Morse Theory for Implicit Surface Modeling

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Abstract. Morse theory describes the relationship between a function's critical points and the homotopy type of the function's domain. The theorems of Morse theory were developed specifically for functions on a manifold. This work adapts these theorems for use with parameterized families of implicit surfaces in computer graphics. The result is a theoretical basis for the determination of the global topology of an implicit surface, and supports the interactive modeling of implicit surfaces by direct manipulation of a topologically-correct triangulated representation.

1 Introduction

Implicit surfaces provide a powerful and versatile shape model in computer graphics by representing geometry as the zero-set of a function over three-space, although displaying such surfaces requires a search through space. The display of an implicit surface is hastened by maintaining a triangulation that can be quickly rendered on modern graphics workstations. However, when the implicit surface changes topological type, the triangulation needs to be updated in the neighborhood of the topology change. A recent technique uses the critical points of the function to detect changes in topology and reconfigures the triangulation to correctly reflect the topology of the new surface [10,11].

The fundamental detail missing from these publications is the connection between a function's critical points and the topology of its implicit surface. This connection can be found in Morse theory, but the theorems of Morse theory do not directly apply to the implicit surfaces used in computer graphics. This paper formalizes this connection with obvious but not entirely trivial extensions of theorems from Morse theory to implicit surface topology.

Section 2 summarizes the implicit surface geometric representation and techniques for modeling with implicit surfaces. Section 3 reviews Morse theory, focusing on the connection between critical points and homotopy type. Section 4 applies the results of Morse theory to implicit surfaces. Section 5 concludes with remarks on further applications of Morse theory in computer graphics.

2 The Problem of Modeling with Implicit Surfaces

An implicit surface is defined as the zero-set of a function $f : \mathbb{R}^3 \to \mathbb{R}$. The implicit surface is often a compact manifold, though not always smooth [7], compact (e.g. the cylinder $f(x, y, z) = x^2 + y^2 - 1$), nor even a manifold [3].

Natural geometric primitives, such as the plane, sphere, cylinder, cone and torus, can be described implicitly as the solutions to linear, quadratic and quartic polynomials. These primitives are commonly treated as solids (3-manifolds-with-boundary) by considering the points where the function is negative (or positive) to be in the interior the set. These solid primitives are combined with binary set operations (union, intersection and difference) to form more complex shapes in a procedure known in computer graphics as *constructive solid geometry* (CSG). Implicit surfaces also facilitate the joining of surfaces with a process called *blending* which smoothes the results of a CSG operation.

Perhaps the most popular blending technique in computer graphics is the *blobby* model [1]. The blobby model represents shapes with implicit surfaces defined by functions of the form

$$f(\mathbf{x}) = T - \sum_{i=1}^{N} e^{-k_i F_i(\mathbf{x})} \tag{1}$$

where the functions $F_i : \mathbb{R}^3 \rightarrow \mathbb{R}$ implicitly define primitive shapes, the k_i are parameters controlling the strength of the primitives and T is a threshold value. The primitive shapes are often quadric spheres

$$F_i(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_i) \cdot (\mathbf{x} - \mathbf{x}_i) \tag{2}$$

centered about so-called *key points* $\mathbf{x}_i$. The implicit surface is the boundary of a solid, and the function f is negative in this solid. As might be clear from (1) and (2), the blobby model originated as a method for visualizing electron densities in molecules with nuclei at $\mathbf{x}_i$, but has matured into a geometric representation capable of synthesizing a variety of natural and man-made forms [2]. Moreover, in addition to points, other primitives such as lines, polygons, curves and patches [4] can be collected together to form a *skeleton*. The primitives composing this skeleton may be thickened (using a suitable function F_i) into implicit surfaces which may then be blended (using a suitable function f) into a single smooth implicit surface.

For example in Figure 1, the shape on the left is composed of the CSG union of eight spheres whereas the shape on the right is composed of the same eight spheres joined with (1).

