Partitioning the permutahedron

Etienne Rassart

Institute for Advanced Study

Experimental Mathematics Seminar

Rutgers University, October 28, 2004

Joint work with Sara Billey and Victor Guillemin

Outline

- The permutahedron
  - Weight multiplicities
  - The Duistermaat-Heckman function
- Vector partition functions
- Counting the regions
- Comparing the two partitions
- Open problems

The permutahedron

The permutahedron

We obtain a $k-1$ dimensional permutahedron by

- picking a point $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$ with distinct coordinates;
- permuting the coordinates of λ in all possible ways to get $k!$ points;
- taking the convex hull of those points.

$$P_\lambda = \mathrm{conv}(\mathfrak{S}_k \cdot \lambda)$$

$$P_\lambda = \mathrm{conv}\left(\{(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(k)}) \ : \ \sigma \in \mathfrak{S}_k\}\right)$$

The permutahedron

- It is $k-1$ dimensional because the points

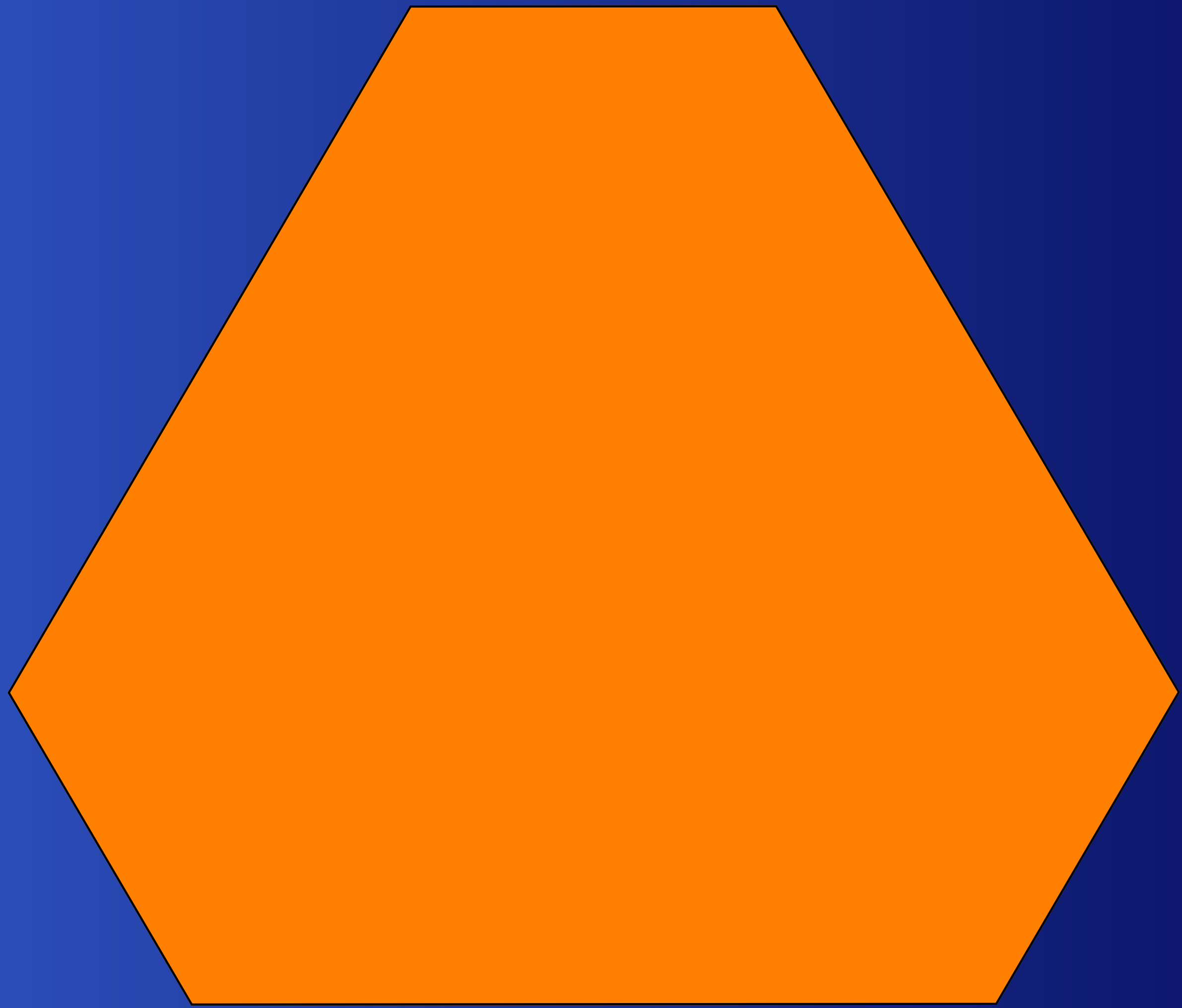
$$\{(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(k)}) \; : \; \sigma \in \mathfrak{S}_k\}$$

all lie on the same subspace of $\mathbb{R}^k$:

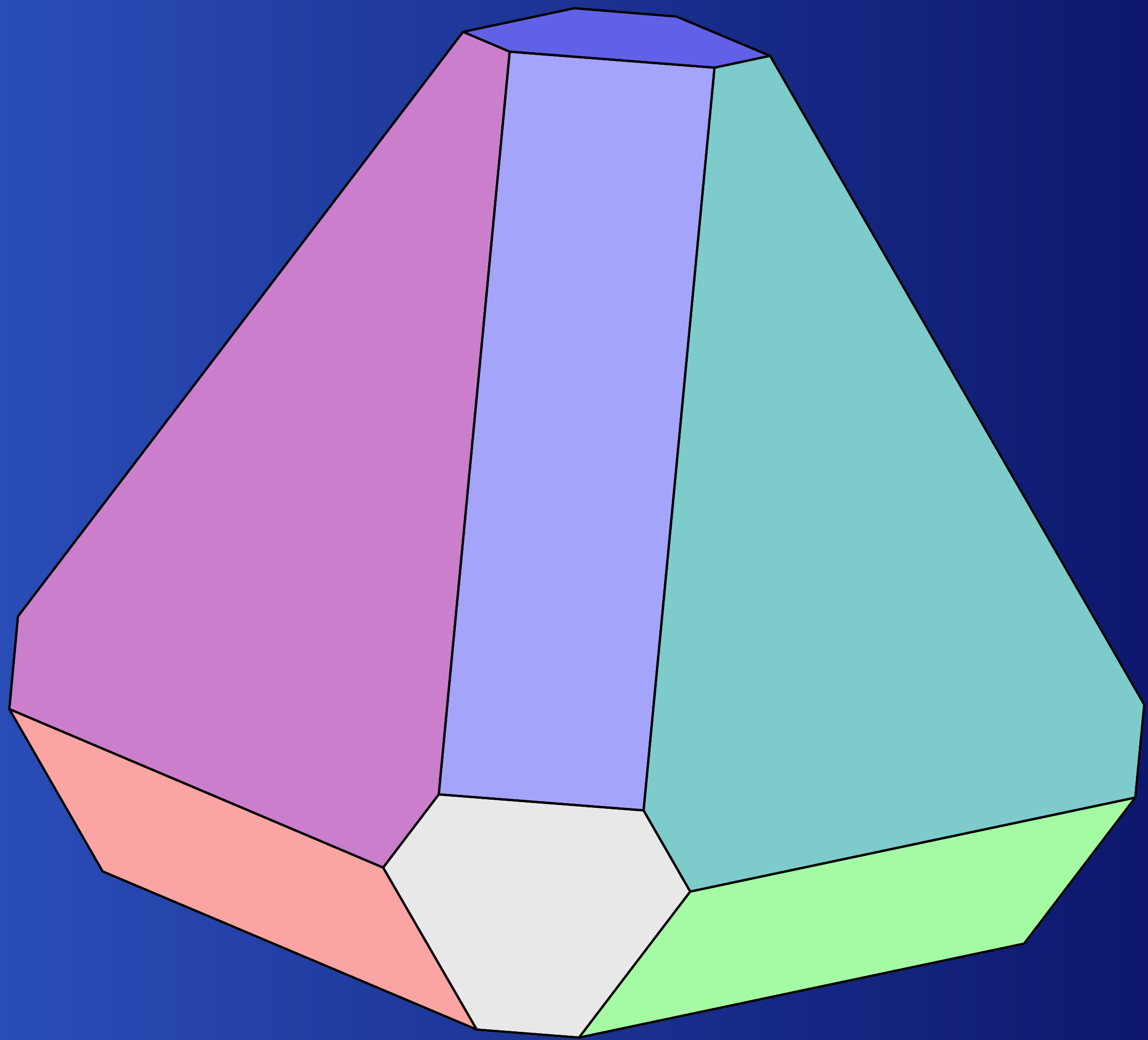
$$x_1 + x_2 + \cdots + x_k = \sum_{i=1}^{k} \lambda_i = \sum_{i=1}^{k} \lambda_{\sigma(i)} .$$

- We will assume that $\lambda_1 + \cdots + \lambda_k = 0$.

Examples of permutahedra



Examples of permutahedra



Integral permutahedra

- Since $P_\lambda = P_{\sigma(\lambda)}$, we will further assume that
$$\lambda_1 > \lambda_2 > \cdots > \lambda_k \,.$$
- We will usually consider integral λ, *i.e.* $\lambda \in \mathbb{Z}^k$.

Integral permutahedra

- Since $P_\lambda = P_{\sigma(\lambda)}$, we will further assume that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_k \, .$$

- We will usually consider integral λ, *i.e.* $\lambda \in \mathbb{Z}^k$.

- Then P_λ is an integral polytope, and $P_\lambda \cap \mathbb{Z}^k$ is the lattice spanned by the vectors

$$\{e_i - e_j \ : \ 1 \leq i < j \leq k\} \, ,$$

or, equivalently, by the vectors

$$\{e_i - e_{i+1} \ : \ 1 \leq i \leq k-1\} \, .$$

Two functions on the permutahedron

We will consider two functions on an integral permutahedron.

- A discrete one: weight multiplicities, defined on the lattice points in the permutahedron.
- A continuous one: the Duistermaat-Heckman function, defined on the whole permutahedron.

Both functions partition the permutahedron into polytopal domains over which they are given by polynomials.

Gelfand-Tsetlin diagrams

A Gelfand-Tsetlin diagram is an array of integers of the form

$$
\begin{array}{ccccccccc}
\lambda_1^{(k)} & & \lambda_2^{(k)} & & \cdots & & \lambda_{k-1}^{(k)} & & \lambda_k^{(k)} \\
& \lambda_1^{(k-1)} & & \lambda_2^{(k-1)} & & \cdots & & \lambda_{k-1}^{(k-1)} & \\
& & \ddots & & \vdots & & \iddots & & \\
& & & \lambda_1^{(2)} & & \lambda_2^{(2)} & & & \\
& & & & \lambda_1^{(1)} & & & &
\end{array}
$$

such that

Gelfand-Tsetlin diagrams

$$\begin{array}{ccccccccc}
\lambda_1^{(k)} & & \lambda_2^{(k)} & & \cdots & & \lambda_{k-1}^{(k)} & & \lambda_k^{(k)} \\
& \lambda_1^{(k-1)} & & \lambda_2^{(k-1)} & & \cdots & & \lambda_{k-1}^{(k-1)} & \\
& & \ddots & & \vdots & & \iddots & & \\
& & & \lambda_1^{(2)} & & \lambda_2^{(2)} & & & \\
& & & & \lambda_1^{(1)} & & & &
\end{array}$$

Gelfand-Tsetlin diagrams

$$\begin{array}{ccccccccc}
\lambda_1 & & \lambda_2 & & \cdots & & \lambda_{k-1} & & \lambda_k \\
& \lambda_1^{(k-1)} & & \lambda_2^{(k-1)} & & \cdots & & \lambda_{k-1}^{(k-1)} & \\
& & \ddots & & \vdots & & \iddots & & \\
& & & \lambda_1^{(2)} & & \lambda_2^{(2)} & & & \\
& & & & \lambda_1^{(1)} & & & &
\end{array}$$

Gelfand-Tsetlin diagrams

$$
\begin{array}{ccccccccc}
\lambda_1 & & \lambda_2 & & \cdots & & \lambda_{k-1} & & \lambda_k \\
 & \lambda_1^{(k-1)} & & \lambda_2^{(k-1)} & & \cdots & & \lambda_{k-1}^{(k-1)} & \\
 & & \ddots & & \vdots & & \iddots & & \\
 & & & \lambda_1^{(2)} & & \lambda_2^{(2)} & & & \\
 & & & & \lambda_1^{(1)} & & & &
\end{array}
$$

and

$$
\begin{array}{ccc}
\lambda_j^{(i+1)} & & \lambda_{j+1}^{(i+1)} \\
 & \lambda_j^{(i+1)} \geq \lambda_j^{(i)} \geq \lambda_{j+1}^{(i+1)} & \\
 & \lambda_j^{(i)} &
\end{array}
$$

for every such triangle in the diagram.

Weight multiplicities

- The weight multiplicity $m_\lambda(\beta)$ of β in P_λ is the number of Gelfand-Tsetlin diagrams with top row λ and row sums satisfying

$$\sum_{i=1}^{m} \lambda_i^{(m)} = \beta_1 + \cdots + \beta_m \qquad \text{for } 1 \le m \le k.$$

Weight multiplicities

- The weight multiplicity $m_\lambda(\beta)$ of β in P_λ is the number of Gelfand-Tsetlin diagrams with top row λ and row sums satisfying

$$\sum_{i=1}^{m} \lambda_i^{(m)} = \beta_1 + \cdots + \beta_m \qquad \text{for } 1 \leq m \leq k.$$

- Weight multiplicities vanish outside the permutahedron.

Weight multiplicities

- The weight multiplicity $m_\lambda(\beta)$ of β in P_λ is the number of Gelfand-Tsetlin diagrams with top row λ and row sums satisfying

$$\sum_{i=1}^{m} \lambda_i^{(m)} = \beta_1 + \cdots + \beta_m \qquad \text{for } 1 \leq m \leq k.$$

- Weight multiplicities vanish outside the permutahedron.

- Weight multiplicities are invariant under the action of the symmetric group $\mathfrak{S}_k$.

GT-diagrams and SSYTs

$$\begin{array}{ccccccc} 7 & & 5 & & 4 & & 1 \\ & 6 & & 5 & & 2 & \\ & & 5 & & 3 & & \\ & & & 3 & & & \end{array}$$

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = 17$$
$$\beta_1 + \beta_2 + \beta_3 = 13$$
$$\beta_1 + \beta_2 = 8$$
$$\beta_1 = 3$$

GT-diagrams and SSYTs

$$\begin{array}{ccccccc} 7 & & 5 & & 4 & & 1 \\ & 6 & & 5 & & 2 & \\ & & 5 & & 3 & & \\ & & & 3 & & & \end{array}$$

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = 17$$
$$\beta_1 + \beta_2 + \beta_3 = 13$$
$$\beta_1 + \beta_2 = 8$$
$$\beta_1 = 3$$

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline \end{array}$$

(3)

GT-diagrams and SSYTs

$$\begin{array}{ccccccc} 7 & & 5 & & 4 & & 1 \\ & 6 & & 5 & & 2 & \\ & & 5 & & 3 & & \\ & & & 3 & & & \end{array}$$

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = 17$$
$$\beta_1 + \beta_2 + \beta_3 = 13$$
$$\beta_1 + \beta_2 = 8$$
$$\beta_1 = 3$$

1	1	1	2	2
2	2	2		

$$(5, 3)$$

GT-diagrams and SSYTs

$$\begin{array}{ccccccc} 7 & & 5 & & 4 & & 1 \\ & 6 & & 5 & & 2 & \\ & & 5 & & 3 & & \\ & & & 3 & & & \end{array}$$

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = 17$$
$$\beta_1 + \beta_2 + \beta_3 = 13$$
$$\beta_1 + \beta_2 = 8$$
$$\beta_1 = 3$$

1	1	1	2	2	3
2	2	2	3	3	
3	3				

$$(6, 5, 2)$$

GT-diagrams and SSYTs

$$\begin{array}{ccccccc} 7 & & 5 & & 4 & & 1 \\ & 6 & & 5 & & 2 & \\ & & 5 & & 3 & & \\ & & & 3 & & & \end{array}$$

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = 17$$
$$\beta_1 + \beta_2 + \beta_3 = 13$$
$$\beta_1 + \beta_2 = 8$$
$$\beta_1 = 3$$

1	1	1	2	2	3	4
2	2	2	3	3		
3	3	4	4			
4						

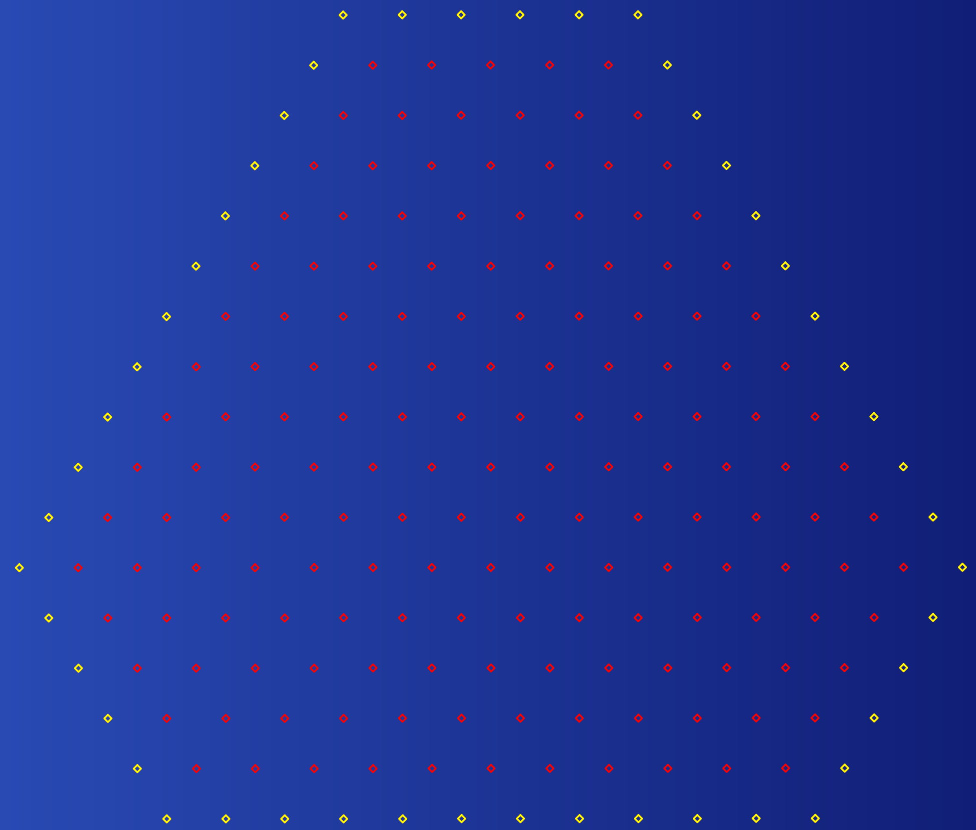
$$(7, 5, 4, 1)$$

Gelfand-Tsetlin polytopes

$$
\begin{array}{ccccccccc}
\lambda_1 & & \lambda_2 & & \cdots & & \lambda_{k-1} & & \lambda_k \\
& \lambda_1^{(k-1)} & & \lambda_2^{(k-1)} & & \cdots & & \lambda_{k-1}^{(k-1)} & \\
& & \ddots & & \vdots & & \iddots & & \\
& & & \lambda_1^{(2)} & & \lambda_2^{(2)} & & & \\
& & & & \lambda_1^{(1)} & & & &
\end{array}
$$

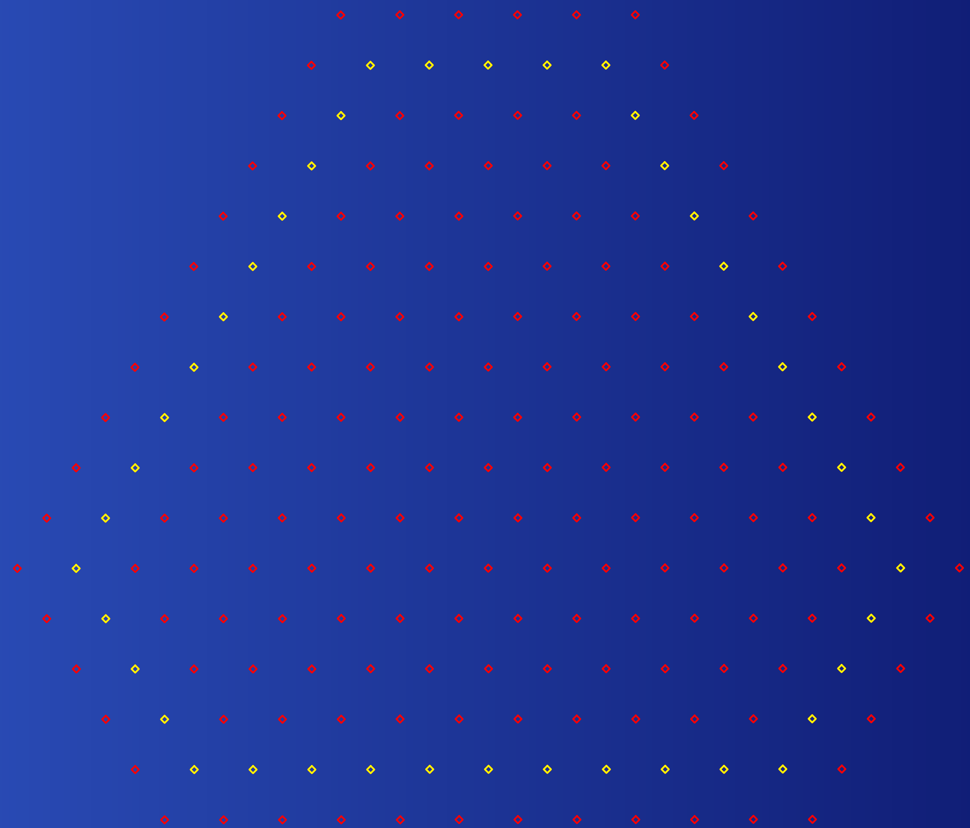
GT_λ $\qquad$ $\mathrm{GT}_{\lambda\beta}$

