Partitioning the permutahedron

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Outline

- The permutahedron
  - Weight multiplicities
  - The Duistermaat-Heckman function
- Vector partition functions
- Counting the regions
- Comparing the two partitions
- Open problems
The permutahedron
The permutahedron

We obtain a $k - 1$ dimensional permutahedron by

- picking a point $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$ with distinct coordinates;
- permuting the coordinates of $\lambda$ in all possible ways to get $k!$ points;
- taking the convex hull of those points.

\[
P_\lambda = \text{conv} \left( \mathfrak{S}_k \cdot \lambda \right)
\]

\[
P_\lambda = \text{conv} \left( \{ (\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(k)}) : \sigma \in \mathfrak{S}_k \} \right)
\]
The permutahedron

It is $k - 1$ dimensional because the points

$$\{(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(k)}) : \sigma \in \mathcal{S}_k\}$$

all lie on the same subspace of $\mathbb{R}^k$:

$$x_1 + x_2 + \cdots + x_k = \sum_{i=1}^{k} \lambda_i = \sum_{i=1}^{k} \lambda_{\sigma(i)}.$$

We will assume that $\lambda_1 + \cdots + \lambda_k = 0$. 
Examples of permutahedra
Examples of permutahedra
Integral permutahedra

Since \( P_\lambda = P_{\sigma(\lambda)} \), we will further assume that

\[ \lambda_1 > \lambda_2 > \cdots > \lambda_k . \]

We will usually consider integral \( \lambda \), i.e. \( \lambda \in \mathbb{Z}^k \).
Integral permutahedra

Since $P_\lambda = P_{\sigma(\lambda)}$, we will further assume that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_k.$$

We will usually consider integral $\lambda$, i.e. $\lambda \in \mathbb{Z}^k$.

Then $P_\lambda$ is an integral polytope, and $P_\lambda \cap \mathbb{Z}^k$ is the lattice spanned by the vectors

$$\left\{ e_i - e_j : 1 \leq i < j \leq k \right\},$$

or, equivalently, by the vectors

$$\left\{ e_i - e_{i+1} : 1 \leq i \leq k - 1 \right\}.$$
Two functions on the permutahedron

We will consider two functions on an integral permutahedron.

- A discrete one: weight multiplicities, defined on the lattice points in the permutahedron.
- A continuous one: the Duistermaat-Heckman function, defined on the whole permutahedron.

Both functions partition the permutahedron into polytopal domains over which they are given by polynomials.
A Gelfand-Tsetlin diagram is an array of integers of the form

\[
\begin{array}{cccc}
\lambda_1^{(k)} & \lambda_2^{(k)} & \cdots & \lambda_{k-1}^{(k)} & \lambda_k^{(k)} \\
\lambda_1^{(k-1)} & \lambda_2^{(k-1)} & \cdots & \lambda_{k-1}^{(k-1)} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{(2)} & \lambda_2^{(2)} \\
\lambda_1^{(1)} \\
\end{array}
\]

such that
Gelfand-Tsetlin diagrams

\[ \lambda_1^{(k)} \quad \lambda_2^{(k)} \quad \ldots \quad \lambda_{k-1}^{(k)} \quad \lambda_k^{(k)} \]

\[ \lambda_1^{(k-1)} \quad \lambda_2^{(k-1)} \quad \ldots \quad \lambda_{k-1}^{(k-1)} \quad \lambda_k^{(k-1)} \]

\[ \ldots \]

\[ \lambda_1^{(2)} \quad \lambda_2^{(2)} \]

\[ \lambda_1^{(1)} \quad \lambda_2^{(1)} \]
Gelfand-Tsetlin diagrams

\[
\begin{array}{cccc}
\lambda_1^{(k-1)} & \lambda_2^{(k-1)} & \cdots & \lambda_{k-1}^{(k-1)} \\
\lambda_1^{(k-1)} & \lambda_2^{(k-1)} & \cdots & \lambda_{k-1}^{(k-1)} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{(2)} & \lambda_2^{(2)} & \cdots & \lambda_{k-1}^{(2)} \\
\lambda_1^{(1)} & \cdots & \cdots & \\
\end{array}
\]
Gelfand-Tsetlin diagrams

\[
\begin{array}{cccc}
\lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\
\lambda_1^{(k-1)} & \lambda_2^{(k-1)} & \cdots & \lambda_{k-1}^{(k-1)} & \\
\vdots & \vdots & \ddots & \vdots & \\
\lambda_1^{(2)} & \lambda_2^{(2)} & & & \\
\lambda_1^{(1)} & & & & \\
\end{array}
\]

and

\[
\lambda_j^{(i+1)} \prec \lambda_j^{(i)} \prec \lambda_{j+1}^{(i+1)}
\]

for every such triangle in the diagram.
The weight multiplicity $m_\lambda(\beta)$ of $\beta$ in $P_\lambda$ is the number of Gelfand-Tsetlin diagrams with top row $\lambda$ and row sums satisfying

$$\sum_{i=1}^{m} \lambda_i^{(m)} = \beta_1 + \cdots + \beta_m$$

for $1 \leq m \leq k$. Weight multiplicities vanish outside the permutahedron. Weight multiplicities are invariant under the action of the symmetric group $S_k$. 