While implicit surfaces serve as a powerful shape representation in computer graphics, they are not well suited for interactive modeling. The main impediment is rendering. Whereas other shape descriptions such as the parametric surface yield a surface as the range of a function, an implicit surface must be found in a given region of space. The increased computation required to find the implicit surface makes displaying them at interactive rates difficult.

A rendering method called *ray tracing* displays shapes by following each ray of light backwards from the eye, through each pixel and into the scene.

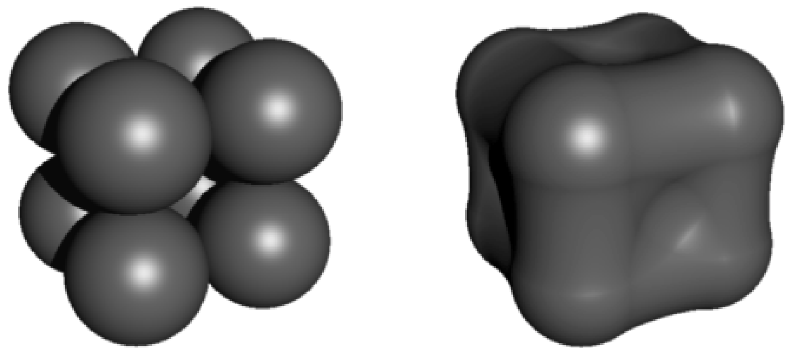

Fig. 1. A ray-traced implicit surface composed of the union of eight spheres (left) and the blended union of eight spheres (right).

Implicit surfaces are well suited for ray tracing. Let $l : \mathbb{R}^+ \to \mathbb{R}^3$ be the parametric definition of a ray. Then the intersection of the ray with the implicit surface of $f(\mathbf{x})$ is determined by finding the zeros of the real function of one variable $f \circ l$. The images in Figure 1 were rendered using such a ray tracing algorithm developed specifically for mathematical visualization [6].

In order to design a blobby model interactively, the implicit surface needs to be displayed in real time. Even with the power of modern graphics workstations, ray tracing remains too costly for interactive applications. Instead, recent techniques visualize the implicit surface in real time by maintaining a simplified approximation. For example, an implicit surface can be interactively manipulated using an efficient visual representation consisting of a system of mutually-repelling particles constrained to the surface, displayed as a collection of disks tangent to the surface [12]. As the surface changes shape due to user interaction, the disks maintain their position on the surface. Figure 2 (left) demonstrates this method of display.

Connecting these particles triangulates the implicit surface, as shown in Figure 2 (right). As the implicit surface changes, the vertices remain on the surface and the triangulation remains intact. However, when the implicit surface changes topological type, the triangulation is no longer a valid representation of the implicit surface. Whereas the particles require only the local tangent information to indicate the surface, the triangulated representation must be aware of any portions of the surface that are newly joined or separated. Morse theory provides the tools necessary to make such a determination.

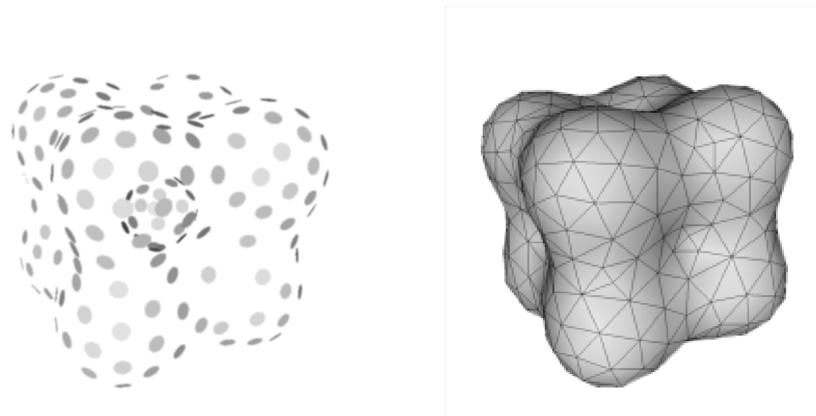

Fig. 2. The blobby cube displayed using a particle system (left) and triangulated (right). Note that the blobby cube is hollow and the particle system rendering reveals the air bubble.