$\lambda = (9, -2, -7)$



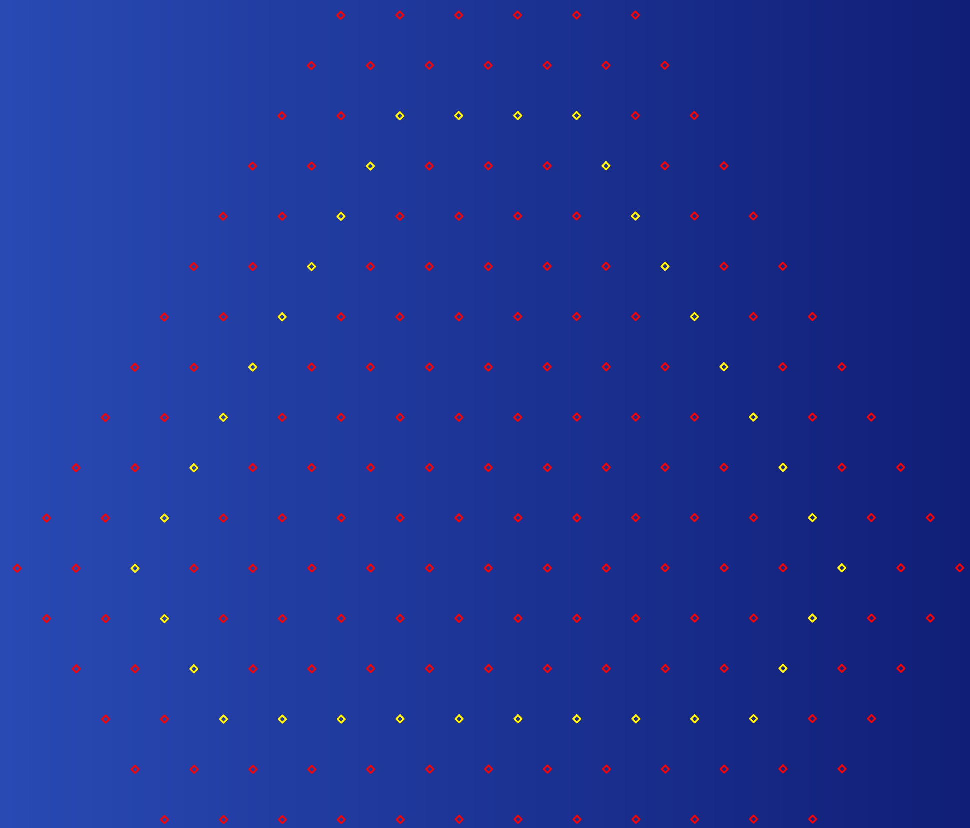
$$m_\lambda(\beta) = 1$$

$\lambda = (9, -2, -7)$



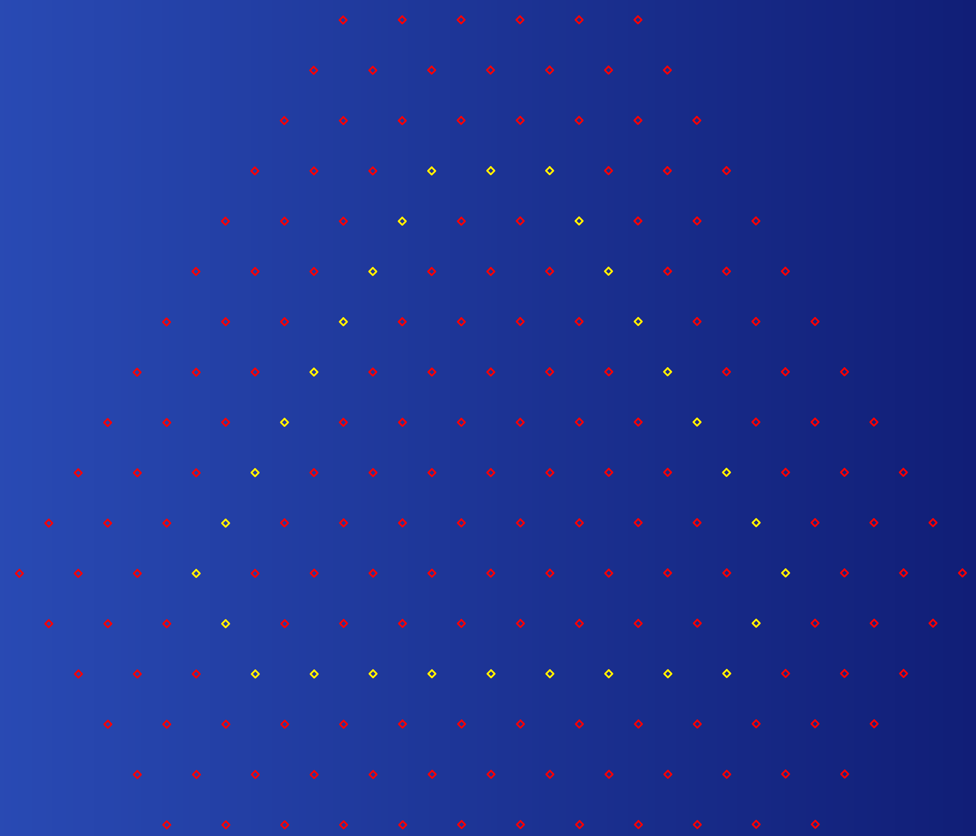
$$m_\lambda(\beta) = 2$$

$\lambda = (9,\ -2,\ -7)$



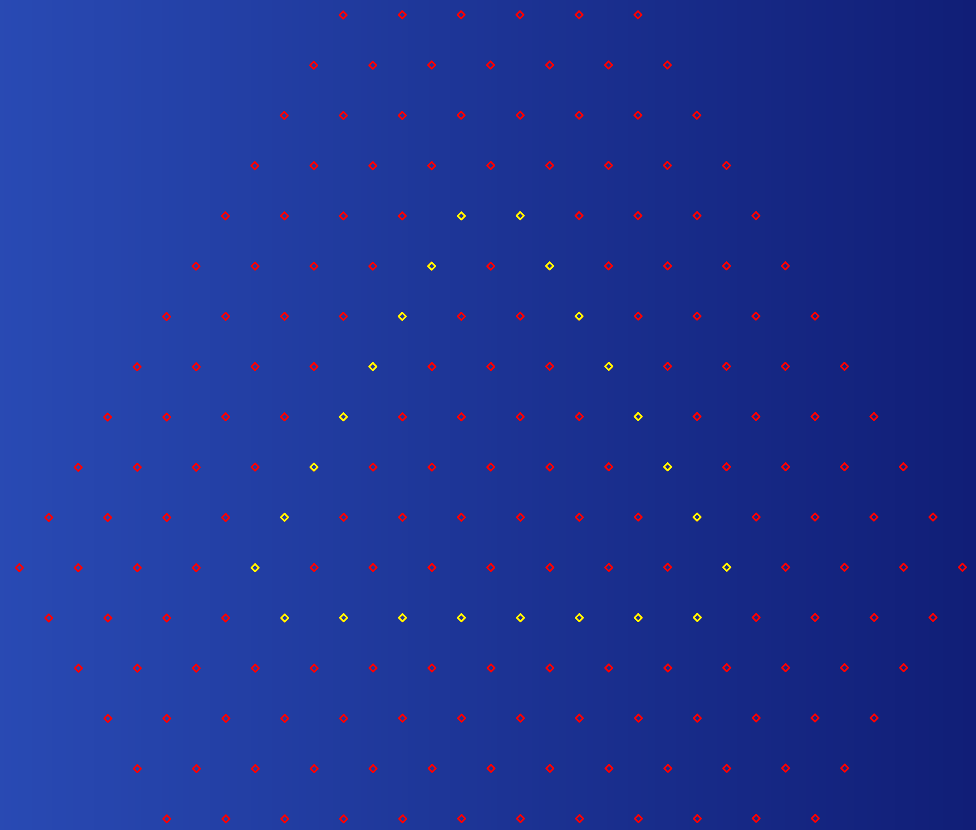
$$m_\lambda(\beta) = 3$$

$\lambda = (9,\ -2,\ -7)$



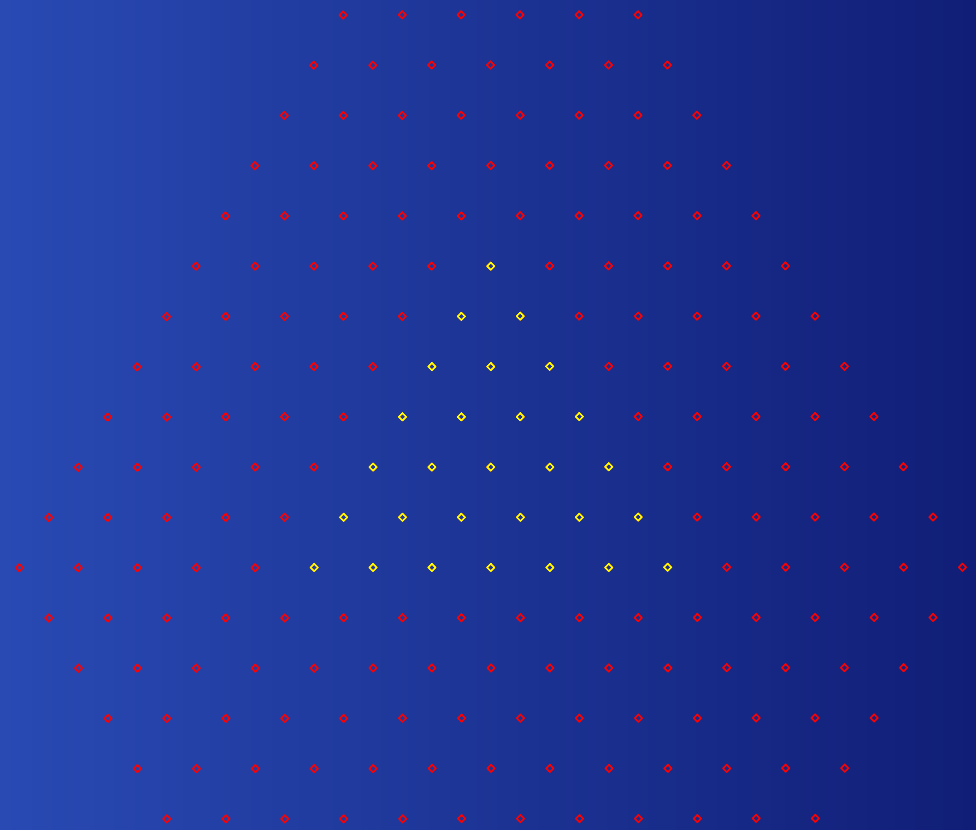
$$m_\lambda(\beta) = 4$$

$\lambda = (9, -2, -7)$



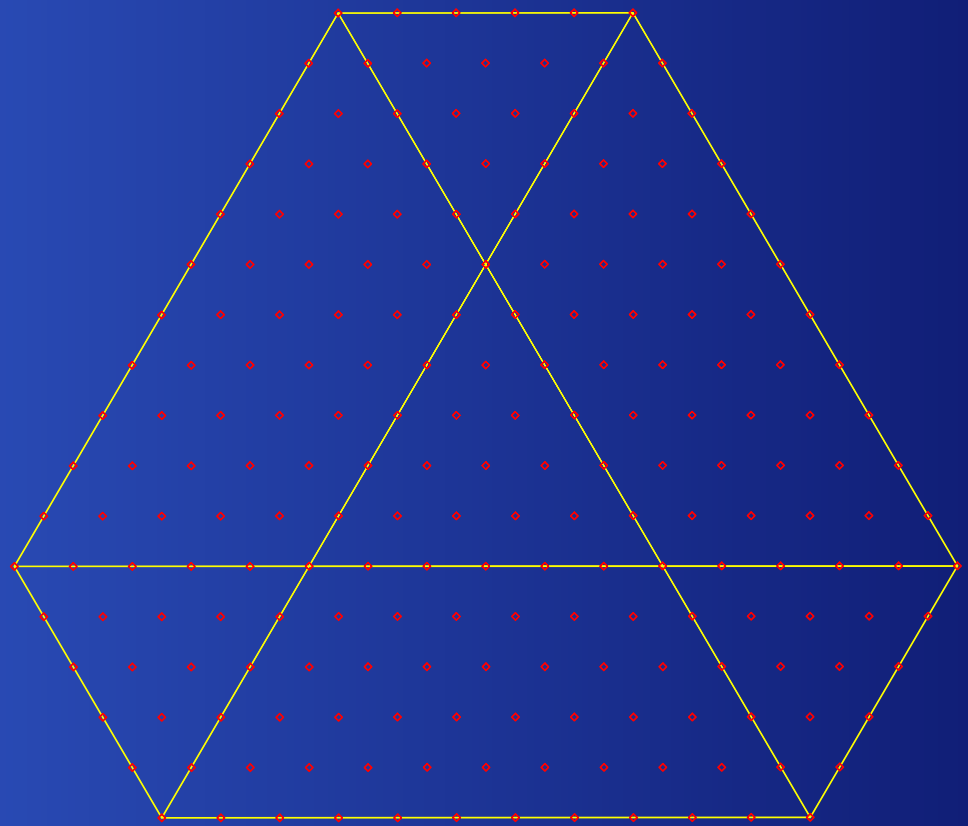
$$m_\lambda(\beta) = 5$$

$\lambda = (9, -2, -7)$

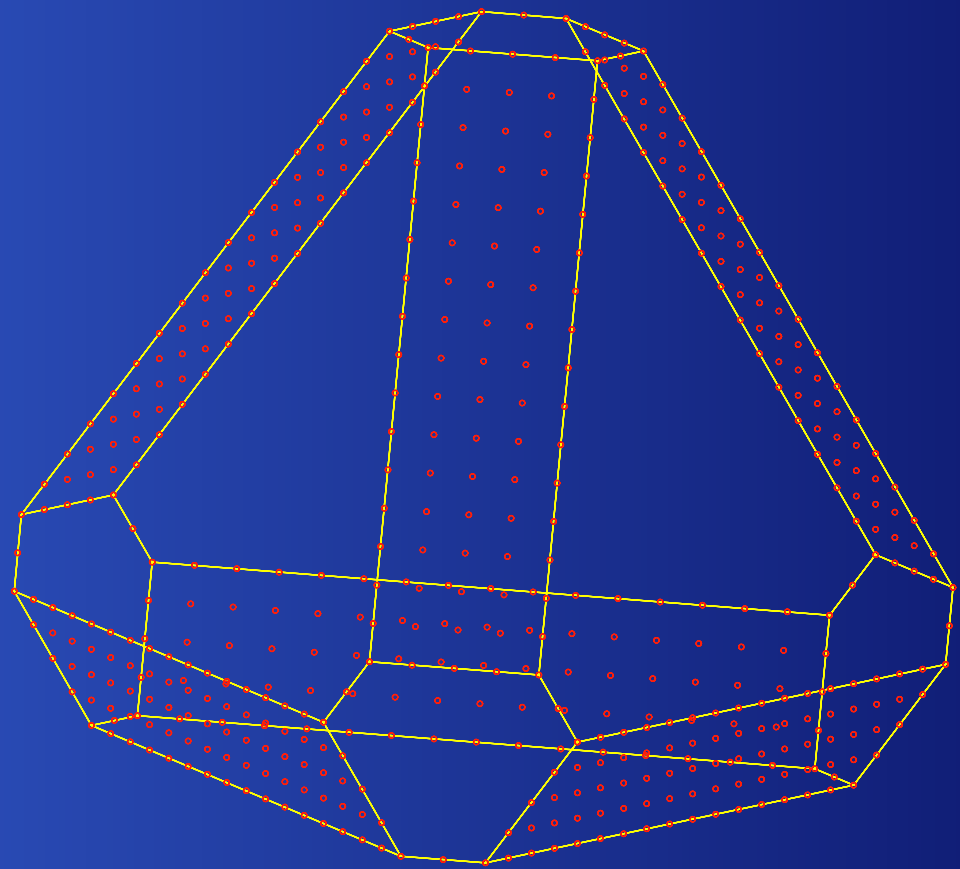


$$m_\lambda(\beta) = 6$$

$$\lambda = (9,\ -2,\ -7)$$

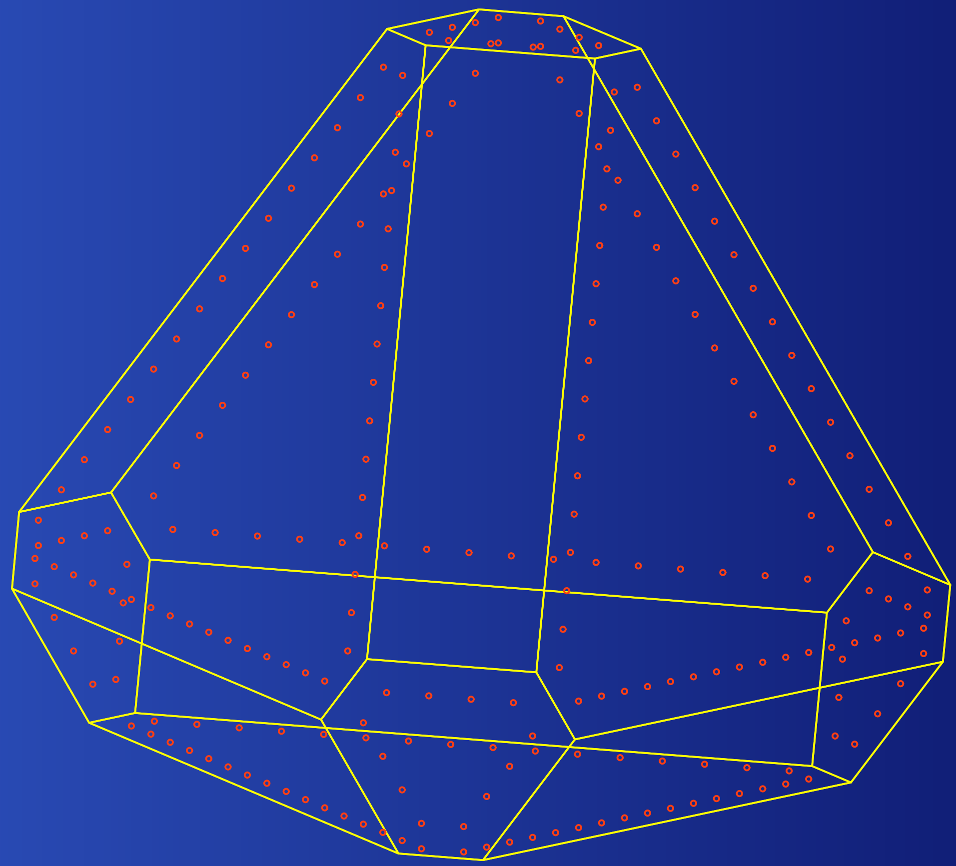


$\lambda = (14, -2, -4, -8)$



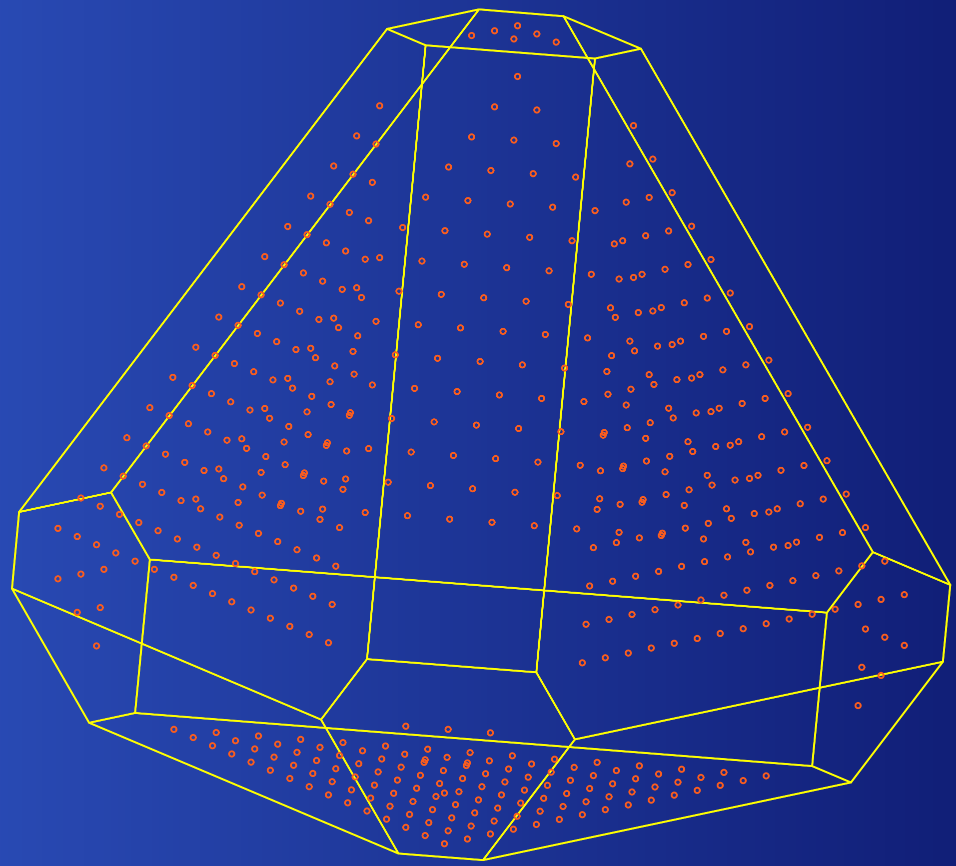
$$m_\lambda(\beta) = 1$$

$\lambda = (14, -2, -4, -8)$



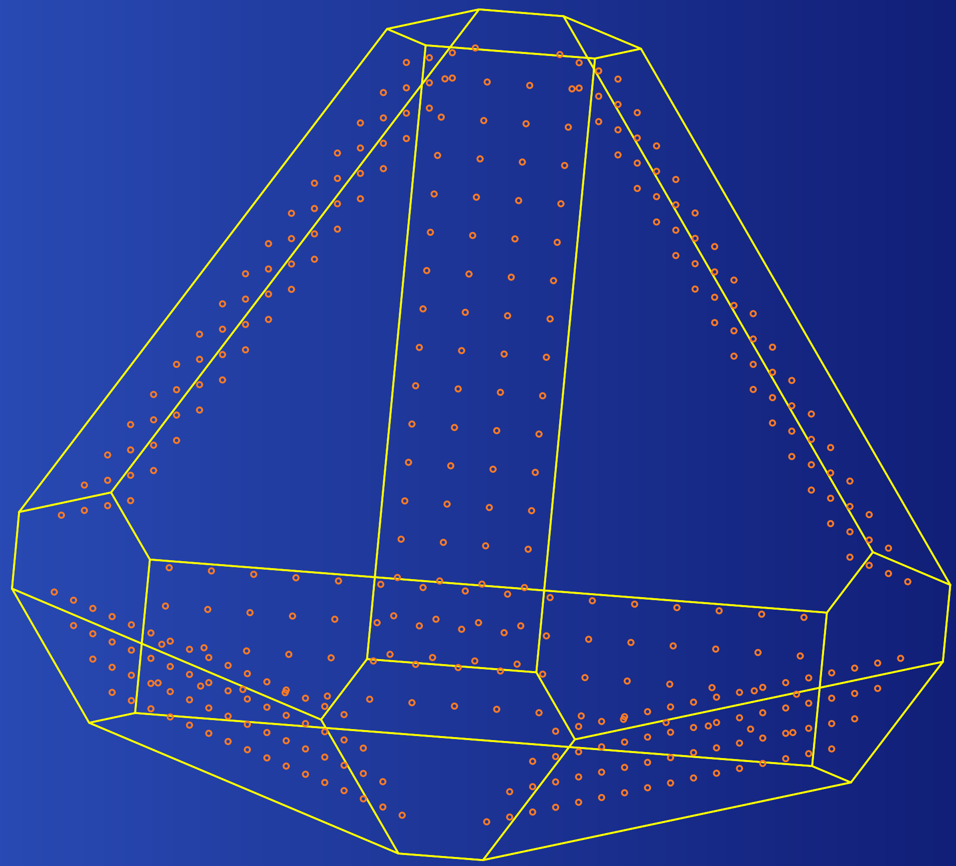
$$m_\lambda(\beta) = 2$$

$\lambda = (14, -2, -4, -8)$



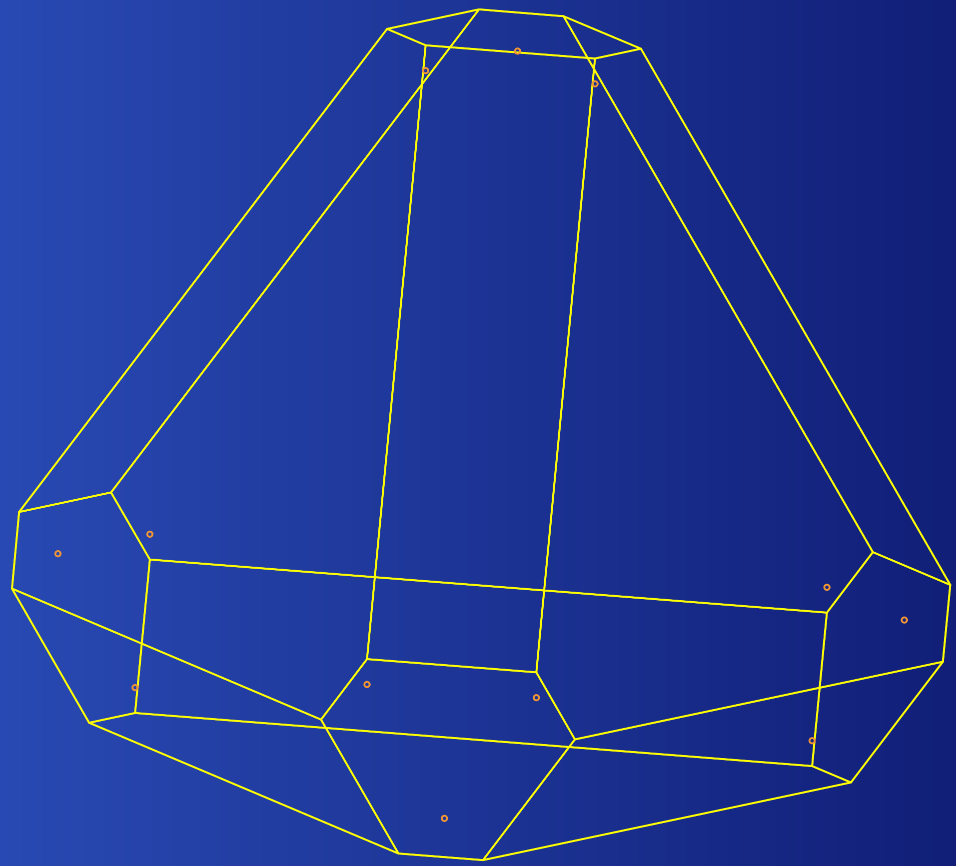
$$m_\lambda(\beta) = 3$$

$\lambda = (14, -2, -4, -8)$



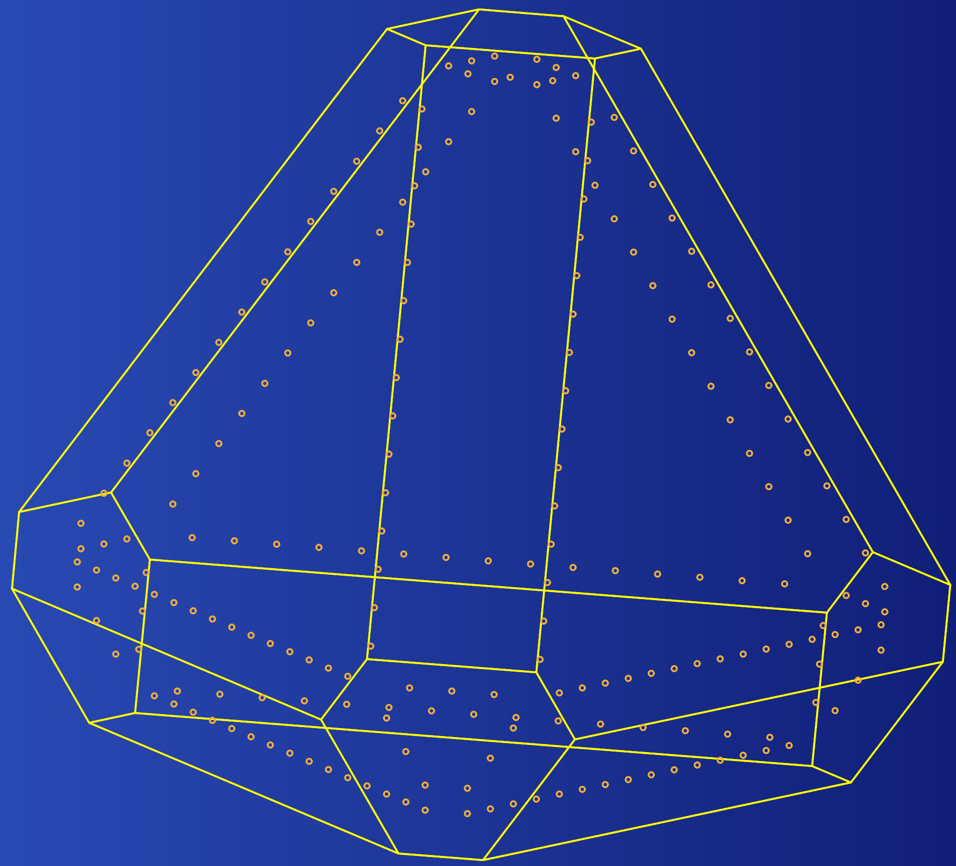
$$m_\lambda(\beta) = 4$$

$\lambda = (14, -2, -4, -8)$



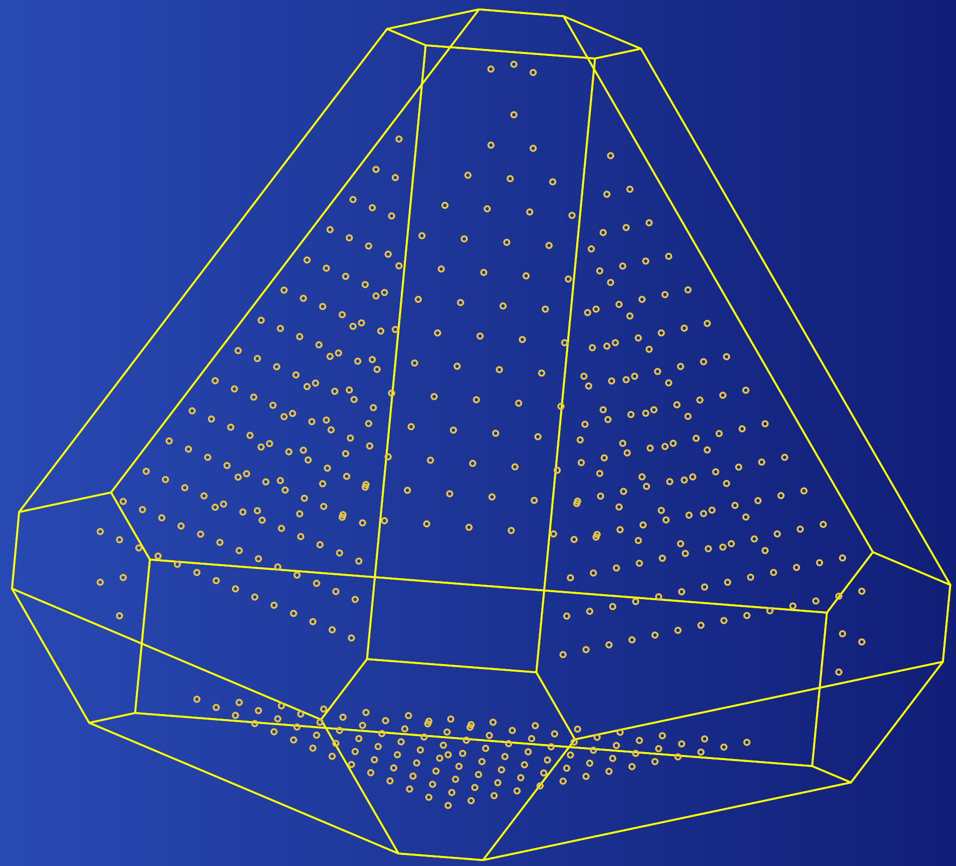
$$m_\lambda(\beta) = 5$$

$$\lambda = (14,\ -2,\ -4,\ -8)$$



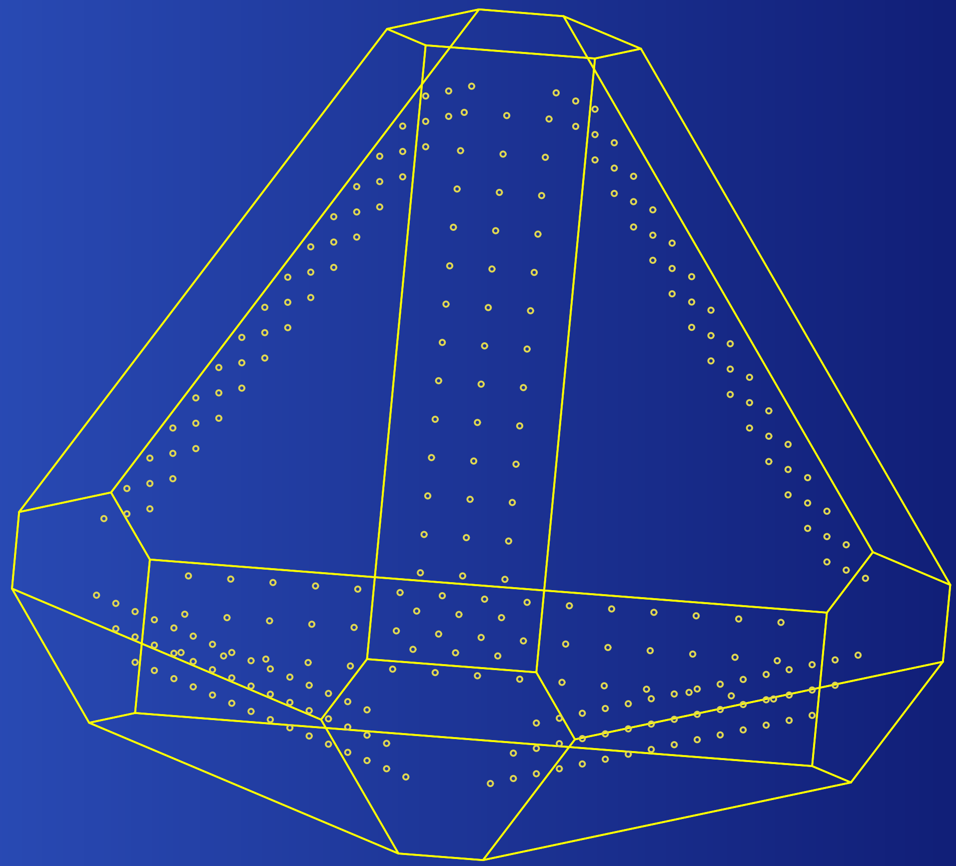
$$m_\lambda(\beta) = 7$$

$\lambda = (14, -2, -4, -8)$



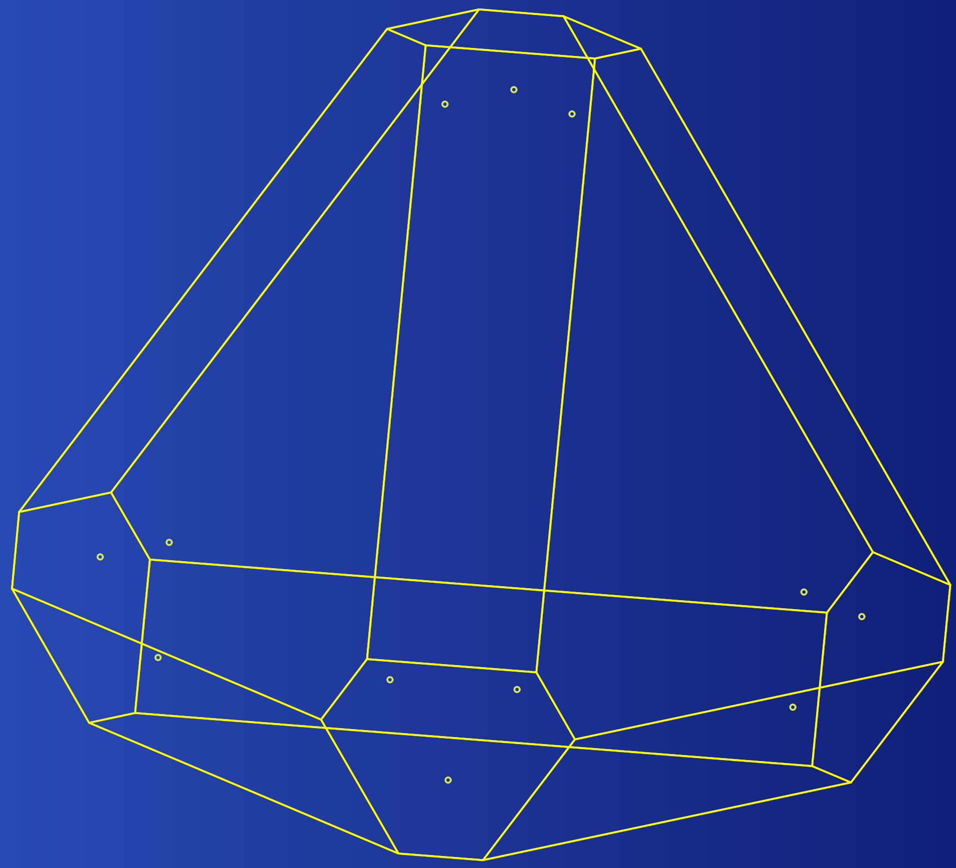
$$m_\lambda(\beta) = 9$$

$\lambda = (14, -2, -4, -8)$



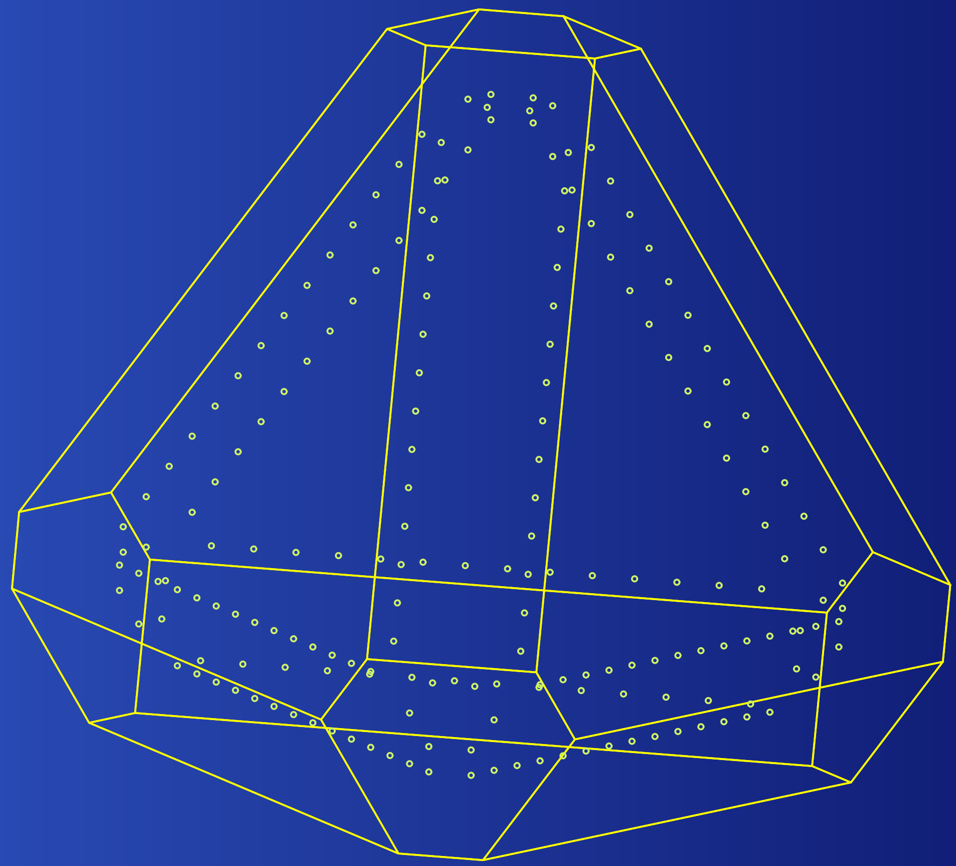
$$m_\lambda(\beta) = 10$$

$\lambda = (14, -2, -4, -8)$



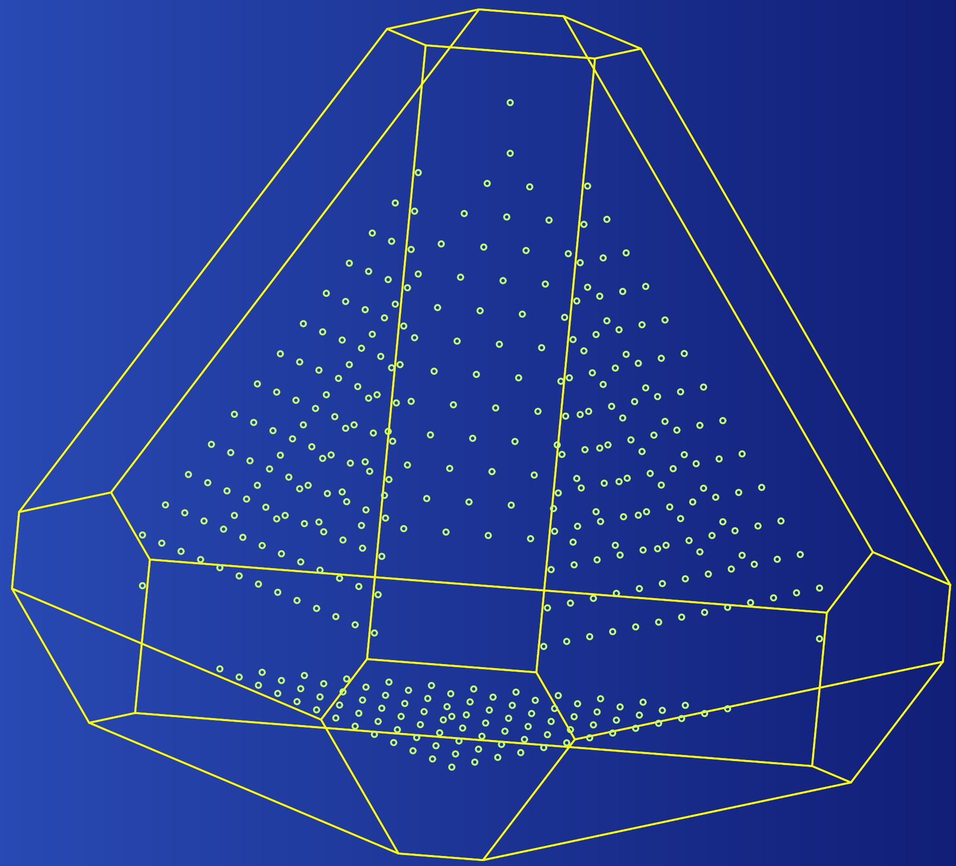
$$m_\lambda(\beta) = 12$$

$\lambda = (14, -2, -4, -8)$



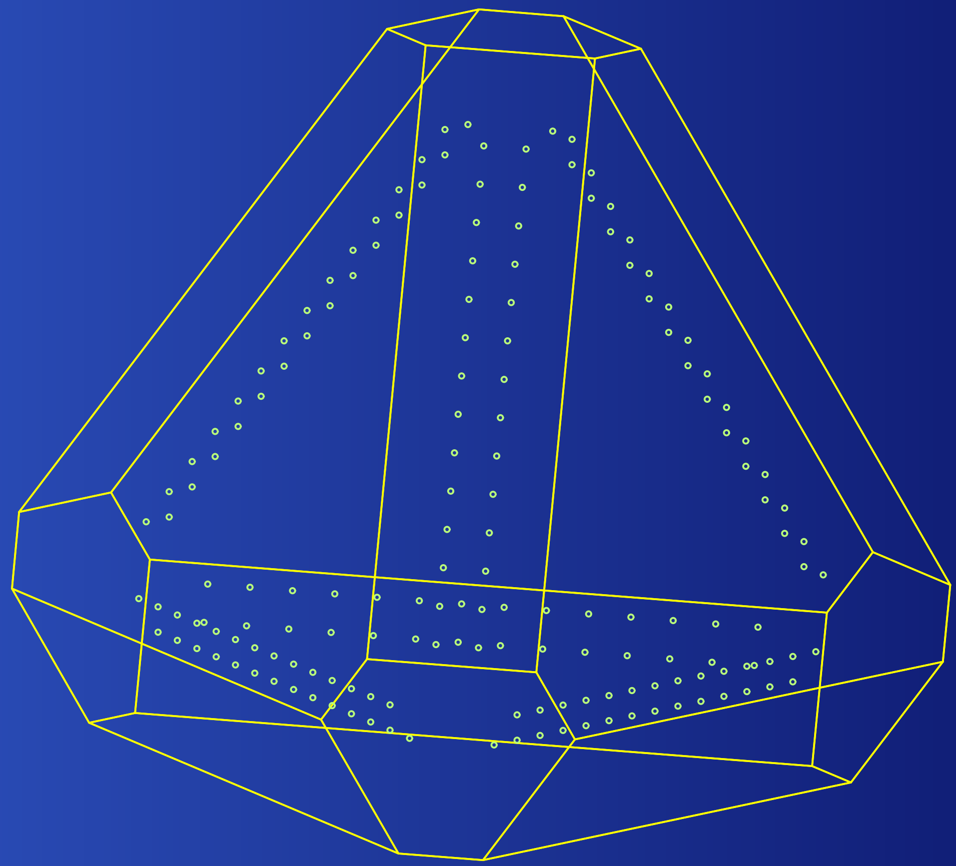
$$m_\lambda(\beta) = 15$$

$\lambda = (14, -2, -4, -8)$



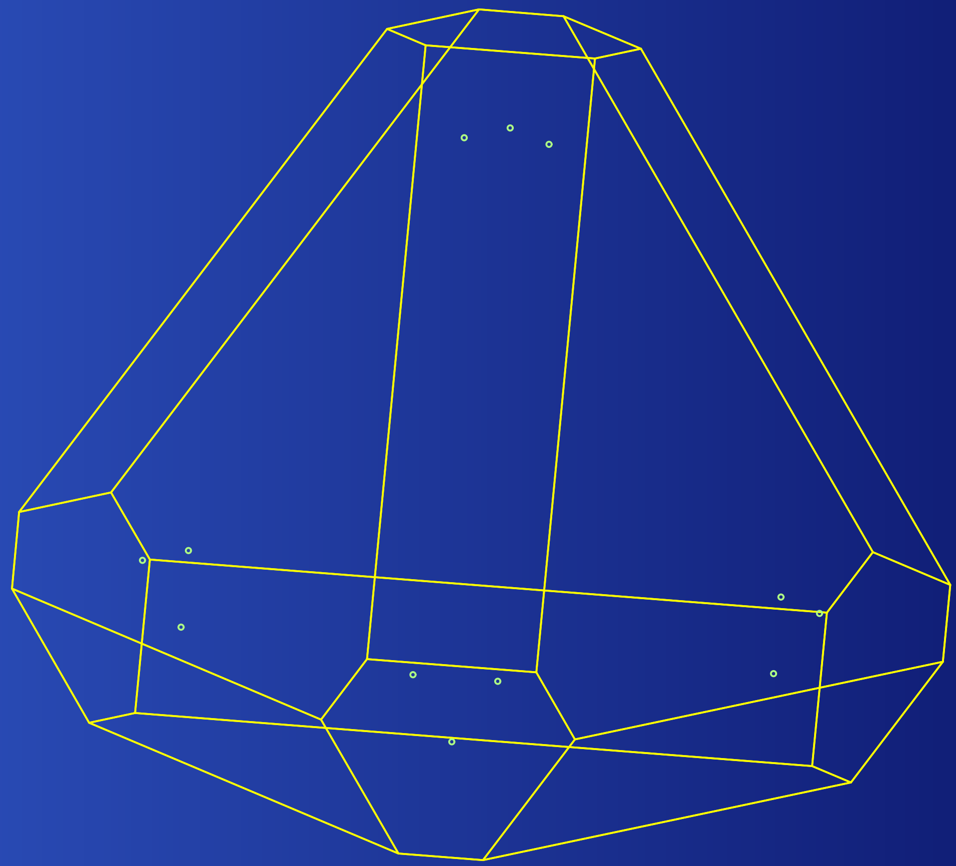
$$m_\lambda(\beta) = 18$$