$$P \text{ VPF} \# 2 \text{ O}$$
Weight multiplicities

The weight multiplicity $m_\lambda(\beta)$ of $\beta$ in $P_\lambda$ is the number of Gelfand-Tsetlin diagrams with top row $\lambda$ and row sums satisfying

$$\sum_{i=1}^{m} \lambda^{(m)}_i = \beta_1 + \cdots + \beta_m \quad \text{for} \quad 1 \leq m \leq k.$$ 

Weight multiplicities vanish outside the permutahedron.
Weight multiplicities

The weight multiplicity $m_\lambda(\beta)$ of $\beta$ in $P_\lambda$ is the number of Gelfand-Tsetlin diagrams with top row $\lambda$ and row sums satisfying

$$\sum_{i=1}^{m} \lambda_i^{(m)} = \beta_1 + \cdots + \beta_m \quad \text{for } 1 \leq m \leq k.$$ 

Weight multiplicities vanish outside the permutahedron.

Weight multiplicities are invariant under the action of the symmetric group $\mathfrak{S}_k$. 
GT-diagrams and SSYT's

\[
\begin{align*}
7 & \quad 5 & \quad 4 & \quad 1 & \quad \beta_1 + \beta_2 + \beta_3 + \beta_4 = 17 \\
6 & \quad 5 & \quad 2 & \quad \beta_1 + \beta_2 + \beta_3 = 13 \\
5 & \quad 3 & \quad \beta_1 + \beta_2 = 8 \\
3 & \quad \beta_1 = 3 \\
\end{align*}
\]
GT-diagrams and SSYTs

\[
\begin{array}{cccc}
7 & 5 & 4 & 1 \\
6 & 5 & 2 & \\
5 & 3 & \\
3 & \\
\end{array}
\]

\[
\beta_1 + \beta_2 + \beta_3 + \beta_4 = 17 \\
\beta_1 + \beta_2 + \beta_3 = 13 \\
\beta_1 + \beta_2 = 8 \\
\beta_1 = 3 \\
\]

(3)
GT-diagrams and SSYTs

\[
\begin{align*}
7 & \quad 5 & \quad 4 & \quad 1 & \quad \beta_1 + \beta_2 + \beta_3 + \beta_4 = 17 \\
6 & \quad 5 & \quad 2 & \quad \beta_1 + \beta_2 + \beta_3 = 13 \\
5 & \quad 3 & \quad \beta_1 + \beta_2 = 8 \\
3 & \quad \beta_1 = 3
\end{align*}
\]

\[
\begin{array}{cccc}
1 & 1 & 1 & 2 & 2 \\
2 & 2 & 2 & &
\end{array}
\]

\((5, 3)\)
GT-diagrams and SSYTs

\[
\begin{array}{cccc}
7 & 5 & 4 & 1 \\
6 & 5 & 2 & \\
5 & 3 & \\
3 & \\
\end{array}
\]

\[
\begin{align*}
\beta_1 + \beta_2 + \beta_3 + \beta_4 &= 17 \\
\beta_1 + \beta_2 + \beta_3 &= 13 \\
\beta_1 + \beta_2 &= 8 \\
\beta_1 &= 3
\end{align*}
\]

\[
\begin{array}{cccccc}
1 & 1 & 1 & 2 & 2 & 3 \\
2 & 2 & 2 & 3 & 3 & \\
3 & 3 & \\
\end{array}
\]

\((6, 5, 2)\)
GT-diagrams and SSYTs

\[
\begin{array}{cccc}
7 & 5 & 4 & 1 \\
6 & 5 & 2 \\
5 & 3 \\
3 \\
\end{array}
\]

\[
\begin{align*}
\beta_1 + \beta_2 + \beta_3 + \beta_4 &= 17 \\
\beta_1 + \beta_2 + \beta_3 &= 13 \\
\beta_1 + \beta_2 &= 8 \\
\beta_1 &= 3
\end{align*}
\]

\[
(7, 5, 4, 1)
\]
Gelfand-Tsetlin polytopes

\[
\begin{array}{cccc}
\lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\
\lambda_1^{(k-1)} & \lambda_2^{(k-1)} & \cdots & \lambda_{k-1}^{(k-1)} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{(2)} & \lambda_2^{(2)} \\
\lambda_1^{(1)} \\
\end{array}
\]

\[\text{GT}_\lambda \quad \text{GT}_{\lambda\beta}\]
\( \lambda = (9, -2, -7) \)

\[ m_{\lambda}(\beta) = 1 \]
\[ \lambda = (9, -2, -7) \]

\[ m_\lambda(\beta) = 2 \]
\[ \lambda = (9, -2, -7) \]

\[ m_\lambda(\beta) = 3 \]
\[ \lambda = (9, -2, -7) \]

\[ m_\lambda(\beta) = 4 \]
\( \lambda = (9, -2, -7) \)

\[ m_\chi(\beta) = 5 \]
\[ \lambda = (9, -2, -7) \]

\[ m_\lambda(\beta) = 6 \]
\[ \lambda = (9, -2, -7) \]
\[ \lambda = (14, -2, -4, -8) \]

\[ m_\lambda(\beta) = 1 \]
\[ \lambda = (14, -2, -4, -8) \]

\[ m_\lambda(\beta) = 2 \]
$\lambda = (14, -2, -4, -8)$

$m_\lambda(\beta) = 3$
$\lambda = (14, -2, -4, -8)$

$m_\lambda(\beta) = 4$
\[ \lambda = (14, -2, -4, -8) \]