3 Morse Theory

This section reviews elementary Morse theory [8], specifically the classification of critical points of a function on a manifold, and the effect of these critical points on the homotopy type of the manifold. The function is commonly smooth, but Morse theory can be applied to functions of varying smoothness, even piecewise linear. The following development of definitions and theorems require only C^2 (second-derivative) continuity which broadens the variety of implicit surfaces accessible by the theorems. The section relies on some prior knowledge of homotopy theory [9].

Definition 1. Let f be a C^2 real map on a manifold M. A point $p \in M$ is a *critical point* iff its derivatives with respect to a local coordinate system on M vanish.

More specifically, since M is an n-manifold, then there exists a C^2 one-to-one correspondence g between a neighborhood about any point $p \in M$ and an open neighborhood of the origin in $\mathbb{R}^n$ such that $g(p) = \mathbf{x} = (x_1, x_2, \ldots, x_n)$. Then the point $p \in M$ is a critical point with respect to f if the gradient

$$\nabla f = \left(\frac{\partial f \circ g^{-1}(\mathbf{x})}{\partial x_1}, \frac{\partial f \circ g^{-1}(\mathbf{x})}{\partial x_2}, \ldots \frac{\partial f \circ g^{-1}(\mathbf{x})}{\partial x_n} \right) = \mathbf{0}. \tag{3}$$

Morse theory focuses only on non-degenerate critical points. Such points, also called Morse points, are critical points where the *Hessian*

$$V(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix} \tag{4}$$

has non-zero determinant. Since $\partial^2 f/\partial x_i \partial x_j = \partial^2 f/\partial x_j \partial x_i$, the matrix $V(f)$ is symmetric with real eigenvalues. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of $V(f)$. If any of the eigenvalues is zero, then the critical point is *degenerate.* Otherwise it is called *non-degenerate.* The *index* of the critical point is the number of negative eigenvalues of $V(f)$.

The Morse Lemma states that the neighborhood about a non-degenerate critical point can be deformed into the neighborhood of the non-degenerate critical point of a quadratic function.

Lemma 2. (Morse Lemma) *Let p be a non-degenerate critical point of f with index λ, and let $c = f(p)$. Then there exists a local coordinate system $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in a neighborhood U of p with p as its origin and*

$$f(\mathbf{y}) = c - y_1^2 - y_2^2 - \cdots - y_\lambda^2 + y_{\lambda+1}^2 + \cdots + y_n^2. \tag{5}$$

Morse theory focuses on determining the homotopy type of a shape based on its critical points. A classic example [5] demonstrates the effects of critical points on homotopy type by observing the portion of a torus below a clipping plane, as the clipping plane moves through the torus. One can observe these same changes by dunking a doughnut into a cup of coffee, as shown in Figure 3.

For this example, let M denote the surface of a vertically-oriented torus and let $f(p)$ return the height of point $p \in M$. Assume the bottom of the torus is at height zero and the top is of height one. In general the notation M^a indicates the points $p \in M$ such that $f(p) \leq a$, in this case the portion of the torus up to a height of a.

As the clipping plane traverses up the torus, Figure 4 shows that the changes in the topology of the torus can be described by attaching the appropriate k-cell to the truncated surface. Notice that the dimension of the attached cell equals the index of the critical point passed by the clipping plane.

The following theorem shows that M^a is topologically similar to $M^b \supset M^a$ if there is no critical point in M^a that is not also in M^b.