$\lambda = (14, -2, -4, -8)$



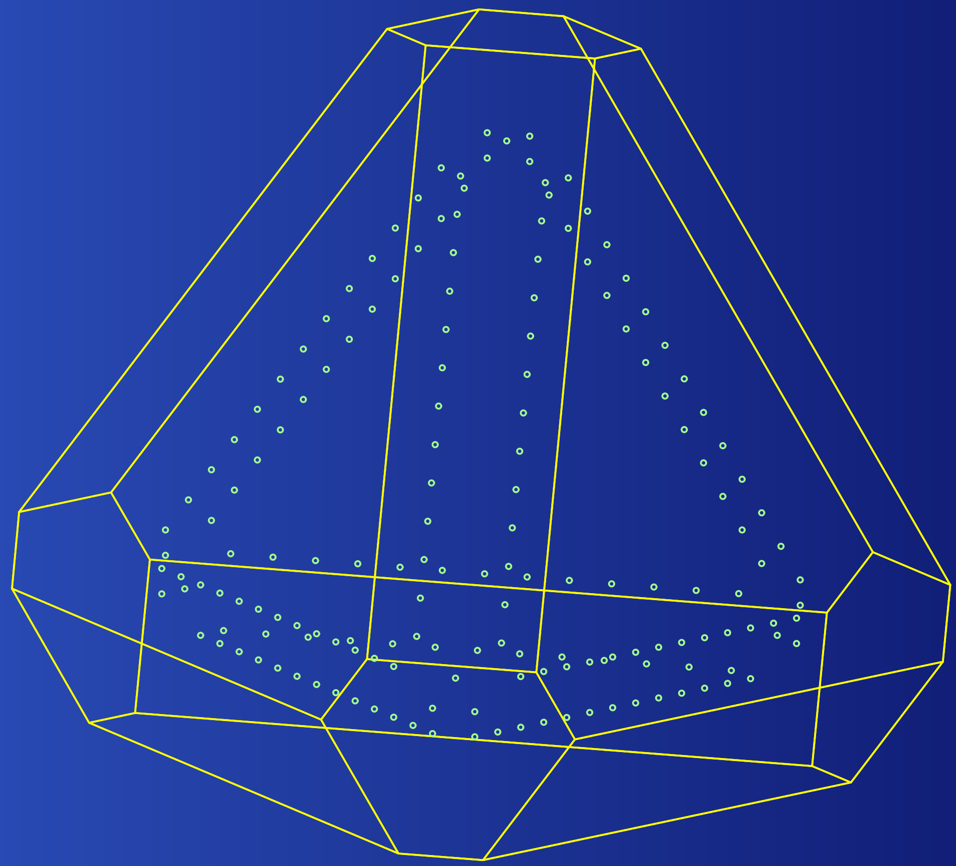
$$m_\lambda(\beta) = 19$$

$$\lambda = (14, -2, -4, -8)$$



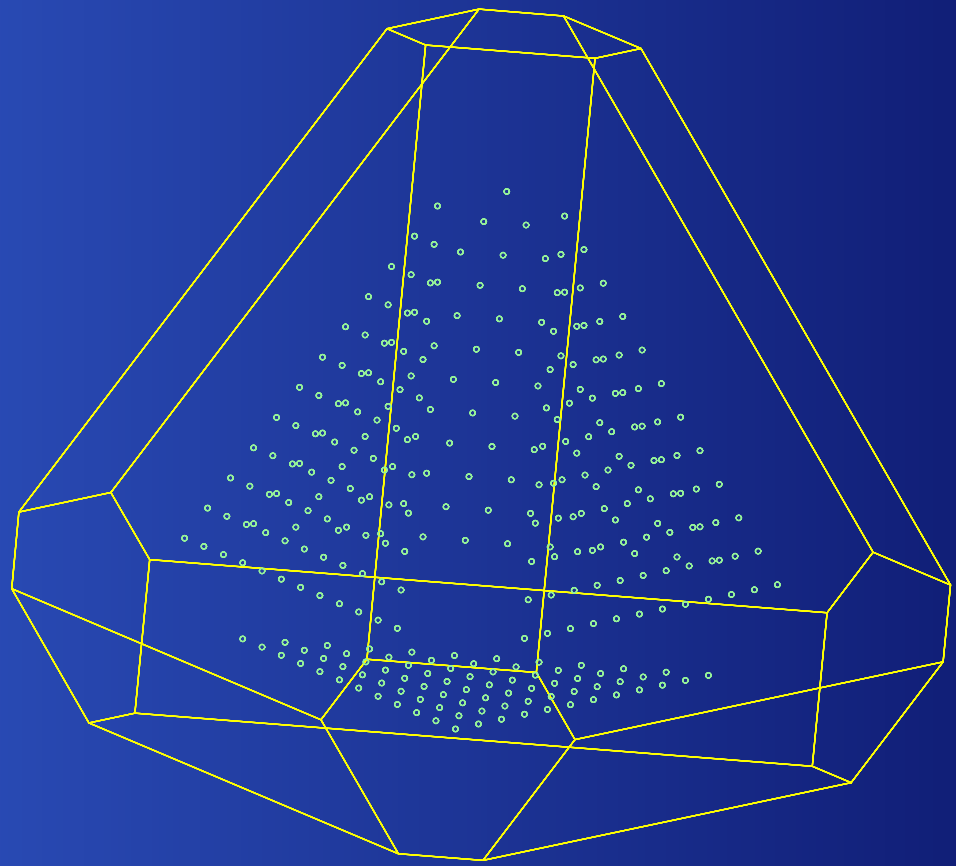
$$m_\lambda(\beta) = 22$$

$\lambda = (14, -2, -4, -8)$



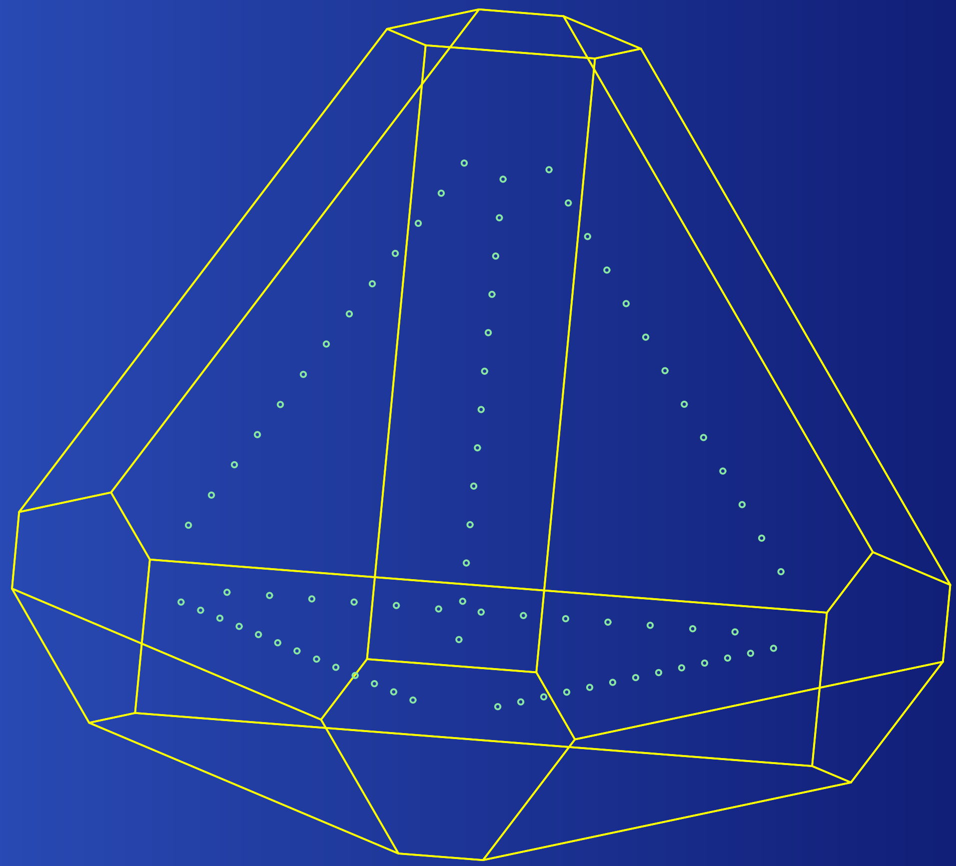
$$m_\lambda(\beta) = 26$$

$\lambda = (14, -2, -4, -8)$



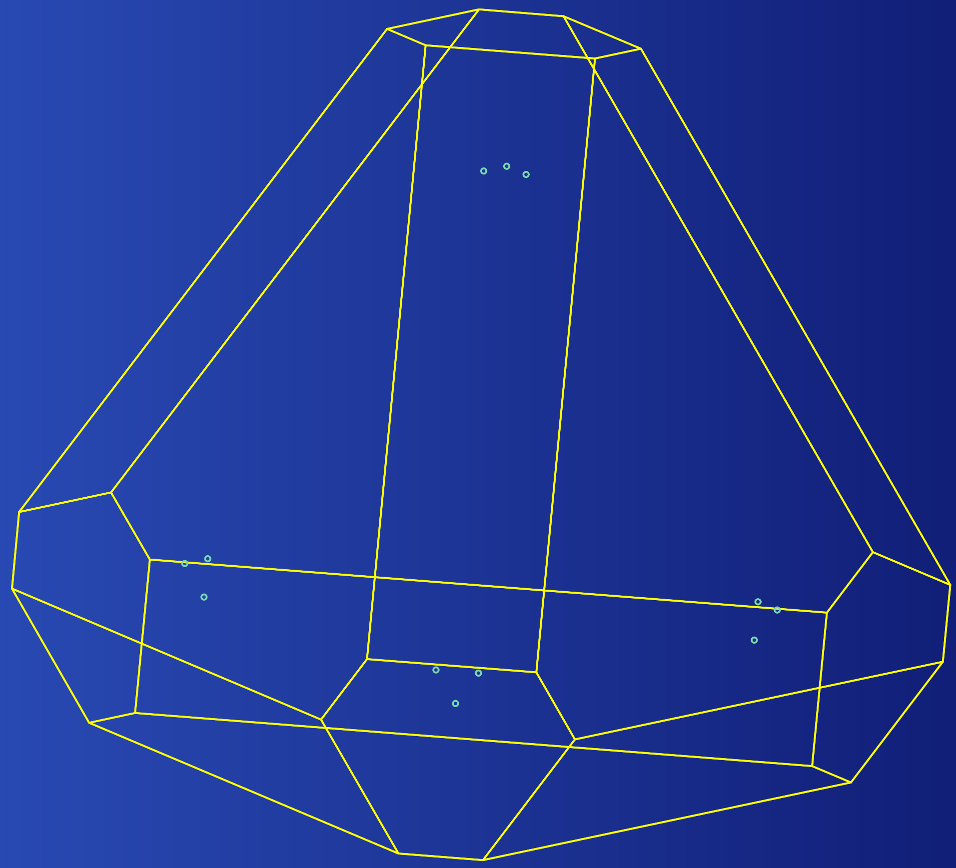
$$m_\lambda(\beta) = 30$$

$\lambda = (14, -2, -4, -8)$



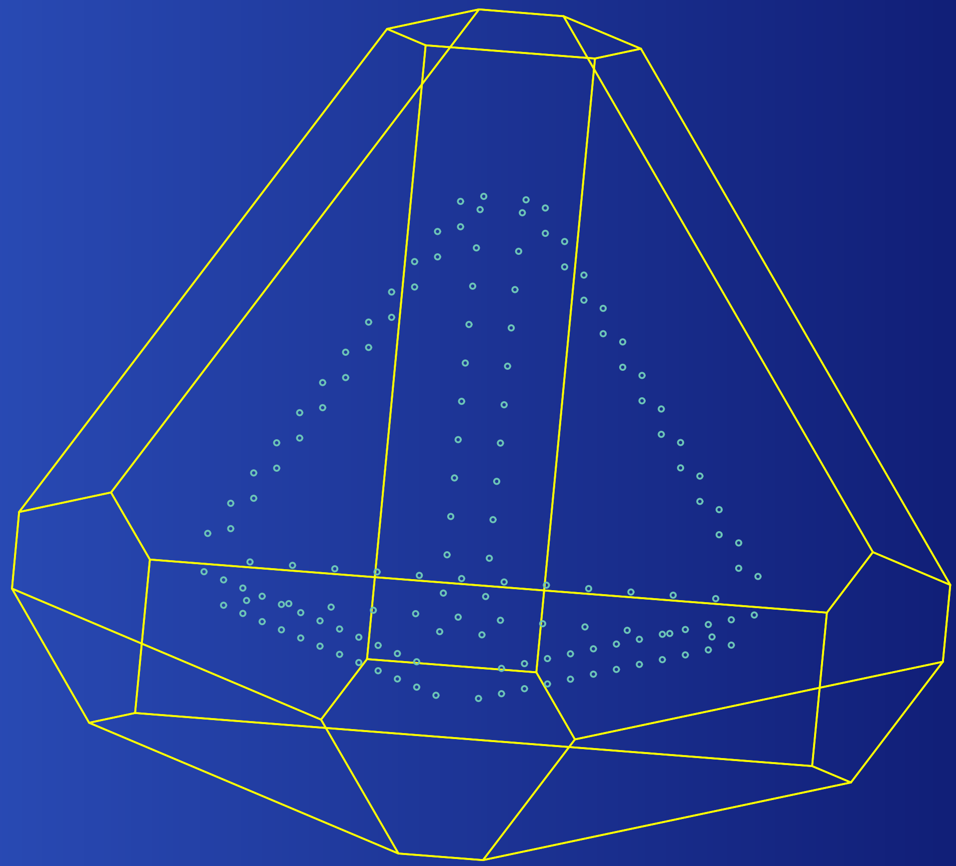
$$m_\lambda(\beta) = 31$$

$\lambda = (14, -2, -4, -8)$



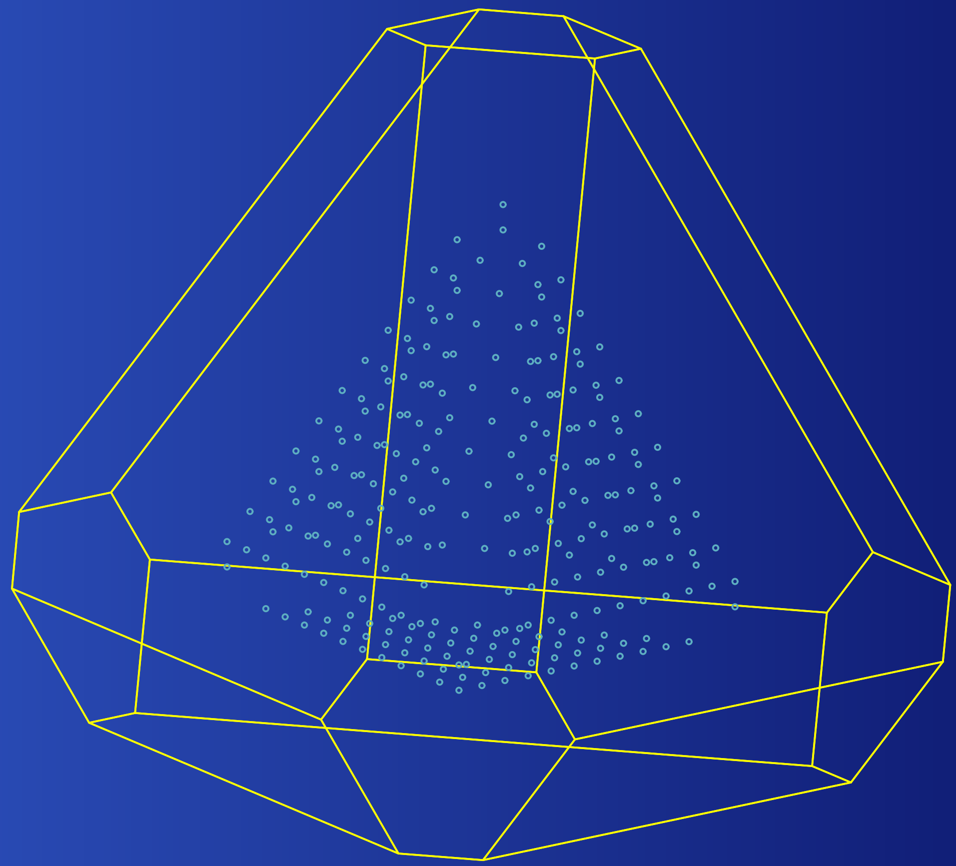
$$m_\lambda(\beta) = 35$$

$$\lambda = (14, -2, -4, -8)$$



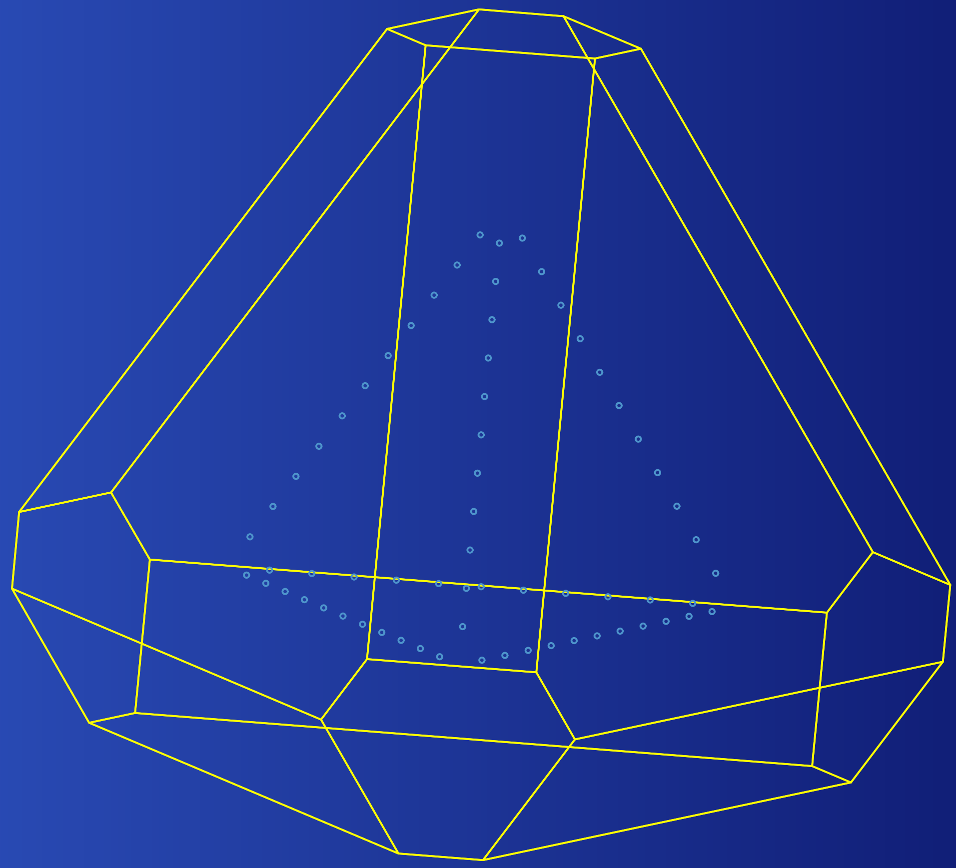
$$m_\lambda(\beta) = 40$$

$\lambda = (14, -2, -4, -8)$



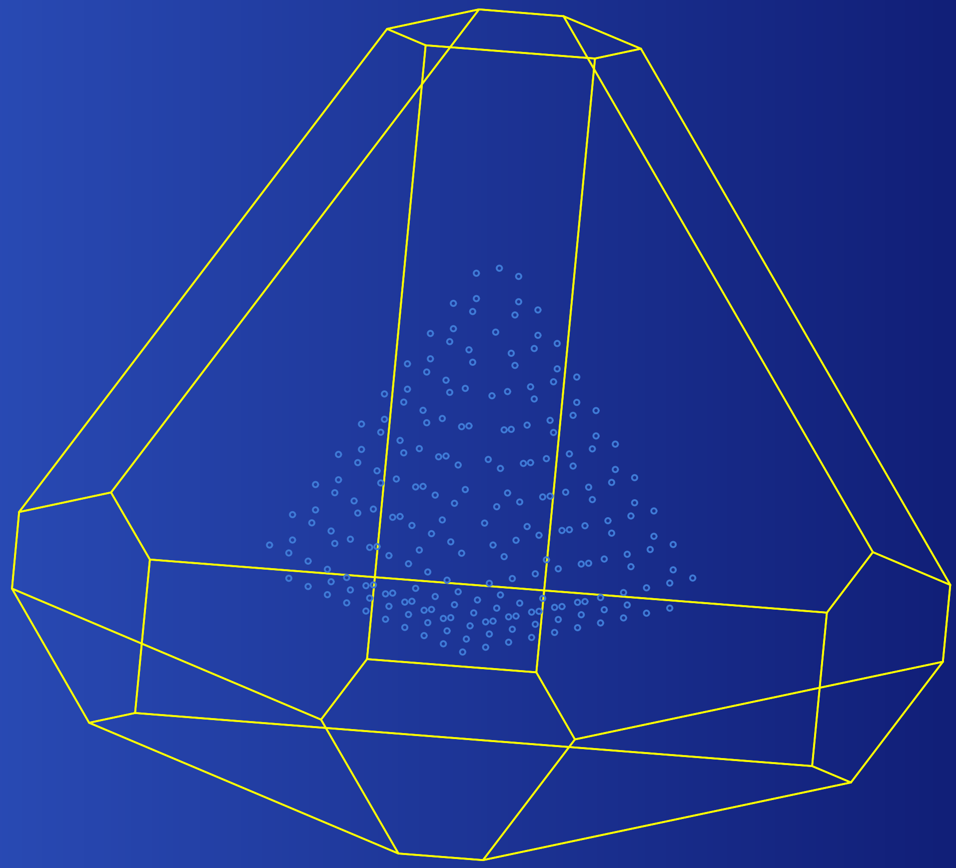
$$m_\lambda(\beta) = 45$$

$\lambda = (14,\ -2,\ -4,\ -8)$



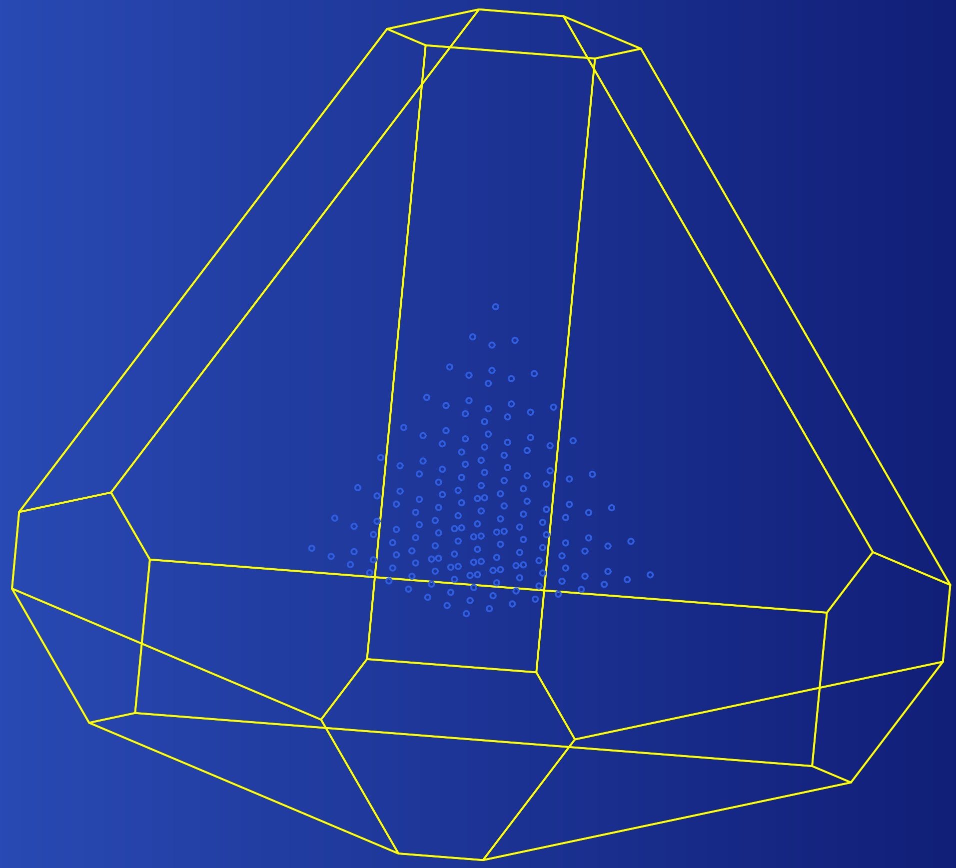
$$m_\lambda(\beta) = 50$$

$\lambda = (14, -2, -4, -8)$



$$m_\lambda(\beta) = 55$$

$$\lambda = (14, -2, -4, -8)$$



$$m_\lambda(\beta) = 60$$

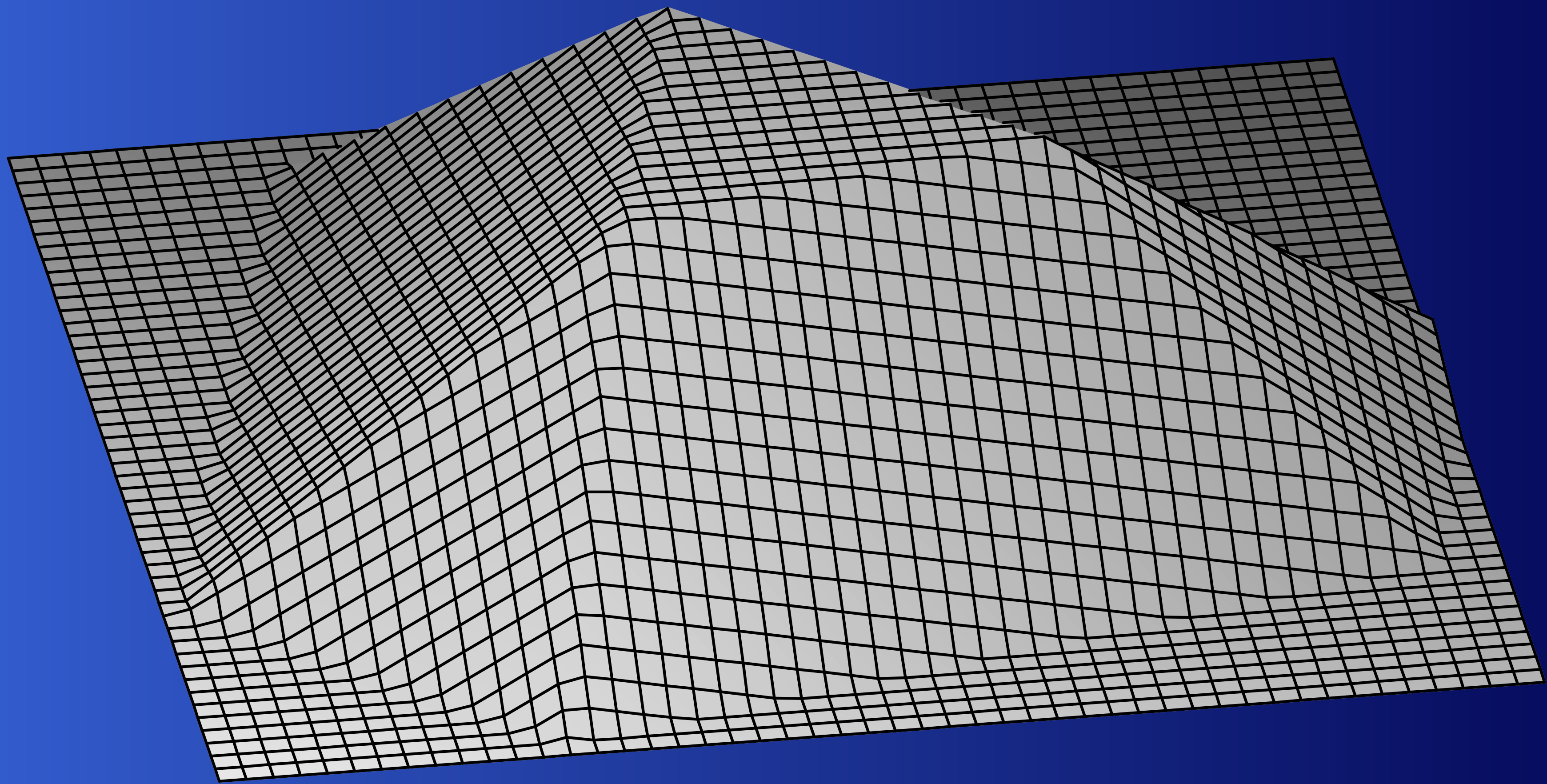
The Duistermaat-Heckman function

- For λ integral there is a function from symplectic geometry, the Duistermaat-Heckman function, that is piecewise polynomial on P_λ.

- It approximates the weight multiplicities.

- The domains of polynomiality form a partition of P_λ into subpolytopes.

The DH function for $k = 3$



The DH function for $k = 3$

Theorem **(Heckman, Guillemin-Lerman-Sternberg)**

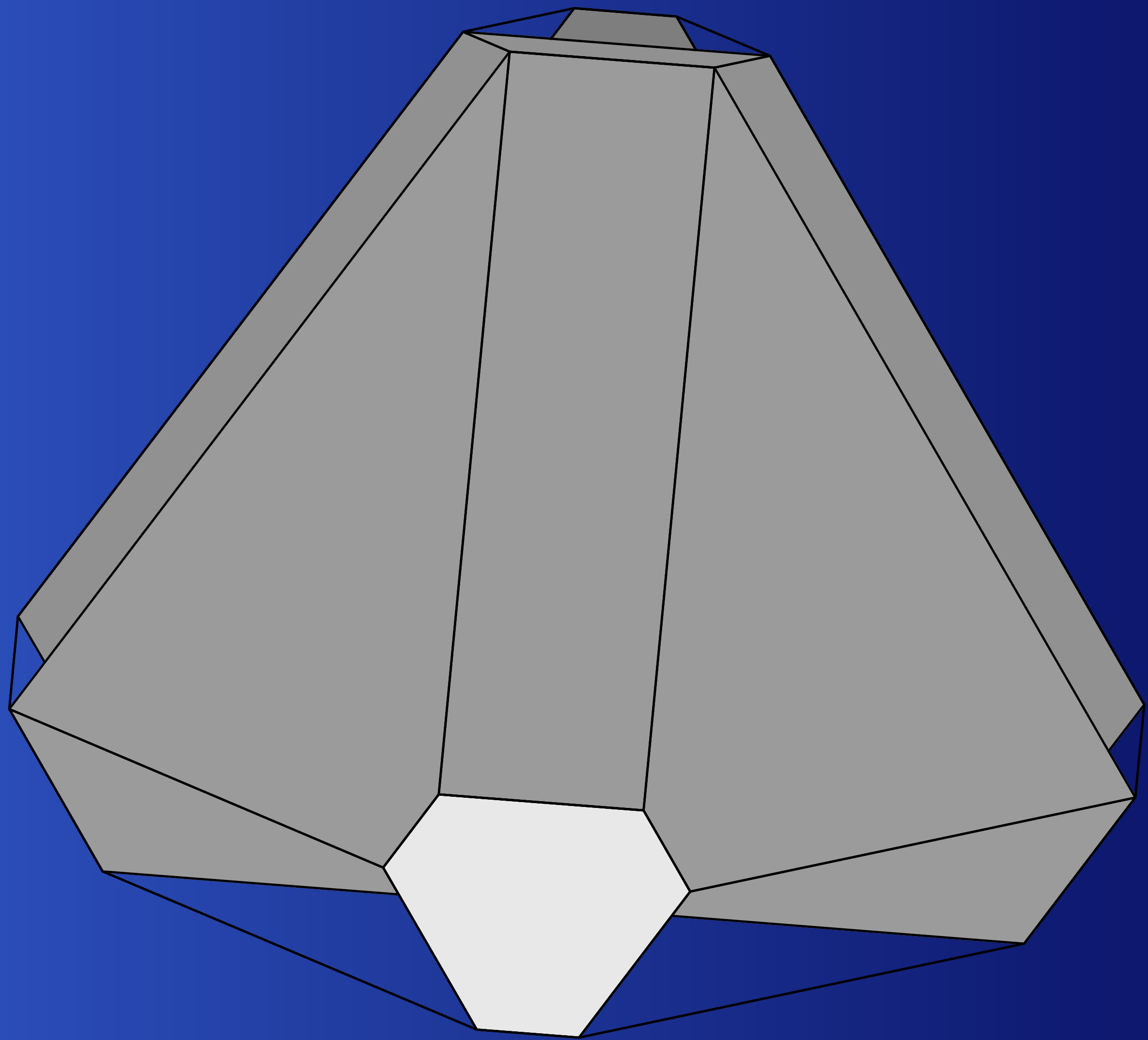
Consider the convex polytopes

$$\mathrm{conv}(W \cdot \sigma(\lambda))$$

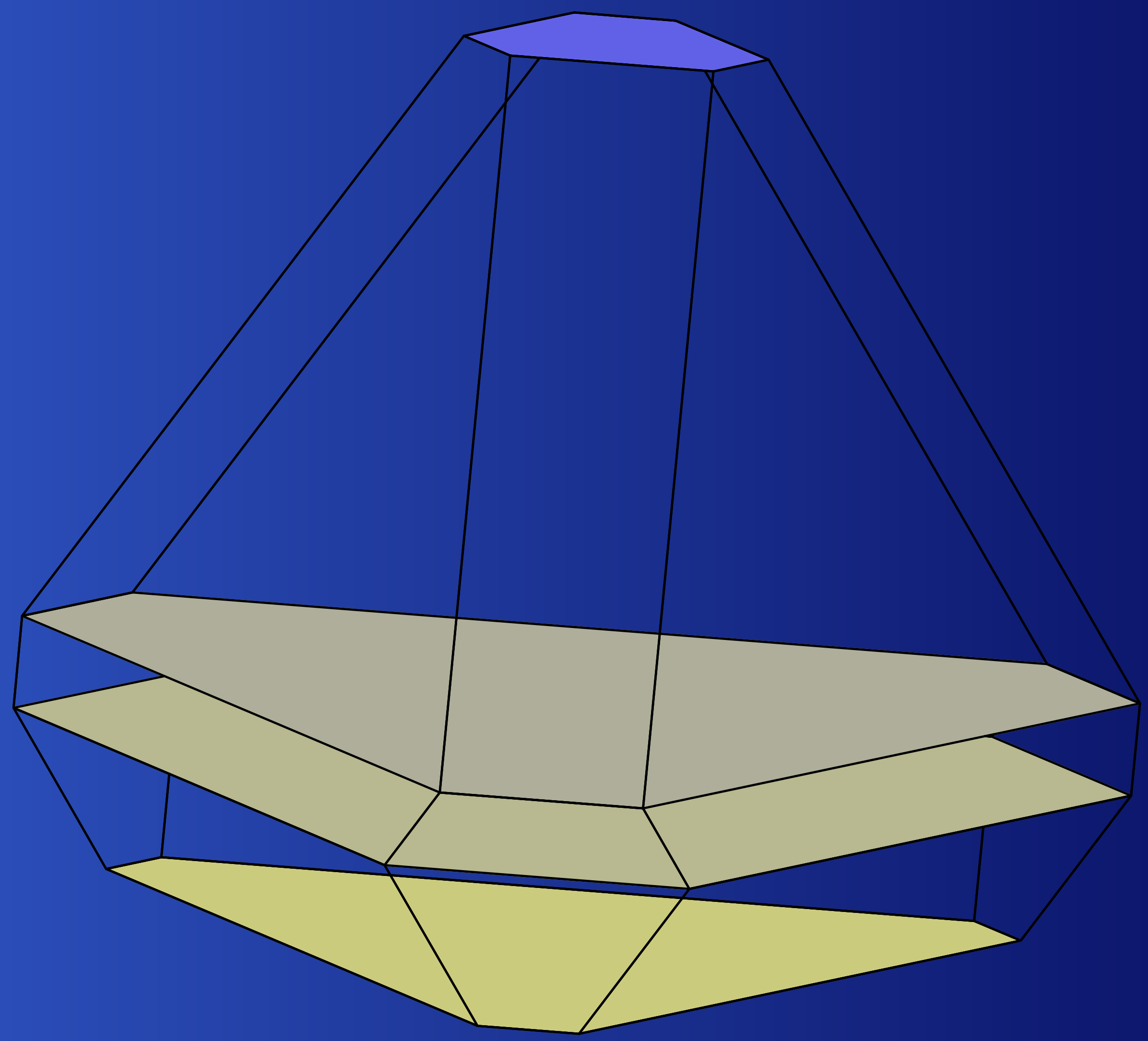
where $\sigma \in \mathfrak{S}_k$ *and* W *is the stabilizer of a facet of* $\mathrm{conv}(\mathfrak{S}_k \cdot \lambda)$.