\[ m_\lambda(\beta) = 5 \]
\[ \lambda = (14, -2, -4, -8) \]

\[ m_\lambda(\beta) = 7 \]
\[ \lambda = (14, -2, -4, -8) \]

\[ m_{\lambda}(\beta) = 9 \]
\[ \lambda = (14, -2, -4, -8) \]

\[ m_{\lambda}(\beta) = 10 \]
\[ \lambda = (14, -2, -4, -8) \]

\[ m_\lambda(\beta) = 12 \]
$\lambda = (14, -2, -4, -8)$

$m_\lambda(\beta) = 15$
$\lambda = (14, -2, -4, -8)$

$m_\lambda(\beta) = 18$
\[ \lambda = (14, -2, -4, -8) \]

\[ m_\lambda(\beta) = 19 \]
$\lambda = (14, -2, -4, -8)$

$m_\lambda(\beta) = 22$
\[ \lambda = (14, -2, -4, -8) \]

\[ m_\lambda(\beta) = 26 \]
\[ \lambda = (14, -2, -4, -8) \]

\[ m_\lambda(\beta) = 30 \]
\[ \lambda = (14, -2, -4, -8) \]

\[ m_\lambda(\beta) = 31 \]
\[ \lambda = (14, -2, -4, -8) \]

\[ m_\lambda(\beta) = 35 \]
\[ \lambda = (14, -2, -4, -8) \]

\[ m_\lambda(\beta) = 40 \]
\[ \lambda = (14, -2, -4, -8) \]

\[ m_\lambda(\beta) = 45 \]
\[ \lambda = (14, -2, -4, -8) \]

\[ m_\lambda(\beta) = 50 \]
\[ \lambda = (14, -2, -4, -8) \]

\[ m_\lambda(\beta) = 55 \]
$$\lambda = (14, -2, -4, -8)$$

$$m_\lambda(\beta) = 60$$
The Duistermaat-Heckman function

For $\lambda$ integral there is a function from symplectic geometry, the Duistermaat-Heckman function, that is piecewise polynomial on $P_{\lambda}$.

It approximates the weight multiplicities.

The domains of polynomiality form a partition of $P_{\lambda}$ into subpolytopes.
The DH function for $k = 3$
The DH function for $k = 3$

**Theorem** (Heckman, Guillemin-Lerman-Sternberg)

Consider the convex polytopes

$$\text{conv}(W \cdot \sigma(\lambda))$$

where $\sigma \in \mathfrak{S}_k$ and $W$ is the stabilizer of a facet of $\text{conv}(\mathfrak{S}_k \cdot \lambda)$.

These polytopes are walls that partition $\text{conv}(\mathfrak{S}_k \cdot \lambda)$ into convex subpolytopes over which the Duistermaat-Heckman function is polynomial.
\[ \lambda = (7, -1, -2, -4) \]
\[ \lambda = (7, -1, -2, -4) \]
\[ \lambda = (7, -1, -2, -4) \]
\[ \lambda = (7, -1, -2, -4) \]
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\( \lambda = (7, -1, -2, -4) \)
$\lambda = (7, -1, -2, -4)$
\[ \lambda = (7, -1, -2, -4) \]
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$\lambda = (7, -1, -2, -4)$
\[ \lambda = (7, -1, -2, -4) \]
\[ \lambda = (7, -1, -2, -4) \]
$\lambda = (7, -1, -2, -4)$
\[ \lambda = (7, -1, -2, -4) \]
$\lambda = (7, -1, -2, -4)$
\lambda = (7, -1, -2, -4)
$$\lambda = (7, -1, -2, -4)$$
\[ \lambda = (7, -1, -2, -4) \]
\[ \lambda = (7, -1, -2, -4) \]
\( k = 4 \)
Vector partition functions
Let $M$ be a $d \times n$ matrix over the integers. The vector partition function associated to $M$ is the function

$$
\phi_M : \mathbb{Z}^d \longrightarrow \mathbb{N}
$$

$$
b \longmapsto |\{x \in \mathbb{N}^n : Mx = b\}|
$$
Vector partition functions

Let $M$ be a $d \times n$ matrix over the integers. The vector partition function associated to $M$ is the function

$$
\phi_M : \mathbb{Z}^d \rightarrow \mathbb{N}
$$

$$
b \mapsto \left| \{ x \in \mathbb{N}^n : Mx = b \} \right|
$$

Example

If $M = \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ then $\phi_M(b) = 3$

since $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. 
If $M$ is such that $\ker M \cap \mathbb{R}^n_{\geq 0} = 0$, then

$$P_b = \{x \in \mathbb{R}^n_{\geq 0} : Mx = b\}$$

is a polytope.

$\phi_M(b)$ is the number of integral points in $P_b$. 
Polytopes and partition functions

- If $M$ is such that $\ker M \cap \mathbb{R}^n_{\geq 0} = 0$, then

$$P_b = \{ x \in \mathbb{R}^n_{\geq 0} : Mx = b \}$$

is a polytope.

$\phi_M(b)$ is the number of integral points in $P_b$.

- $\phi_M$ vanishes outside of $\text{pos}(M)$. 
Quasipolynomials

A quasipolynomial function on a lattice $L$ is a function that is polynomial on each coset of a sublattice $N$ of $L$.

