C_i$ is an irredundant decomposition of A and H_i is the boundary of C_i, then the hyperplane, H_i, is called the *supporting hyperplane* of the facet $H_i \cap A$.

We suspect that the 0-faces of a polyhedron are vertices in the sense of Definition 3.4.1.

This is true and, in fact, the vertices of a polyhedron coincide with its extreme points (see Definition 3.4.3).

Proposition 6.2.6 *Let $A \subseteq \mathbb{E}^n$ be a polyhedron with nonempty interior.*

(1) For any point, $a \in \partial A$, on the boundary of A, the intersection of all the supporting hyperplanes to A at a coincides with the intersection of all the faces that contain a. In particular, points of order k of A are those points in the relative interior of the k-faces of A[2]; thus, 0-faces coincide with the vertices of A.

(2) The vertices of A coincide with the extreme points of A.

We are now ready for the theorem showing the equivalence of $\mathcal{V}$-polytopes and $\mathcal{H}$-polytopes.

[2]Given a convex set, S, in $\mathbb{A}^n$, its *relative interior* is its interior in the affine hull of S (which might be of dimension strictly less than n).

6.3 The Equivalence of $\mathcal{H}$-Polytopes and $\mathcal{V}$-Polytopes

The next result is a nontrivial theorem usually attributed to Weyl and Minkowski (see Barvinok [**?**]).

Theorem 6.3.1 *(Weyl-Minkowski) If A is an $\mathcal{H}$-polytope, then A has a finite number of extreme points (equal to its vertices) and A is the convex hull of its set of vertices; thus, an $\mathcal{H}$-polytope is a $\mathcal{V}$-polytope. Moreover, A has a finite number of k-faces (for $k = 0, \ldots, d-2$, where $d = \dim(A)$). Conversely, the convex hull of a finite set of points is an $\mathcal{H}$-polytope. As a consequence, a $\mathcal{V}$-polytope is an $\mathcal{H}$-polytope.*

In view of Theorem 6.3.1, we are justified in dropping the $\mathcal{V}$ or $\mathcal{H}$ in front of polytope, and will do so from now on.

Theorem 6.3.1 has some interesting corollaries regarding the dual of a polytope.

Corollary 6.3.2 *If A is any polytope in $\mathbb{E}^n$ such that the interior of A contains the origin, O, then the dual, A^*, of A is also a polytope whose interior contains O and $A^{**} = A$.*

Corollary 6.3.3 *If A is any polytope in $\mathbb{E}^n$ whose interior contains the origin, O, then the k-faces of A are in bijection with the $(n-k-1)$-faces of the dual polytope, A^*. This correspondence is as follows: If $Y = \mathrm{aff}(F)$ is the k-dimensional subspace determining the k-face, F, of A then the subspace, $Y^* = \mathrm{aff}(F^*)$, determining the corresponding face, F^*, of A^*, is the intersection of the polar hyperplanes of points in Y.*

We also have the following proposition whose proof would not be that simple if we only had the notion of an $\mathcal{H}$-polytope.

Proposition 6.3.4 *If $A \subseteq \mathbb{E}^n$ is a polytope and $f\colon \mathbb{E}^n \to \mathbb{E}^m$ is an affine map, then $f(A)$ is a polytope in $\mathbb{E}^m$.*

The reader should check that the Minkowski sum of polytopes is a polytope.

We were able to give a short proof of Theorem 6.3.1 because we relied on a powerful theorem, namely, Krein and Milman.

A drawback of this approach is that it bypasses the interesting and important problem of designing algorithms for finding the vertices of an $\mathcal{H}$-polyhedron from the sets of inequalities defining it.

A method for doing this is Fourier-Motzkin elimination, see Ziegler [**?**] (Chapter 1). This is also a special case of *linear programming*.

It is also possible to generalize the notion of $\mathcal{V}$-polytope to polyhedra using the notion of cone.

6.4 The Equivalence of $\mathcal{H}$-Polyhedra and $\mathcal{V}$-Polyhedra

The equivalence of $\mathcal{H}$-polytopes and $\mathcal{V}$-polytopes can be generalized to polyhedral sets, *i.e.*, finite intersections of half-spaces that are not necessarily bounded. This equivalence was first proved by Motzkin in the early 1930's.