These polytopes are walls that partition $\mathrm{conv}(\mathfrak{S}_k \cdot \lambda)$ *into convex subpolytopes over which the Duistermaat-Heckman function is polynomial.*

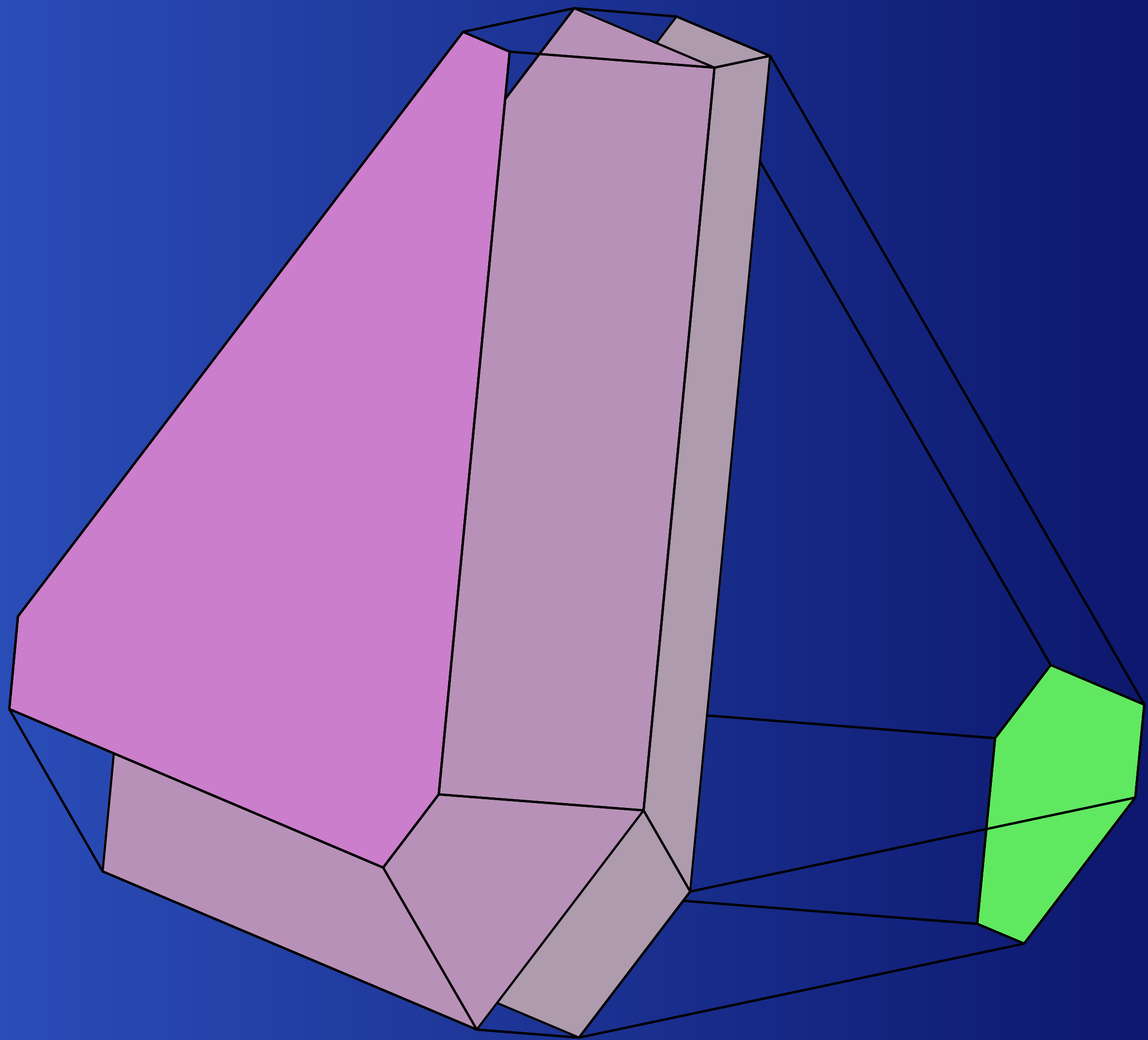
$\lambda = (7, -1, -2, -4)$



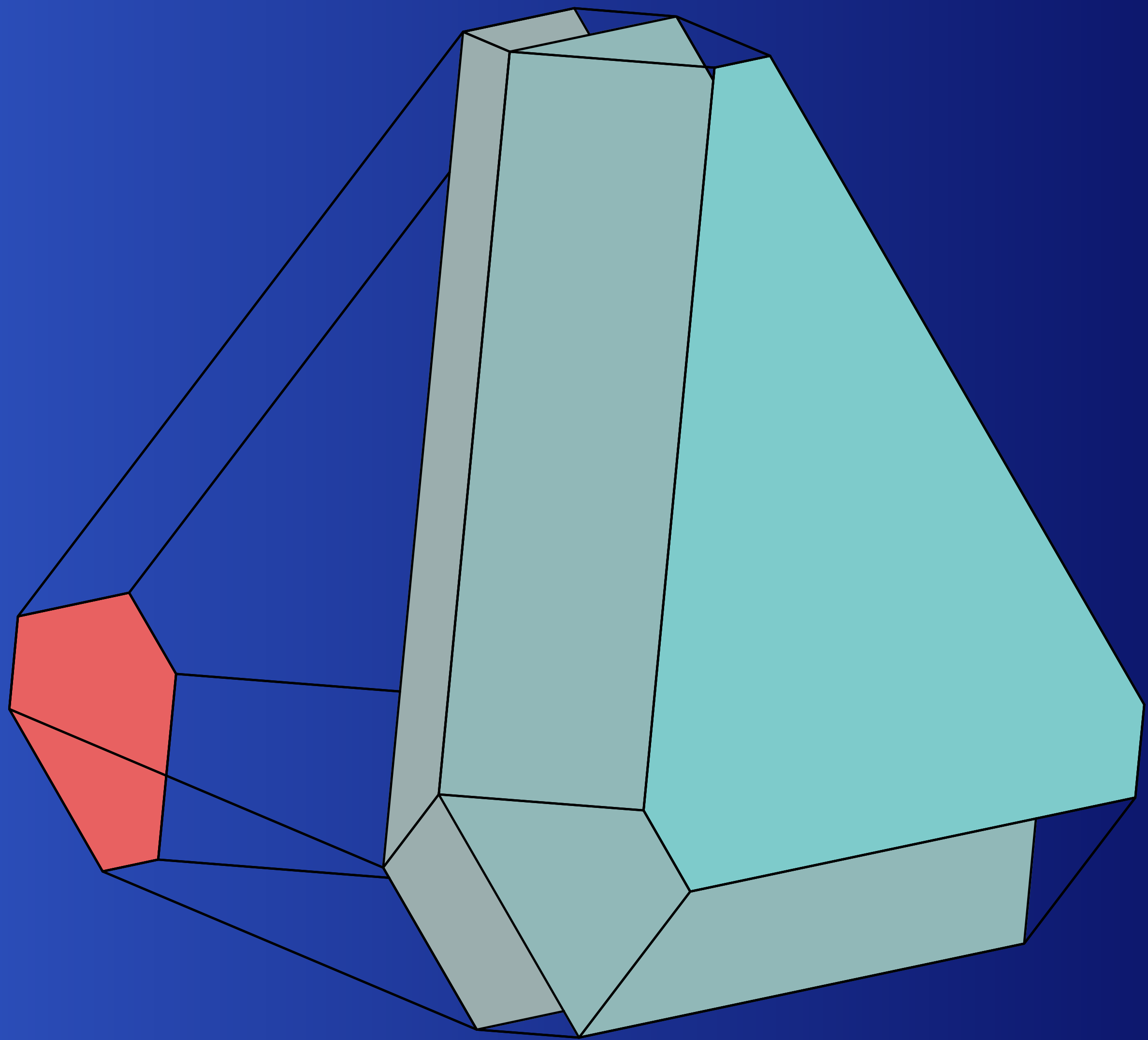
$\lambda = (7, -1, -2, -4)$



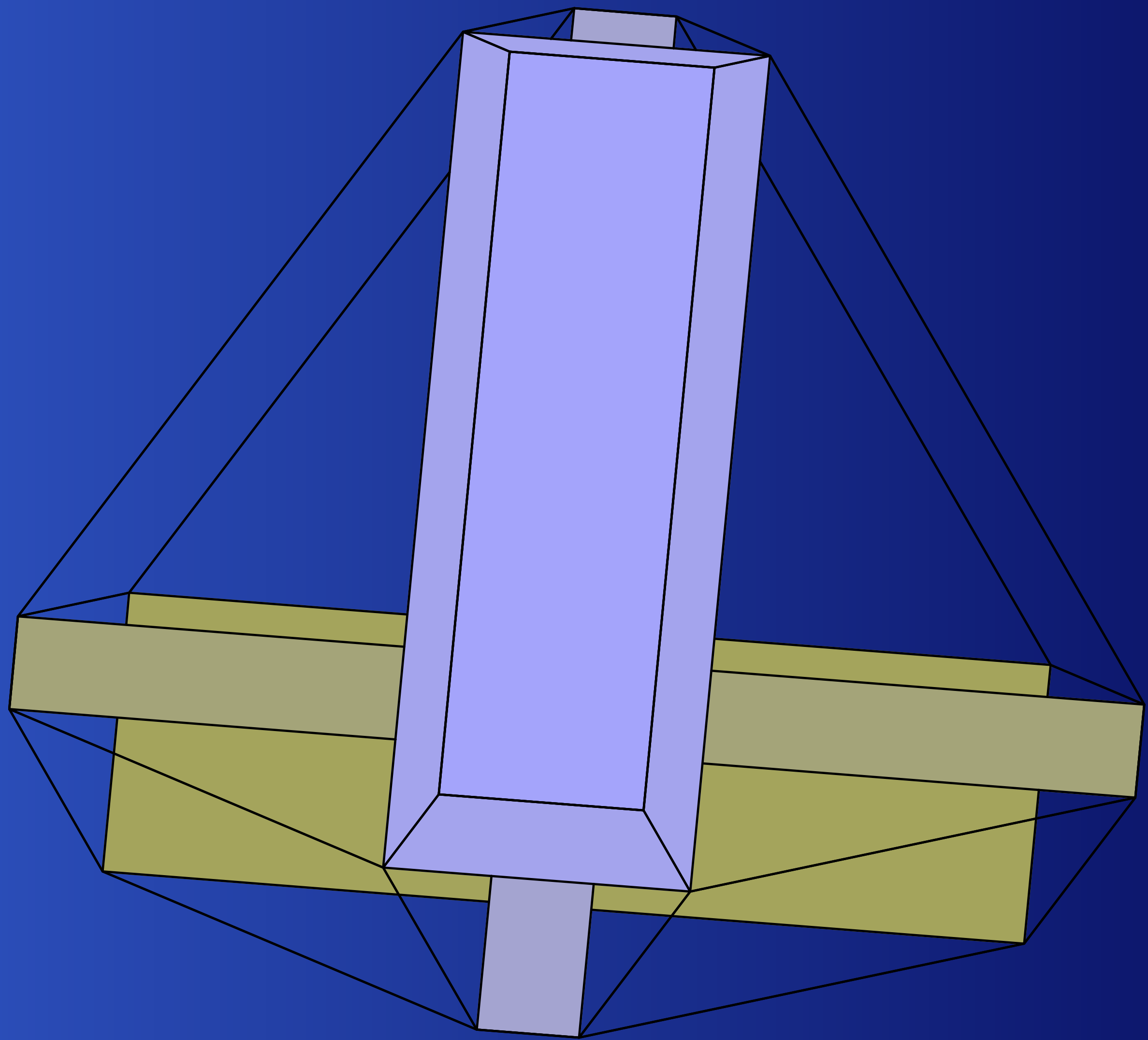
$\lambda = (7,\ -1,\ -2,\ -4)$



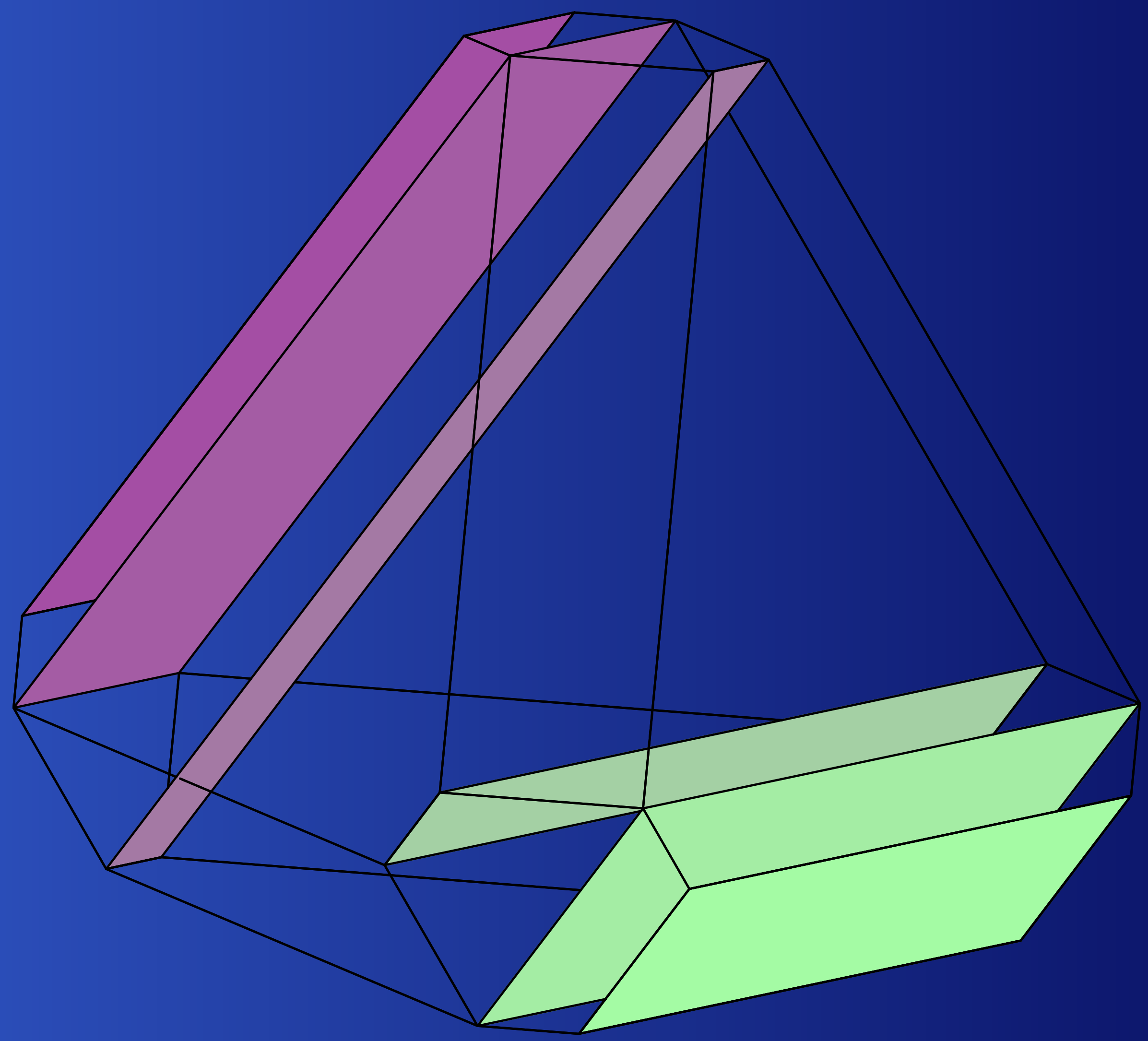
$\lambda = (7,\ -1,\ -2,\ -4)$



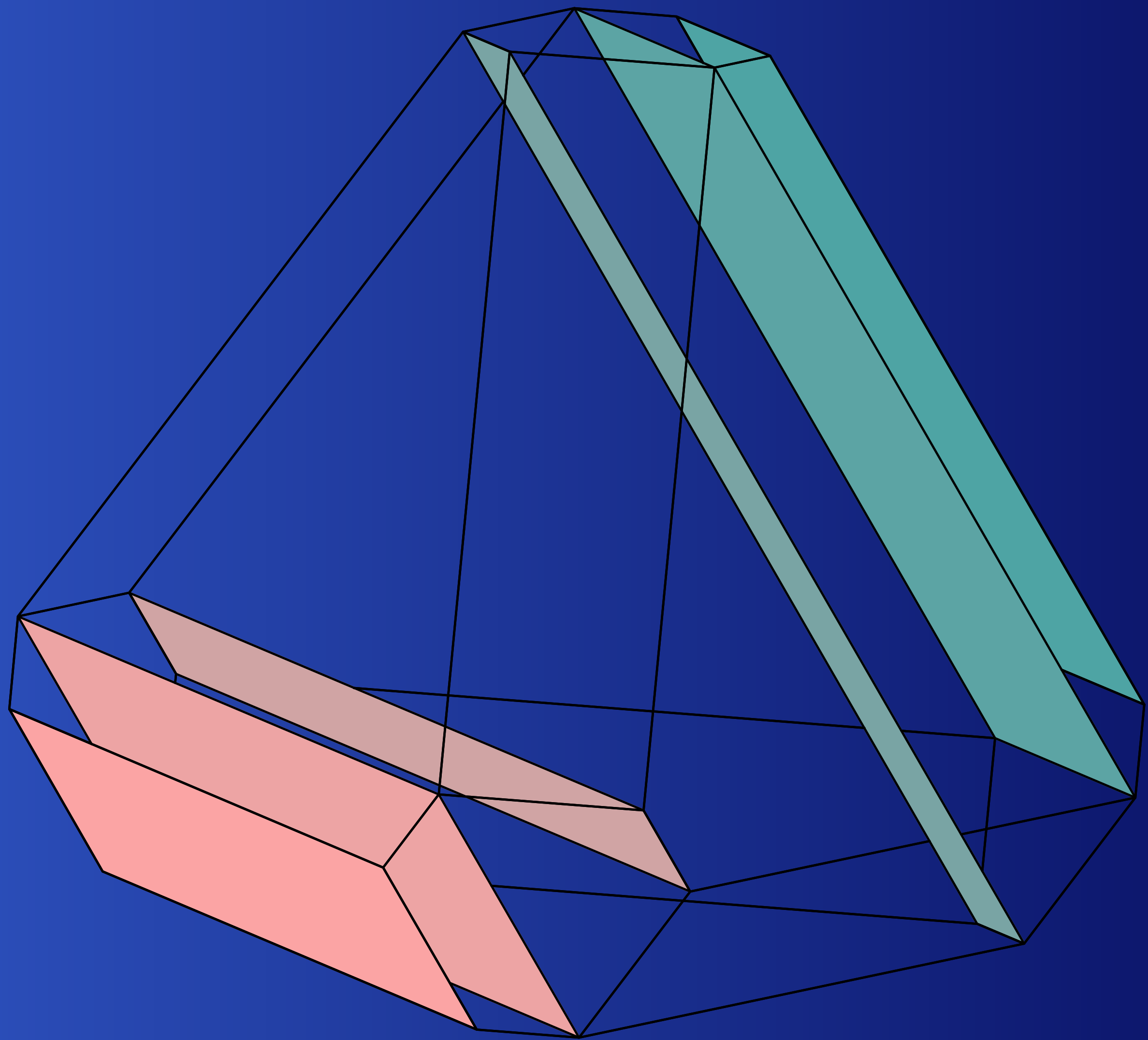
$\lambda = (7, -1, -2, -4)$



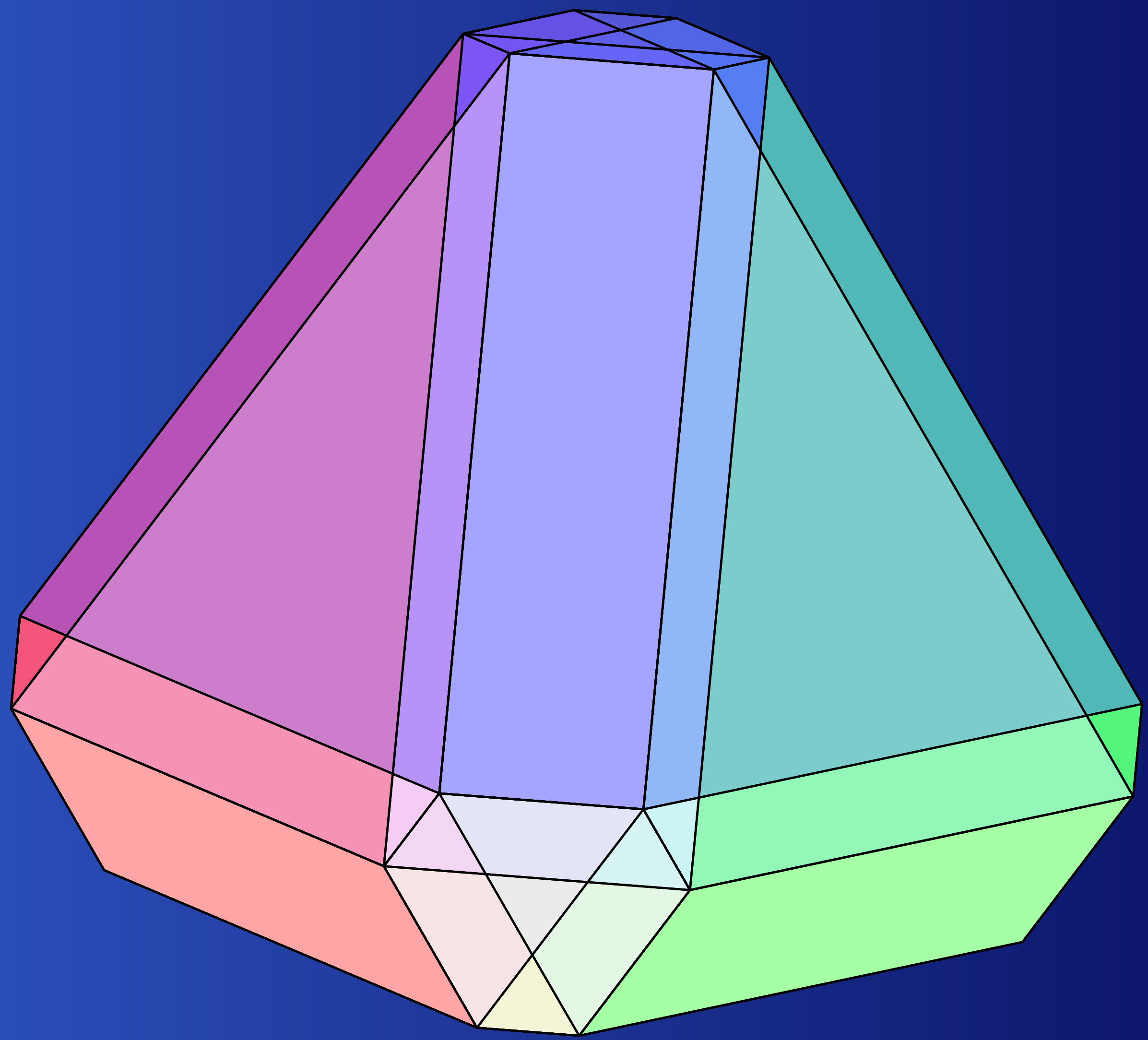
$\lambda = (7, -1, -2, -4)$



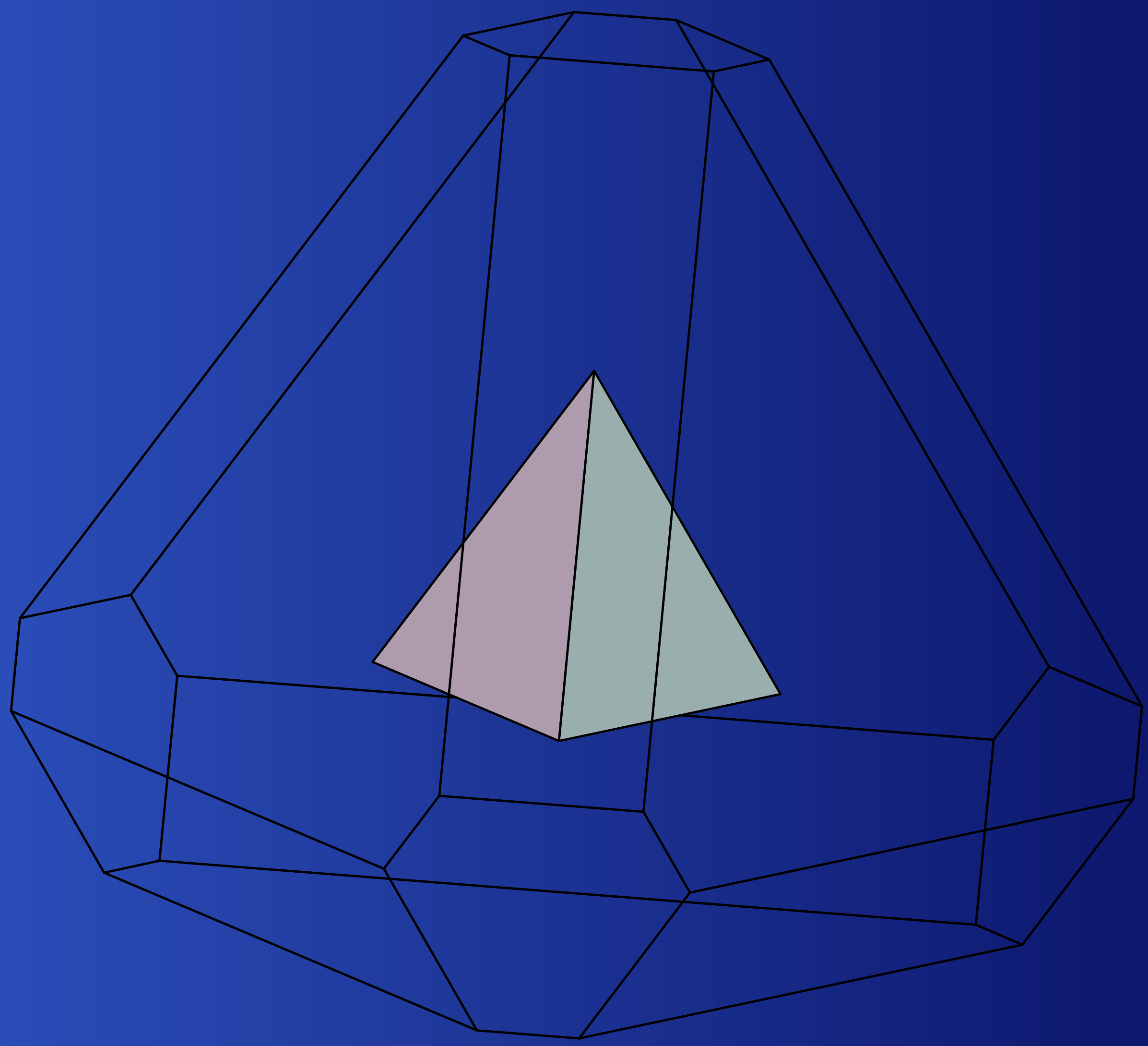
$\lambda = (7, -1, -2, -4)$



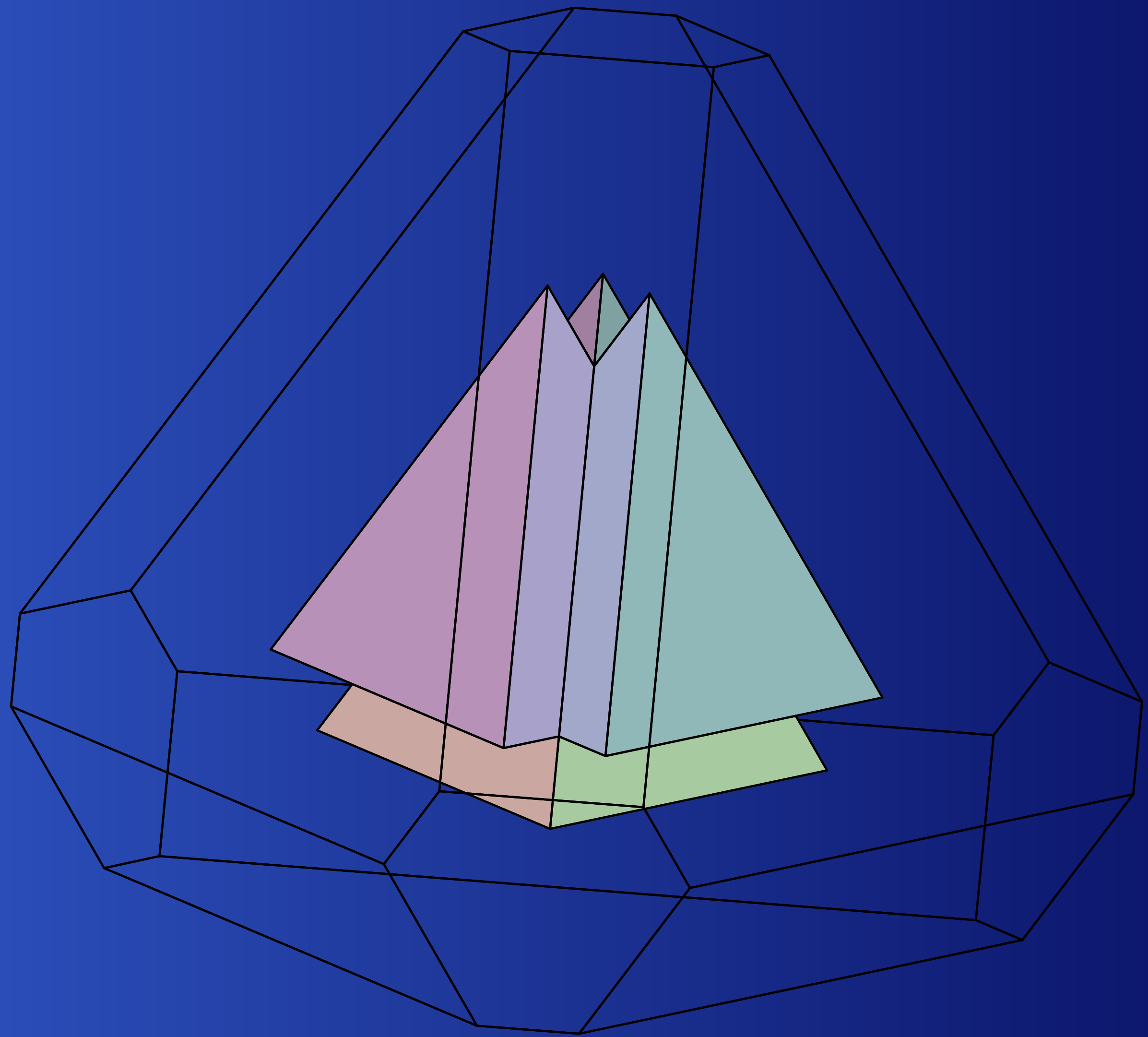
$\lambda = (7, -1, -2, -4)$



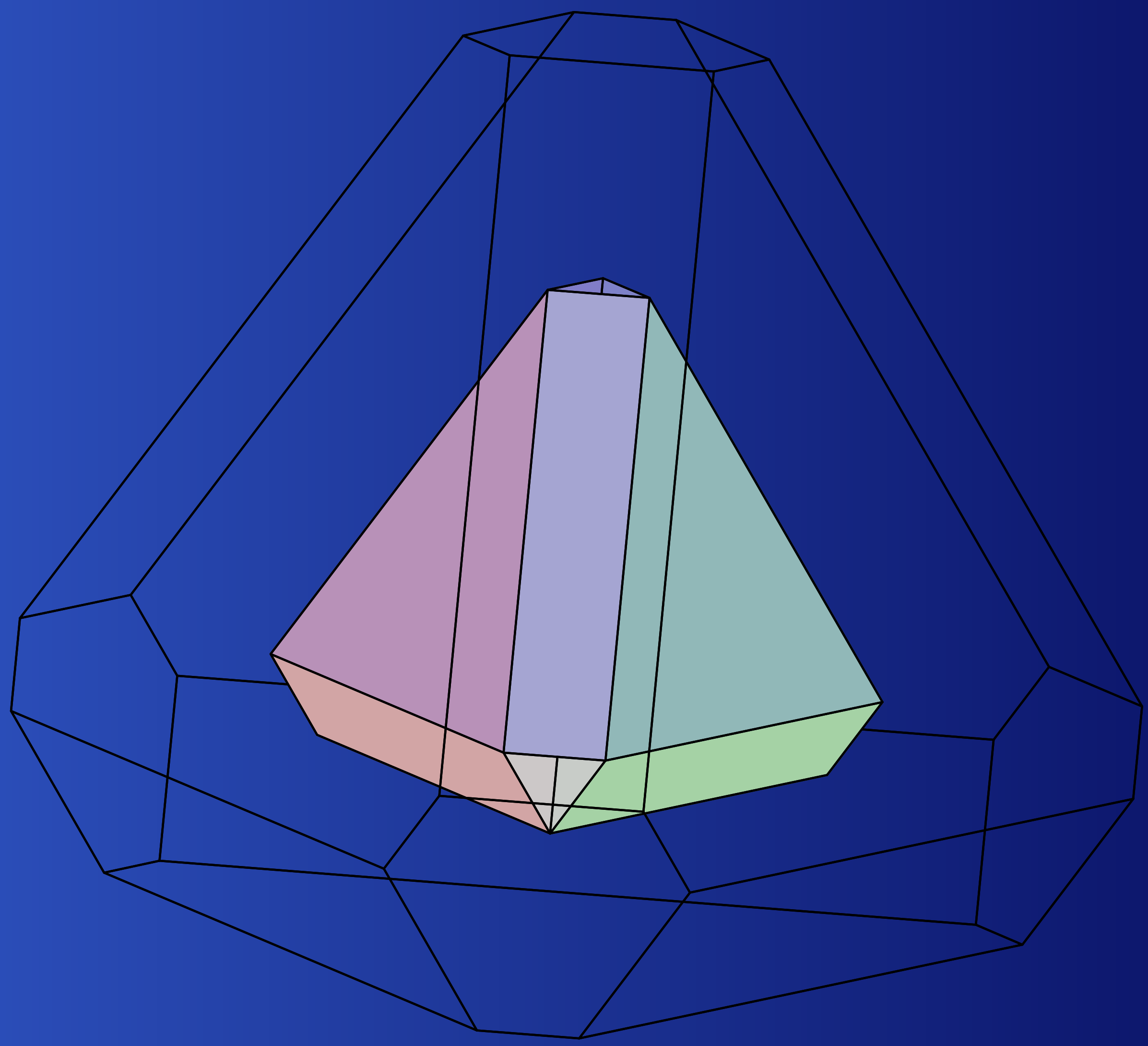
$\lambda = (7,\ -1,\ -2,\ -4)$



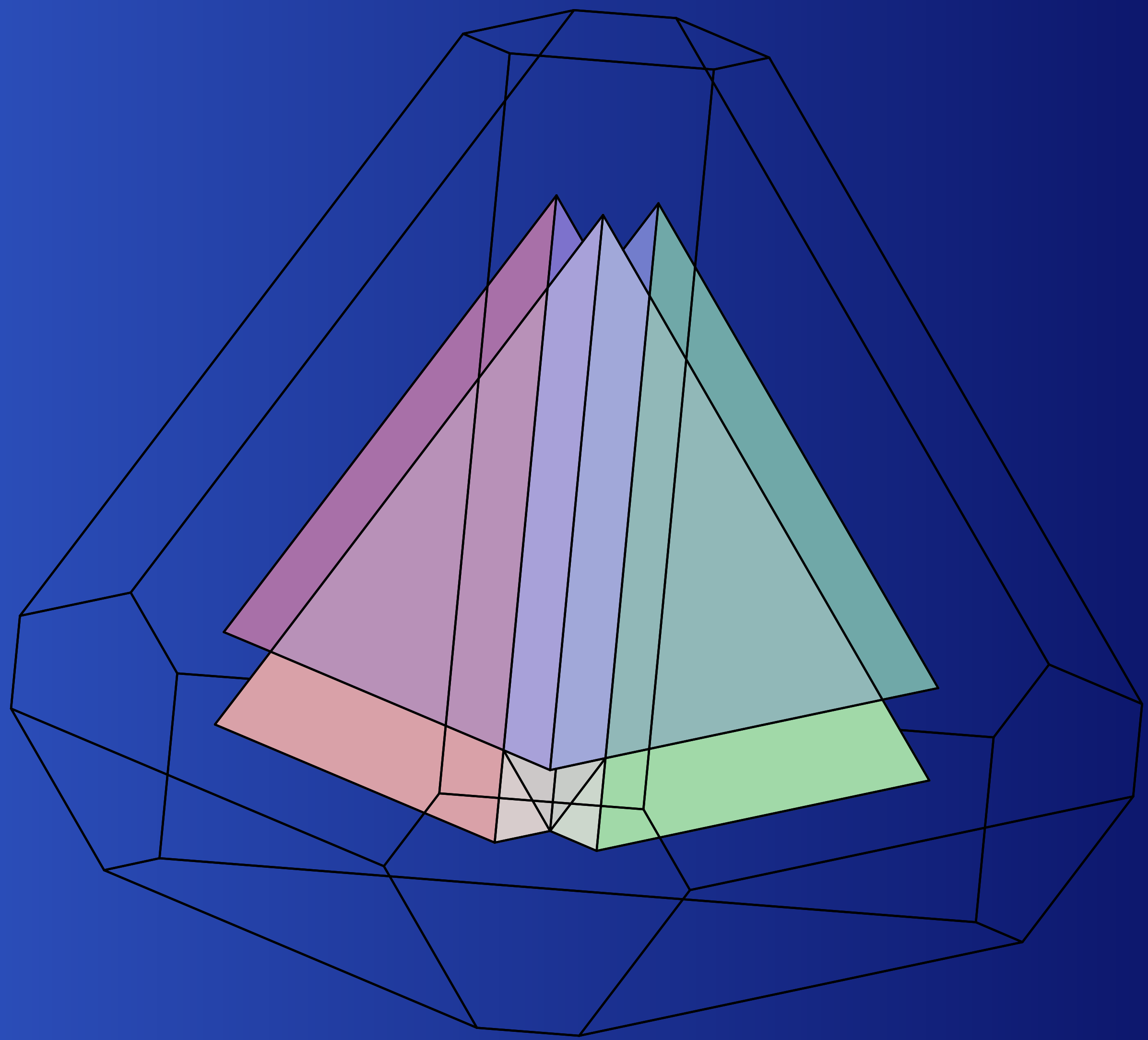
$\lambda = (7,\ -1,\ -2,\ -4)$



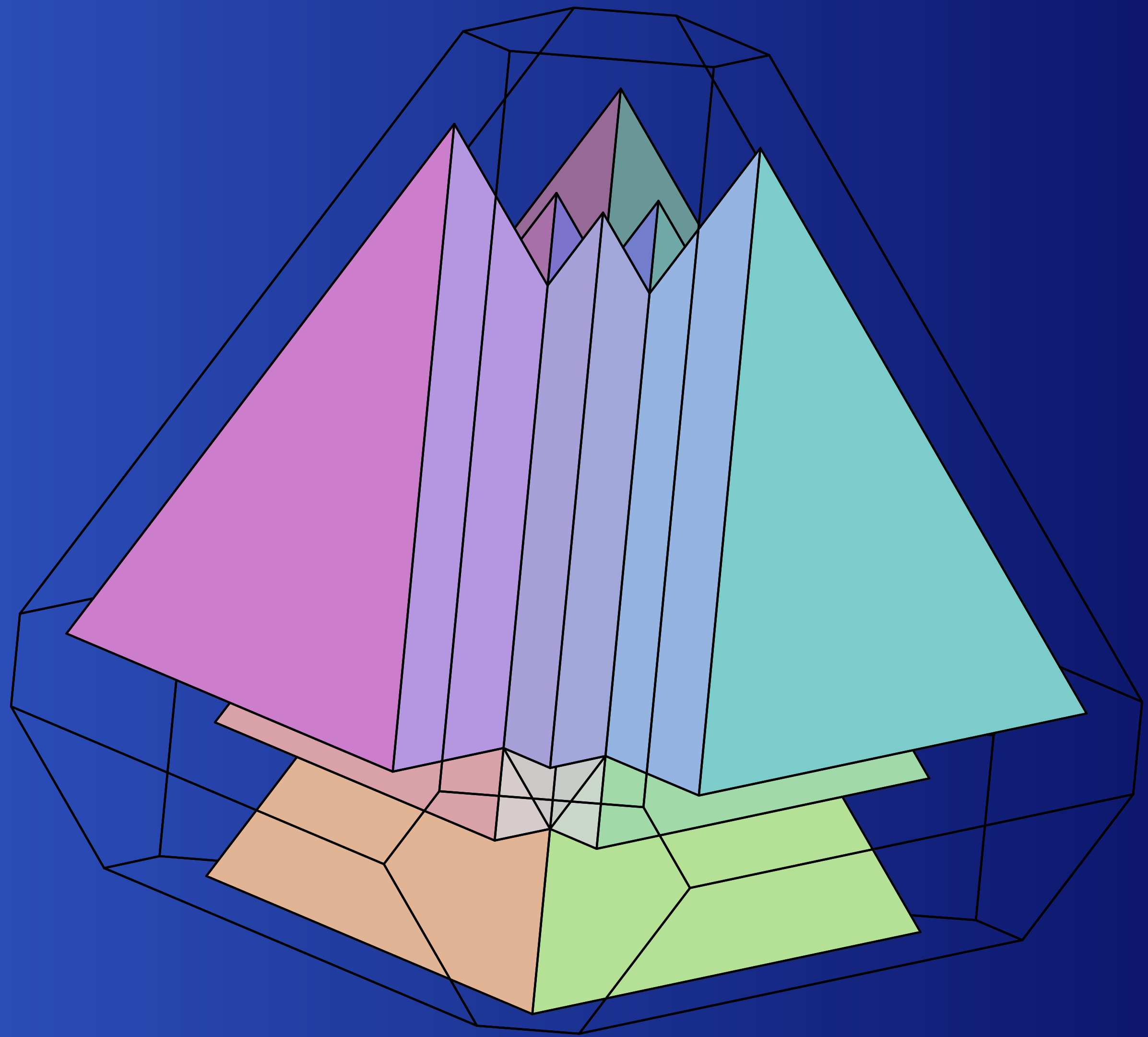
$\lambda = (7, -1, -2, -4)$



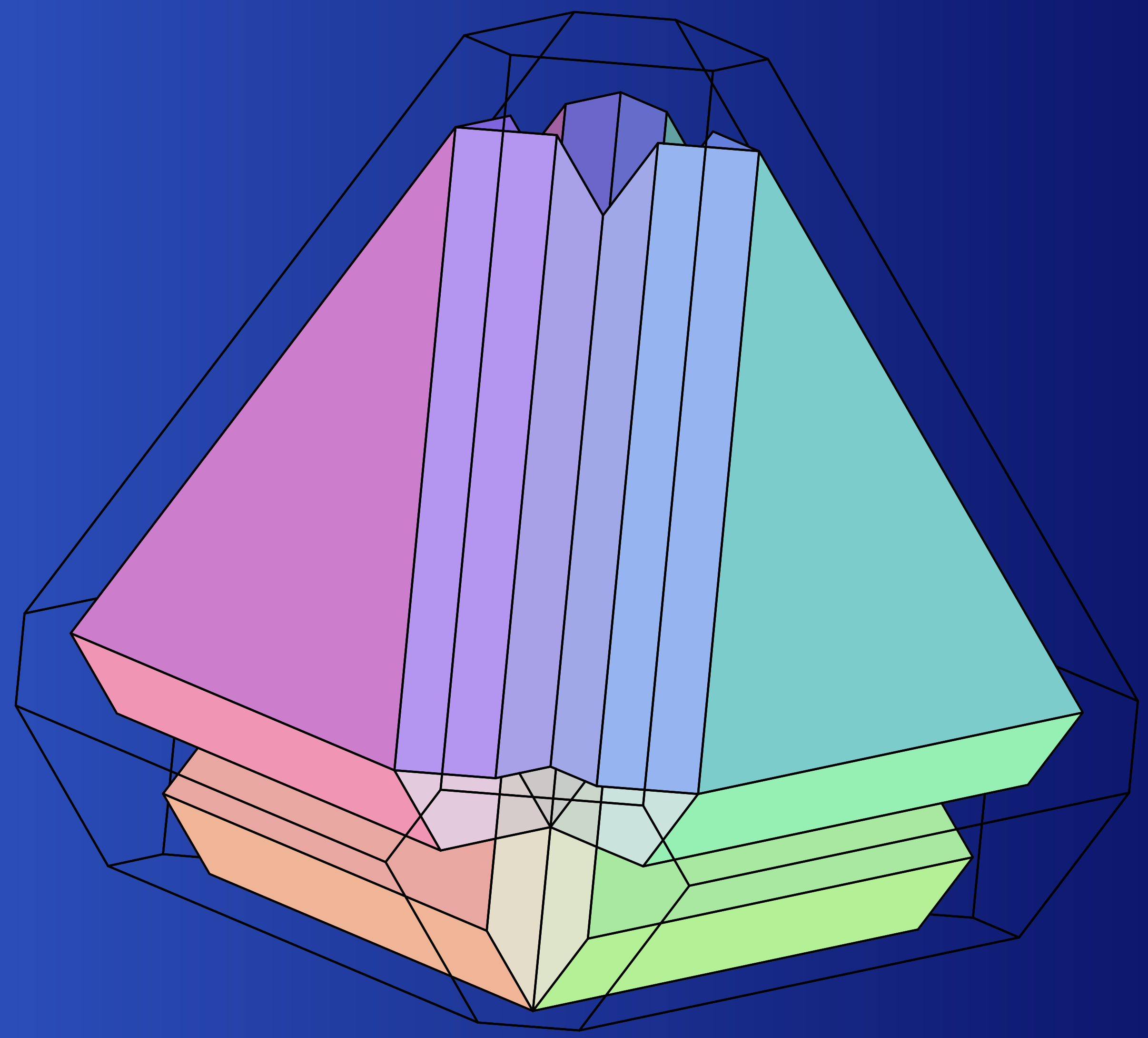
$\lambda = (7,\ -1,\ -2,\ -4)$



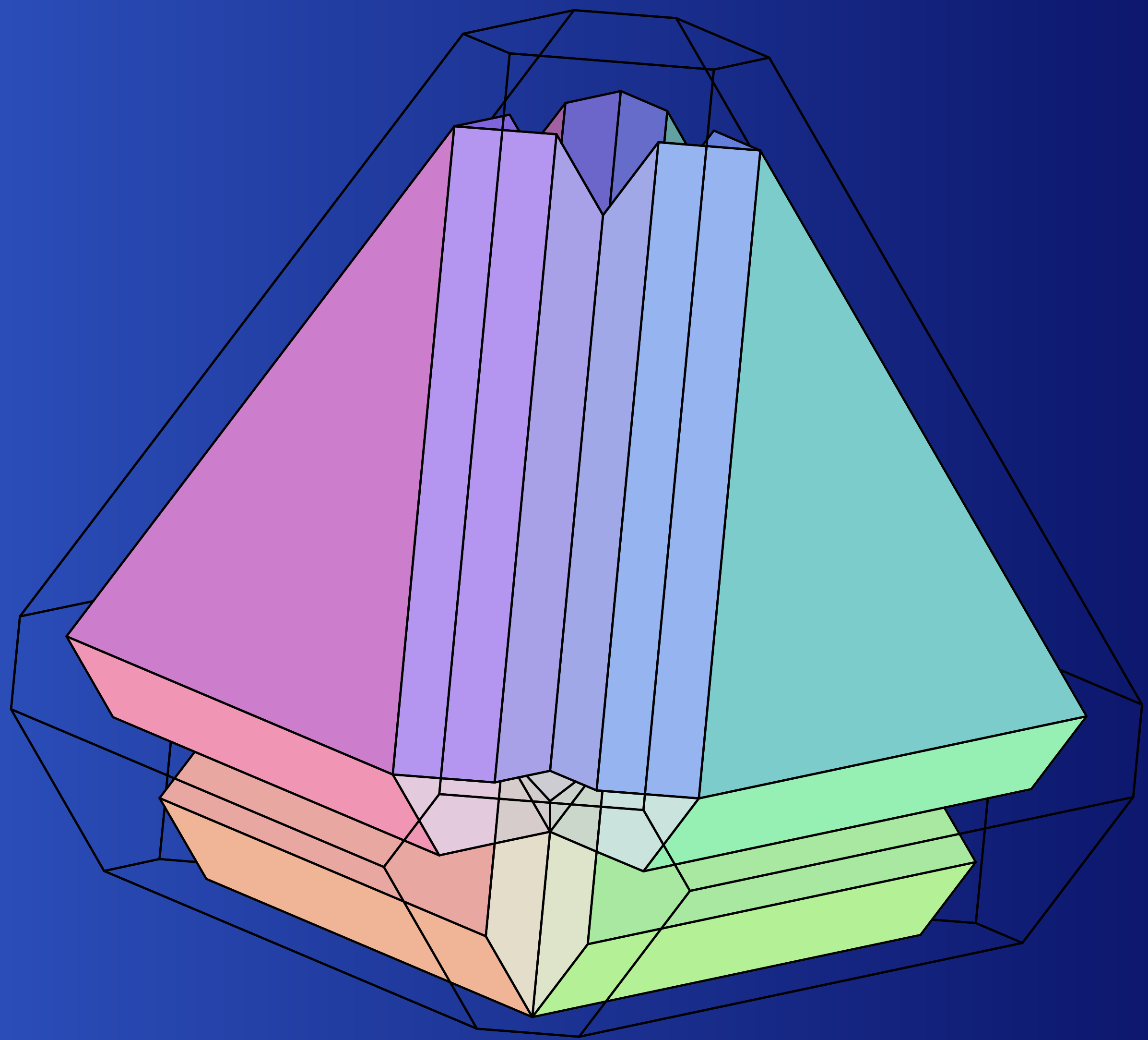
$\lambda = (7, -1, -2, -4)$



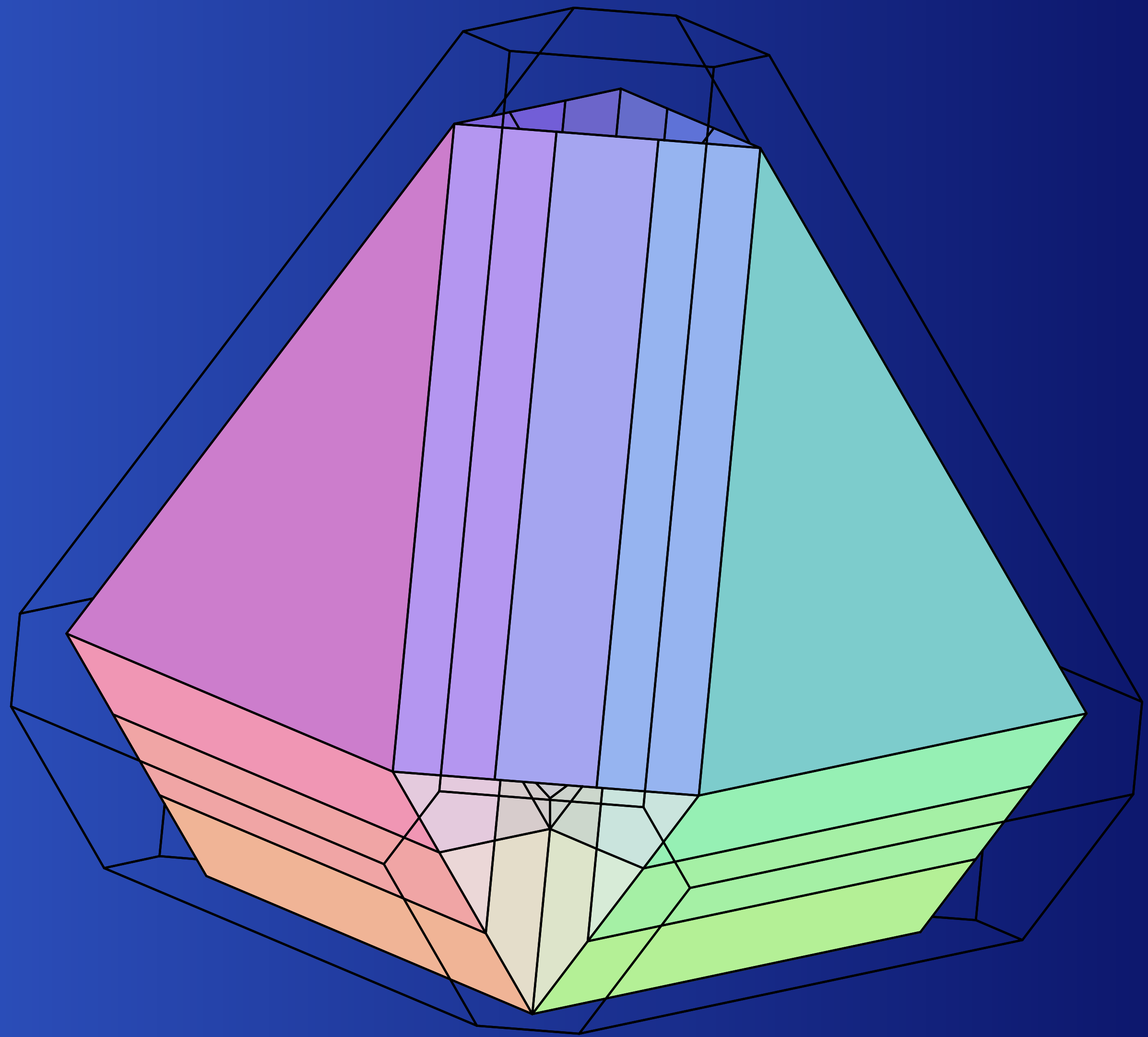
$$\lambda = (7,\ -1,\ -2,\ -4)$$



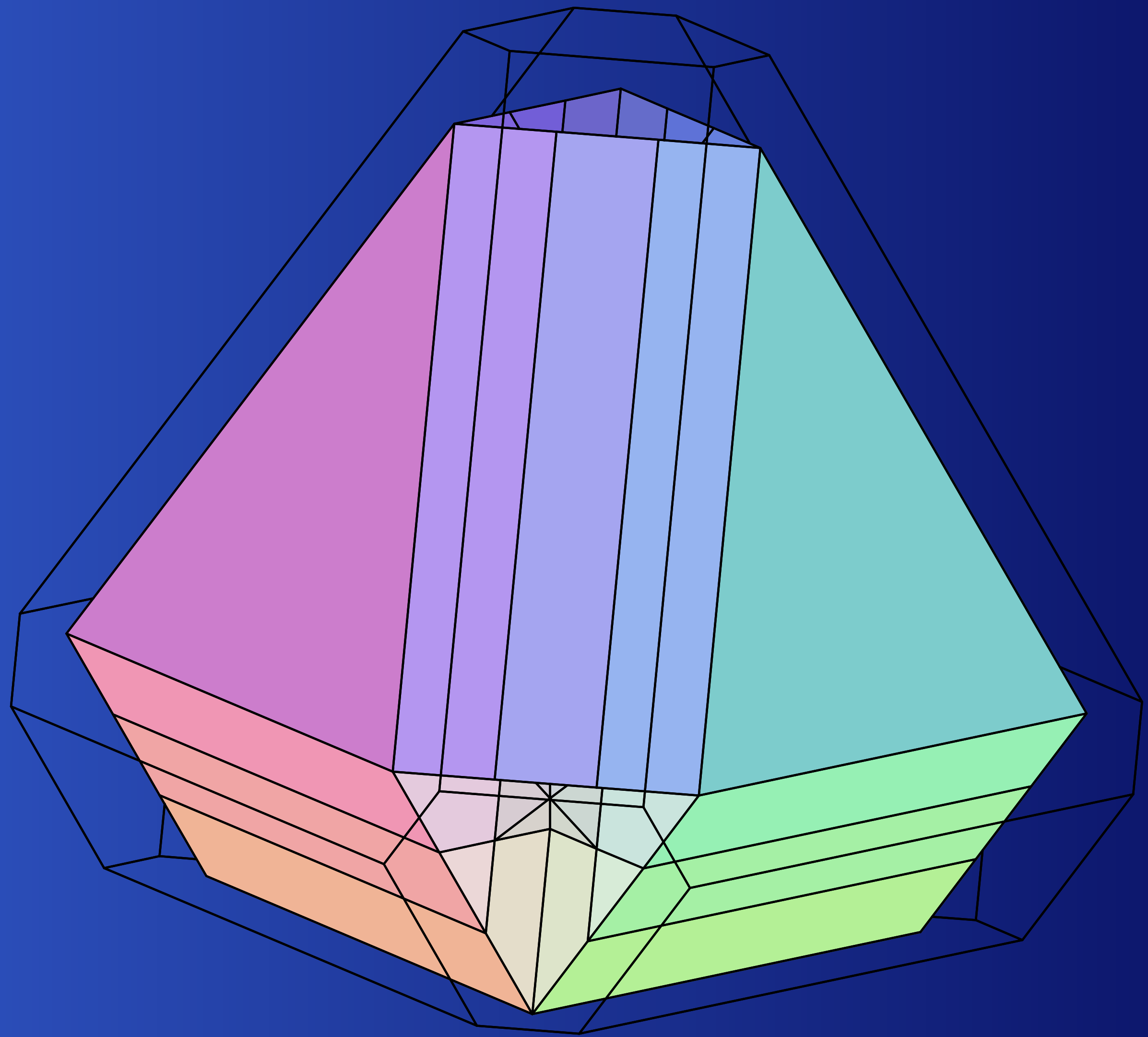
$\lambda = (7,\ -1,\ -2,\ -4)$



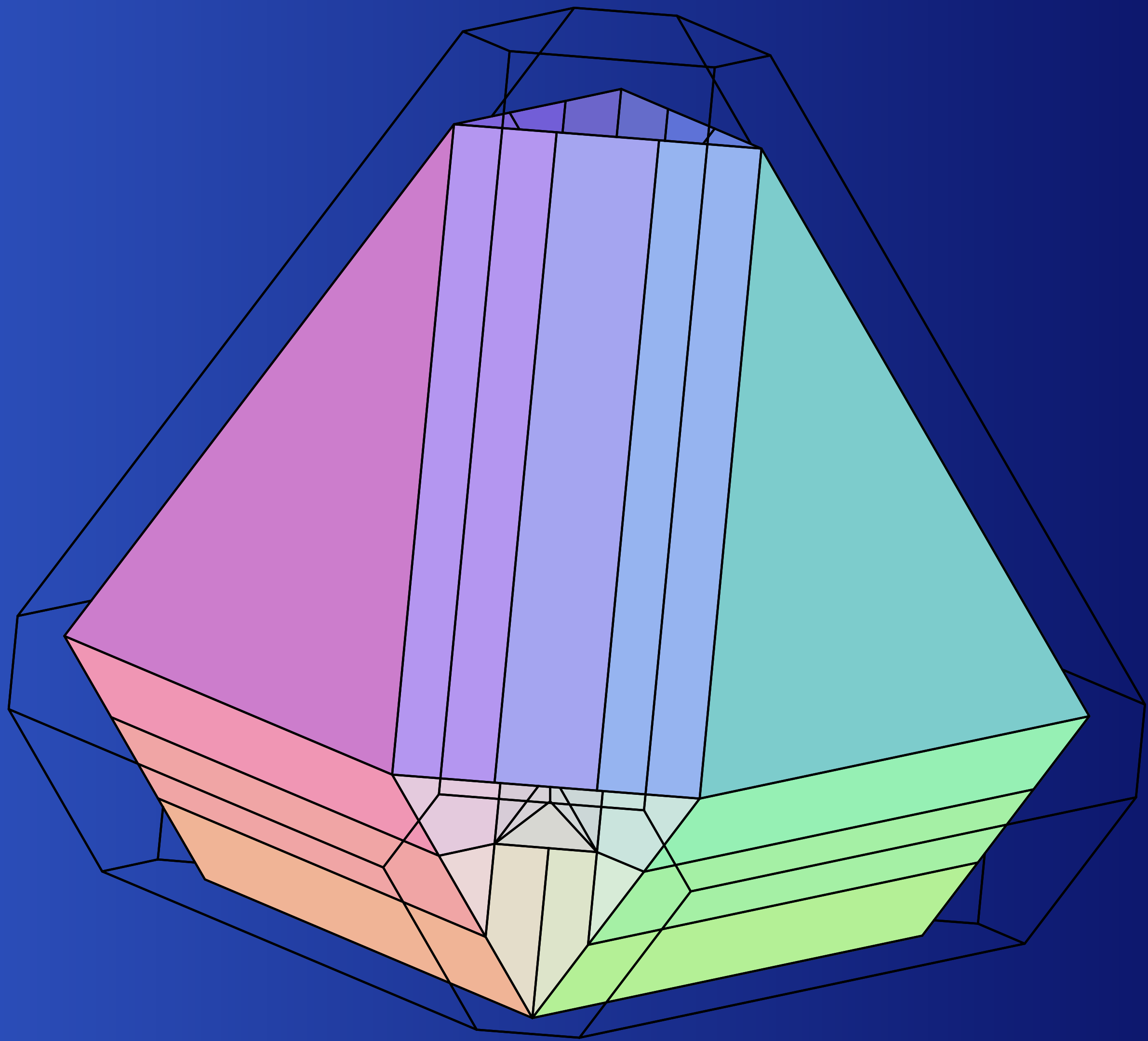
$\lambda = (7, -1, -2, -4)$



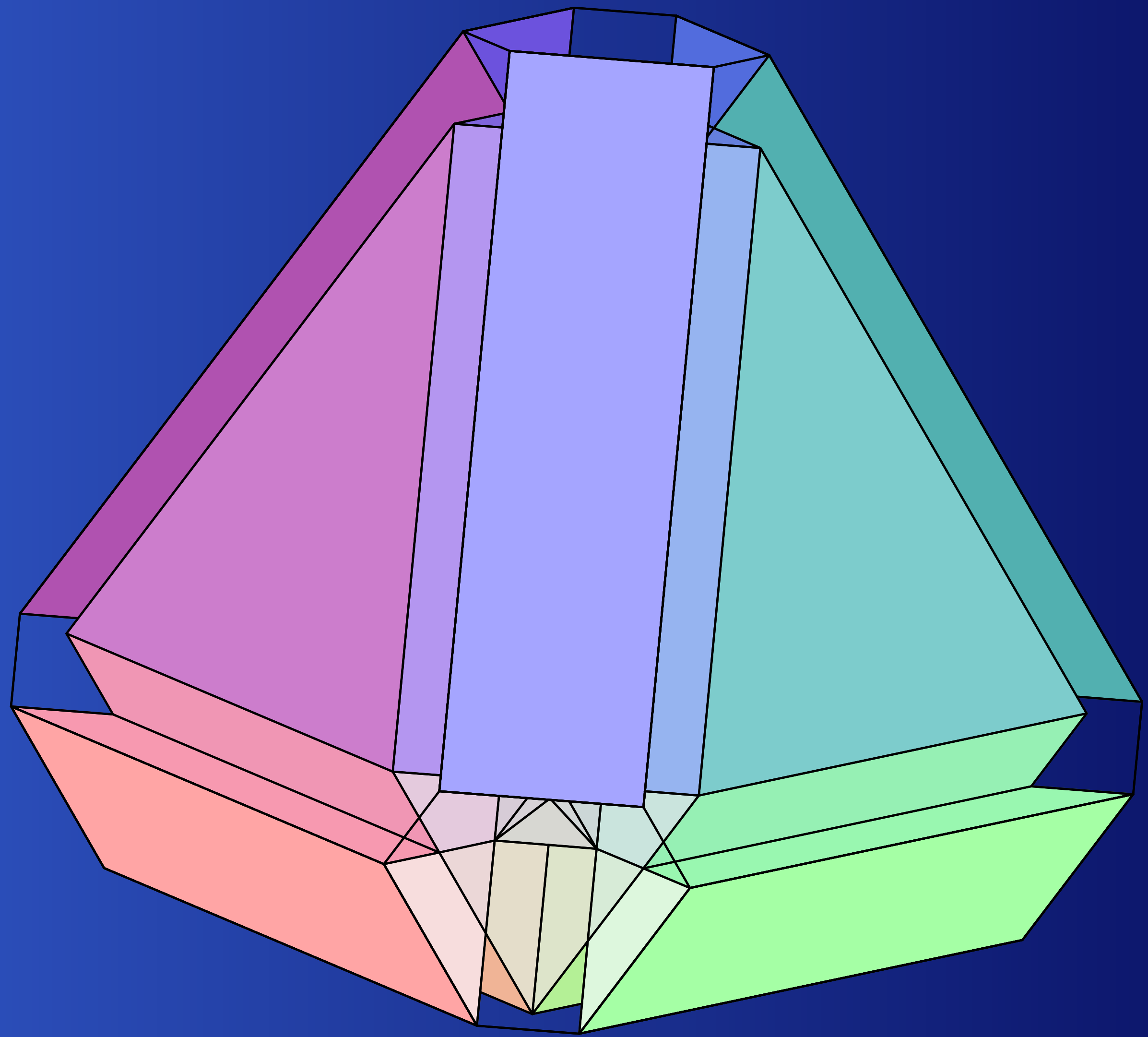
$\lambda = (7, -1, -2, -4)$



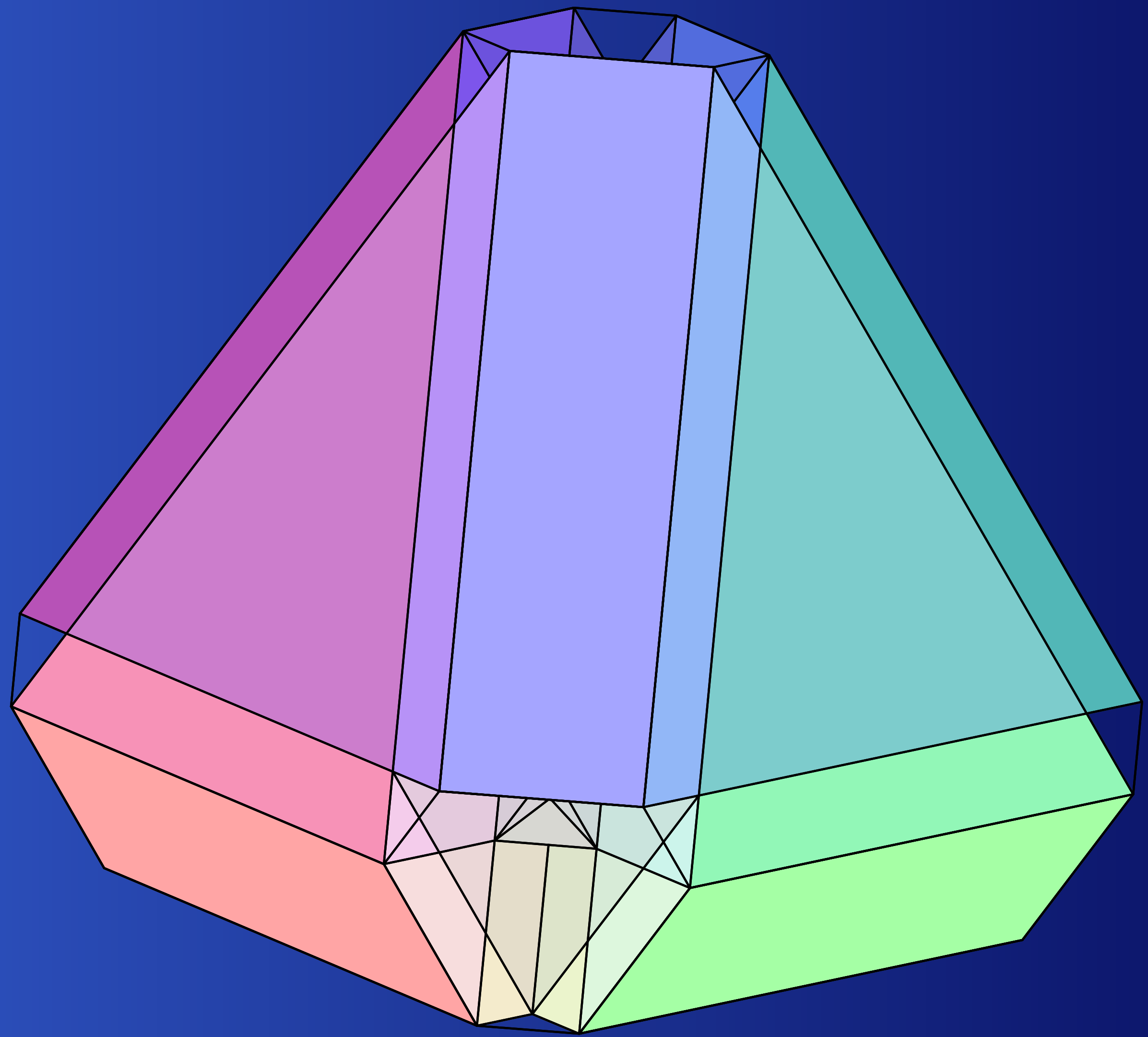
$\lambda = (7, -1, -2, -4)$



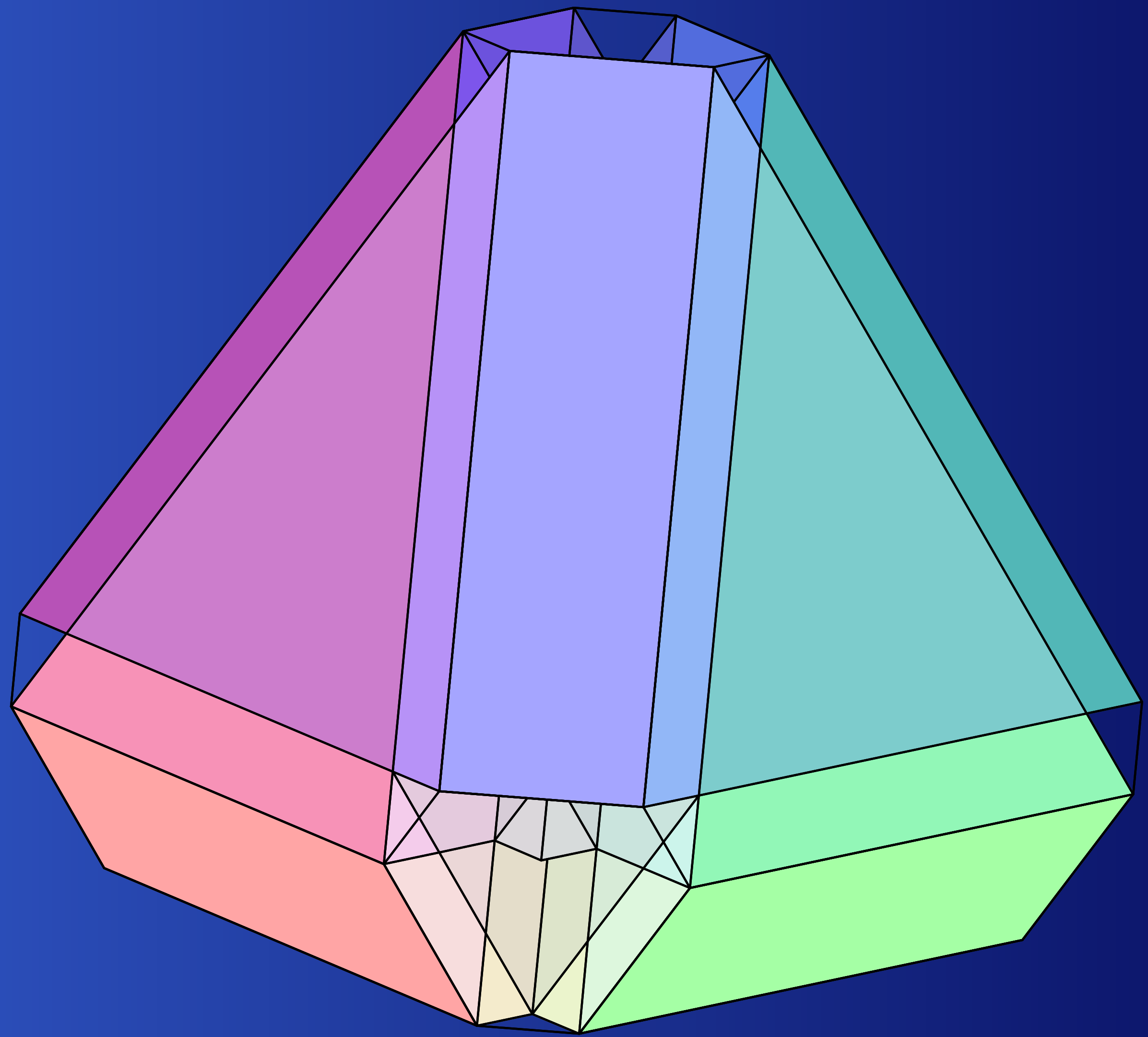
$\lambda = (7, -1, -2, -4)$



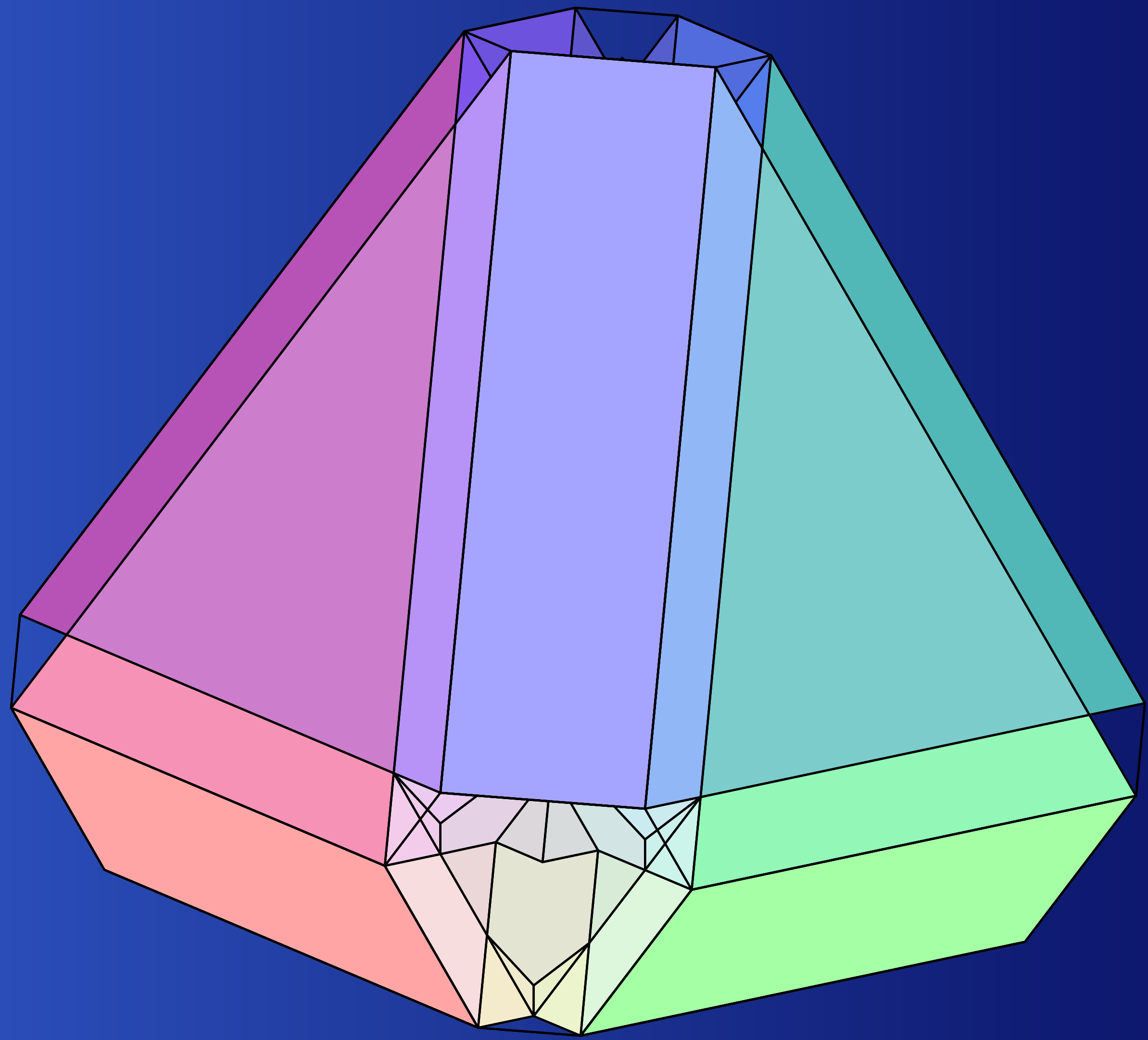
$\lambda = (7, -1, -2, -4)$



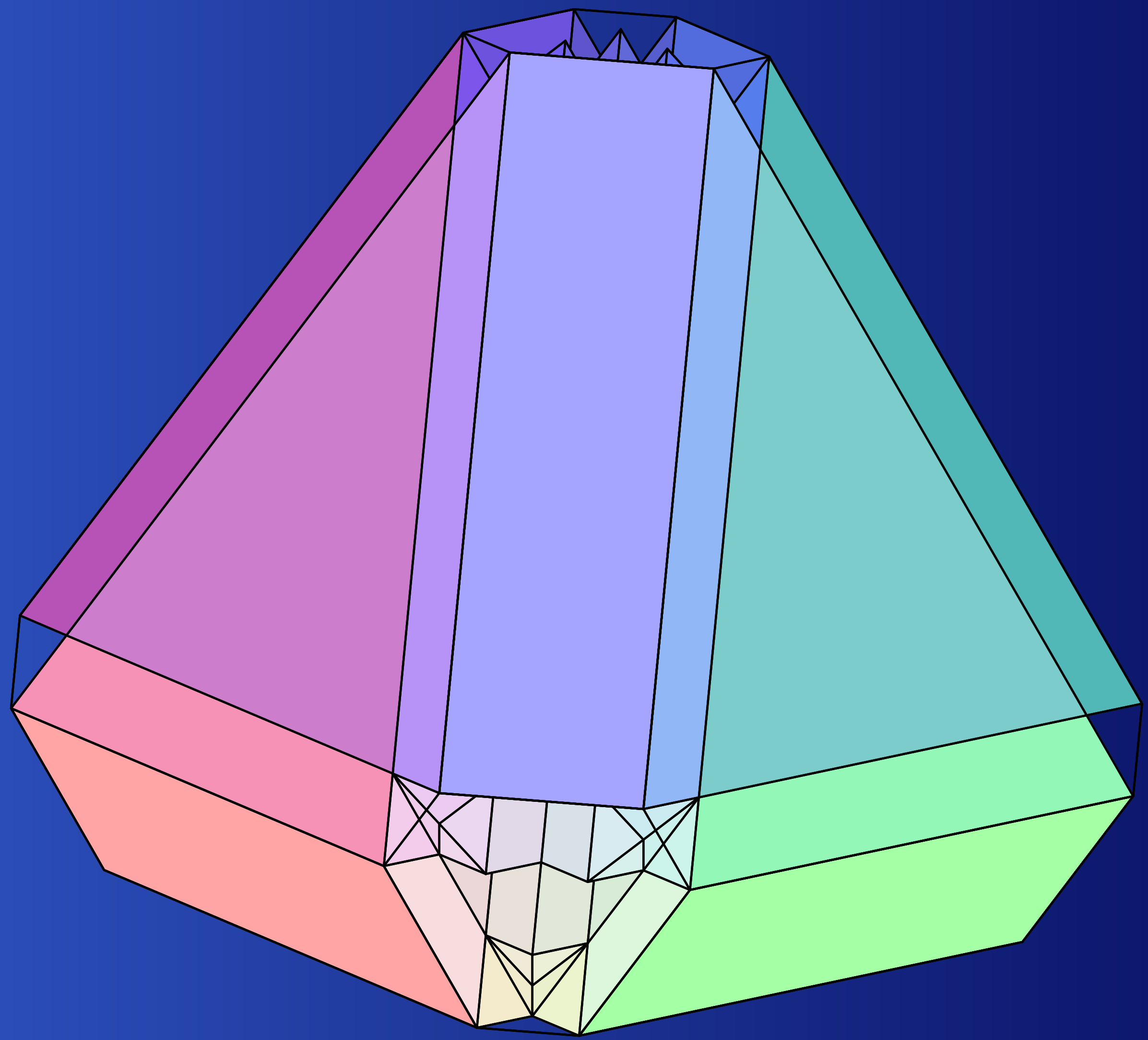
$\lambda = (7, -1, -2, -4)$



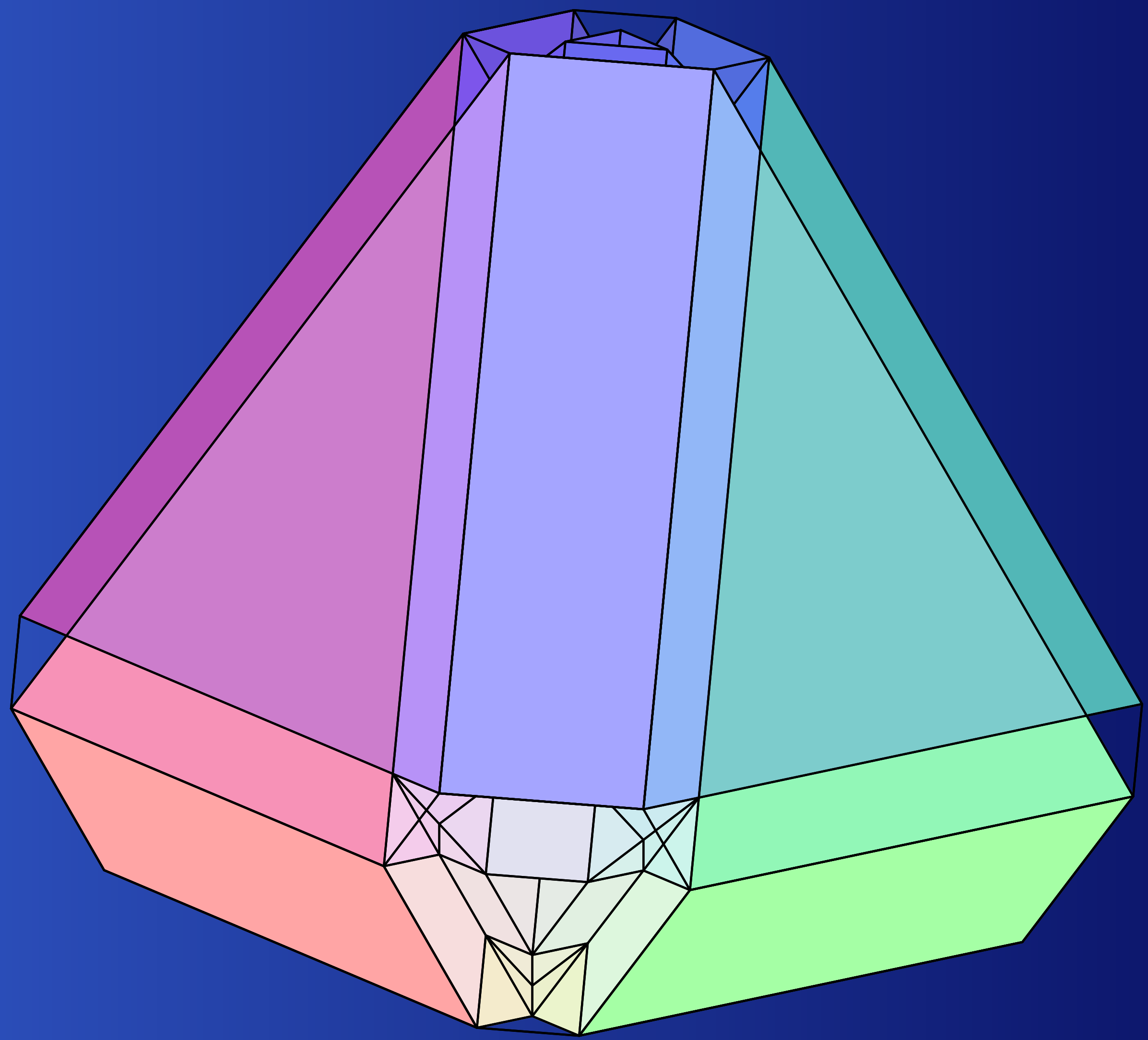
$\lambda = (7, -1, -2, -4)$



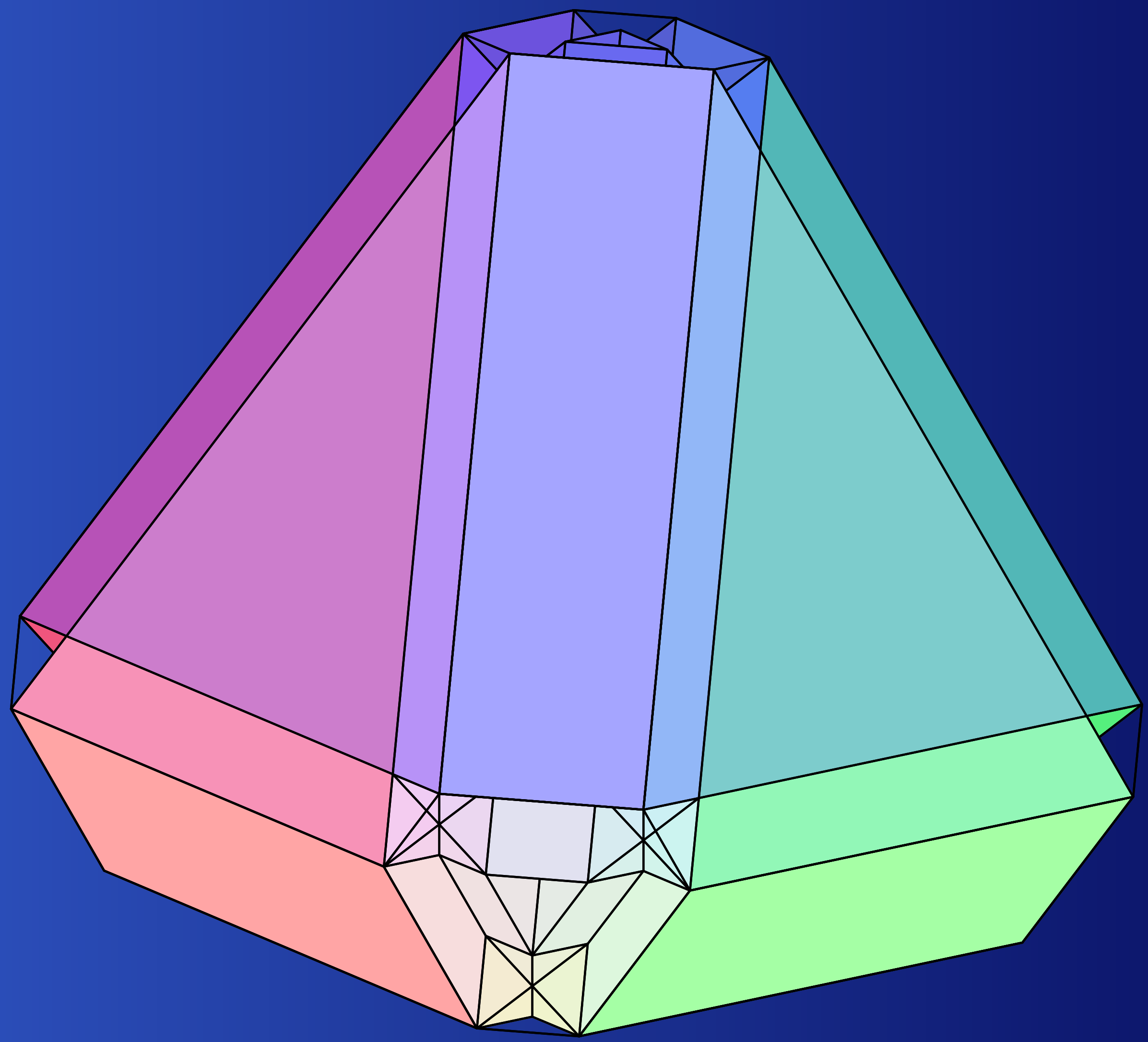
$\lambda = (7, -1, -2, -4)$



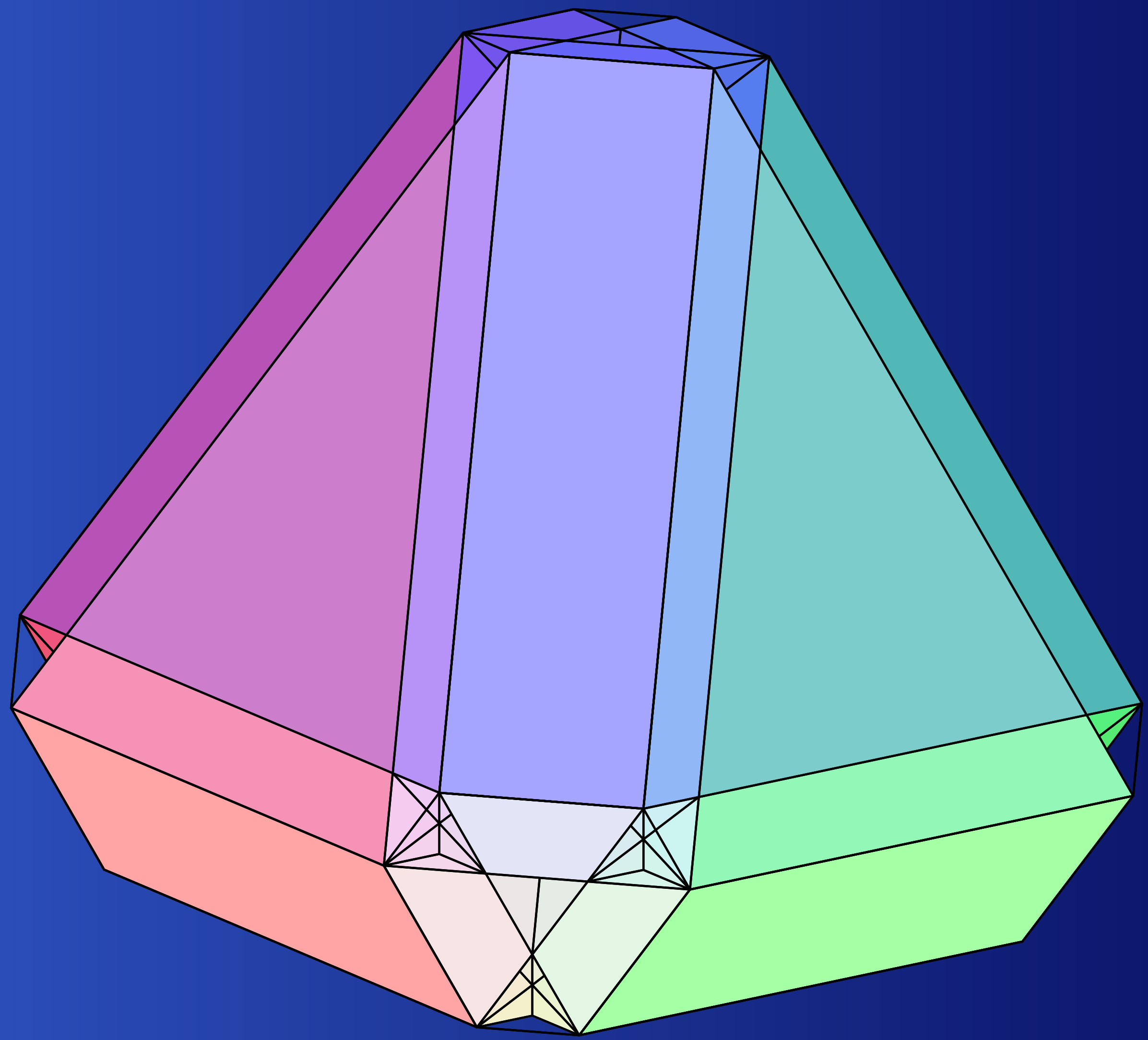
$\lambda = (7, -1, -2, -4)$



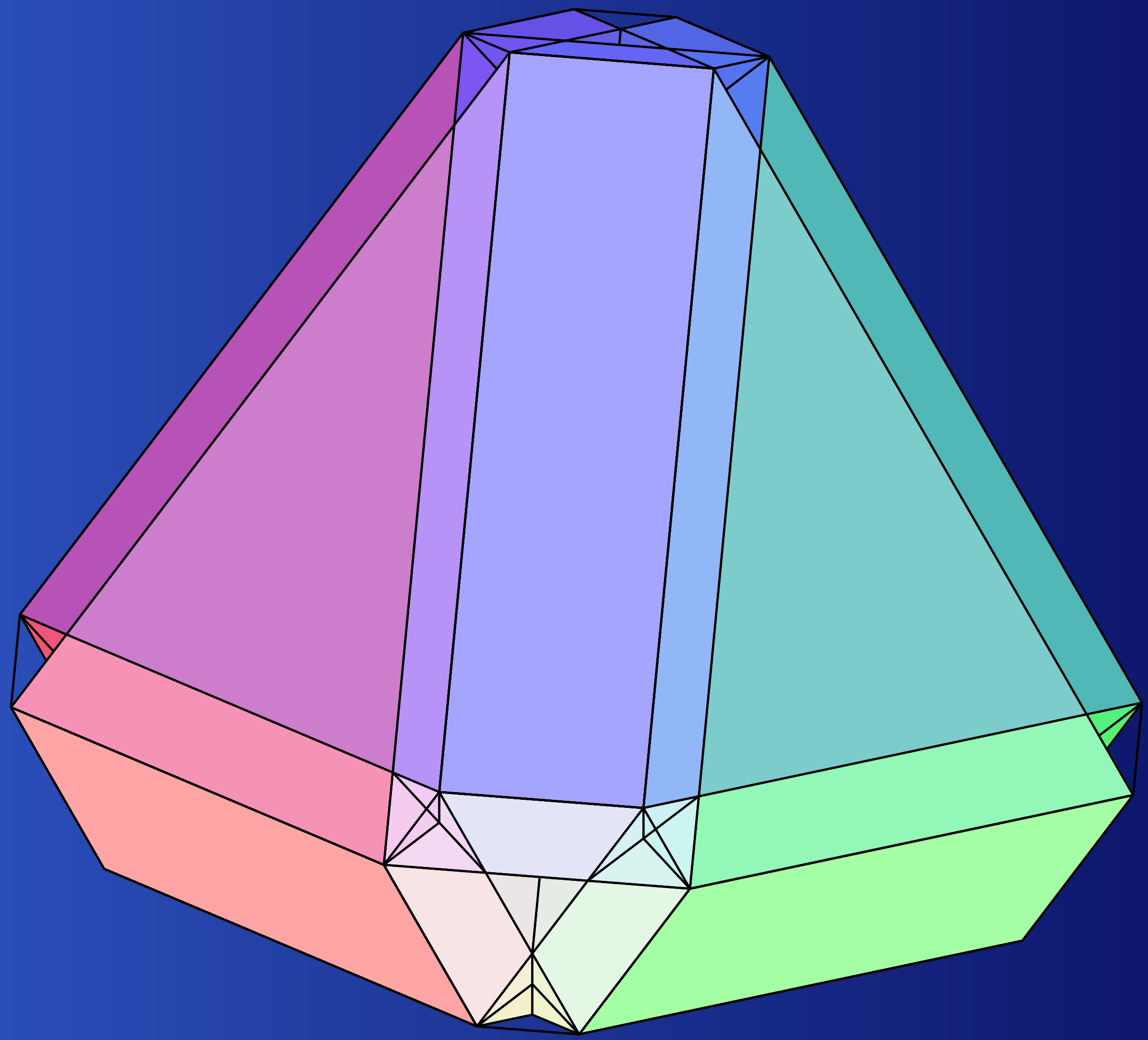
$\lambda = (7, -1, -2, -4)$



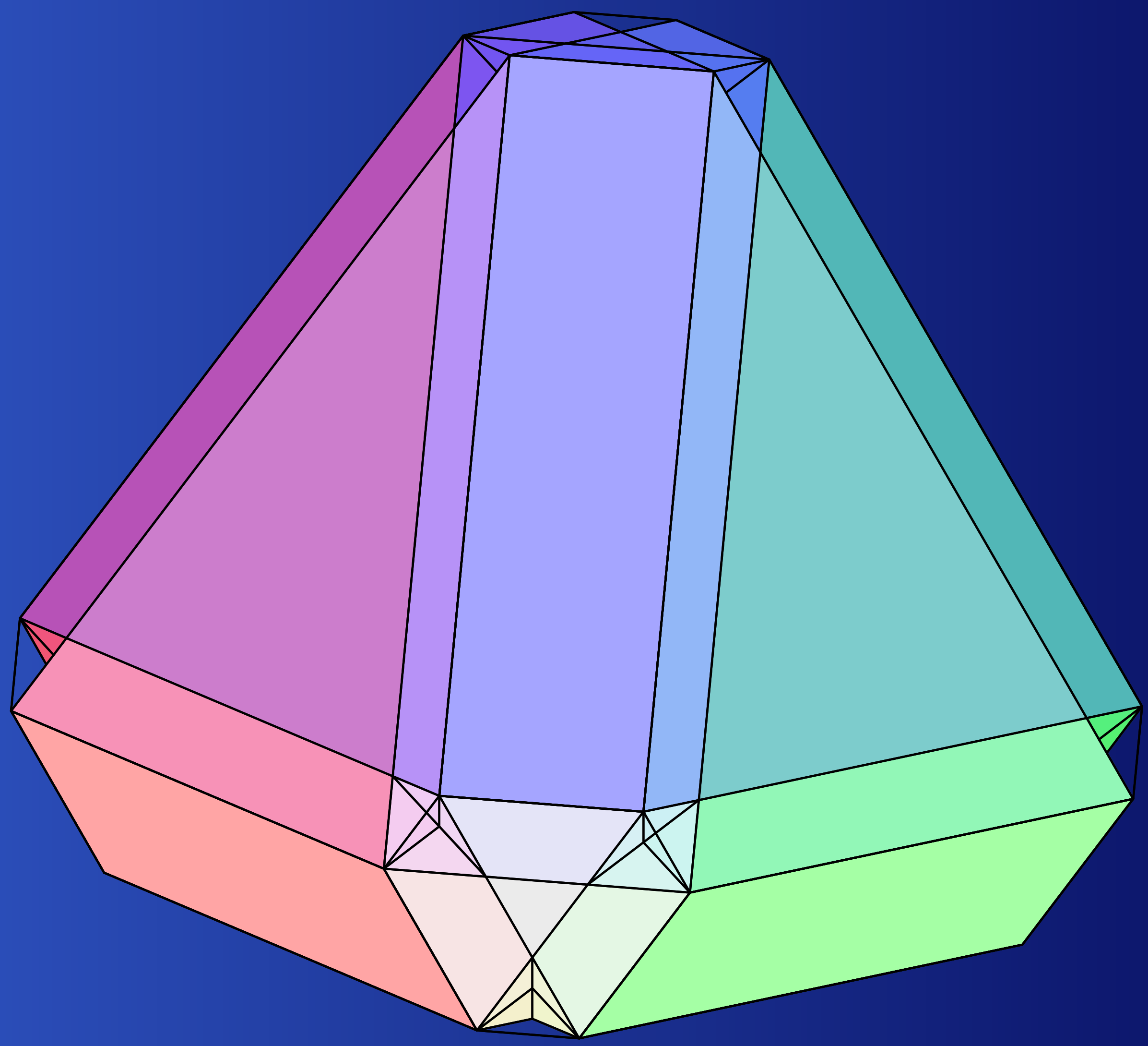
$\lambda = (7, -1, -2, -4)$



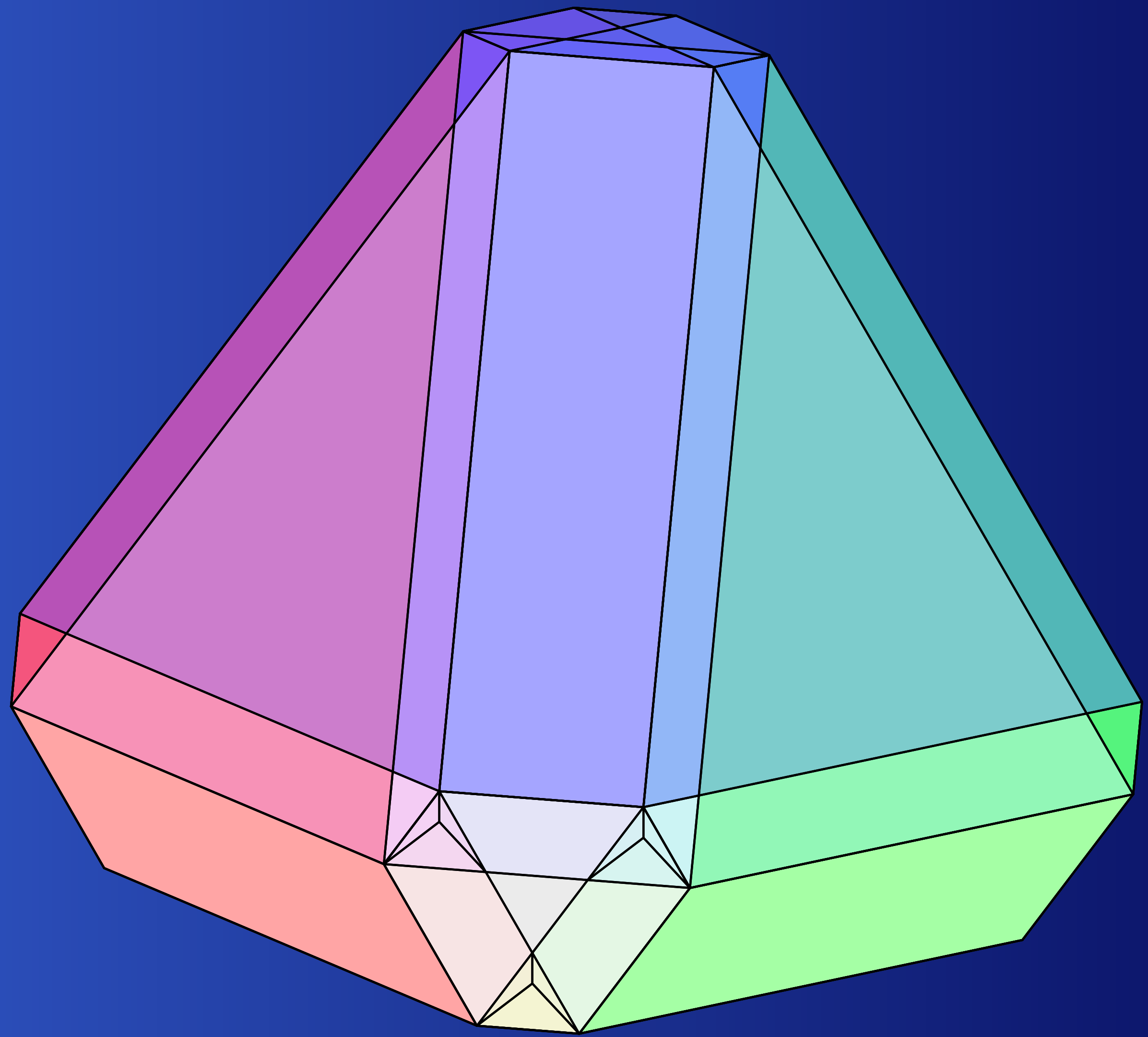
$\lambda = (7, -1, -2, -4)$



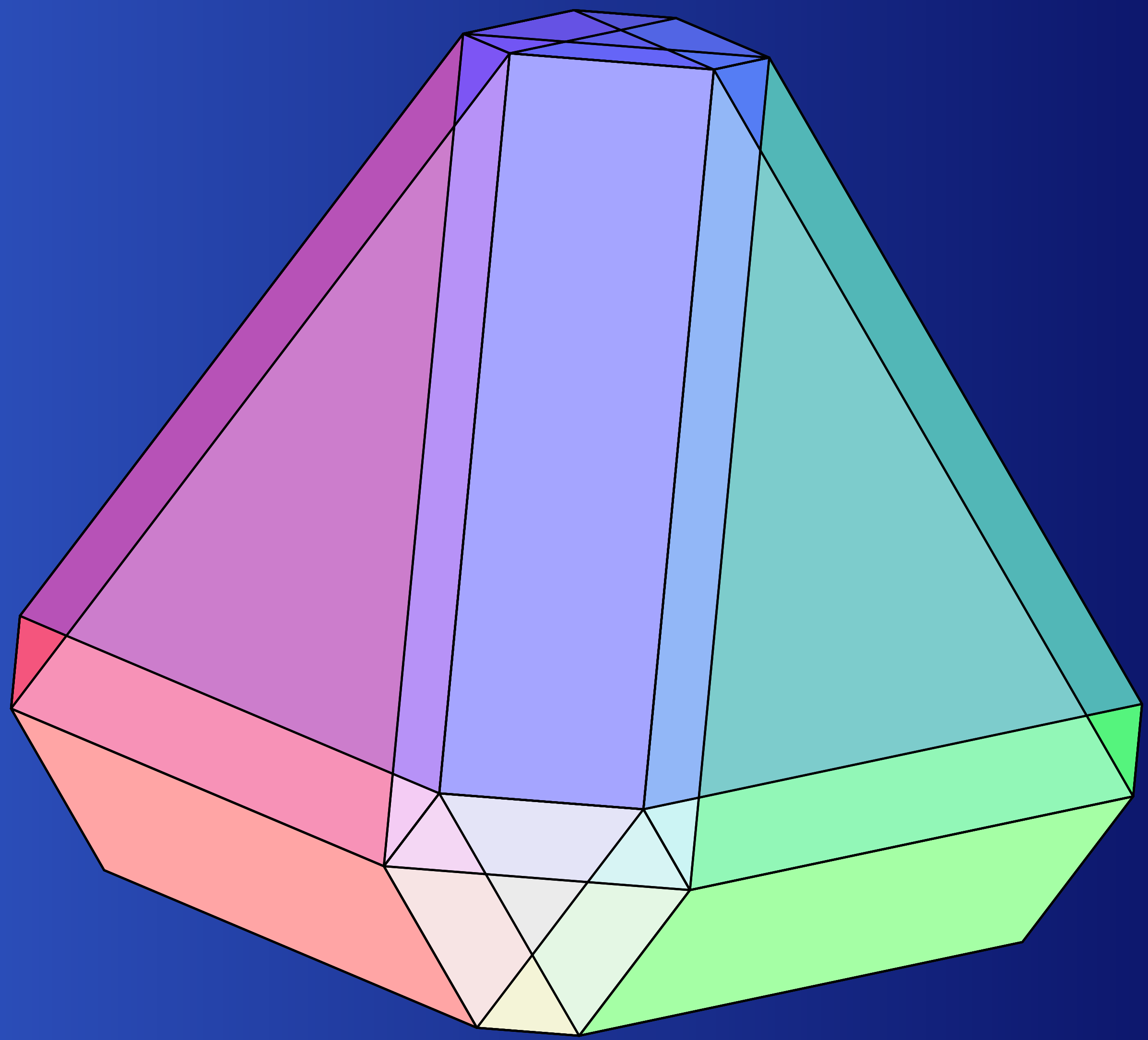
$\lambda = (7,\ -1,\ -2,\ -4)$



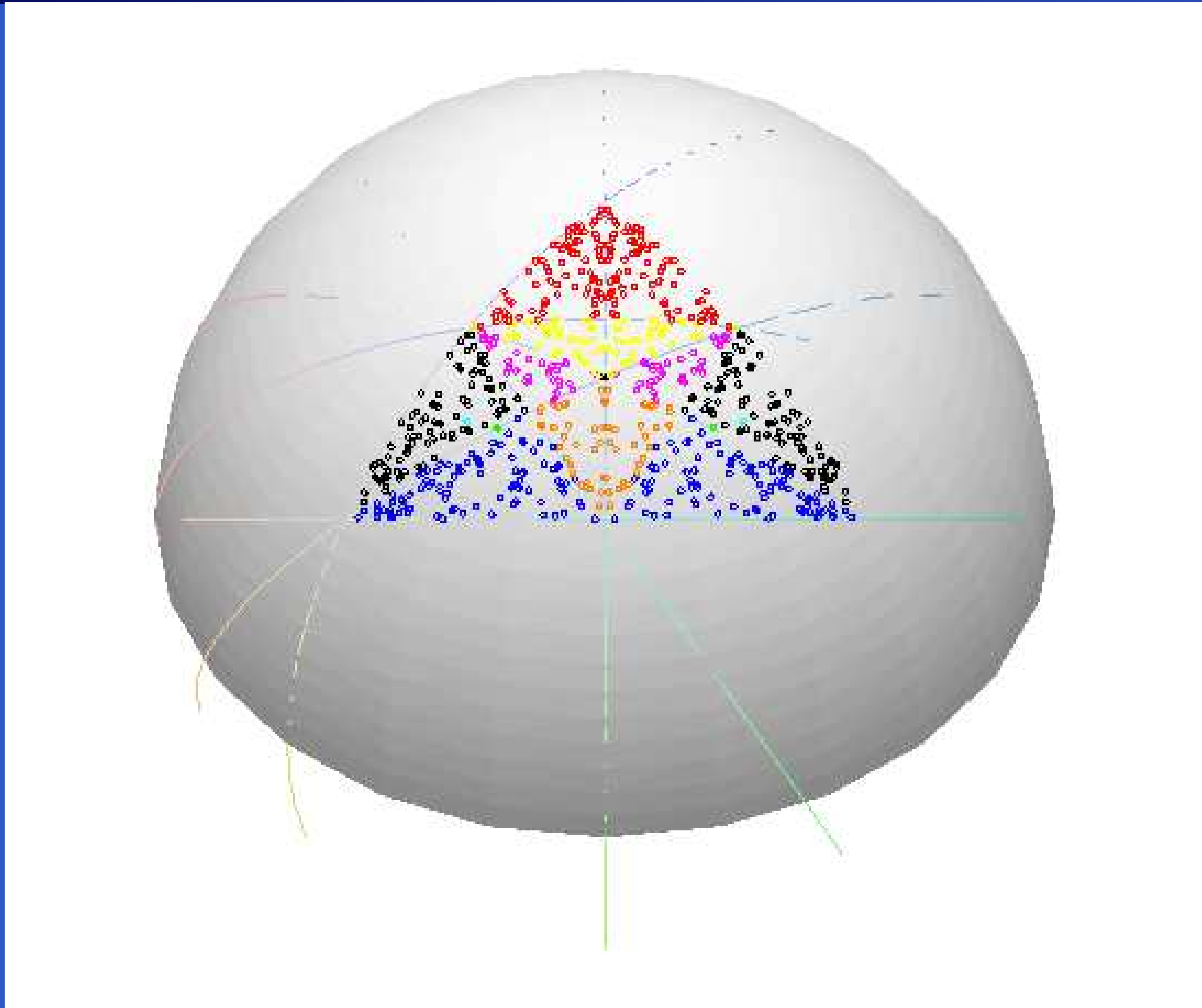
$\lambda = (7, -1, -2, -4)$



$\lambda = (7, -1, -2, -4)$



$k = 4$



Vector partition functions

Vector partition functions

Let M be a $d \times n$ matrix over the integers. The vector partition function associated to M is the function

$$\begin{array}{rccc} \phi_M : & \mathbb{Z}^d & \longrightarrow & \mathbb{N} \\ & b & \mapsto & |\{x \in \mathbb{N}^n \ : \ Mx = b\}| \end{array}$$

Vector partition functions

Let M be a $d \times n$ matrix over the integers. The **vector partition function** associated to M is the function

$$\begin{array}{rcccl} \phi_M : & \mathbb{Z}^d & \longrightarrow & \mathbb{N} \\ & b & \mapsto & |\{x \in \mathbb{N}^n & : \; Mx = b\}| \end{array}$$

Example

If $M = \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ then $\phi_M(b) = 3$

since $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

Polytopes and partition functions

- If M is such that $\ker M \cap \mathbb{R}^n_{\geq 0} = 0$, then

$$P_b = \{x \in \mathbb{R}^n_{\geq 0} \ : \ Mx = b\}$$

is a polytope.

$\phi_M(b)$ is the number of integral points in P_b .

Polytopes and partition functions

- If M is such that $\ker M \cap \mathbb{R}^n_{\geq 0} = 0$, then

$$P_b = \{x \in \mathbb{R}^n_{\geq 0} \ : \ Mx = b\}$$

  is a polytope.

  $\phi_M(b)$ is the number of integral points in P_b .

- ϕ_M vanishes outside of $\mathrm{pos}(M)$.

Quasipolynomials