Example

$$f(n_1, n_2) = \begin{cases} 
    n_1^2 + 3n_1n_2 + 2n_2 & \text{if } n_1 + n_2 \equiv 0 \pmod{3}, \\
    2n_2^3 + n_1^2n_2^2 + 3 & \text{if } n_1 + n_2 \equiv 1 \pmod{3}, \\
    0 & \text{if } n_1 + n_2 \equiv 2 \pmod{3}.
\end{cases}$$
A chamber (or cone) complex is a collection $\mathcal{C}$ of convex polyhedral cones such that

- if $C \in \mathcal{C}$ and $F$ is a face of $C$, then $F \in \mathcal{C}$; ($\mathcal{C}$ is closed under taking faces.)

- if $C_1, C_2 \in \mathcal{C}$, then $C_1 \cap C_2$ is a common face to $C_1$ and $C_2$. (Cones touch along whole faces. Note that $\{0\}$ is a 0-dimensional face of any cone.)
The structure of partition functions

\[ \phi_M \text{ is piecewise quasipolynomial of degree } n - \text{rank}(M) \]. (Sturmfels)
The structure of partition functions

$\phi_M$ is piecewise quasipolynomial of degree $n - \text{rank}(M)$. (Sturmfels)

The domains of quasipolynomiality form a complex of convex polyhedral cones, the chamber complex of $\phi_M$. 
The structure of partition functions

- $\phi_M$ is piecewise quasipolynomial of degree $n - \text{rank}(M)$. (Sturmfels)

- The domains of quasipolynomiality form a complex of convex polyhedral cones, the chamber complex of $\phi_M$.

- Alekseevskaya, Gelfand and Zelevinsky described how to determine the chamber complex of a partition function from its matrix.
Determining the chamber complex

We can assume without loss of generality that $M$ has full rank $d$.

Find all the $d \times d$ nonsingular submatrices $M_{\sigma}$ of $M$. 
Determining the chamber complex

We can assume without loss of generality that $M$ has full rank $d$.

- Find all the $d \times d$ nonsingular submatrices $M_\sigma$ of $M$.
- Determine the cone $\tau_\sigma = \text{pos}(M_\sigma)$ spanned by the columns of $M_\sigma$. 
Determining the chamber complex

We can assume without loss of generality that $M$ has full rank $d$.

- Find all the $d \times d$ nonsingular submatrices $M_\sigma$ of $M$.

- Determine the cone $\tau_\sigma = \text{pos}(M_\sigma)$ spanned by the columns of $M_\sigma$.

- The chamber complex of $\varphi_M$ is the common refinement of the $\tau_\sigma$. 
Example

Consider

\[
M = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 \\
\end{pmatrix}
\]
Example

Consider

\[ M = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 \\
\end{pmatrix} \]

\[ B = \{123, 125, 126, 134, 135, 136, 145, 146, 234, 236, 245, 246, 256, 345, 356, 456\} \]
Example
Example

\[ \alpha_2 \]

[Diagram showing a triangle with vertices labeled \( \alpha_1 \), \( \alpha_2 \), and \( \alpha_3 \), with additional labels \( \alpha_1 + \alpha_2 \), \( \alpha_2 + \alpha_3 \), and \( \alpha_1 + \alpha_2 + \alpha_3 \).]
A partition function for the $m_\lambda(\beta)$

**Theorem**

For every $k$, we can find integer matrices $E_k$ and $B_k$ such that the weight multiplicities can be written as

$$m_\lambda(\beta) = \phi_{E_k} \left( B_k \begin{pmatrix} \lambda \\ \beta \end{pmatrix} \right).$$
Example: $k = 3$

Gelfand-Tsetlin diagrams for $k = 3$ have the form

\[
\begin{array}{ccc}
\lambda_1 & \lambda_2 & \lambda_3 \\
\mu_1 & \mu_2 & \nu \\
\end{array}
\]
Example: \( k = 3 \)

Gelfand-Tsetlin diagrams for \( k = 3 \) have the form

\[
\begin{array}{ccc}
\lambda_1 & \lambda_2 & \lambda_3 \\
\mu_1 & \mu_2 & \\
\nu & & \\
\end{array}
\]

Row sums:

\[
\begin{align*}
\nu &= \beta_1 \\
\mu_1 + \mu_2 &= \beta_1 + \beta_2 \\
\lambda_1 + \lambda_2 + \lambda_3 &= \beta_1 + \beta_2 + \beta_3.
\end{align*}
\]
Example: $k = 3$

\[
\begin{align*}
\mu_1 &\leq \lambda_1 \\
-\mu_1 &\leq -\lambda_2 \\
-\mu_1 &\leq \lambda_2 - \beta_1 - \beta_2 \\
\mu_1 &\leq \beta_1 + \beta_2 + \lambda_1 + \lambda_2 \\
-\mu_1 &\leq -\beta_1 \\
-\mu_1 &\leq -\beta_2 .
\end{align*}
\]
Example: \( k = 3 \)

\[
\begin{align*}
\mu_1 + s_1 &= \lambda_1 \\
-\mu_1 + s_2 &= -\lambda_2 \\
-\mu_1 + s_3 &= \lambda_2 - \beta_1 - \beta_2 \\
\mu_1 + s_4 &= \beta_1 + \beta_2 + \lambda_1 + \lambda_2 \\
-\mu_1 + s_5 &= -\beta_1 \\
-\mu_1 + s_6 &= -\beta_2.
\end{align*}
\]
Example: $k = 3$

\[
\begin{align*}
\mu_1 + s_1 &= \lambda_1 \\
-\mu_1 + s_2 &= -\lambda_2 \\
-\mu_1 + s_3 &= \lambda_2 - \beta_1 - \beta_2 \\
\mu_1 + s_4 &= \beta_1 + \beta_2 + \lambda_1 + \lambda_2 \\
-\mu_1 + s_5 &= -\beta_1 \\
-\mu_1 + s_6 &= -\beta_2.
\end{align*}
\]

