Math 603, Spring 2003, HW 5, due 3/31/2003

Part A

AI) Write A for an integral domain, $K = \operatorname{Frac} A$, and set $\widetilde{A} = \operatorname{Int}_K(A)$. The domain $\widetilde{A}$ is called the *normalization* of A. Now set
$$\mathcal{F} = (\widetilde{A} \to A) = \{\xi \in A \mid \xi\widetilde{A} \subseteq A\}.$$
Of course, $\mathcal{F}$ is an ideal of A, called the *conductor* of A in $\widetilde{A}$ (German: Führer). Check that $\mathcal{F}$ is also an ideal of $\widetilde{A}$.

(a) If S is a multiplicative subset in A, show $S^{-1}\widetilde{A} = \operatorname{Int}_K(S^{-1}A)$. Prove further, $S^{-1}A$ is normal if $\mathcal{F} \cap S \neq \emptyset$.

(b) Assume $\widetilde{A}$ is a finitely generated A-module (frequently the case). Show that the conductor of $S^{-1}A$ in $S^{-1}\widetilde{A}$ is the extended ideal $\mathcal{F}^e$. Show also in this case $S^{-1}A$ is normal if and only if $\mathcal{F} \cap S \neq \emptyset$.

(c) If $\widetilde{A}$ is a finitely generated A-module, then
$$\{\mathfrak{p} \in \operatorname{Spec} A \mid A_{\mathfrak{p}} \text{ is normal}\}$$
is open in $\operatorname{Spec} A$, in fact it is a dense open of $\operatorname{Spec} A$.

AII) A *discrete valuation*, ν, on a (commutative) ring A, is a function $\nu : A \to \mathbb{Z} \cup \{\infty\}$ satisfying

(a) $\nu(xy) = \nu(x) + \nu(y)$

(b) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$

(c) $\nu(x) = \infty \iff x = 0$.

A pair (A, ν) where A a commutative ring and ν is a discrete valuation is called a *discrete valuation ring* (DVR). Prove the following are equivalent:

(a) A is a DVR

(b) A is a local PID

(c) A is a local, noetherian, normal domain of Krull dimension 1

(d) A is a local, noetherian, normal domain and $(\mathfrak{m}_A \to a)\big(= \{\xi \in \operatorname{Frac} A \mid \xi\mathfrak{m}_A \subseteq A\}\big) \neq A$. Here, $\mathfrak{m}_A$ is the maximal ideal of A.

AIII) Let A be a commutative ring with unity and assume A is semi-local (it possesses just finitely many maximal ideals). Write $\mathcal{J}$ for the Jacobson radical of A and give A its $\mathcal{J}$-adic topology.

(a) Prove that A is noetherian iff each maximal ideal of A is finitely generated and each ideal is closed in the $\mathcal{J}$-adic topology.

(b) Assume A is noetherian, then the map $A \to A_{\mathrm{rad}}$ gives A_{rad} its $\mathcal{J}$-adic topology. If A_{rad} is complete prove that A is complete.

AIV) (a) Let A be a local ring, give A its $\mathfrak{m}$-adic topology ($\mathfrak{m} = \mathfrak{m}_A$ is the maximal ieal of A) and assume A is complete. Given an A-algebra, B, suppose B is finitely generated as an A-module. Prove that B is a finite product of A-algebras each of which is a local ring. Give an example to show that some hypothesis like completness is necessary for the conclusion to be valid.

(b) Again A is complete and local, assume $f(X) \in A[X]$ is a monic polynomial. Write $\overline{f}(X)$ for the image of f in $(A/\mathfrak{m})[X]$. If $\overline{f}(X)$ factors as $g(X)h(X)$ where g and h are relatively prime in $(A/\mathfrak{m})[X]$, show that f factors as $G(X)H(X)$ where $\overline{G}(X) = g(X)$; $\overline{H}(X) = h(X)$. What can you say about $\deg G$, $\deg H$ and uniqueness of this factorization? Compare parts (a) and (b).

