Math 603, Spring 2003, HW 4, due 3/17/2003

Part A

AI) If A is a noetherian ring, write $X = \operatorname{Spec} A$ with the Zariski topology. Prove the following are equivalent:

(a) X is T_1

(b) X is T_2

(c) X is discrete

(d) X is finite and T_1.

AII) Call a ring *semi-local* iff it possesses just finitely many maximal ideals.

(a) If $\mathfrak{p}_1, \dots, \mathfrak{p}_t \in \operatorname{Spec} A$ and $S = A - \bigcup_{j=1}^{t} \mathfrak{p}_j$, then $S^{-1}A$ is semi-local.

(b) Say A is semi-local and $\mathfrak{m}_1, \dots, \mathfrak{m}_t$ are its maximal ideals. Show that the natural map of rings

$$A/\mathcal{J}(A) \to \prod_{i=1}^{t} A/\mathfrak{m}_i$$

is an isomorphism.

(c) If A is semi-local, show $\operatorname{Pic}(A) = (0)$.

AIII) Let A be a domain. An element $a \in A$, not a unit, is called *irreducible* iff it is *not* the product $a = bc$ in which neither b nor c is a unit. The element a is a *prime* iff the principal ideal, Aa, is a prime ideal. Of course, prime $\Longleftrightarrow$ irreducible.

(a) Assume A is noetherian, show each non-unit of A is a finite product of irreducible elements. (A need not be a domain for this.)

(b) Prove that the factorization of (a) is unique (when it exists) iff every irreducible element of A is prime.

(c) Say A is a UFD and S a multiplicative subset of A. Show that $S^{-1}A$ is a UFD. If A is locally a UFD is A a UFD?

(d) Prove: if A is noetherian then A is a PID $\Longleftrightarrow$ A is a UFD *and* $\dim A = 1$.

(e) Assume A is just a domain. A *weight function*, w, *on* A is a function $A - \{0\} \to \mathbb{Z}_{\geq 0}$ so that

i. $a \mid b \implies w(a) \leq w(b)$, with equality $\Longleftrightarrow$ $b \mid a$ too

ii. a and $b \in A$ and say $a \nmid b$ and $b \nmid a$. Then $\exists\, p, q, r \in A$ so that $r = pa + qb$ and $w(r) < \min\{w(a), w(b)\}$.

Prove: a domain is a PID $\Longleftrightarrow$ it possesses a weight function. Can you characterize the fields among the PIDs by their weight functions?

AIV) We will prove in class the following two theorems for *noetherian* rings, A.

(a) Every prime ideal of A has finite height

(b) If A is also a domain and $\mathfrak{a}$ is a principal ideal then each isolated prime ideal of $\mathfrak{a}$ has height 1.

Assume (a) and (b) and prove: a noetherian domain is a UFD iff each height 1 prime is principal.

AV) Let $A \subseteq B$ be commutative rings. Recall that $\xi \in B$ is *integral over* A iff ξ satisfies a MONIC equation with coefficients in A, i.e.,

$$(\exists\, r > 0)(\xi^r + a_1\xi^{r-1} + \cdots + a_{r-1}\xi + a_r = 0, \quad a_j \in A).$$

Note that ξ is integral over $A \implies \xi$ is algebraic over A and the concepts coalesce when A is a field. If all $\xi \in B$ are integral over A, we say B is *integral over* A (old terminology: B is *integrally dependent on* A) or $A \to B$ is an *integral (homo)morphism*.

(a) Prove the following are equivalent:

i. $\xi \in B$ is integral over A

ii. The subring $A[\xi]$ of B is a finitely generated A-module

iii. The subring $A[\xi]$ of B is contained in a SUBRING, C, of B and this C is a finitely generated A-module.

iv. $\exists$ a finitely generated A-module, M, contained in B satisfying $\xi M \subseteq M$ and the annihilator of M in $A[\xi]$ is (0)

v. If A is assumed noetherian, $A[\xi]$ is contained in a finitely generated A-module.