The $s_i$ are constrained to be nonnegative.
Example: $k = 3$

\[
\begin{align*}
\mu_1 + s_1 &= \lambda_1 \\
-\mu_1 + s_2 &= -\lambda_2 \\
-\mu_1 + s_3 &= \lambda_2 - \beta_1 - \beta_2 \\
\mu_1 + s_4 &= \beta_1 + \beta_2 + \lambda_1 + \lambda_2 \\
-\mu_1 + s_5 &= -\beta_1 \\
-\mu_1 + s_6 &= -\beta_2.
\end{align*}
\]

- The $s_i$ are constrained to be nonnegative.
- Finally we can use $\mu_1 = \lambda_1 - s_1$ to get rid of $\mu_1$. 
Example: $k = 3$

\[ s_1 + s_2 = \lambda_1 - \lambda_2 \]
\[ -s_2 + s_3 = 2\lambda_2 - \beta_1 - \beta_2 \]
\[ s_2 + s_4 = \beta_1 + \beta_2 + \lambda_1 \]
\[ -s_2 + s_5 = \lambda_2 - \beta_1 \]
\[ -s_2 + s_6 = \lambda_2 - \beta_2 \]
Example: $k = 3$

\[
\begin{align*}
  s_1 + s_2 &= \lambda_1 - \lambda_2 \\
  -s_2 + s_3 &= 2\lambda_2 - \beta_1 - \beta_2 \\
  s_2 + s_4 &= \beta_1 + \beta_2 + \lambda_1 \\
  -s_2 + s_5 &= \lambda_2 - \beta_1 \\
  -s_2 + s_6 &= \lambda_2 - \beta_2 
\end{align*}
\]

Solving for $s_i \geq 0 \; \forall i$. 
Example: $k = 3$

\[
\begin{align*}
    s_1 + s_2 &= \lambda_1 - \lambda_2 \\
    -s_2 + s_3 &= 2\lambda_2 - \beta_1 - \beta_2 \\
    s_2 + s_4 &= \beta_1 + \beta_2 + \lambda_1 \\
    -s_2 + s_5 &= \lambda_2 - \beta_1 \\
    -s_2 + s_6 &= \lambda_2 - \beta_2 
\end{align*}
\]

- Solving for $s_i \geq 0 \ \forall \ i$.

- Requiring the $s_i$'s to be integers yields all integer solutions to the Gelfand-Tsetlin constraints.
**Example: \( k = 3 \)**

So we are solving

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
\lambda_1 - \lambda_2 \\
2\lambda_2 - \beta_1 - \beta_2 \\
\beta_1 + \beta_2 + \lambda_1 \\
\lambda_2 - \beta_1 \\
\lambda_2 - \beta_2
\end{pmatrix}
\]

for \( \vec{s} \in \mathbb{N}^6 \).
Example: $k = 3$

So we are solving

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 1
\end{pmatrix}
\cdot
\begin{pmatrix}
s_1 \\
s_2 \\
s_3 \\
s_4 \\
s_5 \\
s_6
\end{pmatrix}
= 
\begin{pmatrix}
\lambda_1 - \lambda_2 \\
2\lambda_2 - \beta_1 - \beta_2 \\
\beta_1 + \beta_2 + \lambda_1 \\
\lambda_2 - \beta_1 \\
\lambda_2 - \beta_2
\end{pmatrix}
$$

for $\vec{s} \in \mathbb{N}^6$. Hence

$$m_\lambda(\beta) = \phi_{E_2} \left( B_2(\frac{\lambda}{\beta}) \right).$$
The chamber complex

$\phi E_k$ takes values in $\mathbb{R}^{(k-1)(k+2)/2}$.
The chamber complex

\( \phi_{E_k} \) takes values in \( \mathbb{R}^{(k-1)(k+2)/2} \).

Let \( \tilde{B} = \left\{ B_k (\lambda/\beta) : \lambda \in \mathbb{R}^k, \beta \in \mathbb{R}^k, \sum \lambda_i = \sum \beta_i = 0 \right\} \).
The chamber complex

- $\phi_{E_k}$ takes values in $\mathbb{R}^{(k-1)(k+2)/2}$.

- Let $\tilde{B} = \left\{ B_k \left( \begin{array}{c} \lambda \\ \beta \end{array} \right) : \lambda \in \mathbb{R}^k, \beta \in \mathbb{R}^k, \sum \lambda_i = \sum \beta_i = 0 \right\}$.