Part B

BI) In this problem, A is an integral domain and $k = \operatorname{Frac} A$. If ν and ω are two discrete valuations of k (c.f. AII, the functions ν and ω are defined on A and extended to k *via* $\nu(a/b) = \nu(a) - \nu(b)$, etc.), let's call ν, ω inequivalent iff one is not a constant multiple of the other. Write $\mathcal{S}$ for a set of *inequivalent* discrete valuations of k and say that A is adapted to $\mathcal{S}$ provided

$$A = \big\{x \in k \mid (\forall \nu \in \mathcal{S})(\nu(x) \geq 0)\big\}.$$

(a) Prove the following are equivalent:

i. A is a Dedekind domain

ii. ($\forall$ ideals, $\mathfrak{a}$, of A)$(\forall x, x \neq 0, x \in \mathfrak{a})(\exists\, y \in \mathfrak{a})(\mathfrak{a} = (x, y))$.

iii. There is a family of discrete valuations of k, say $\mathcal{S}$, for which A is adapted to $\mathcal{S}$ and so that the following holds:

$$(\forall \nu, \omega \in \mathcal{S})\big(\nu \neq \omega \implies (\exists\, a \in A)(\nu(a) \geq 1 \text{ and } \omega(a-1) \geq 1)\big).$$

(b) *Vis a vis* part (a), describe a one-to-one correspondence $\mathcal{S} \leftrightarrow \operatorname{Max}(A)$.

(c) Take $k = \mathbb{Q}$, consider all prime numbers p with $p \equiv 1 \pmod 4$, write $\operatorname{ord}_p(n)$ for the highest exponent, e, so that $p^e \mid n$. Then ord_p is a discrete valuation of $\mathbb{Q}$, and we set $\mathcal{S} = \big\{\operatorname{ord}_p \mid p \equiv 1 \pmod 4\big\}$. Illustrate iii in part (a) above with this $\mathcal{S}$. What is A, in concrete terms? It is pretty clear now how to make many Dedekind domains.

(d) Say A is a Dedekind domain and $\mathfrak{a}$, $\mathfrak{b}$ are two non-zero ideals of A. Show $\exists\, x \in k (= \operatorname{Frac} A)$, so that $\mathfrak{a} + x\mathfrak{b} = A$.

(e) Again let A be a Dedekind domain and let L be a *finite* subset of $\operatorname{Max}(A)$. Write $A^L = \bigcap\{A_\mathfrak{p} \mid \mathfrak{p} \notin L\}$, then $A \subseteq A^L$ and so $\mathbb{G}_m(A) \subseteq \mathbb{G}_m(A^L)$. Recall, $\mathbb{G}_m(B)$ is the group of units of the ring B. Prove that $\operatorname{Pic}(A)$ is a torsion group $\iff$ $\mathbb{G}_m(A^L)/\mathbb{G}_m(A)$ is a free abelian group of rank $\#(L)$ for every finite set L of $\operatorname{Max}(A)$.

BII) Here, k is a field and $A = k[X_\alpha]_{\alpha \in I}$. The index set, I, may possibly be infinite. Write $\mathfrak{m}$ for the fractional ideal generated by all the X_α, $\alpha \in I$. Set $A_i = A/\mathfrak{m}^{i+1}$, so $A_0 = k$. These A_i form a left mapping system and we set

$$\widehat{A} = \varprojlim A_i$$

and call $\widehat{A}$ the *completion of A in the* $\mathfrak{m}$*-adic topology.* Note that the kernel of $\widehat{A} \to A_j$ is the closure of $\mathfrak{m}^{j+1}$ in $\widehat{A}$.

(a) Show that $\widehat{A}$ is canonically isomorphic to the ring of formal power series in the X_α in which only finitely many monomials of each degree occur.

(b) Now let $I = \mathbb{N}$ (the counting numbers) and write $\widehat{\mathfrak{m}}$ for the closure of $\mathfrak{m}$ in $\widehat{A}$. By adapting Cantor's diagonal argument, prove that $\widehat{\mathfrak{m}}$ is NOT $\widehat{A}\mathfrak{m}$. Which is bigger?

(c) Again, I as in (b). Let k be a finite field, prove the
Lemma. If k is a finite field and $\lambda > 0$, $(\exists\, n_\lambda)(\forall n \geq n_\lambda)$, $\exists$ a *homogeneous* polynomial, $F_n \in k[n^2 \text{ variables}]$, so that $\deg F_n = n$ and F_n *cannot* be written as the sum of terms of degree n of *any* polynomial $P_1Q_1 + \cdots + P_\lambda Q_\lambda$, where P_j, Q_j are in $k[n^2 \text{ variables}]$ and have no constant term.
Use the lemma to prove $(\widehat{m})^2 \neq \widehat{(\mathfrak{m}^2)}$.

(d) Use (b) and (c) to prove that $\widehat{A}$ is NOT complete in the $\widehat{\mathfrak{m}}$-adic topology.