Definition 6.4.1 Let $\mathcal{E}$ be any affine Euclidean space of finite dimension, d (with associated vector space, $\overrightarrow{\mathcal{E}}$). A subset, $C \subseteq \overrightarrow{\mathcal{E}}$, is a *cone* if C is closed under linear combinations involving only nonnnegative scalars. Given a subset, $V \subseteq \overrightarrow{\mathcal{E}}$, the *conical hull* or *positive hull* of V is the set

$$\mathrm{cone}(V) = \{\sum_I \lambda_i v_i \mid \{v_i\}_{i \in I} \subseteq V, \lambda_i \geq 0 \quad \text{for all } i \in I\}.$$

A *$\mathcal{V}$-polyhedron* or *polyhedral set* is a subset, $A \subseteq \mathcal{E}$, such that

$$\begin{aligned} A = \mathrm{conv}(Y) + \mathrm{cone}(V) & \\ &= \{a + v \mid a \in \mathrm{conv}(Y),\ v \in \mathrm{cone}(V)\}, \end{aligned}$$

where $V \subseteq \overrightarrow{\mathcal{E}}$ is a finite set of vectors and $Y \subseteq \mathcal{E}$ is a finite set of points.

A set, $C \subseteq \overrightarrow{\mathcal{E}}$, is a *$\mathcal{V}$-cone* or *polyhedral cone* if C is the positive hull of a finite set of vectors, that is,

$$C = \text{cone}(\{u_1, \ldots, u_p\}),$$

for some vectors, $u_1, \ldots, u_p \in \overrightarrow{\mathcal{E}}$. An *$\mathcal{H}$-cone* is any subset of $\overrightarrow{\mathcal{E}}$ given by a finite intersection of closed half-spaces cut out by hyperplanes through 0.

The positive hull, cone(V), of V is also denoted pos(V).

Observe that a $\mathcal{V}$-cone can be viewed as a polyhedral set for which $Y = \{O\}$, a single point.

However, if we take the point O as the origin, we may view a $\mathcal{V}$-polyhedron, A, for which $Y = \{O\}$, as a $\mathcal{V}$-cone.

We will switch back and forth between these two views of cones as we find it convenient

As a consequence, a ($\mathcal{V}$ or $\mathcal{H}$)-cone always contains 0, sometimes called an *apex* of the cone.

We can prove that we may always assume that $\mathcal{E} = \mathbb{E}^d$ and that our polyhedra have nonempty interior. It will be convenient to decree that $\mathbb{E}^d$ is an $\mathcal{H}$-polyhedron.

The generalization of Theorem 6.3.1 is that every $\mathcal{V}$-polyhedron is an $\mathcal{H}$-polyhedron and conversely.

Ziegler proceeds as follows: First, he shows that the equivalence of $\mathcal{V}$-polyhedra and $\mathcal{H}$-polyhedra reduces to the equivalence of $\mathcal{V}$-cones and $\mathcal{H}$-cones using an "old trick" of projective geometry, namely, "homogenizing" [**?**] (Chapter 1).

Then, he uses two dual versions of Fourier-Motzkin elimination to pass from $\mathcal{V}$-cones to $\mathcal{H}$-cones and conversely.

Since the homogenization method is an important technique we will describe it in some detail.

However, it turns out that the double dualization technique used in the proof of Theorem 6.3.1 can be easily adapted to prove that every $\mathcal{V}$-polyhedron is an $\mathcal{H}$-polyhedron.

Moreover, it can also be used to prove that every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron!

So, we will not describe the version of Fourier-Motzkin elimination used to go from $\mathcal{V}$-cones to $\mathcal{H}$-cones.

However, we will present the Fourier-Motzkin elimination method used to go from $\mathcal{H}$-cones to $\mathcal{V}$-cones.

In order to avoid confusion between the zero vector and the origin of $\mathbb{E}^d$, we will denote the origin by O and the center of polar duality by Ω.