- The only part of the chamber complex of $\phi_{E_k}$ that is relevant for the weight multiplicities is its intersection with $\tilde{B}$.
\[ \phi_{E_k} \text{ takes values in } \mathbb{R}^{(k-1)(k+2)/2} . \]

Let \( \tilde{B} = \left\{ B_k(\lambda) : \lambda \in \mathbb{R}^k, \beta \in \mathbb{R}^k, \sum \lambda_i = \sum \beta_i = 0 \right\} . \)

The only part of the chamber complex of \( \phi_{E_k} \) that is relevant for the weight multiplicities is its intersection with \( \tilde{B} . \)

We can intersect the base cones with \( \tilde{B} \) before taking the common refinement.
On \( \tilde{B} \), we can work in \((\lambda, \beta)\)-coordinates.
The chamber complex

- On $\tilde{B}$, we can work in $(\lambda, \beta)$-coordinates.
- We will call $\mathcal{C}^{(k)}$ this $(2k - 2)$-dim complex in $(\lambda, \beta)$-coordinates.
The chamber complex

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- We will call $\mathcal{C}^{(k)}$ this $(2k - 2)$-dim complex in $(\lambda, \beta)$-coordinates.

- For fixed $\lambda$, let
  \[ L(\lambda) = \{(\lambda_1, \ldots, \lambda_k, \beta_1, \ldots, \beta_k) : \beta_i \in \mathbb{R}, \sum \beta_i = \sum \lambda_i = 0\} \]
The chamber complex

On \( \tilde{B} \), we can work in \((\lambda, \beta)\)-coordinates.

We will call \( \mathcal{C}^{(k)} \) this \((2k - 2)\)-dim complex in \((\lambda, \beta)\)-coordinates.

For fixed \( \lambda \), let
\[
L(\lambda) = \{ (\lambda_1, \ldots, \lambda_k, \beta_1, \ldots, \beta_k) : \beta_i \in \mathbb{R}, \sum \beta_i = \sum \lambda_i = 0 \}.
\]

The intersection of \( \mathcal{C}^{(k)} \) with \( L(\lambda) \) gives domains of (quasi)polynomiality for the multiplicities.
Corollary

Let $C_{\Lambda}^{(k)}$ be the chamber complex given by the common refinement of the projections $p_{\Lambda}(\tau)$ of the cones of $C^{(k)}$ onto $\mathbb{R}^k$.

Then $C_{\Lambda}^{(k)}$ classifies the $\lambda$’s, in the sense that if $\lambda$ and $\lambda'$ belong to the same cell of $C_{\Lambda}^{(k)}$, then all their domains are indexed by the same subsets of cones from $C_{\Lambda}^{(k)}$, and therefore have the same corresponding polynomials.
Counting the regions
The complex for $k = 3$

- The complex $C^{(3)}$ consists of eight 4-dimensional cones and their faces.

- The complex is invariant under permuting the $\beta$ coordinates.

- We project the cones of $C^{(3)}$ on the $\lambda$ coordinates and take the common refinement.

- $C^{(3)}_\Lambda$ has two top-dimensional cones.
The projected complex

\[
C_1 = \text{pos}\left((2, -1, -1), (1, 0, -1)\right)
\]

\[
C_2 = \text{pos}\left((1, 0, -1), (1, 1, -2)\right)
\]

There are two generic cases for $\lambda$ in this case ($\lambda_2 < 0$ or $\lambda_2 > 0$), each with 7 domains.
The complex for $k = 4$

For $k = 4$, we find that $C^{(4)}$ has 1202 top dimensional cones (6-dim cones in 8-dim ambient $(\lambda, \beta)$-space).

However, it is not invariant under the action of $S_4$ on the $\beta$ coordinates.

This means that the complex is not optimal and can be coarsened further.
The union of the top dimensional cones of $C^{(4)}$ with the same weight polynomial is again a convex polyhedral cone.
The union of the top dimensional cones of $C^{(4)}$ with the same weight polynomial is again a convex polyhedral cone.

This means we can glue these cones together.

We can then verify that we get a chamber complex that way, the glued complex $G$. 
Idea of the proof

1. Select the cones with a same weight polynomial. (Yellow)
2. Consider their rays.
3. Construct the cone spanned by those rays.
Idea of the proof

4. Find a transversal affine halfspace and intersect it with the cones to get polytopes.

5. Compare volumes.
Idea of the proof

- This process was automated symbolically, so that there are no roundoff errors which might cause us to miss a small cone.

- The complex $G$ has 612 top dimensional cones.

- It is invariant under the action of $S_4$ on $\beta$. There are 64 orbits of cones.
Counting the regions
Counting the regions
Counting the regions
Is this partition for weight multiplicities the same as the one we have in terms of walls for the Duistermaat-Heckman function?
Comparing the two partitions
The DH function and the $m_\lambda(\beta)$

- The partitions agree for all the random $\lambda$ we tried.

- The difficulty is intersecting $\mathcal{G}$ with $L(\lambda)$ not for a specific $\lambda$ but for a general (symbolic) one.
The DH function and the $m_\lambda(\beta)$

- The partitions agree for all the random $\lambda$ we tried.

- The difficulty is intersecting $\mathcal{G}$ with $L(\lambda)$ not for a specific $\lambda$ but for a general (symbolic) one.

**Theorem**

*For $k = 4$, the partitions of the permutahedron into its domains of polynomiality for the weight multiplicities and for the Duistermaat-Heckman function are the same.*
Strategy

The partition for the DH function is given in terms of walls rather than a description of the subpolytopes themselves.