(e) All the pathology exhibited in (b), (c) and (d) arises as I is not finite, indeed when I is finite, prove:

i. $\widehat{\mathfrak{m}}$ is $\widehat{A}\mathfrak{m}$;

ii. $\widehat{\mathfrak{m}}^2 = \widehat{(\mathfrak{m}^2)}$;

iii. $\widehat{A}$ *is* complete in the $\widehat{\mathfrak{m}}$-adic topology.

BIII) Say $\mathcal{X}$ denotes the category TOP (topological spaces and continuous maps) and Haus$(\mathcal{X})$ the full subcategory of Hausdorff topological spaces.

(a) At first, use the ordinary Cartesian product in $\mathcal{X}$, with the product topology. Denote this $Y \times Z$. Show that $Y \in \text{Haus}(\mathcal{X}) \iff$ the diagonal map $\Delta : Y \to Y \times Y$ is closed.

(b) For $X, Y \in \text{Haus}(\mathcal{X})$, recall that $X \xrightarrow{f} Y$ is called a *proper* map $\iff f^{-1}(\text{compact}) = \text{compact}$. (Of course, any map $f : X \to Y$ will be proper if X is compact.) Show that $f : X \to Y$ is proper iff $(\forall T \in \text{Haus}(\mathcal{X}))(f_T : X \underset{Y}{\times} T \to Y \underset{Y}{\times} T$ is a closed map.)

(c) With (a) and (b) as background look at another subcategory, $\mathcal{X}_A$ of $\mathcal{X}$: here A is a commutative ring, $\mathcal{X}_A$ consists of the topological spaces $\operatorname{Spec} B$, where B is an A-algebra. Maps in $\mathcal{X}_A$ are those coming from homomorphisms of A-algebras, viz: $B \to C$ gives $\operatorname{Spec} C \to \operatorname{Spec} B$. Define

$$(\operatorname{Spec} B)\ \Pi\ (\operatorname{Spec} C) = \operatorname{Spec}(B \otimes_A C)$$

and prove that $\mathcal{X}_A$ possesses products.
NB:

i. The topology on $\operatorname{Spec} B\ \Pi\ \operatorname{Spec} C$ is NOT the product topology—it is stronger (more opens and closeds)

ii. $\operatorname{Spec} B\ \Pi\ \operatorname{Spec} C \neq \operatorname{Spec} B \times \operatorname{Spec} C$ as sets.

Prove: the diagonal map $\Delta_Y : Y \to Y \underset{\operatorname{Spec} A}{\Pi} Y$ is closed ($Y = \operatorname{Spec} B$). This recaptures (a) in the non-Hausdorff setting of $\mathcal{X}_A$.

(d) Given $f : \operatorname{Spec} C \to \operatorname{Spec} B$ (arising from an A-algebra map $B \to C$) call f *proper* $\iff$ i) C is a finitely generated B-algebra and ii)
$(\forall T = \operatorname{Spec} D)(f_T : \operatorname{Spec} C \underset{\operatorname{Spec} A}{\Pi} \operatorname{Spec} D \to \operatorname{Spec} B \underset{\operatorname{Spec} A}{\Pi} \operatorname{Spec} D$ is a closed map.)

Prove: if C is integral over B, then f is proper. However, prove also, $\operatorname{Spec}\big(B[T]\big) \to \operatorname{Spec} B$ is *never* proper.

(e) Say $A = \mathbb{C}$. For which A-algebras B is the map $\operatorname{Spec} B \to \operatorname{Spec} A$ proper?

BIV) A is *noetherian* local, $\mathfrak{m}_A$ its maximal ideal, and

$$\widehat{A} = \varprojlim_n A/\mathfrak{m}^{n+1} = \text{completion of } A \text{ in the } \mathfrak{m}\text{-adic topology.}$$

Let B, $\mathfrak{m}_B$ be another noetherian local ring and its maximal ideal. Assume $f : A \to B$ is a ring homomorphism and we *always assume* $f(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$.

(a) Prove: f gives rise to a homomorphism $\widehat{A} \xrightarrow{\widehat{f}} \widehat{B}$ (and $\mathfrak{m}_{\widehat{A}} \to \mathfrak{m}_{\widehat{B}}$).

(b) Prove: $\widehat{f}$ is an isomorphism $\iff$

i. B is flat over A

ii. $f(\mathfrak{m}_A) \cdot B = \mathfrak{m}_B$

iii. $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ is an isomorphism.