Given any nonzero vector, $u \in \mathbb{R}^d$, let $u^\dagger_-$ be the closed half-space

$$u^\dagger_- = \{x \in \mathbb{R}^d \mid x \cdot u \leq 0\}.$$

In other words, $u^\dagger_-$ is the closed-half space bounded by the hyperplane through Ω normal to u and on the "opposite side" of u.

Proposition 6.4.2 *Let $A = \mathrm{conv}(Y) + \mathrm{cone}(V) \subseteq \mathbb{E}^d$ be a $\mathcal{V}$-polyhedron with $Y = \{y_1, \ldots, y_p\}$ and $V = \{v_1, \ldots, v_q\}$. Then, for any point, Ω, if $A \neq \{\Omega\}$, then the polar dual, A^*, of A w.r.t. Ω is the $\mathcal{H}$-polyhedron given by*

$$A^* = \bigcap_{i=1}^{p} (y_i^\dagger)_- \cap \bigcap_{j=1}^{q} (v_j^\dagger)_-.$$

Furthermore, if A has nonempty interior and Ω belongs to the interior of A, then A^* is bounded, that is, A^* is an $\mathcal{H}$-polytope. If $A = \{\Omega\}$, then A^* is the special polyhedron, $A^* = \mathbb{E}^d$.

It is fruitful to restate Proposition 6.4.2 in terms of matrices (as we did for Proposition 6.2.2).

First, observe that

$$(\mathrm{conv}(Y) + \mathrm{cone}(V))^* = (\mathrm{conv}(Y \cup \{\Omega\}) + \mathrm{cone}(V))^*.$$

If we pick Ω as an origin then we can represent the points in Y as vectors. The old origin is still denoted O and Ω is now denoted 0. The zero vector is denoted $\mathbf{0}$.

If Y is the $d \times p$ matrix whose i^{th} column is y_i and V is the $d \times q$ matrix whose j^{th} column is v_j, then A^* is given by:

$$A^* = \{x \in \mathbb{R}^d \mid Y^\top x \leq \mathbf{1},\ V^\top x \leq \mathbf{0}\}.$$

We write
$P(Y^\top, \mathbf{1}; V^\top, \mathbf{0}) = \{x \in \mathbb{R}^d \mid Y^\top x \leq \mathbf{1},\ V^\top x \leq \mathbf{0}\}.$

Proposition 6.4.3 *Let $\{y_1, \ldots, y_p\}$ be any set of points in $\mathbb{E}^d$ and let $\{v_1, \ldots, v_q\}$ be any set of nonzero vectors in $\mathbb{R}^d$. If Y is the $d \times p$ matrix whose i^{th} column is y_i and V is the $d \times q$ matrix whose j^{th} column is v_j, then*

$$(\mathrm{conv}(\{y_1, \ldots, y_p\}) \cup \mathrm{cone}(\{v_1, \ldots, v_q\}))^* = P(Y^\top, \mathbf{1}; V^\top, \mathbf{0}),$$

with

$$P(Y^\top, \mathbf{1}; V^\top, \mathbf{0}) = \{x \in \mathbb{R}^d \mid Y^\top x \leq \mathbf{1},\ V^\top x \leq \mathbf{0}\}.$$

Conversely, given any $p \times d$ matrix, Y, and any $q \times d$ matrix, V, we have

$$P(Y, \mathbf{1}; V, \mathbf{0})^* = \mathrm{conv}(\{y_1, \ldots, y_p\} \cup \{0\}) \cup \mathrm{cone}(\{v_1, \ldots, v_q\}),$$

where $y_i \in \mathbb{R}^n$ is the i^{th} row of Y and $v_j \in \mathbb{R}^n$ is the j^{th} row of V or, equivalently,

$$P(Y, \mathbf{1}; V, \mathbf{0})^* = \{x \in \mathbb{R}^d \mid x = Y^\top u + V^\top t,\ u \in \mathbb{R}^p,\ t \in \mathbb{R}^q,\ u, t \geq 0,\ \mathbb{I}u = 1\},$$

where $\mathbb{I}$ is the row vector of length p whose coordinates are all equal to 1.

We can now use Proposition 6.4.2, Proposition 6.1.3 and Krein and Millman's Theorem to prove that every $\mathcal{V}$-polyhedron is an $\mathcal{H}$-polyhedron.