A quasipolynomial function on a lattice L is a function that is polynomial on each coset of a sublattice N of L.

Example

$$f(n_1, n_2) = \begin{cases} n_1^2 + 3n_1n_2 + 2n_2 & \text{if } n_1 + n_2 \equiv 0 \pmod 3, \\ 2n_2^3 + n_1^2n_2^2 + 3 & \text{if } n_1 + n_2 \equiv 1 \pmod 3, \\ 0 & \text{if } n_1 + n_2 \equiv 2 \pmod 3. \end{cases}$$

Chamber (cone) complexes

A chamber (or cone) complex is a collection $\mathcal{C}$ of convex polyhedral cones such that

- if $C \in \mathcal{C}$ and F is a face of C, then $F \in \mathcal{C}$;

  ($\mathcal{C}$ is closed under taking faces.)

- if $C_1, C_2 \in \mathcal{C}$, then $C_1 \cap C_2$ is a common face to C_1 and C_2.

  (Cones touch along whole faces. Note that $\{0\}$ is a 0-dimensional face of any cone.)

The structure of partition functions

- ϕ_M is piecewise quasipolynomial of degree $n - \mathrm{rank}(M)$. (Sturmfels)

The structure of partition functions

- ϕ_M is piecewise quasipolynomial of degree $n - \mathrm{rank}(M)$. (Sturmfels)

- The domains of quasipolynomiality form a complex of convex polyhedral cones, the chamber complex of ϕ_M .

The structure of partition functions

- ϕ_M is piecewise quasipolynomial of degree $n - \mathrm{rank}(M)$. (Sturmfels)
- The domains of quasipolynomiality form a complex of convex polyhedral cones, the chamber complex of ϕ_M.
- Alekseevskaya, Gelfand and Zelevinsky described how to determine the chamber complex of a partition function from its matrix.

Determining the chamber complex

We can assume without loss of generality that M has full rank d.

- Find all the $d \times d$ nonsingular submatrices M_σ of M.

Determining the chamber complex

We can assume without loss of generality that M has full rank d.

- Find all the $d \times d$ nonsingular submatrices M_σ of M.

- Determine the cone $\tau_\sigma = \mathrm{pos}(M_\sigma)$ spanned by the columns of M_σ.

Determining the chamber complex

We can assume without loss of generality that M has full rank d .

- Find all the $d \times d$ nonsingular submatrices M_σ of M .
- Determine the cone $\tau_\sigma = \mathrm{pos}(M_\sigma)$ spanned by the columns of M_σ .
- The chamber complex of ϕ_M is the common refinement of the τ_σ .

Example

Consider

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

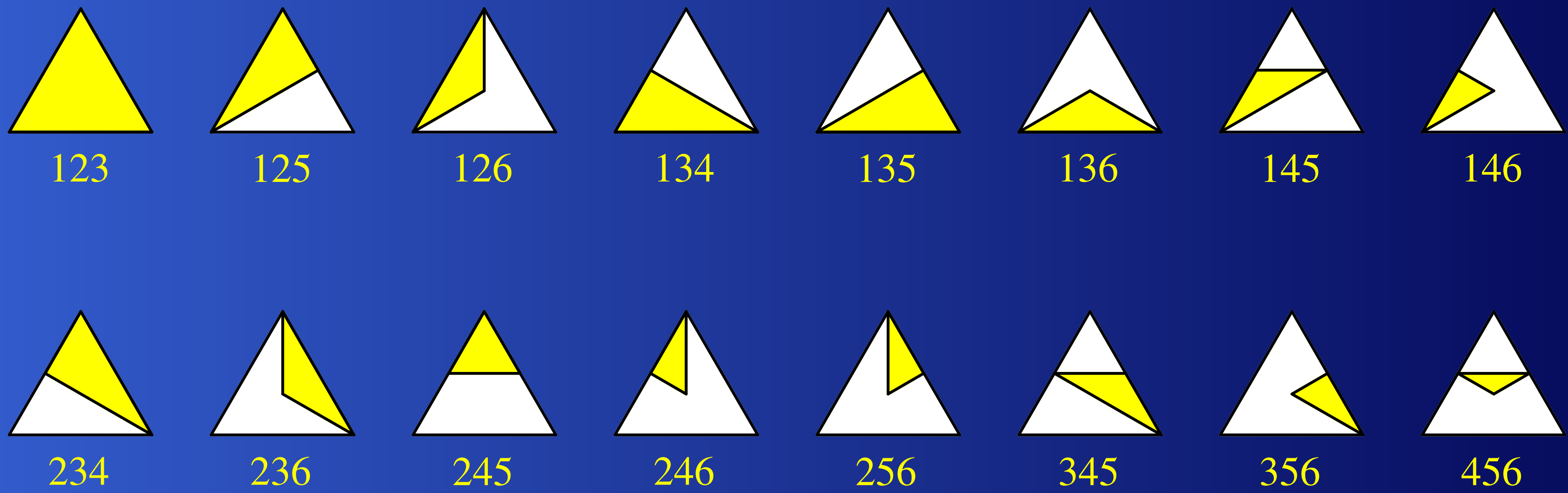
Example

Consider

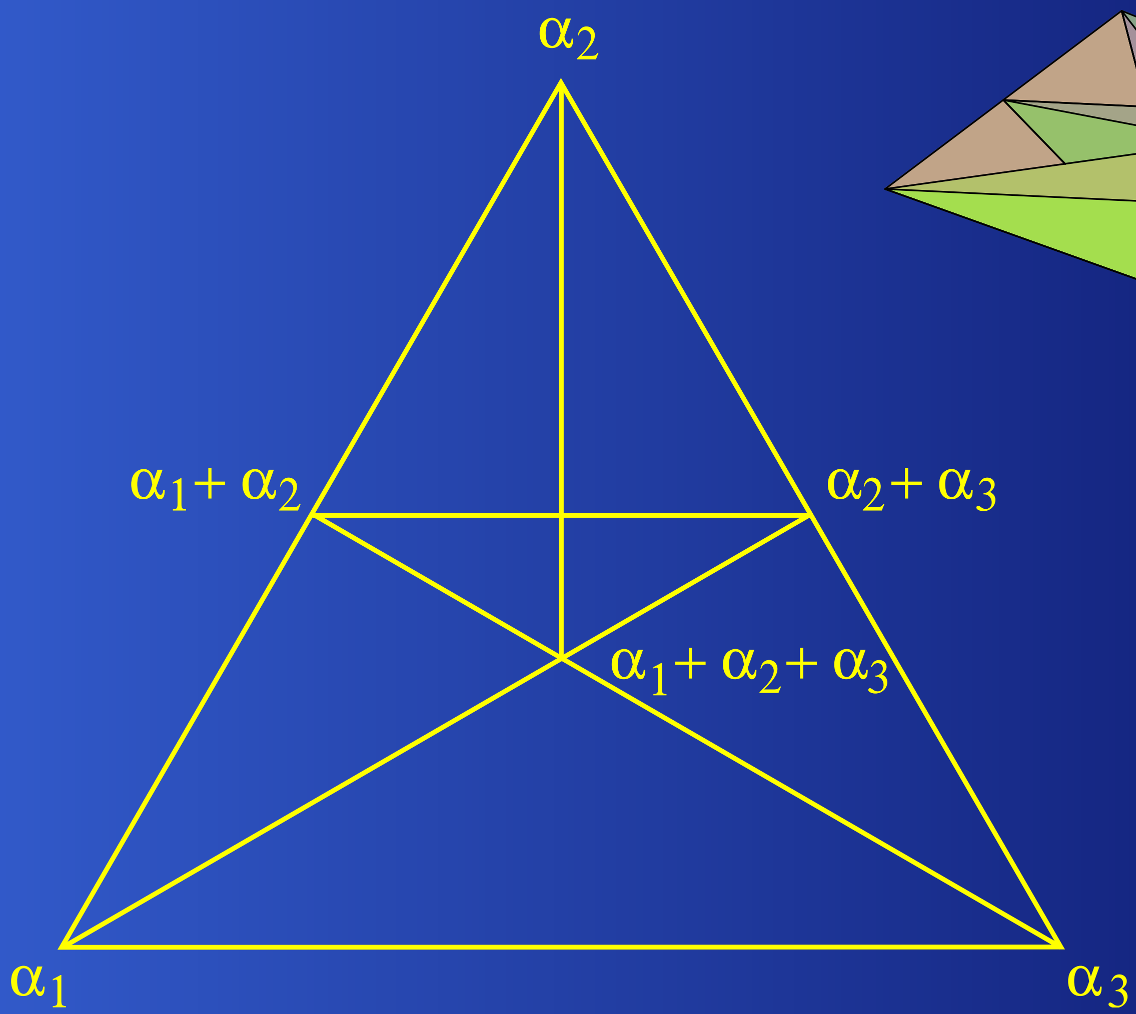
$$M = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{aligned} \mathcal{B} \;&=\; \{123, 125, 126, 134, 135, 136, 145, 146, \\ &\quad\; 234, 236, 245, 246, 256, 345, 356, 456\}\,. \end{aligned}$$

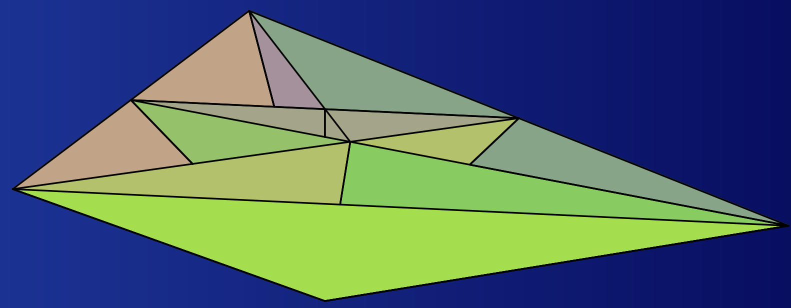
Example



Example



α_2
$\alpha_1 + \alpha_2$
$\alpha_2 + \alpha_3$
$\alpha_1 + \alpha_2 + \alpha_3$
α_1
α_3



A partition function for the $m_\lambda(\beta)$

Theorem

For every k, we can find integer matrices E_k and B_k such that the weight multiplicities can be written as

$$m_\lambda(\beta) = \phi_{E_k}\left(B_k\begin{pmatrix}\lambda\\ \beta\end{pmatrix}\right).$$

Example: $k = 3$

Gelfand-Tsetlin diagrams for $k = 3$ have the form

$$\begin{matrix} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \mu_1 & & \mu_2 & \\ & & \nu & & \end{matrix}$$

Example: $k = 3$

Gelfand-Tsetlin diagrams for $k = 3$ have the form

$$\begin{array}{ccccc} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \mu_1 & & \mu_2 & \\ & & \nu & & \end{array}$$

Row sums:

$$\begin{aligned} \nu &= \beta_1 \\ \mu_1 + \mu_2 &= \beta_1 + \beta_2 \\ \lambda_1 + \lambda_2 + \lambda_3 &= \beta_1 + \beta_2 + \beta_3 \,. \end{aligned}$$

Example: $k = 3$

$$
\begin{aligned}
\mu_1 &\leq \lambda_1 \\
-\mu_1 &\leq -\lambda_2 \\
-\mu_1 &\leq \lambda_2 - \beta_1 - \beta_2 \\
\mu_1 &\leq \beta_1 + \beta_2 + \lambda_1 + \lambda_2 \\