Theorem 3. [8] *Let $f : M \to \mathbb{R}$ be C^2, let $a < b$ and suppose that the set $f^{-1}[a, b]$ is compact and contains no critical points of f. Then M^a is homeomorphic to M^b.*

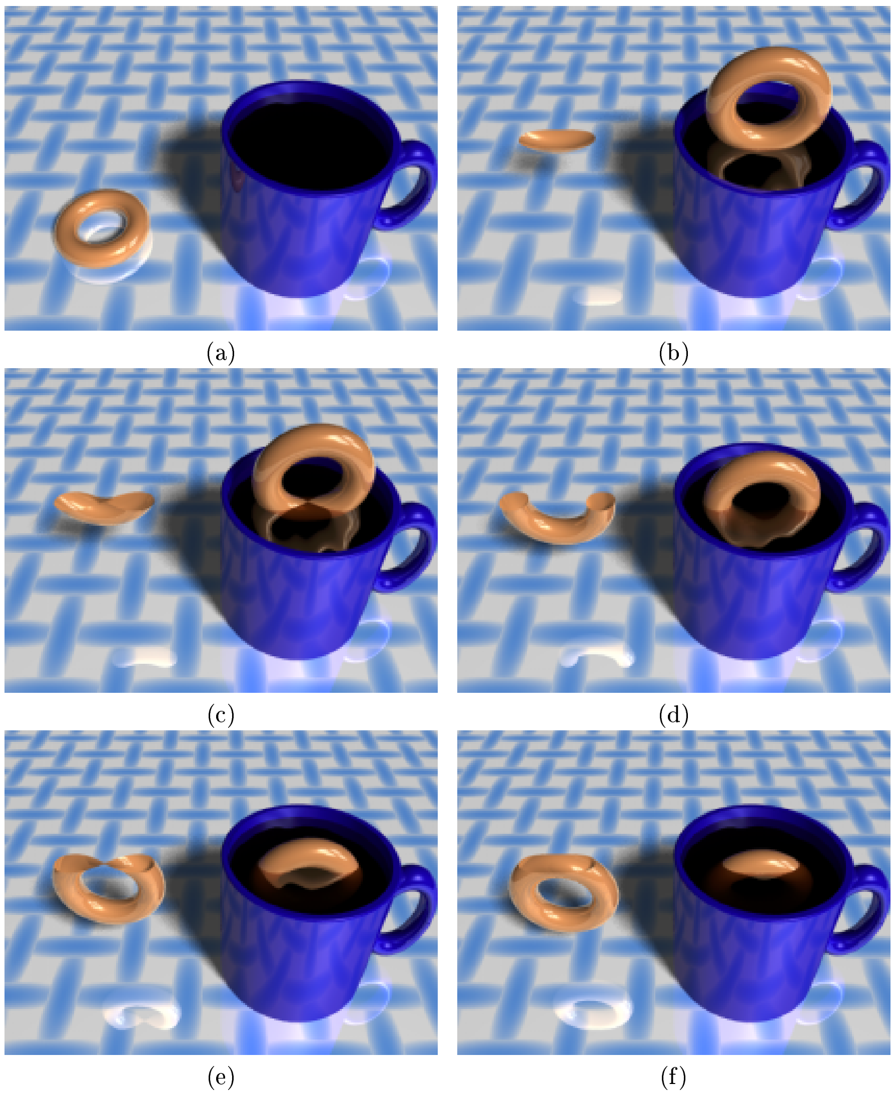

Fig. 3. Dunking a doughnut. A shiny doughnut and a cup of coffee (a). The dunked portion of the doughnut's surface changes from the empty set to a shape homeomorphic to a disk (b). The dunked portion changes (c) from a disk to a truncated cylinder (d). The dunked portion changes (e) from a cylinder to a truncated torus (f).

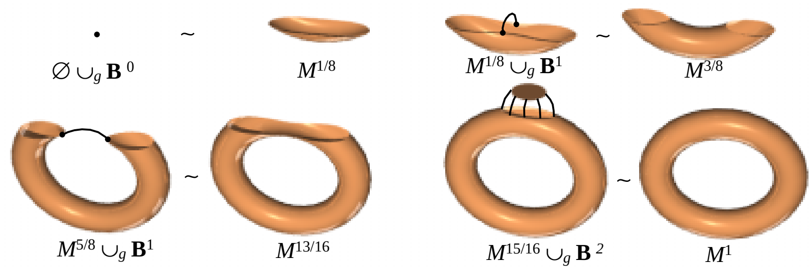

Fig. 4. Homotopy classes of the clipped torus.