Proposition 6.4.4 *Every $\mathcal{V}$-polyhedron, A, is an $\mathcal{H}$-polyhedron. Furthermore, if $A \neq \mathbb{E}^d$, then A is of the form $A = P(Y, \mathbf{1})$.*

Interestingly, we can now prove easily that every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron.

Proposition 6.4.5 *Every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron.*

Putting together Propositions 6.4.4 and 6.4.5 we obtain our main theorem:

Theorem 6.4.6 *(Equivalence of $\mathcal{H}$-polyhedra and $\mathcal{V}$-polyhedra) Every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron and conversely.*

Even though we proved the main result of this section, it is instructive to consider a more computational proof making use of cones and an elimination method known as *Fourier-Motzkin elimination*.

The problem with the converse of Proposition 6.4.4 when A is unbounded (*i.e.*, not compact) is that Krein and Millman's Theorem does not apply.

We need to take into account "points at infinity" corresponding to certain vectors.

The trick we used in Proposition 6.4.4 is that the polar dual of a $\mathcal{V}$-polyhedron with nonempty interior is an $\mathcal{H}$-*polytope*.

This reduction to polytopes allowed us to use Krein and Millman to convert an $\mathcal{H}$-polytope to a $\mathcal{V}$-polytope and then again we took the polar dual.

Another trick is to switch to cones by “homogenizing”.

Given any subset, $S \subseteq \mathbb{E}^d$, we can form the cone, $C(S) \subseteq \mathbb{E}^{d+1}$, by “placing” a copy of S in the hyperplane, $H_{d+1} \subseteq \mathbb{E}^{d+1}$, of equation $x_{d+1} = 1$, and drawing all the half lines from the origin through any point of S.

Let $P \subseteq \mathbb{E}^d$ be an $\mathcal{H}$-polyhedron. Then, P is cut out by m hyperplanes, H_i, and for each H_i, there is a nonzero vector, a_i, and some $b_i \in \mathbb{R}$ so that

$$H_i = \{x \in \mathbb{E}^d \mid a_i \cdot x = b_i\}$$

and P is given by

$$P = \bigcap_{i=1}^{m} \{x \in \mathbb{E}^d \mid a_i \cdot x \leq b_i\}.$$

If A denotes the $m \times d$ matrix whose i-th row is a_i and b is the vector $b = (b_1, \ldots, b_m)$, then we can write

$$P = P(A, b) = \{x \in \mathbb{E}^d \mid Ax \leq b\}.$$

We "homogenize" $P(A,b)$ as follows: Let $C(P)$ be the subset of $\mathbb{E}^{d+1}$ defined by

$$
\begin{aligned}
C(P) &= \left\{ \begin{pmatrix} x \\ x_{d+1} \end{pmatrix} \in \mathbb{R}^{d+1} \mid Ax \le x_{d+1}b,\ x_{d+1} \ge 0 \right\} \\
&= \left\{ \begin{pmatrix} x \\ x_{d+1} \end{pmatrix} \mid Ax - x_{d+1}b \le 0,\ -x_{d+1} \le 0 \right\}.
\end{aligned}
$$

Thus, we see that $C(P)$ is the $\mathcal{H}$-cone given by the system of inequalities

$$
\begin{pmatrix} A & -b \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ x_{d+1} \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \end{pmatrix}
$$

and that

$$
P = C(P) \cap H_{d+1}.
$$

Conversely, if Q is any $\mathcal{H}$-cone in $\mathbb{E}^{d+1}$ (in fact, any $\mathcal{H}$-polyhedron), it is clear that $P = Q \cap H_{d+1}$ is an $\mathcal{H}$-polyhedron in $\mathbb{E}^d$.

Let us now assume that
$P \subseteq \mathbb{E}^d$ is a $\mathcal{V}$-polyhedron, $P = \mathrm{conv}(Y) + \mathrm{cone}(V)$, where $Y = \{y_1, \ldots, y_p\}$ and $V = \{v_1, \ldots, v_q\}$.