Therefore our goal will be to identify which pieces of the chamber complex $G$ give rise to the walls when intersected with $L(\lambda)$. 
The intersection of a top dimensional cone with $L(\lambda)$, if it is not empty, gives a domain.

So the intersection of the facets of that cone with $L(\lambda)$ give the facets of the domain (generically).

We build the walls by glueing such facets together.
1. Let $\mathcal{F}$ be the set of all facets of the top dimensional cones $G_1, \ldots, G_{612}$ of $\mathcal{G}$. 
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2. Group the facets in $\mathcal{F}$ according to their normals. 37 normal directions: $\{n_1, \ldots, n_{37}\}$. 
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Call $\mathcal{F}_i$ the subset of $\mathcal{F}$ consisting of all the facets with normals in direction $n_i$. 
Proof/Algorithm

1. Let $\mathcal{F}$ be the set of all facets of the top dimensional cones $G_1, \ldots, G_{612}$ of $\mathcal{G}$.

2. Group the facets in $\mathcal{F}$ according to their normals. 37 normal directions: $\{n_1, \ldots, n_{37}\}$.

   - Call $\mathcal{F}_i$ the subset of $\mathcal{F}$ consisting of all the facets with normals in direction $n_i$.

   - Each facet lies on a unique hyperplane.
Proof/Algorithm

1. Let \( \mathcal{F} \) be the set of all facets of the top dimensional cones \( G_1, \ldots, G_{612} \) of \( \mathcal{G} \).

2. Group the facets in \( \mathcal{F} \) according to their normals. 37 normal directions: \( \{n_1, \ldots, n_{37}\} \).
   - Call \( \mathcal{F}_i \) the subset of \( \mathcal{F} \) consisting of all the facets with normals in direction \( n_i \).
   - Each facet lies on a unique hyperplane.
   - These hyperplanes go through the origin.
Proof/Algorithm

1. Let $\mathcal{F}$ be the set of all facets of the top dimensional cones $G_1, \ldots, G_{612}$ of $G$.

2. Group the facets in $\mathcal{F}$ according to their normals. 37 normal directions: $\{n_1, \ldots, n_{37}\}$.

   Call $\mathcal{F}_i$ the subset of $\mathcal{F}$ consisting of all the facets with normals in direction $n_i$.

   Each facet lies on a unique hyperplane.

   These hyperplanes go through the origin.

   So two facets lie on the same hyperplane if and only if they have the same normals up to scalar multiple.
The facets in $F_i$, when intersected with $L(\lambda)$ will therefore give all the facets of domains lying on a given wall.
The facets in $\mathcal{F}_i$, when intersected with $L(\lambda)$ will therefore give all the facets of domains lying on a given wall.

3. We glue these facets together by setting for each $i$

$$K_i = \bigcup_{F \in \mathcal{F}_i} F.$$  

We verify that $K_i$ is again a convex polyhedral cone by a truncation and volume comparison method (as before).
The intersections of the $K_i$ with $L(\lambda)$ will be the walls partitioning the permutahedron.
Proof/Algorithm

The intersections of the $K_i$ with $L(\lambda)$ will be the walls partitioning the permutahedron.

The facets of $K_i$, when intersected with $L(\lambda)$, will correspond to the edges of the walls generically (or be empty).
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The facets of $K_i$, when intersected with $L(\lambda)$, will correspond to the edges of the walls generically (or be empty).

The facets of those facets will correspond to the vertices of the walls generically (or be empty).
4. For each $i$, let $V_i$ be the set of facets of facets of $K_i$. 
4. For each $i$, let $\mathcal{V}_i$ be the set of facets of facets of $K_i$.

5. For each $i$, identify the $f \in \mathcal{V}_i$ whose intersection with $L(\lambda)$ (for generic $\lambda$) is a point and find that point. [Explanations later]
4. For each $i$, let $V_i$ be the set of facets of facets of $K_i$.

5. For each $i$, identify the $f \in V_i$ whose intersection with $L(\lambda)$ (for generic $\lambda$) is a point and find that point. [Explanations later]

6. For each $i$, verify that the vertices thus found define the same wall as the one of the Duistermaat-Heckman partition.
Step 5 explained

- Things would break down in Step 5 if not for a remarkable fact.

- The subset \( \mathcal{W} \) of the hyperplane
  \[ x_1 + x_2 + x_3 + x_4 = 0 \]
  where \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \)
  is a simplicial cone.

- Denote by \( \{\omega_1, \omega_2, \omega_3\} \) its basis (rays), so that
  \[ \mathcal{W} = \text{pos}(\omega_1, \omega_2, \omega_3). \]