Proof. Let the family of continuous maps $\phi_t : M \to M$ be defined as the solution of the ordinary differential equation

$$\dot{\phi}_t(p) = \frac{\nabla f(\phi_t(p))}{||\nabla f(\phi_t(p))||^2} \tag{6}$$

(where $\dot{\phi} = d\phi/dt$) with the initial value $\phi_0(p) = p$ on $f^{-1}[a,b]$, and let ϕ continuously go to the identity ($\dot{\phi}_t(p) = \mathbf{0}$) outside a compact neighborhood of $f^{-1}[a,b]$ not containing any critical points, such that each map ϕ_t is bijective and continuous with continuous inverse.

The function value of f on the curve $\phi_t(p)$ generates on M fixing p and varying t changes at the same rate as t changes, since the directional derivative

$$\frac{df(\phi_t(p))}{dt} = \dot{\phi}_t(p) \cdot \nabla f(\phi_t(p)) = 1. \tag{7}$$

Hence the homeomorphism ϕ_{b-a} carries M^a onto M^b. □

Theorem 4. [8] *Let $f : M \to \mathbb{R}$ be C^2, and let $p \in M$ be a non-degenerate critical point with index λ. Setting $f(p) = c$, suppose that $f^{-1}[c-\epsilon, c+\epsilon]$ is compact, and contains no critical point of f other than p, for some $\epsilon > 0$. Then, for all sufficiently small ϵ, the set $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a λ-cell attached.*

Elements of the proof of this theorem will be needed to prove a later proposition. The following is a brief summary of a classic proof [8], which should be consulted for details.

Proof. Using Morse's Lemma, choose a coordinate system $u_1, \ldots, u_n$ in a neighborhood U of p such that

$$f(p) = c - u_1(p)^2 - \ldots - u_\lambda(p)^2 + u_{\lambda+1}(p)^2 + \ldots + u_n(p)^2. \tag{8}$$

We abbreviate $\zeta = u_1^2 + \ldots + u_\lambda^2$ and $\eta = u_{\lambda+1}^2 + \ldots + u_n^2$ such that $f(p) = c - \zeta + \eta$.

Let $\epsilon > 0$ be sufficiently small such that $f^{-1}[c-\epsilon, c+\epsilon]$ is compact and contains no other critical points other than p, and U contains a ball of radius 2ϵ.

The function $\mu : \mathbb{R} \to \mathbb{R}$ (any differentiable function such that $\mu(0) > \epsilon, \mu(r) = 0$ for $r \geq 2\epsilon$, and $-1 < \mu'(r) \leq 0$) locally warps the effects of the function on the manifold as $F(p) = f(p) - \mu(\zeta(p) + 2\eta(p))$.

Given the definition of the new function F, the following four assertions follow and suffice to prove the theorem. Proof of each of the assertions can be found in [8].

Assertion 1. $F^{-1}(-\infty, c+\epsilon] = M^{c+\epsilon}$.

Assertion 2. F shares the same critical points as f.

Assertion 3. $F^{-1}(-\infty, c-\epsilon] \cong M^{c+\epsilon}$.

Let $e^\lambda \subset M$ be the λ-cell

$$e^\lambda = \{(u_1, \ldots, u_n) : u_1^2 + \cdots + u_\lambda^2 \leq \epsilon, u_{\lambda+1}^2 + \cdots + u_n^2 = 0\}. \tag{9}$$

Denote $\mathcal{H} = \text{closure}(F^{-1}(-\infty, c-\epsilon] - M^{c-\epsilon})$. Note that $e^\lambda \subset \mathcal{H}$.