Define $\widehat{Y} = \{\widehat{y}_1, \ldots, \widehat{y}_p\} \subseteq \mathbb{E}^{d+1}$, and
$\widehat{V} = \{\widehat{v}_1, \ldots, \widehat{v}_q\} \subseteq \mathbb{E}^{d+1}$, by

$$\widehat{y}_i = \begin{pmatrix} y_i \\ 1 \end{pmatrix}, \qquad \widehat{v}_j = \begin{pmatrix} v_j \\ 0 \end{pmatrix}.$$

We check immediately that

$$C(P) = \mathrm{cone}(\{\widehat{Y} \cup \widehat{V}\})$$

is a $\mathcal{V}$-cone in $\mathbb{E}^{d+1}$ such that

$$C = C(P) \cap H_{d+1},$$

where H_{d+1} is the hyperplane of equation $x_{d+1} = 1$.

Conversely, if $C = \mathrm{cone}(W)$ is a $\mathcal{V}$-cone in $\mathbb{E}^{d+1}$, with $w_{i\,d+1} \geq 0$ for every $w_i \in W$, we prove next that
$P = C \cap H_{d+1}$ is a $\mathcal{V}$-polyhedron.

Proposition 6.4.7 *(Polyhedron–Cone Correspondence)* *We have the following correspondence between polyhedra in $\mathbb{E}^d$ and cones in $\mathbb{E}^{d+1}$:*

(1) For any $\mathcal{H}$-polyhedron, $P \subseteq \mathbb{E}^d$, if $P = P(A, b) = \{x \in \mathbb{E}^d \mid Ax \leq b\}$, where A is an $m \times d$-matrix and $b \in \mathbb{R}^m$, then $C(P)$ given by

$$\begin{pmatrix} A & -b \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ x_{d+1} \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is an $\mathcal{H}$-cone in $\mathbb{E}^{d+1}$ and $P = C(P) \cap H_{d+1}$, where H_{d+1} is the hyperplane of equation $x_{d+1} = 1$. Conversely, if Q is any $\mathcal{H}$-cone in $\mathbb{E}^{d+1}$ (in fact, any $\mathcal{H}$-polyhedron), then $P = Q \cap H_{d+1}$ is an $\mathcal{H}$-polyhedron in $\mathbb{E}^d$.

(2) Let $P \subseteq \mathbb{E}^d$ *be any* $\mathcal{V}$*-polyhedron, where* $P = \mathrm{conv}(Y) + \mathrm{cone}(V)$ *with* $Y = \{y_1, \ldots, y_p\}$ *and* $V = \{v_1, \ldots, v_q\}$*. Define* $\widehat{Y} = \{\widehat{y}_1, \ldots, \widehat{y}_p\} \subseteq \mathbb{E}^{d+1}$*, and* $\widehat{V} = \{\widehat{v}_1, \ldots, \widehat{v}_q\} \subseteq \mathbb{E}^{d+1}$*, by*

$$\widehat{y}_i = \begin{pmatrix} y_i \\ 1 \end{pmatrix}, \qquad \widehat{v}_j = \begin{pmatrix} v_j \\ 0 \end{pmatrix}.$$

Then,

$$C(P) = \mathrm{cone}(\{\widehat{Y} \cup \widehat{V}\})$$

is a $\mathcal{V}$*-cone in* $\mathbb{E}^{d+1}$ *such that*

$$C = C(P) \cap H_{d+1},$$

Conversely, if $C = \mathrm{cone}(W)$ *is a* $\mathcal{V}$*-cone in* $\mathbb{E}^{d+1}$*, with* $w_{i\,d+1} \geq 0$ *for every* $w_i \in W$*, then* $P = C \cap H_{d+1}$ *is a* $\mathcal{V}$*-polyhedron in* $\mathbb{E}^d$*.*

By Proposition 6.4.7, if P is an $\mathcal{H}$-polyhedron, then $C(P)$ is an $\mathcal{H}$-cone. If we can prove that every $\mathcal{H}$-cone is a $\mathcal{V}$-cone, then again, Proposition 6.4.7 shows that
$P = C(P) \cap H_{d+1}$ is a $\mathcal{V}$-polyhedron.

Therefore, in order to prove that every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron it suffices to show that every $\mathcal{H}$-cone is a $\mathcal{V}$-cone.