-\mu_1 &\leq -\beta_1 \\
-\mu_1 &\leq -\beta_2 \,.
\end{aligned}
$$

Example: $k = 3$

$$
\begin{array}{rcrcl}
\mu_1 & + & s_1 & = & \lambda_1 \\
-\mu_1 & + & s_2 & = & -\lambda_2 \\
-\mu_1 & + & s_3 & = & \lambda_2 - \beta_1 - \beta_2 \\
\mu_1 & + & s_4 & = & \beta_1 + \beta_2 + \lambda_1 + \lambda_2 \\
-\mu_1 & + & s_5 & = & -\beta_1 \\
-\mu_1 & + & s_6 & = & -\beta_2 \,.
\end{array}
$$

Example: $k = 3$

$$\begin{aligned}
\mu_1 &+ s_1 &= \lambda_1 \\
-\mu_1 &+ s_2 &= -\lambda_2 \\
-\mu_1 &+ s_3 &= \lambda_2 - \beta_1 - \beta_2 \\
\mu_1 &+ s_4 &= \beta_1 + \beta_2 + \lambda_1 + \lambda_2 \\
-\mu_1 &+ s_5 &= -\beta_1 \\
-\mu_1 &+ s_6 &= -\beta_2 \,.
\end{aligned}$$

- The s_i are constrained to be nonnegative.

Example: $k = 3$

$$
\begin{array}{rcrcl}
\mu_1 & + & s_1 & = & \lambda_1 \\
-\mu_1 & + & s_2 & = & -\lambda_2 \\
-\mu_1 & + & s_3 & = & \lambda_2 - \beta_1 - \beta_2 \\
\mu_1 & + & s_4 & = & \beta_1 + \beta_2 + \lambda_1 + \lambda_2 \\
-\mu_1 & + & s_5 & = & -\beta_1 \\
-\mu_1 & + & s_6 & = & -\beta_2 \,.
\end{array}
$$

- The s_i are constrainted to be nonnegative.
- Finally we can use $\mu_1 = \lambda_1 - s_1$ to get rid of μ_1.

Example: $k = 3$

$$\begin{aligned}
s_1 + s_2 &= \lambda_1 - \lambda_2 \\
-s_2 + s_3 &= 2\lambda_2 - \beta_1 - \beta_2 \\
s_2 + s_4 &= \beta_1 + \beta_2 + \lambda_1 \\
-s_2 + s_5 &= \lambda_2 - \beta_1 \\
-s_2 + s_6 &= \lambda_2 - \beta_2
\end{aligned}$$

Example: $k = 3$

$$
\begin{aligned}
s_1 + s_2 &= \lambda_1 - \lambda_2 \\
-s_2 + s_3 &= 2\lambda_2 - \beta_1 - \beta_2 \\
s_2 + s_4 &= \beta_1 + \beta_2 + \lambda_1 \\
-s_2 + s_5 &= \lambda_2 - \beta_1 \\
-s_2 + s_6 &= \lambda_2 - \beta_2
\end{aligned}
$$

- Solving for $s_i \geq 0 \ \forall\, i$.

Example: $k = 3$

$$
\begin{aligned}
s_1 + s_2 &= \lambda_1 - \lambda_2 \\
-s_2 + s_3 &= 2\lambda_2 - \beta_1 - \beta_2 \\
s_2 + s_4 &= \beta_1 + \beta_2 + \lambda_1 \\
-s_2 + s_5 &= \lambda_2 - \beta_1 \\
-s_2 + s_6 &= \lambda_2 - \beta_2
\end{aligned}
$$

- Solving for $s_i \geq 0 \ \forall\, i$.
- Requiring the s_i's to be integers yields all integer solutions to the Gelfand-Tsetlin constraints.

Example: $k = 3$

So we are solving

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}}_{E_2} \cdot \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{pmatrix} = \underbrace{\begin{pmatrix} \lambda_1 - \lambda_2 \\ 2\lambda_2 - \beta_1 - \beta_2 \\ \beta_1 + \beta_2 + \lambda_1 \\ \lambda_2 - \beta_1 \\ \lambda_2 - \beta_2 \end{pmatrix}}_{B_2\binom{\lambda}{\beta}}$$

for $\vec{s} \in \mathbb{N}^6$.

Example: $k = 3$

So we are solving

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}}_{E_2} \cdot \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{pmatrix} = \underbrace{\begin{pmatrix} \lambda_1 - \lambda_2 \\ 2\lambda_2 - \beta_1 - \beta_2 \\ \beta_1 + \beta_2 + \lambda_1 \\ \lambda_2 - \beta_1 \\ \lambda_2 - \beta_2 \end{pmatrix}}_{B_2\binom{\lambda}{\beta}}$$

for $\vec{s} \in \mathbb{N}^6$. Hence $\boxed{m_\lambda(\beta) = \phi_{E_2}\left(B_2\binom{\lambda}{\beta}\right).}$

The chamber complex

- ϕ_{E_k} takes values in $\mathbb{R}^{(k-1)(k+2)/2}$.

The chamber complex

- ϕ_{E_k} takes values in $\mathbb{R}^{(k-1)(k+2)/2}$.
- Let $\tilde{B} = \left\{ B_k\binom{\lambda}{\beta} \;:\; \lambda \in \mathbb{R}^k, \beta \in \mathbb{R}^k, \sum \lambda_i = \sum \beta_i = 0 \right\}$.

The chamber complex

- ϕ_{E_k} takes values in $\mathbb{R}^{(k-1)(k+2)/2}$.
- Let $\tilde{B} = \left\{ B_k \binom{\lambda}{\beta} \ : \ \lambda \in \mathbb{R}^k, \beta \in \mathbb{R}^k, \sum \lambda_i = \sum \beta_i = 0 \right\}$.
- The only part of the chamber complex of ϕ_{E_k} that is relevant for the weight multiplicities is its intersection with $\tilde{B}$.