(b) Say $A \subseteq B$ and write

$$\mathrm{Int}_B(A) = \{\xi \in B \mid \xi \text{ is integral over } A\}.$$

Show that $\mathrm{Int}_B(A)$ is a *subring* of B, that $A \subseteq \mathrm{Int}_B(A)$.

The ring $\mathrm{Int}_B(A)$ is called the *integral closure of A in B*. A is *integrally closed in B* iff $\mathrm{Int}_B(A) = A$. If B is the total fraction ring of A, then $\mathrm{Int}_B(A)$ is called the *normalization of* A and A is a *normal ring* $\iff$ A equals its normalization.

(c) If B is integral over A and S is a multiplicative subset in A, show $S^{-1}B$ is integral over $S^{-1}A$. Say $(\forall\, \mathfrak{p} \in \operatorname{Spec} A)(B_{\mathfrak{p}}$ is integral over $A_{\mathfrak{p}})$. Is B integral over A?

(d) Prove: every UFD is a normal domain. Prove further: S^{-1}(normal ring) is again a normal ring.

Part B

BI) Examples and Counterexamples:

(a) Let $A = k[X,Y]$ with k a field; write $\mathfrak{m} = (X,Y)$. Show that $\mathfrak{q} = (X, Y^2)$ is $\mathfrak{m}$-primary, but $\mathfrak{q}$ is NOT a power of any prime ideal of A. Therefore, primary ideals need not be powers of prime ideals.

(b) Let $A = k[X,Y,Z]/(XY - Z^2) = k[x,y,z]$. Write $\mathfrak{p}$ for the ideal (x,z) of A. Prove that $\mathfrak{p} \in \operatorname{Spec} A$, but $\mathfrak{p}^2$ is not primary. Hence, powers of non-maximal prime ideals need not be primary. What is the primary decomposition of $\mathfrak{p}^2$?

(c) Say $A = k[X,Y]$ as in part (a) and write $\mathfrak{a} = (X^2, XY)$. Show that $\mathfrak{a}$ is NOT primary yet $\sqrt{\mathfrak{a}}$ is a prime ideal—which one? So, here a non-primary ideal has a prime radical. What is the primary decomposition of $\mathfrak{a}$?

(d) If A is a UFD and p is a prime element of A, then $\mathfrak{q} = Ap^n$ is always primary. Conversely, show if $\mathfrak{q}$ is primary and $\sqrt{\mathfrak{q}} = Ap$, then $(\exists\, n \geq 1)(\mathfrak{q} = Ap^n)$. Compare with (c) above.

BII) In this problem, A is a noetherian integral domain.

(a) Assume A is a normal domain, not a field, let $\mathfrak{p} \in \mathrm{Max}(A)$ and suppose $\mathfrak{p}$ is a prime ideal of a principal ideal $(a) = Aa$. Show that as an A-module, $\mathfrak{p}$ is an element of $\mathrm{Pic}(A)$. This gives a way of making elements of $\mathrm{Pic}(A)$. (Suggestions: look at $(\mathfrak{p} \to A)$ in $\operatorname{Frac} A$, that is, $\{\xi \in \operatorname{Frac} A \mid \xi\mathfrak{p} \subseteq A\}$. First show $\big(\mathfrak{p} \to (a)\big) \neq (a)$ (in A) and, from some $x \in \big(\mathfrak{p} \to (a)\big)$ and a, make an element $\xi \in (\operatorname{Frac} A) \cap (\mathfrak{p} \to A)$. Deduce $(\mathfrak{p} \to A) > A$. If $\xi \in (\mathfrak{p} \to A)$ show ξ is integral (if something goes wrong) over A, now use the maximality of $\mathfrak{p}$ to finish.)

(b) Now forget A being normal and assume A has two properties

i. $A_{\mathfrak{p}}$ is a PID if $\mathfrak{p}$ is a non-zero *minimal* prime of A and

ii. $A = \bigcap_{\mathrm{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$.

Under these conditions, prove A is a normal domain.