By a similar argument, Proposition 6.4.7 show that in order to prove that every $\mathcal{V}$-polyhedron is an $\mathcal{H}$-polyhedron it suffices to show that every $\mathcal{V}$-cone is an $\mathcal{H}$-cone.

We will not prove this direction again since we already have it by Proposition 6.4.4.

It remains to prove that every $\mathcal{H}$-cone is a $\mathcal{V}$-cone.

Let $C \subseteq \mathbb{E}^d$ be an $\mathcal{H}$-cone. Then, C is cut out by m hyperplanes, H_i, through 0.

For each H_i, there is a nonzero vector, u_i, so that

$$H_i = \{x \in \mathbb{E}^d \mid u_i \cdot x = 0\}$$

and C is given by

$$C = \bigcap_{i=1}^{m} \{x \in \mathbb{E}^d \mid u_i \cdot x \leq 0\}.$$

If A denotes the $m \times d$ matrix whose i-th row is u_i, then we can write

$$C = P(A, 0) = \{x \in \mathbb{E}^d \mid Ax \leq 0\}.$$

Observe that $C = C_0(A) \cap H_w$, where

$$C_0(A) = \left\{ \begin{pmatrix} x \\ w \end{pmatrix} \in \mathbb{R}^{d+m} \mid Ax \leq w \right\}$$

is an $\mathcal{H}$-cone in $\mathbb{E}^{d+m}$ and

$$H_w = \left\{ \begin{pmatrix} x \\ w \end{pmatrix} \in \mathbb{R}^{d+m} \mid w = 0 \right\}$$

is a hyperplane in $\mathbb{E}^{d+m}$.

We claim that $C_0(A)$ is a $\mathcal{V}$-cone.

This follows by observing that for every $\begin{pmatrix} x \\ w \end{pmatrix}$ satisfying $Ax \leq w$, we can write

$$\begin{pmatrix} x \\ w \end{pmatrix} = \sum_{i=1}^{d} |x_i|(\mathrm{sign}(x_i)) \begin{pmatrix} e_i \\ Ae_i \end{pmatrix} + \sum_{j=1}^{m} (w_j - (Ax)_j) \begin{pmatrix} 0 \\ e_j \end{pmatrix},$$

and then

$$C_0(A) = \mathrm{cone}\left(\left\{ \pm \begin{pmatrix} e_i \\ Ae_i \end{pmatrix} \mid 1 \leq i \leq d \right\} \cup \left\{ \begin{pmatrix} 0 \\ e_j \end{pmatrix} \mid 1 \leq j \leq m \right\}\right).$$

Since $C = C_0(A) \cap H_w$ is now the intersection of a $\mathcal{V}$-cone with a hyperplane, to prove that C is a $\mathcal{V}$-cone it is enough to prove that the intersection of a $\mathcal{V}$-cone with a hyperplane is also a $\mathcal{V}$-cone.

For this, we use *Fourier-Motzkin elimination*. It suffices to prove the result for a hyperplane, H_k, in $\mathbb{E}^{d+m}$ of equation $y_k = 0$ $(1 \leq k \leq d+m)$.

Proposition 6.4.8 *(Fourier-Motzkin Elimination) Say $C = \mathrm{cone}(Y) \subseteq \mathbb{E}^d$ is a $\mathcal{V}$-cone. Then, the intersection $C \cap H_k$ (where H_k is the hyperplane of equation $y_k = 0$) is a $\mathcal{V}$-cone, $C \cap H_k = \mathrm{cone}(Y^{/k})$, with*

$$Y^{/k} = \{y_i \mid y_{ik} = 0\} \cup \{y_{ik}y_j - y_{jk}y_i \mid y_{ik} > 0,\ y_{jk} < 0\},$$

the set of vectors obtained from Y by "eliminating the k-th coordinate". Here, each y_i is a vector in $\mathbb{R}^d$.

As discussed above, Proposition 6.4.8 implies (again!)

Corollary 6.4.9 *Every $\mathcal{H}$-polyhedron is a $\mathcal{V}$-polyhedron.*

Another way of proving that every $\mathcal{V}$-polyhedron is an $\mathcal{H}$-polyhedron is to use cones.