Assertion 4. $M^{c-\epsilon} \cup e^\lambda$ is a deformation retract of $M^{c-\epsilon} \cup \mathcal{H}$. □

4 Application to Implicit Surfaces

The following proposition is a first step at applying the theorems from the previous section to implicit surfaces. It essentially states that two isosurfaces of the same function are topologically similar if there is no critical point in any isosurface between them.

Proposition 5. *Let $f : M \to \mathbb{R}$ be C^2, and such that $f^{-1}[a,b]$ is compact and contains no critical points. Then $f^{-1}(a) \cong f^{-1}(b)$.*

Proof. From Theorem 3 we have that $M^a \cong M^b$. The boundary of M^a (w.r.t. M) is $f^{-1}(a)$ and likewise $\partial M^b = f^{-1}(b)$. The boundaries of two homeomorphic sets must themselves be homeomorphic. □

Proposition 5 can be applied to implicit surfaces, but must be restricted to non-intersecting implicit surfaces such that one implicit surface completely surrounds the other.

In order to show a homeomorphism between two implicit surfaces in general, we must define a family of implicit surfaces and define a height function on this family. Then the properties of the manifold due to the height function will also apply to the family of implicit surfaces.

Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ define a family of functions $f(\mathbf{x};\mathbf{q})$ parameterized by an m-vector $\mathbf{q}$ that defines a family of implicit surfaces as the collection of $(n-1)$-manifolds $f_{\mathbf{q}}^{-1}(0) = \{\mathbf{x} : f(\mathbf{x};\mathbf{q}) = 0\}$. Note that the domain of the instance $f_{\mathbf{q}} : \mathbb{R}^n \to \mathbb{R}$ differs from the domain of the family f. (The latter includes the parameter space.)

Consider two $(n-1)$-manifolds $M_0 = f_{\mathbf{q}_0}^{-1}(0)$ and $M_1 = f_{\mathbf{q}_1}^{-1}(0)$. Let $\mathbf{q}(t), t \in \mathbb{R}$, denote a linear interpolation of parameters such that $\mathbf{q}(0) = \mathbf{q}_0$ and $\mathbf{q}(1) = \mathbf{q}_1$. Let $\mathbf{q} : \mathbb{R} \to \mathbb{R}^m$ be parameterized such that $M \in \mathbb{R}^n \times \mathbb{R}^m = \{(\mathbf{x}, \mathbf{q}(t)) : f(\mathbf{x};\mathbf{q}(t)) = 0, t \in \mathbb{R}\}$ is an n-manifold. Define the height map $h : M \to \mathbb{R}$ as $h(\mathbf{x}, \mathbf{q}) = t$.

Proposition 6. *Let $p = (\mathbf{x}_p, \mathbf{q}_p) \in M$. Then p is a critical point of h ($\nabla h(p) = \mathbf{0}$) if and only if $\nabla f_{\mathbf{q}_p}(\mathbf{x}_p) = (\partial f_{\mathbf{q}}/\mathbf{x}_1, \ldots, \partial f_{\mathbf{q}}/\mathbf{x}_n) = \mathbf{0}$.*

Proof. If p is a critical point of h, then it's value ($= t$) is locally constant along M and vice versa. Hence M is locally perpendicular to the t axis and orthogonal to the $\mathbf{q}$ hyperplane. The $\mathbf{x}$ coordinate system serves as a local coordinate system for M at p. $\square$

Proposition 5 combines with this family of implicit surfaces to assert the following proposition that implicit surfaces do not change homotopy type if they do not intersect a critical point.

Proposition 7. *If the family of implicit surfaces $f_{\mathbf{q}(t)}^{-1}(0)$ is compact for every $t \in [t_0, t_1]$ and none contain a point $\mathbf{x}$ such that $\nabla f_{\mathbf{q}(t)}(\mathbf{x}) = 0$, then $f_{\mathbf{q}_0}^{-1}(0)$ is homeomorphic to $f_{\mathbf{q}_1}^{-1}(0)$.*