The chamber complex

- ϕ_{E_k} takes values in $\mathbb{R}^{(k-1)(k+2)/2}$.
- Let $\tilde{B} = \left\{ B_k \binom{\lambda}{\beta} \;:\; \lambda \in \mathbb{R}^k, \beta \in \mathbb{R}^k, \sum \lambda_i = \sum \beta_i = 0 \right\}$.
- The only part of the chamber complex of ϕ_{E_k} that is relevant for the weight multiplicities is its intersection with $\tilde{B}$.
- We can intersect the base cones with $\tilde{B}$ before taking the common refinement.

The chamber complex

- On $\tilde{B}$, we can work in (λ, β)-coordinates.

The chamber complex

- On $\tilde{B}$, we can work in (λ, β)-coordinates.
- We will call $\mathcal{C}^{(k)}$ this $(2k-2)$-dim complex in (λ, β)-coordinates.

The chamber complex

- On $\tilde{B}$, we can work in (λ, β)-coordinates.
- We will call $\mathcal{C}^{(k)}$ this $(2k-2)$-dim complex in (λ, β)-coordinates.
- For fixed λ, let
$L(\lambda) = \{(\lambda_1, \ldots, \lambda_k, \beta_1, \ldots, \beta_k) \;:\; \beta_i \in \mathbb{R}, \sum \beta_i = \sum \lambda_i = 0\}$.

The chamber complex

- On $\tilde{B}$, we can work in (λ, β)-coordinates.

- We will call $\mathcal{C}^{(k)}$ this $(2k-2)$-dim complex in (λ, β)-coordinates.

- For fixed λ, let
$L(\lambda) = \{(\lambda_1, \ldots, \lambda_k, \beta_1, \ldots, \beta_k) \;:\; \beta_i \in \mathbb{R}, \sum \beta_i = \sum \lambda_i = 0\}\,.$

- The intersection of $\mathcal{C}^{(k)}$ with $L(\lambda)$ gives domains of (quasi)polynomiality for the multiplicities.

Classifying the λ

Corollary

Let $\mathcal{C}_\Lambda^{(k)}$ be the chamber complex given by the common refinement of the projections $p_\Lambda(\tau)$ of the cones of $\mathcal{C}^{(k)}$ onto $\mathbb{R}^k$.

Then $\mathcal{C}_\Lambda^{(k)}$ classifies the λ's, in the sense that if λ and λ' belong to the same cell of $\mathcal{C}_\Lambda^{(k)}$, then all their domains are indexed by the same subsets of cones from $\mathcal{C}^{(k)}$, and therefore have the same corresponding polynomials.

Counting the regions

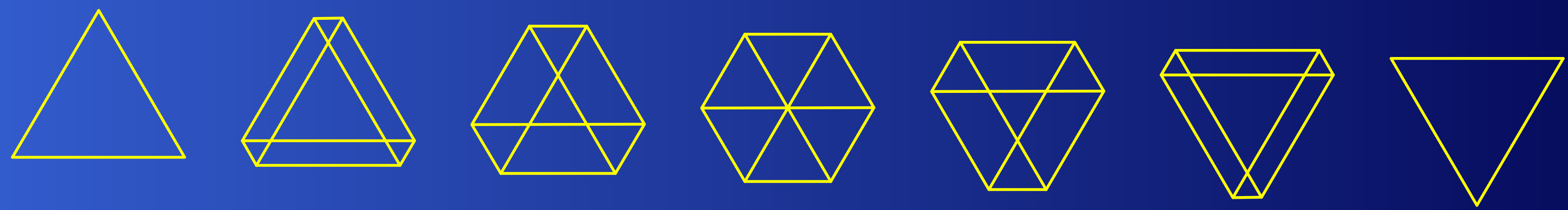
The complex for $k = 3$

- The complex $\mathcal{C}^{(3)}$ consists of eight 4-dimensional cones and their faces.
- The complex is invariant under permuting the β coordinates.
- We project the cones of $\mathcal{C}^{(3)}$ on the λ coordinates and take the common refinement.
- $\mathcal{C}_{\Lambda}^{(3)}$ has two top-dimensional cones.

The projected complex

$$C_1 = \operatorname{pos}\big((2,-1,-1),(1,0,-1)\big)$$

$$C_2 = \operatorname{pos}\big((1,0,-1),(1,1,-2)\big)$$



- There are two generic cases for λ in this case ($\lambda_2 < 0$ or $\lambda_2 > 0$), each with 7 domains.

The complex for $k = 4$

- For $k = 4$, we find that $\mathcal{C}^{(4)}$ has 1202 top dimensional cones (6-dim cones in 8-dim ambient (λ, β)-space).

- However, it is not invariant under the action of $\mathcal{S}_4$ on the β coordinates.

- This means that the complex is not optimal and can be coarsened further.

Glueing the complex

Theorem

The union of the top dimensional cones of $\mathcal{C}^{(4)}$ *with the same weight polynomial is again a convex polyhedral cone.*

Glueing the complex

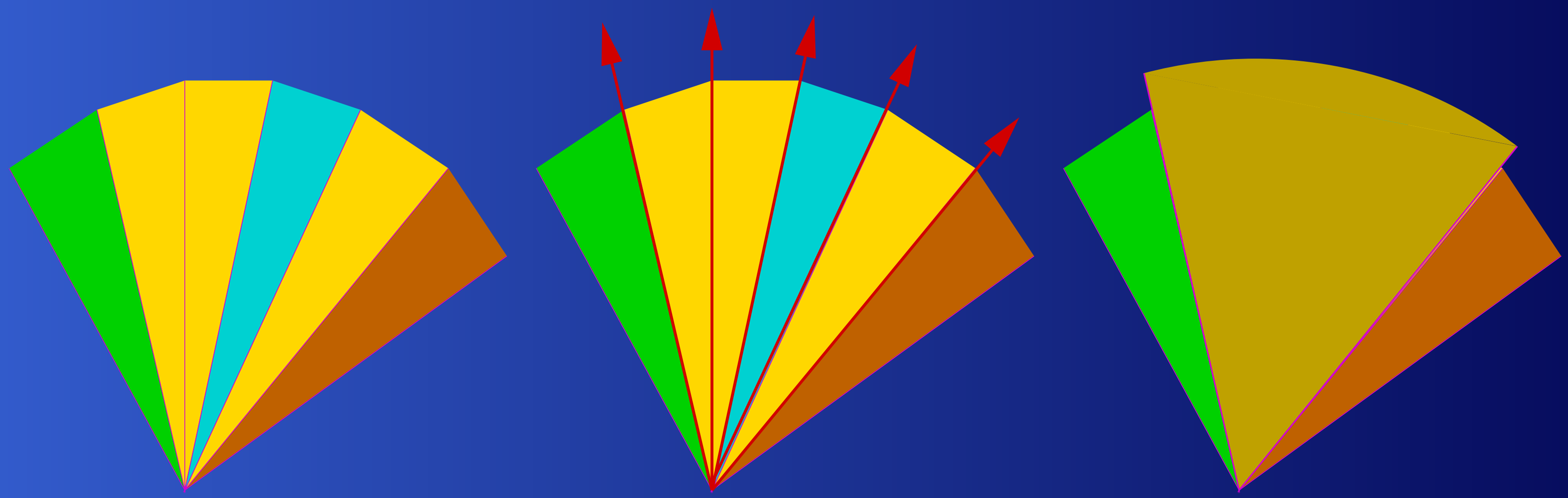
Theorem

The union of the top dimensional cones of $\mathcal{C}^{(4)}$ *with the same weight polynomial is again a convex polyhedral cone.*

- This means we can glue these cones together.
- We can then verify that we get a chamber complex that way, the glued complex $\mathcal{G}$.

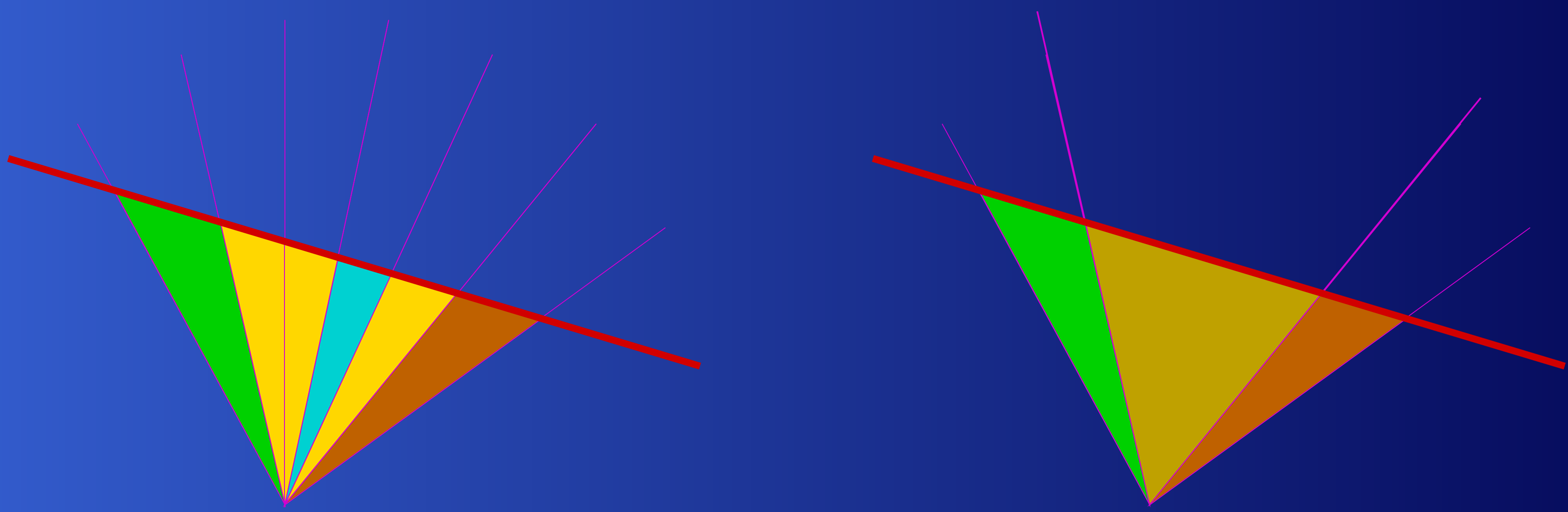
Idea of the proof

1. Select the cones with a same weight polynomial. (Yellow)
2. Consider their rays.
3. Construct the cone spanned by those rays.