Let $P = \mathrm{conv}(Y) + \mathrm{cone}(V) \subseteq \mathbb{E}^d$ be a $\mathcal{V}$-polyhedron.

We can view Y as a $d \times p$ matrix whose ith column is the ith vector in Y and V as $d \times q$ matrix whose jth column is the jth vector in V.

Then, we can write

$$\begin{aligned} P = \{x \in \mathbb{R}^d \mid (\exists u \in \mathbb{R}^p)(\exists t \in \mathbb{R}^d) \\ (x = Yu + Vt,\ u \geq 0,\ \mathbb{I}u = 1,\ t \geq 0)\}, \end{aligned}$$

where $\mathbb{I}$ is the row vector

$$\mathbb{I} = \underbrace{(1, \ldots, 1)}_{p}.$$

Now, observe that P can be interpreted as the projection of the $\mathcal{H}$-polyhedron, $\widetilde{P} \subseteq \mathbb{E}^{d+p+q}$, given by

$$\begin{aligned} \widetilde{P} = \{(x, u, t) \in \mathbb{R}^{d+p+q} \mid x = Yu + Vt, \\ u \geq 0,\ \mathbb{I}u = 1,\ t \geq 0\} \end{aligned}$$

onto $\mathbb{R}^d$.

Consequently, if we can prove that the projection of an $\mathcal{H}$-polyhedron is also an $\mathcal{H}$-polyhedron, then we will have proved that every $\mathcal{V}$-polyhedron is an $\mathcal{H}$-polyhedron.

In view of Proposition 6.4.7 and the discussion that followed, it is enough to prove that the projection of any $\mathcal{H}$-cone is an $\mathcal{H}$-cone.

This can be done by using a type of Fourier-Motzkin elimination dual to the method used in Proposition 6.4.8.

We state the result without proof and refer the interested reader to Ziegler [**?**], Section 1.2–1.3, for full details.

Proposition 6.4.10 *If $C = P(A, 0) \subseteq \mathbb{E}^d$ is an $\mathcal{H}$-cone, then the projection,* $\mathrm{proj}_k(C)$*, onto the hyperplane, H_k, of equation $y_k = 0$ is given by*
$\mathrm{proj}_k(C) = \mathrm{elim}_k(C) \cap H_k$*, with*

$$\begin{aligned}\mathrm{elim}_k(C) &= \{x \in \mathbb{R}^d \mid (\exists t \in \mathbb{R})(x + te_k \in P)\}\\ &= \{z - te_k \mid z \in P,\ t \in \mathbb{R}\} = P(A^{/k}, 0)\end{aligned}$$

and where the rows of $A^{/k}$ are given by

$$A^{/k} = \{a_i \mid a_{i\,k} = 0\} \cup \{a_{i\,k}a_j - a_{j\,k}a_i \mid a_{i\,k} > 0, a_{j\,k} < 0\}.$$

It should be noted that both Fourier-Motzkin elimination methods generate a quadratic number of new vectors or inequalities at each step and thus they lead to a combinatorial explosion.

Therefore, these methods become intractable rather quickly.

The problem is that many of the new vectors or inequalities are redundant. Thereore, it is important to find ways of detecting redundancies and there are various methods for doing so.

Again, the interested reader should consult Ziegler [**?**], Chapter 1.

We conclude this section with a version of Farkas Lemma for polyhedral sets.

Lemma 6.4.11 *(Farkas Lemma, Version IV) Let Y be any $d \times p$ matrix and V be any $d \times q$ matrix. For every $z \in \mathbb{R}^d$, exactly one of the following alternatives occurs:*

(a) There exist $u \in \mathbb{R}^p$ and $t \in \mathbb{R}^q$, with $u \geq 0$, $t \geq 0$, $\mathbb{I}u = 1$ and $z = Yu + Vt$.

(b) There is some vector, $(\alpha, c) \in \mathbb{R}^{d+1}$, such that $c^\top y_i \geq \alpha$ for all i with $1 \leq i \leq p$, $c^\top v_j \geq 0$ for all j with $1 \leq j \leq q$, and $c^\top z < \alpha$.

Observe that Farkas IV can be viewed as a separation criterion for polyhedral sets.