Proof. The surfaces $h^{-1}(t_0) = f_{\mathbf{q}_0}^{-1}(0)$ and $h^{-1}(t_1) = f_{\mathbf{q}_1}^{-1}(0)$. Proposition 6 asserts there are no critical values on M between t_0 and t_1, which allows Proposition 5 to show $h^{-1}(t_0) \cong h^{-1}(t_1)$. $\square$

Proposition 8. *Let $\mathbf{x}_p$ be a non-degenerate critical point with index λ of $f_{\mathbf{q}(t_p)}$. If there exists some $\epsilon > 0$ such that the set $\{\mathbf{x} : f(\mathbf{x};\mathbf{q}(t)) = 0, t \in [t_p - \epsilon, t_p + \epsilon]\}$ is compact and contains no other critical points than $(\mathbf{x}_p, \mathbf{q}_p)$, and assuming without loss of generality that $\partial f(\mathbf{x}_p;\mathbf{q}(t_p))/\partial t < 0$, then the n-manifold-with-boundary $f_{\mathbf{q}(t_p+\epsilon)}^{-1}(-\infty, 0]$ has the same homotopy type as $f_{\mathbf{q}(t_p-\epsilon)}^{-1}(-\infty, 0]$ with a λ-cell attached.*

The following proof follows the same logic as the proof of Theorem 4 but also uses a projection to show that the regions bounded by homeomorphic sets are also homeomorphic.

Proof. Following the proof of Theorem 4, choose a coordinate system such that $h = -\zeta + \eta$ in a neighborhood $U \subset M$ of p. Let $H = -\zeta + \eta - \mu(\zeta + 2\eta)$ inside U and $H = h$ outside U. As before, H has the same critical points as h, and the manifold-with-boundary $H^{-1}(-\infty, t_p + \epsilon] = h^{-1}(-\infty, t_p + \epsilon]$, but the critical point $(\mathbf{x}_p, \mathbf{q}_p)$ is in $H^{-1}(-\infty, t_0 - \epsilon]$. Since there is no critical point in $H^{-1}[t_p - \epsilon, t_p + \epsilon]$, we have $h^{-1}(t_p + \epsilon) \cong H^{-1}(t_p - \epsilon)$ by Proposition 5.

Let $\pi : M \to \mathbb{R}^n$ be the projection $(\mathbf{x}, \mathbf{q}) \mapsto \mathbf{x}$. Proposition 6 shows us that near p, the manifold M is orthogonal to the $\mathbf{q}$ hyperplane, so ϵ can be set small enough such that the projection π is one-to-one in the neighborhood U.

Recalling the map ϕ from the proof of Theorem 3, we have

$$\begin{aligned} H^{-1}(t_p - \epsilon) &\cong H^{-1}(t_p + \epsilon), && (10)\\ &= h^{-1}(t_p + \epsilon), && (11)\\ &\cong \pi \circ h^{-1}(t_p + \epsilon), && (12)\\ &= f^{-1}_{\mathbf{q}(t_p+\epsilon)}(0). && (13) \end{aligned}$$

Hence the homeomorphism $\pi \circ \phi_{2\epsilon}$ maps $H^{-1}(t_p - \epsilon)$ to $f^{-1}_{\mathbf{q}(t_p+\epsilon)}(0)$. The latter implicit surface is the boundary of the implicit solid $f^{-1}_{\mathbf{q}(t_p+\epsilon)}(-\infty, 0]$. This region is mapped via the homeomorphism $\phi_{-2\epsilon} \circ \pi^{-1} : \mathbb{R}^n \to M$ into a subset of M with $H^{-1}(t_p - \epsilon)$ as its boundary.

As before, the handle

$$\mathcal{H} = \text{closure}(H^{-1}(t_p - \epsilon) - h^{-1}(t_p - \epsilon)) \tag{14}$$

is the subset that creates the change in homotopy type, and this handle is homotopic to a λ-cell. Both the handle and the boundary of the λ-cell extend to $h^{-1}(t_p - \epsilon)$ and hence their projections extend to $f^{-1}_{\mathbf{q}(t_p-\epsilon)}(-\infty, 0]$. □

The disconnection direction $(\partial f(\mathbf{x}_0, q_0)/\partial q > 0)$ is not defined since there is no mechanism available to us to "remove a λ-cell." Instead, we must invert the t parameter about the critical point to treat the problem in the connection direction, or consider the closure of the complement of the implicit solid and attach an $(n - \lambda)$-cell.