- It turns out that all the \( f \in \mathcal{V}_i \) have very nice expressions in terms of the \( \omega_j \).
Example

\[
\begin{align*}
 f_1 &= \text{pos}( (\omega_1, \omega_1), (\omega_2, \pi \cdot \omega_2), (\omega_3, \omega_3)) \\
 f_2 &= \text{pos}( (\omega_1, \omega_1), (\omega_2, \pi \cdot \omega_2), (\omega_3, \phi \cdot \omega_3)) \\
 f_3 &= \text{pos}( (\omega_1, \sigma \cdot \omega_1), (\omega_2, \pi \cdot \omega_2), (\omega_3, \phi \cdot \omega_3)) \\
 f_4 &= \text{pos}( (\omega_1, \sigma \cdot \omega_1), (\omega_2, \pi \cdot \omega_2), (\omega_3, \omega_3)) \\
 f_5 &= \text{pos}( (\omega_1, \sigma \cdot \omega_1), (\omega_3, \phi \cdot \omega_3), (\omega_3, \omega_3)) \\
 f_6 &= \text{pos}( (\omega_2, \pi \cdot \omega_2), (\omega_3, \phi \cdot \omega_3), (\omega_3, \omega_3)) \\
 f_7 &= \text{pos}( (\omega_1, \sigma \cdot \omega_1), (\omega_1, \omega_1), (\omega_3, \phi \cdot \omega_3)) \\
 f_8 &= \text{pos}( (\omega_1, \omega_1), (\omega_3, \phi \cdot \omega_3), (\omega_3, \omega_3)) \\
 f_9 &= \text{pos}( (\omega_1, \sigma \cdot \omega_1), (\omega_1, \omega_1), (\omega_3, \omega_3)) \\
 f_{10} &= \text{pos}( (\omega_1, \sigma \cdot \omega_1), (\omega_1, \omega_1), (\omega_2, \pi \cdot \omega_2))
\end{align*}
\]

where \( \sigma = (1 \ 3), \pi = (2 \ 3), \phi = (2 \ 4). \)
Example

\[ p_\Lambda(f_1) = \text{pos}(\omega_1, \omega_2, \omega_3) \quad p_\Lambda(f_5) = \text{pos}(\omega_1, \omega_3) \]
\[ p_\Lambda(f_2) = \text{pos}(\omega_1, \omega_2, \omega_3) \quad p_\Lambda(f_6) = \text{pos}(\omega_2, \omega_3) \]
\[ p_\Lambda(f_3) = \text{pos}(\omega_1, \omega_2, \omega_3) \quad p_\Lambda(f_7) = \text{pos}(\omega_1, \omega_3) \]
\[ p_\Lambda(f_4) = \text{pos}(\omega_1, \omega_2, \omega_3) \quad p_\Lambda(f_8) = \text{pos}(\omega_1, \omega_3) \]
\[ p_\Lambda(f_9) = \text{pos}(\omega_1, \omega_3) \quad p_\Lambda(f_{10}) = \text{pos}(\omega_1, \omega_2) \]

- Only the first four projected cones span \( \mathcal{W} \). The others will miss a generic \( \lambda \) (they won’t intersect \( L(\lambda) \)).
Example

We can rewrite the cone $f_j$ so that the $\beta$ coordinates of its rays are always the results of applying the same permutation to their $\lambda$ coordinates.

$$f_1 = \text{pos}((\omega_1, (2 \ 3) \cdot \omega_1), (\omega_2, (2 \ 3) \cdot \omega_2), (\omega_3, (2 \ 3) \cdot \omega_3))$$

$$f_2 = \text{pos}((\omega_1, (2 \ 4 \ 3) \cdot \omega_1), (\omega_2, (2 \ 4 \ 3) \cdot \omega_2), (\omega_3, (2 \ 4 \ 3) \cdot \omega_3))$$

$$f_3 = \text{pos}((\omega_1, (1 \ 2 \ 4 \ 3) \cdot \omega_1), (\omega_2, (1 \ 2 \ 4 \ 3) \cdot \omega_2), (\omega_3, (1 \ 2 \ 4 \ 3) \cdot \omega_3))$$

$$f_4 = \text{pos}((\omega_1, (1 \ 2 \ 3) \cdot \omega_1), (\omega_2, (1 \ 2 \ 3) \cdot \omega_2), (\omega_3, (1 \ 2 \ 3) \cdot \omega_3))$$
Example

\[ f_1 \cap L(\lambda) = (\lambda, (2\ 3) \cdot \lambda) \]
\[ f_2 \cap L(\lambda) = (\lambda, (2\ 4\ 3) \cdot \lambda) \]
\[ f_3 \cap L(\lambda) = (\lambda, (1\ 2\ 4\ 3) \cdot \lambda) \]
\[ f_4 \cap L(\lambda) = (\lambda, (1\ 2\ 3) \cdot \lambda) \]

This means there will be a wall with vertices

\[ (2\ 3) \cdot \lambda = (\lambda_1, \lambda_3, \lambda_2, \lambda_4) = \lambda' \]
\[ (1\ 2\ 3) \cdot \lambda = (\lambda_3, \lambda_1, \lambda_2, \lambda_4) = (1\ 2)\lambda' \]
\[ (2\ 4\ 3) \cdot \lambda = (\lambda_1, \lambda_3, \lambda_4, \lambda_2) = (3\ 4)\lambda' \]
\[ (1\ 2\ 4\ 3) \cdot \lambda = (\lambda_3, \lambda_1, \lambda_4, \lambda_2) = (1\ 2)(3\ 4)\lambda' \]
Open problems
Open problems

- What are the regions counts for higher dimensional permutahedra? (15230 regions for $\lambda = (2, 1, 0, -1, -2)$)

- Do the partitions for the DH function and weight multiplicities keep coinciding in higher dimension?

- What about the permutahedra for other groups?

- Are there fast ways to compute weight multiplicities (Kostka numbers)?
Permutahedron for $B_2$
Permutahedron for $B_3$
Permutahedron for $G_2$