Idea of the proof

4. Find a transversal affine halfspace and intersect it with the cones to get polytopes.
5. Compare volumes.



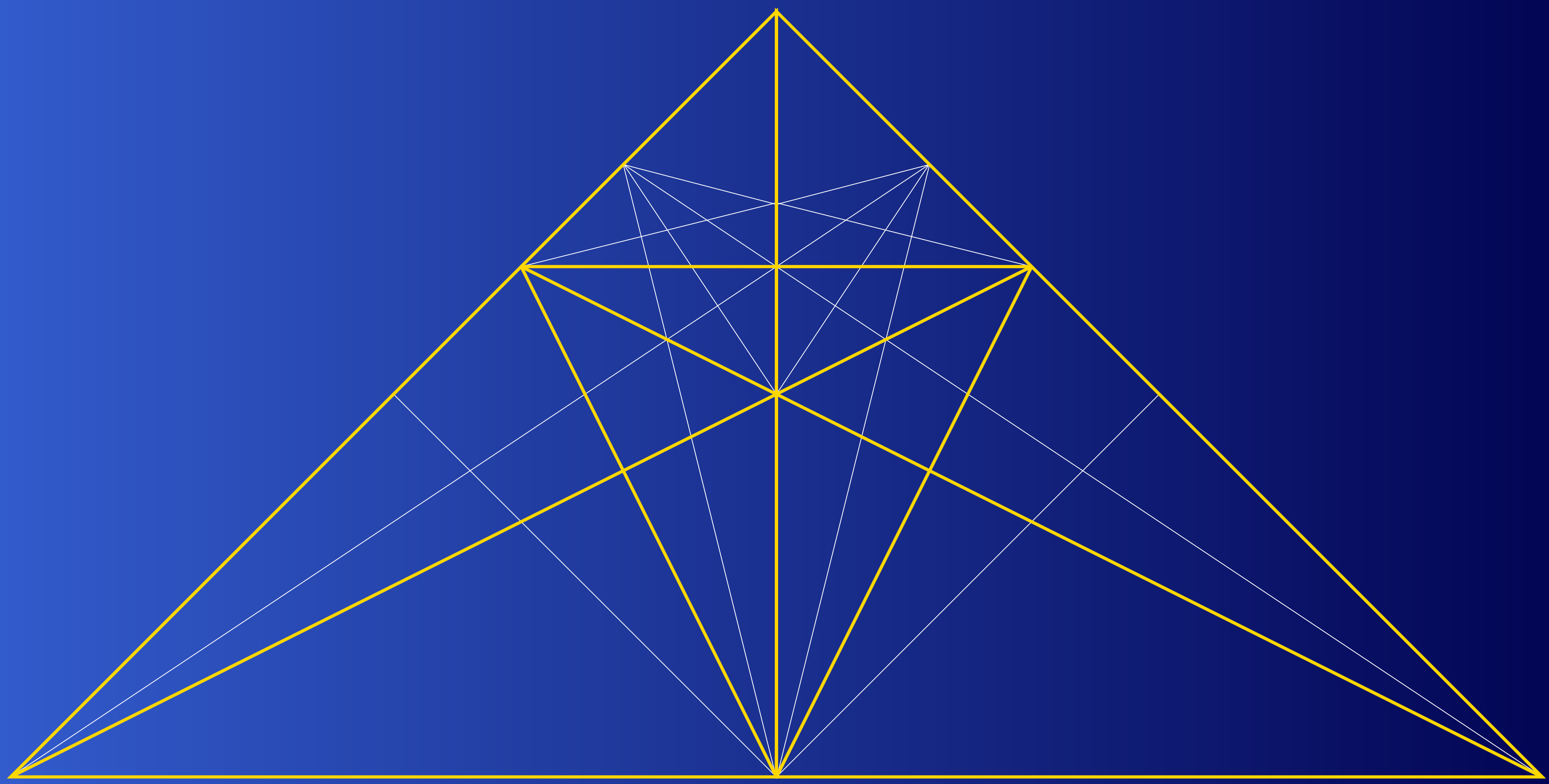
Idea of the proof

- This process was automated symbolically, so that there are no roundoff errors which might cause us to miss a small cone.

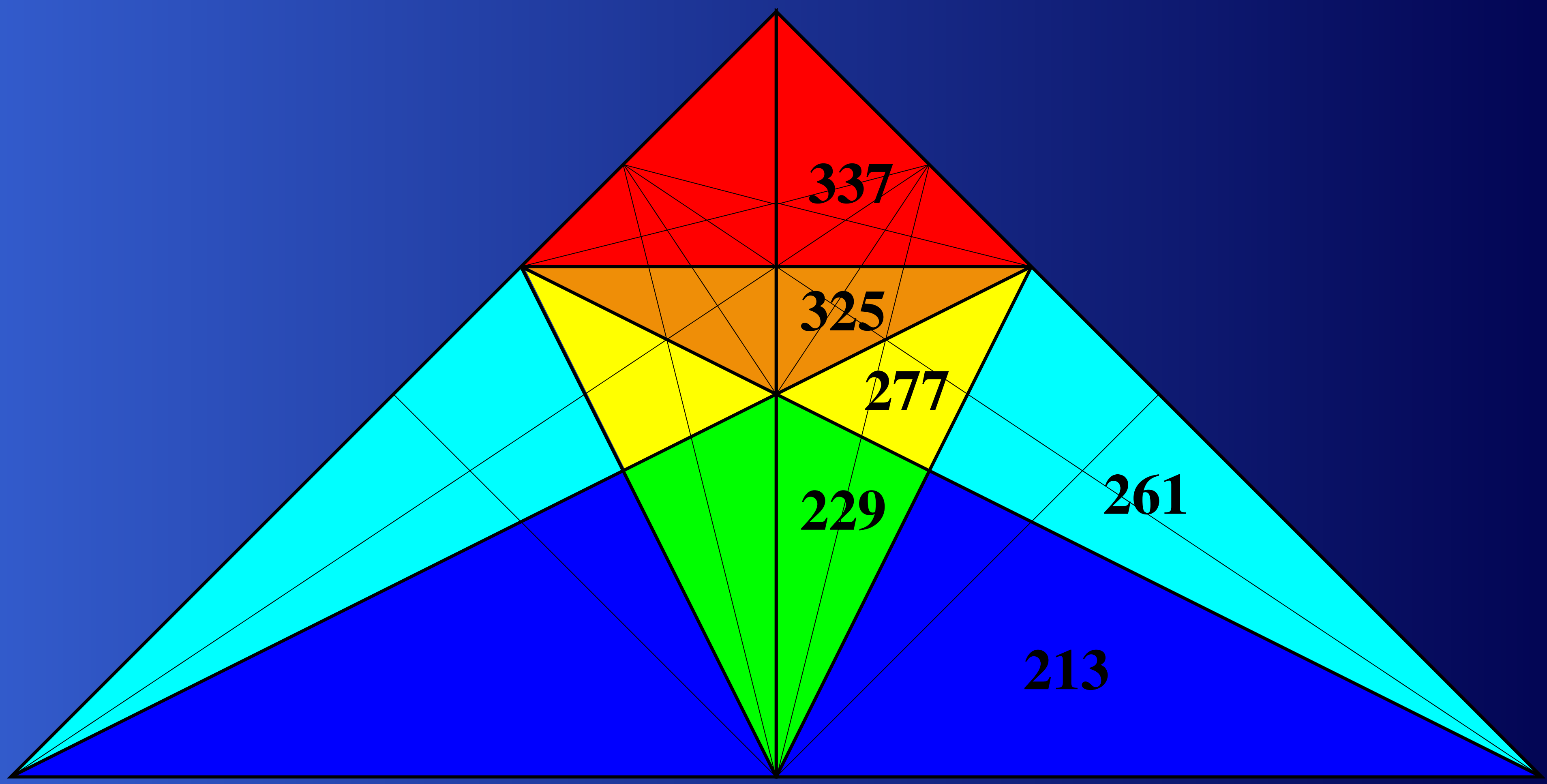
- The complex $\mathcal{G}$ has 612 top dimensional cones.

- It is invariant under the action of $\mathfrak{S}_4$ on β. There are 64 orbits of cones.

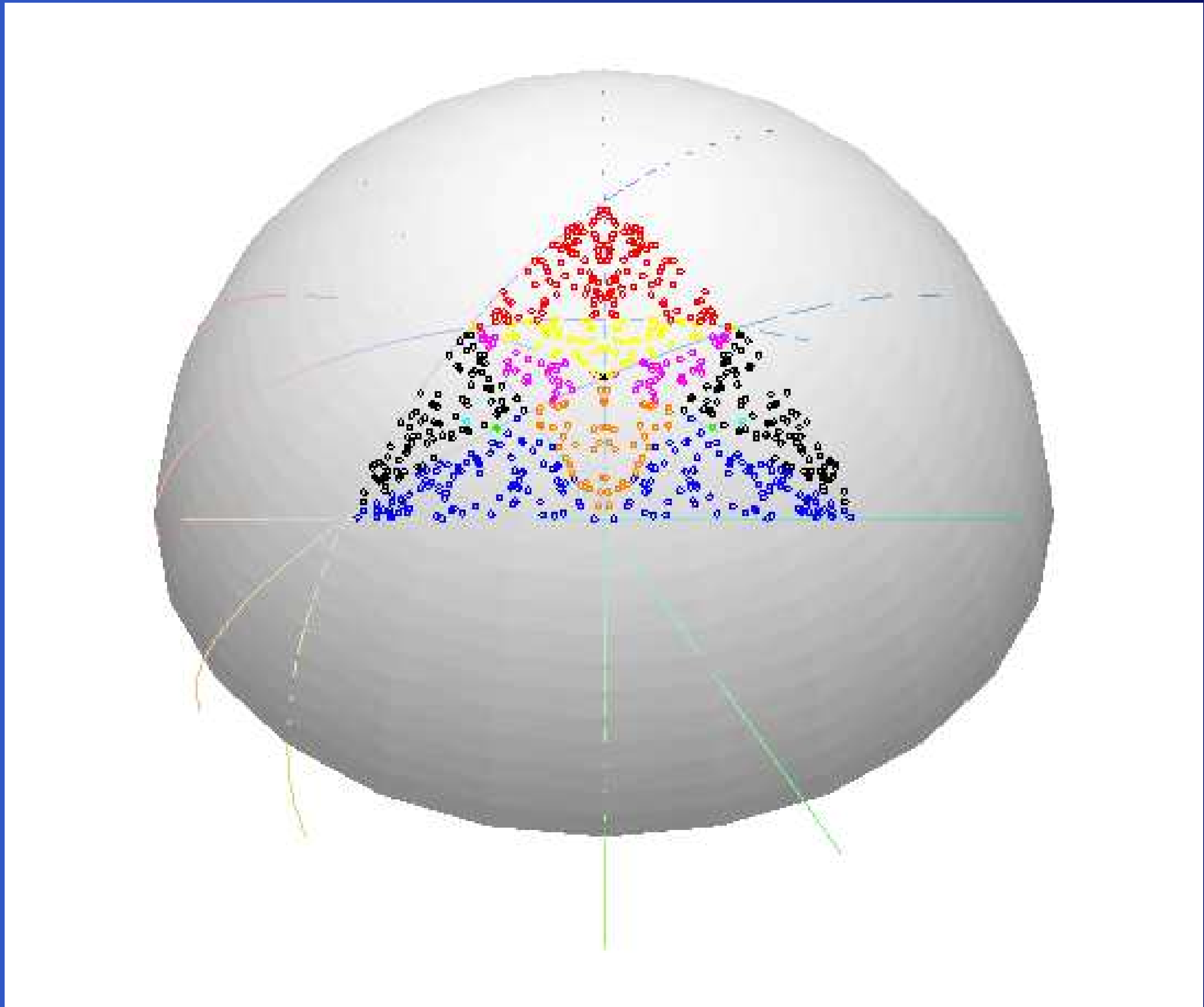
Counting the regions



Counting the regions

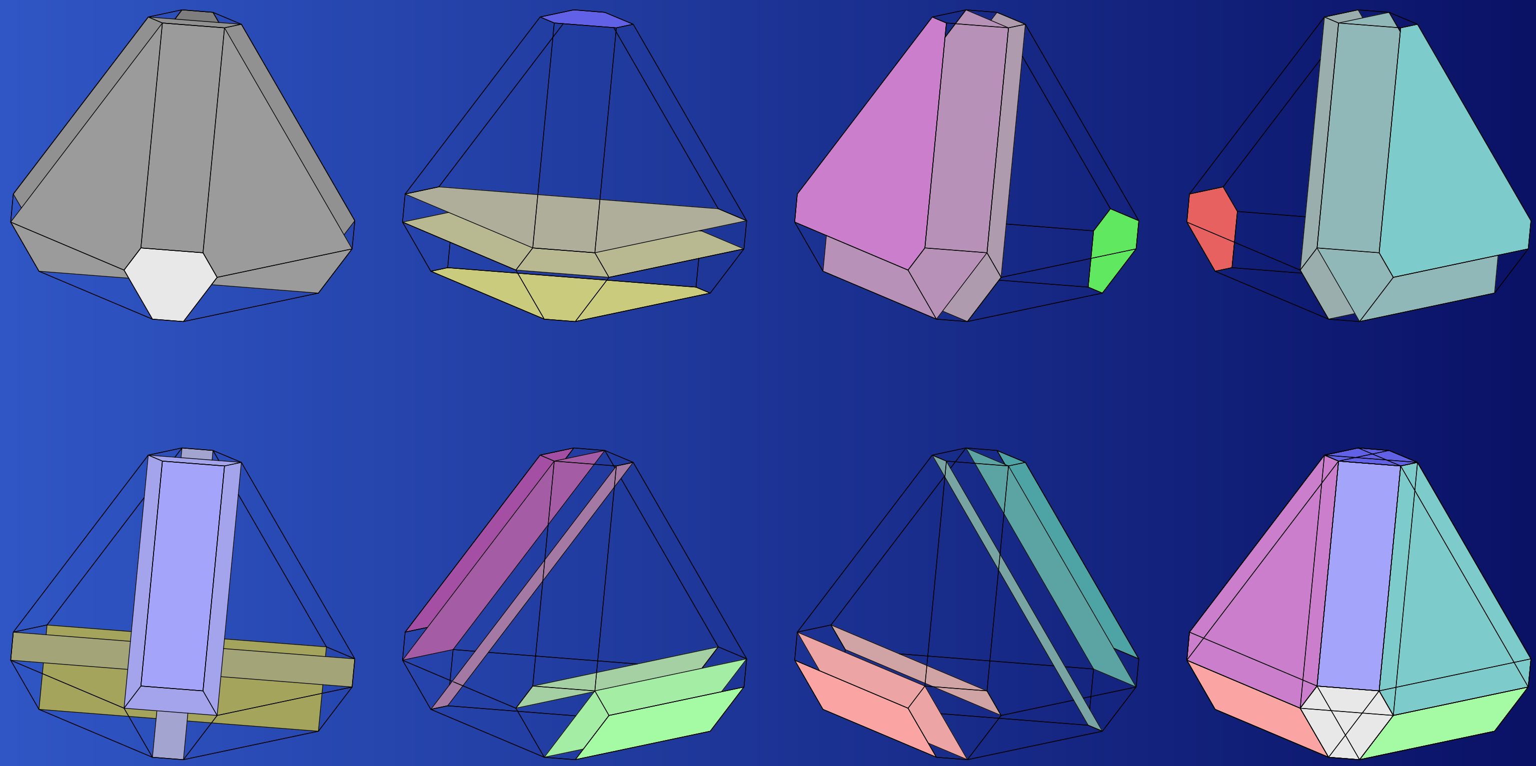


Counting the regions



Two partitions

- Is this partition for weight multiplicities the same as the one we have in terms of walls for the Duistermaat-Heckman function?



Comparing the two partitions

The DH function and the $m_\lambda(\beta)$

- The partitions agree for all the random λ we tried.
- The difficulty is intersecting $\mathcal{G}$ with $L(\lambda)$ not for a specific λ but for a general (symbolic) one.

The DH function and the $m_\lambda(\beta)$

- The partitions agree for all the random λ we tried.
- The difficulty is intersecting $\mathcal{G}$ with $L(\lambda)$ not for a specific λ but for a general (symbolic) one.

Theorem

For $k = 4$, the partitions of the permutahedron into its domains of polynomiality for the weight multiplicities and for the Duistermaat-Heckman function are the same.

Strategy

- The partition for the DH function is given in terms of walls rather than a description of the subpolytopes themselves.
- Therefore our goal will be to identify which pieces of the chamber complex $\mathcal{G}$ give rise to the walls when intersected with $L(\lambda)$.

Strategy

- The intersection of a top dimensional cone with $L(\lambda)$, if it is not empty, gives a domain.
- So the intersection of the facets of that cone with $L(\lambda)$ give the facets of the domain (generically).
- We build the walls by glueing such facets together.

Proof/Algorithm

1. Let $\mathcal{F}$ be the set of all facets of the top dimensional cones $G_1, \ldots, G_{612}$ of $\mathcal{G}$.

Proof/Algorithm

1. Let $\mathcal{F}$ be the set of all facets of the top dimensional cones $G_1, \ldots, G_{612}$ of $\mathcal{G}$.
2. Group the facets in $\mathcal{F}$ according to their normals. 37 normal directions: $\{n_1, \ldots, n_{37}\}$.

Proof/Algorithm

1. Let $\mathcal{F}$ be the set of all facets of the top dimensional cones $G_1, \ldots, G_{612}$ of $\mathcal{G}$.
2. Group the facets in $\mathcal{F}$ according to their normals. 37 normal directions: $\{n_1, \ldots, n_{37}\}$.
   - Call $\mathcal{F}_i$ the subset of $\mathcal{F}$ consisting of all the facets with normals in direction n_i .

Proof/Algorithm

1. Let $\mathcal{F}$ be the set of all facets of the top dimensional cones $G_1, \ldots, G_{612}$ of $\mathcal{G}$.
2. Group the facets in $\mathcal{F}$ according to their normals. 37 normal directions: $\{n_1, \ldots, n_{37}\}$.
   - Call $\mathcal{F}_i$ the subset of $\mathcal{F}$ consisting of all the facets with normals in direction n_i .
   - Each facet lies on a unique hyperplane.

Proof/Algorithm

1. Let $\mathcal{F}$ be the set of all facets of the top dimensional cones $G_1, \ldots, G_{612}$ of $\mathcal{G}$.
2. Group the facets in $\mathcal{F}$ according to their normals. 37 normal directions: $\{n_1, \ldots, n_{37}\}$.
   - Call $\mathcal{F}_i$ the subset of $\mathcal{F}$ consisting of all the facets with normals in direction n_i .
   - Each facet lies on a unique hyperplane.
   - These hyperplanes go through the origin.

Proof/Algorithm

1. Let $\mathcal{F}$ be the set of all facets of the top dimensional cones $G_1, \ldots, G_{612}$ of $\mathcal{G}$.
2. Group the facets in $\mathcal{F}$ according to their normals. 37 normal directions: $\{n_1, \ldots, n_{37}\}$.
   - Call $\mathcal{F}_i$ the subset of $\mathcal{F}$ consisting of all the facets with normals in direction n_i .
   - Each facet lies on a unique hyperplane.
   - These hyperplanes go through the origin.
   - So two facets lie on the same hyperplane if and only if they have the same normals up to scalar multiple.

Proof/Algorithm

The facets in $\mathcal{F}_i$, when intersected with $L(\lambda)$ will therefore give all the facets of domains lying on a given wall.

Proof/Algorithm

The facets in $\mathcal{F}_i$, when intersected with $L(\lambda)$ will therefore give all the facets of domains lying on a given wall.

3. We glue these facets together by setting for each i
$$K_i = \bigcup_{F \in \mathcal{F}_i} F \, .$$
We verify that K_i is again a convex polyhedral cone by a truncation and volume comparison method (as before).

Proof/Algorithm

The intersections of the K_i with $L(\lambda)$ will be the walls partitioning the permutahedron.

Proof/Algorithm

The intersections of the K_i with $L(\lambda)$ will be the walls partitioning the permutahedron.

The facets of K_i, when intersected with $L(\lambda)$, will correspond to the edges of the walls generically (or be empty).

Proof/Algorithm

The intersections of the K_i with $L(\lambda)$ will be the walls partitioning the permutahedron.

The facets of K_i, when intersected with $L(\lambda)$, will correspond to the edges of the walls generically (or be empty).

The facets of those facets will correspond to the vertices of the walls generically (or be empty).

Proof/Algorithm

4. For each i, let $\mathcal{V}_i$ be the set of facets of facets of K_i.

Proof/Algorithm

4. For each i, let $\mathcal{V}_i$ be the set of facets of facets of K_i.

5. For each i, identify the $f \in \mathcal{V}_i$ whose intersection with $L(\lambda)$ (for generic λ) is a point and find that point. [Explanations later]

Proof/Algorithm

4. For each i, let $\mathcal{V}_i$ be the set of facets of facets of K_i.

5. For each i, identify the $f \in \mathcal{V}_i$ whose intersection with $L(\lambda)$ (for generic λ) is a point and find that point. [Explanations later]

6. For each i, verify that the vertices thus found define the same wall as the one of the Duistermaat-Heckman partition.

Step 5 explained

- Things would break down in Step 5 if not for a remarkable fact.

- The subset $\mathcal{W}$ of the hyperplane $x_1 + x_2 + x_3 + x_4 = 0$ where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ is a simplicial cone.

- Denote by $\{\omega_1, \omega_2, \omega_3\}$ its basis (rays), so that $\mathcal{W} = \mathrm{pos}(\omega_1, \omega_2, \omega_3)$.

- It turns out that all the $f \in \mathcal{V}_i$ have very nice expressions in terms of the ω_j.

Example

$$
\begin{aligned}
f_1 &= \operatorname{pos}((\omega_1,\omega_1),\ (\omega_2,\pi\cdot\omega_2),\ (\omega_3,\omega_3))\\
f_2 &= \operatorname{pos}((\omega_1,\omega_1),\ (\omega_2,\pi\cdot\omega_2),\ (\omega_3,\phi\cdot\omega_3))\\
f_3 &= \operatorname{pos}((\omega_1,\sigma\cdot\omega_1),\ (\omega_2,\pi\cdot\omega_2),\ (\omega_3,\phi\cdot\omega_3))\\
f_4 &= \operatorname{pos}((\omega_1,\sigma\cdot\omega_1),\ (\omega_2,\pi\cdot\omega_2),\ (\omega_3,\omega_3)\\
f_5 &= \operatorname{pos}((\omega_1,\sigma\cdot\omega_1),\ (\omega_3,\phi\cdot\omega_3),\ (\omega_3,\omega_3))\\
f_6 &= \operatorname{pos}((\omega_2,\pi\cdot\omega_2),\ (\omega_3,\phi\cdot\omega_3),\ (\omega_3,\omega_3))\\
f_7 &= \operatorname{pos}((\omega_1,\sigma\cdot\omega_1),\ (\omega_1,\omega_1),\ (\omega_3,\phi\cdot\omega_3))\\
f_8 &= \operatorname{pos}((\omega_1,\omega_1),\ (\omega_3,\phi\cdot\omega_3),\ (\omega_3,\omega_3))\\
f_9 &= \operatorname{pos}((\omega_1,\sigma\cdot\omega_1),\ (\omega_1,\omega_1),\ (\omega_3,\omega_3))\\
f_{10} &= \operatorname{pos}((\omega_1,\sigma\cdot\omega_1),\ (\omega_1,\omega_1),\ (\omega_2,\pi\cdot\omega_2))
\end{aligned}
$$

where $\sigma = (1\ 3)$, $\pi = (2\ 3)$, $\phi = (2\ 4)$.

Example

$$
\begin{aligned}
p_\Lambda(f_1) &= \operatorname{pos}(\omega_1, \omega_2, \omega_3) & p_\Lambda(f_5) &= \operatorname{pos}(\omega_1, \omega_3)\\
p_\Lambda(f_2) &= \operatorname{pos}(\omega_1, \omega_2, \omega_3) & p_\Lambda(f_6) &= \operatorname{pos}(\omega_2, \omega_3)\\
p_\Lambda(f_3) &= \operatorname{pos}(\omega_1, \omega_2, \omega_3) & p_\Lambda(f_7) &= \operatorname{pos}(\omega_1, \omega_3)\\
p_\Lambda(f_4) &= \operatorname{pos}(\omega_1, \omega_2, \omega_3) & p_\Lambda(f_8) &= \operatorname{pos}(\omega_1, \omega_3)\\
 & & p_\Lambda(f_9) &= \operatorname{pos}(\omega_1, \omega_3)\\
 & & p_\Lambda(f_{10}) &= \operatorname{pos}(\omega_1, \omega_2)
\end{aligned}
$$

- Only the first four projected cones span $\mathcal{W}$. The others will miss a generic λ (they won't intersect $L(\lambda)$).

Example

- We can rewrite the cone f_j so that the β coordinates of its rays are always the results of applying the same permutation to their λ coordinates.

$$
\begin{aligned}
f_1 &= \mathrm{pos}((\omega_1, (2\,3)\cdot\omega_1),\ (\omega_2, (2\,3)\cdot\omega_2),\ (\omega_3, (2\,3)\cdot\omega_3)) \\
f_2 &= \mathrm{pos}((\omega_1, (2\,4\,3)\cdot\omega_1),\ (\omega_2, (2\,4\,3)\cdot\omega_2),\ (\omega_3, (2\,4\,3)\cdot\omega_3)) \\
f_3 &= \mathrm{pos}((\omega_1, (1\,2\,4\,3)\cdot\omega_1),\ (\omega_2, (1\,2\,4\,3)\cdot\omega_2),\ (\omega_3, (1\,2\,4\,3)\cdot\omega_3)) \\
f_4 &= \mathrm{pos}((\omega_1, (1\,2\,3)\cdot\omega_1),\ (\omega_2, (1\,2\,3)\cdot\omega_2),\ (\omega_3, (1\,2\,3)\cdot\omega_3))
\end{aligned}
$$

Example

$$
\begin{aligned}
f_1 \cap L(\lambda) &= (\lambda, (2\,3) \cdot \lambda) \\
f_2 \cap L(\lambda) &= (\lambda, (2\,4\,3) \cdot \lambda) \\
f_3 \cap L(\lambda) &= (\lambda, (1\,2\,4\,3) \cdot \lambda) \\
f_4 \cap L(\lambda) &= (\lambda, (1\,2\,3) \cdot \lambda)
\end{aligned}
$$

This means there will be a wall with vertices

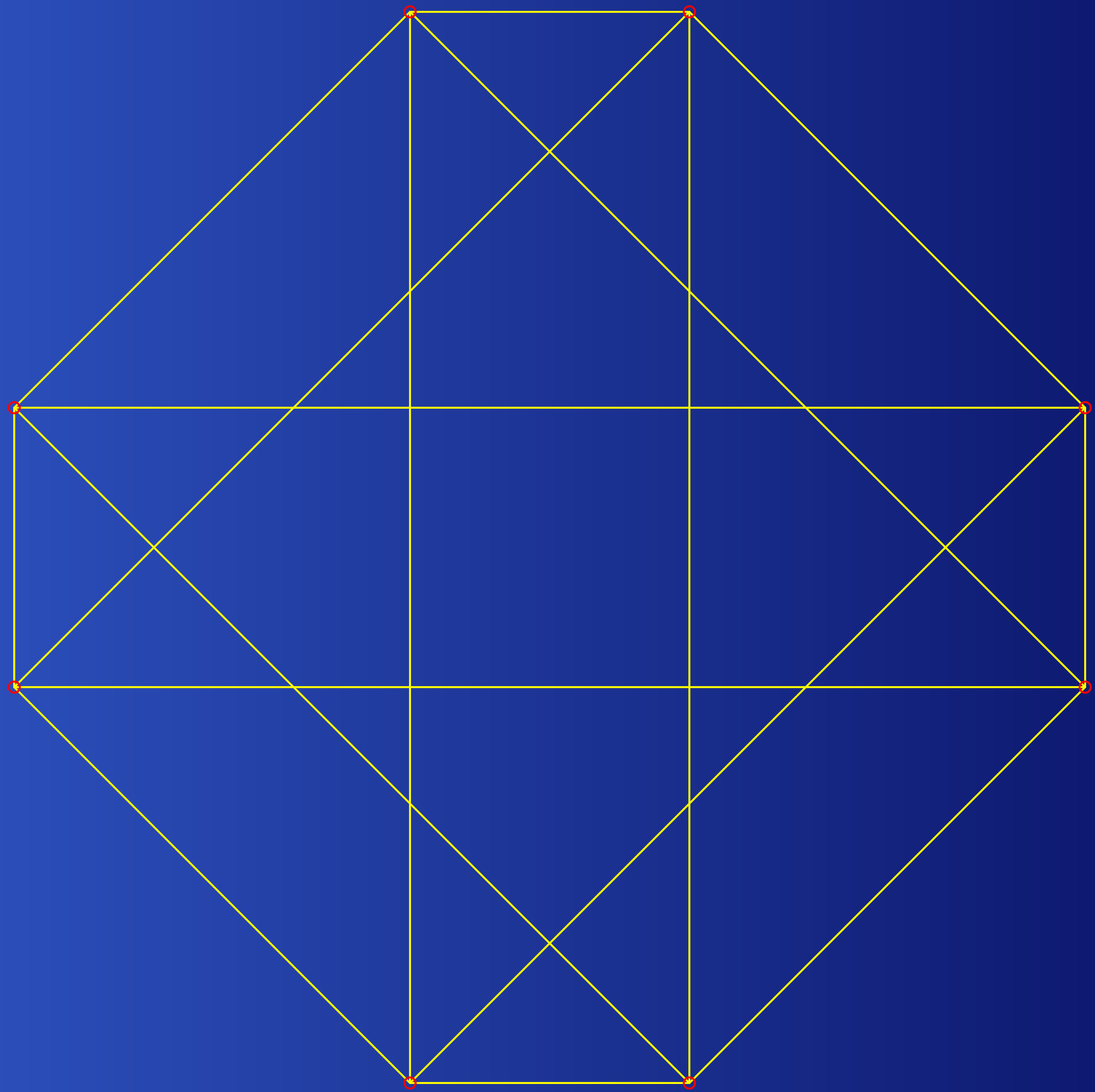
$$
\begin{aligned}
(2\,3) \cdot \lambda &= (\lambda_1, \lambda_3, \lambda_2, \lambda_4) &= \lambda' \\
(1\,2\,3) \cdot \lambda &= (\lambda_3, \lambda_1, \lambda_2, \lambda_4) &= (1\,2)\lambda' \\
(2\,4\,3) \cdot \lambda &= (\lambda_1, \lambda_3, \lambda_4, \lambda_2) &= (3\,4)\lambda' \\
(1\,2\,4\,3) \cdot \lambda &= (\lambda_3, \lambda_1, \lambda_4, \lambda_2) &= (1\,2)(3\,4)\lambda'
\end{aligned}
$$

Open problems

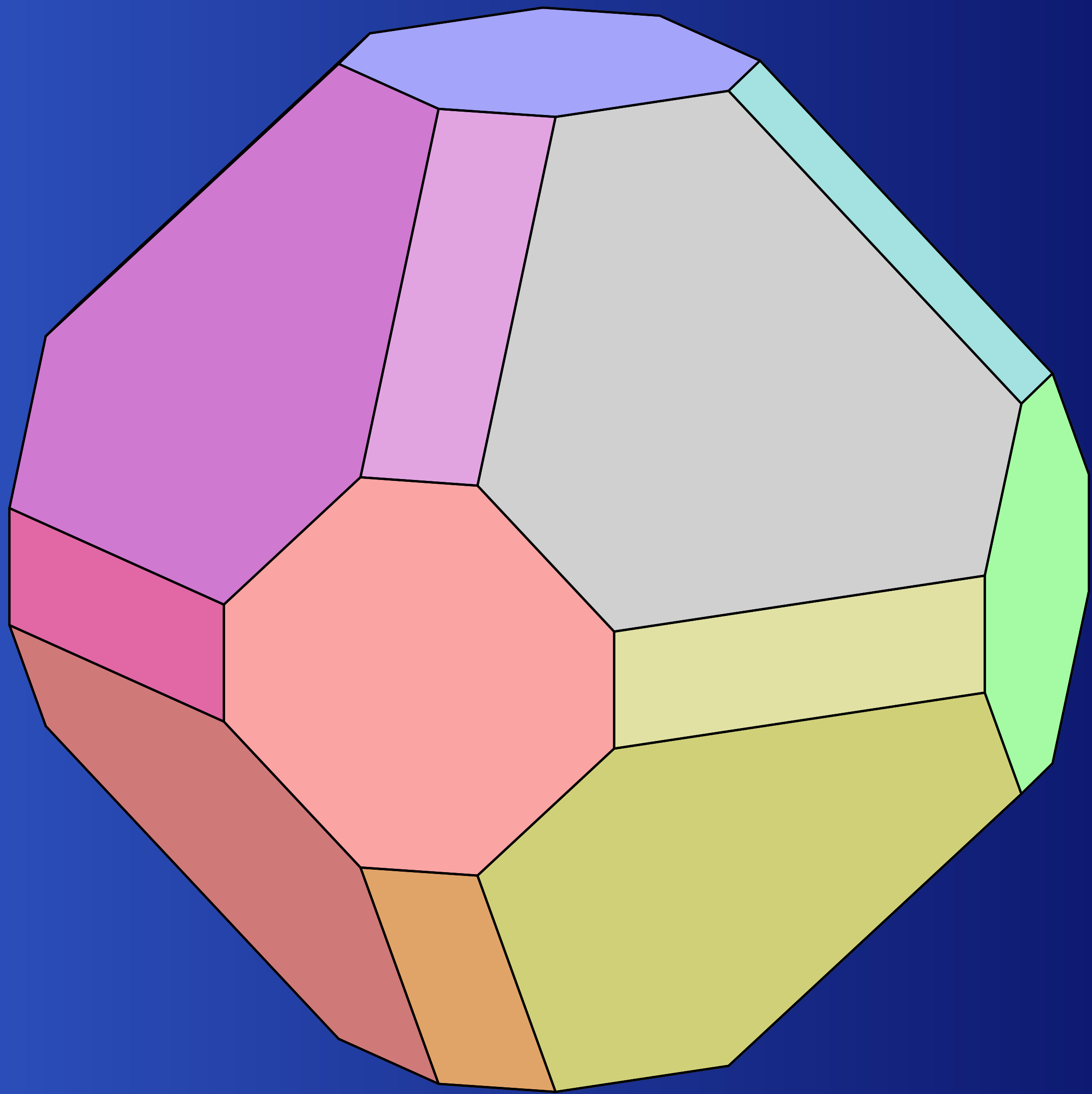
Open problems

- What are the regions counts for higher dimensional permutahedra ? (15230 regions for $\lambda = (2, 1, 0, -1, -2)$)
- Do the partitions for the DH function and weight multiplicities keep coinciding in higher dimension ?
- What about the permutahedra for other groups ?
- Are there fast ways to compute weight multiplicities (Kostka numbers) ?

Permutahedron for B_2



Permutahedron for B_3



Permutahedron for G_2