These propositions allow us to classify changes in the topological type of implicit surfaces. The eight possible topological-type changes are listed in Table 1.

	Critical value	
Index	$+ \to -$	$- \to +$
0	Create	Destroy
1	Connect	Cut
2	Spackle	Pierce
3	Burst	Bubble

Table 1. The eight possible homotopy equivalence class changes in 3-D at a non-degenerate critical point.

When a minimum value becomes negative, a new implicit surface component is created. This can be considered attaching a 0-cell to the empty set. When a minimum value becomes positive, the component is destroyed.

When an index 1 critical value becomes negative, a new connection is formed between two components. In terms of homotopy type, a 1-cell has been attached to the two solid components. When an index 1 critical value becomes positive, a connection is cut. These cases are shown in Figure 5.

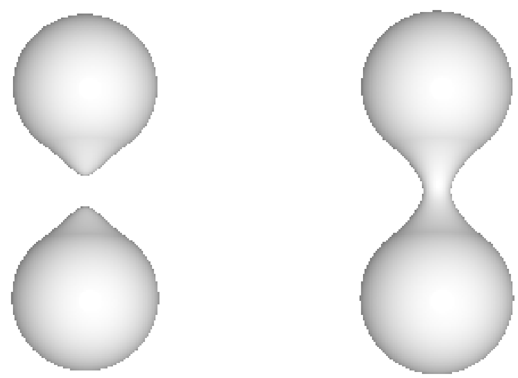

Fig. 5. An index 1 critical point with critical value positive (left) and negative (right).

When an index 2 critical value becomes negative, a hole in the solid is filled in. In terms of homotopy type, a 2-cell has been attached. When an index 2 critical value becomes negative, a new hole is pierced in the solid. These cases are shown in Figure 6.

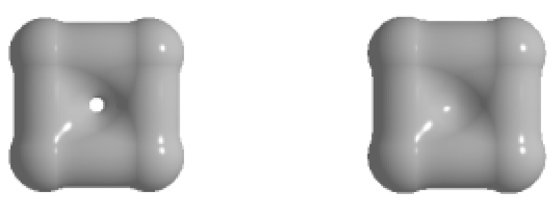

Fig. 6. An index 2 critical point with critical value positive (left) and negative (right).

When a maximum value becomes positive, a hollow region is formed in a solid. Such an air bubble can be seen in the particle system rendering of the blobby cube in Figure 2. When a minimum critical value becomes negative, the bubble bursts. In terms of homotopy type, a 3-cell has been attached in place of the air bubble.

When the implicit solid changes homotopy type, simple algorithms exist to locally reconfigure the triangulation to reflect the new topology [11].

The only remaining problem for maintaining a triangulated version of a dynamic implicit surface is tracking all of the critical points of the function. Several techniques have been explored [11]. The most effective technique performs an interval Newton's method search across a given region of space over a given time interval for critical points that intersect the implicit surface. Such search methods based on interval analysis can be guaranteed not to miss any solutions, resulting in a guarantee that the triangulation is homotopy equivalent to the implicit surface it represents.

5 Conclusion

This document serves to provide a theoretical basis for the alteration of implicit surface topology in the presence of critical points. It ultimately shows that the topological type of an implicit surface before and after a non-degenerate critical value changes sign can be described through the attachment of an appropriate-dimension cell.

Morse theory might also add insight to current problems in shape transformation. The determination that the initial and final shapes share the same topological type would be the first step toward finding a possible topological-type preserving shape transformation. Likewise, the characterization of neighborhoods of critical points occurring during shape transformation may provide new insight into the maintenance of consistent texture coordinates through changes in homotopy type.

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