(c) If A is normal, show conditions i and ii must hold.

(d) Use these results to show Theorem (b) of AIV) in the case of a normal domain.

(e) Prove: if A is a noetherian normal domain, then A is a UFD $\iff$ $\mathrm{Pic}(A) = (0)$. This settles the computation of $\mathrm{Pic}(\mathbb{Z}[X_1, \dots, X_n])$, and $\mathrm{Pic}(k[X_1, \dots, X_n])$ where k is a field.

BIII) A Little Number Theory.

Let $\mathbb{Q}$ be the rational numbers, and consider fields $k = \mathbb{Q}[X]/(f(X))$ where $f(X)$ is an irreducible polynomial over $\mathbb{Q}$. (Each finite extension of $\mathbb{Q}$ has this form, as we'll show.) Such a k will be called a "number field" and we write $\mathcal{O}_k$ for $\mathrm{Int}_k(\mathbb{Z})$.

(a) Show $\mathcal{O}_k$ is a noetherian normal domain with $\dim \mathcal{O}_k = 1$.

(b) If $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_k$, then $(\mathcal{O}_k)_{\mathfrak{p}}$ is a PID and $\mathcal{O}_k$ is a UFD iff $\mathrm{Pic}(\mathcal{O}_k) = (0)$ iff $\mathcal{O}_k$ is a PID.

(c) Say $\mathfrak{a}$ is an ideal of $\mathcal{O}_k$, write $\mathfrak{a} = \bigcap_i \mathfrak{q}_i$ for an irredundant primary decomposition and set $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$. Show that in fact $\mathfrak{a} = \prod_i \mathfrak{p}_i^{e_i}$, $e_i \in \mathbb{Z}_{\geq 0}$, and this is a unique factorization of ideals.

(d) Let k be the fields: $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(\sqrt{-5})$, $\mathbb{Q}(\zeta)$, where ζ is a primitive 7th root of 1. In each case, find $\mathcal{O}_k$ and compute $\mathrm{Pic}(\mathcal{O}_k)$. Make a table.

(e) In $\mathbb{Q}(\sqrt{-3})$, look at $\mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} \mid a, b \in \mathbb{Z}\}$. Is $\mathbb{Z}[\sqrt{-3}] = \mathcal{O}_k$? If not, what is $\mathrm{Pic}(\mathbb{Z}[\sqrt{-3}])$? Same question for $\mathbb{Z}[\sqrt{-5}]$.

(f) Let A be a noetherian, normal domain of dimension 1, write $k = \operatorname{Frac} A$ (e.g., $\mathcal{O}_k = A$ by (a)). We examine submodules (for A) of k. Call one of these, M, a *fractional ideal* iff $(\exists\, b \in A)(b \neq 0)(bM \subseteq A)$. Prove that the following are equivalent for A-submodules of k:

i. M is a fractional ideal

ii. M is a finitely generated A-module

iii. M is a rank one projective A-module.

(g) Under multiplication, MN, the fractional ideals form a group, denote it $\mathcal{I}(A)$. (MN goes over to $M \otimes_A N$ in $\mathrm{Pic}(A)$). Let $\mathcal{C}_A$ be the (localizing) category of finite length modules over A and write $\widetilde{K}(A)$ for the Grothendieck group, $K_0(\mathcal{C}_A)$ of $\mathcal{C}_A$. By the theory of associated primes, each M in $\mathcal{C}_A$ has a composition series

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_{n+1} = (0)$$

and

$$M_i/M_{i+1} \cong A/\mathfrak{p}_i \text{ for some } \mathfrak{p}_i \in \mathrm{Max}(A).$$

These $\mathfrak{p}_i$ are unique up to order and we set

$$\chi_A(M) = \prod_{i=0}^{n} \mathfrak{p}_i \in \mathcal{I}(A).$$

Prove that χ_A is an isomorphism (first prove homomorphism) of the abelian groups $\widetilde{K}(A) \to \mathcal{I}(A)$. What is the kernel of the map $\widetilde{K}(A) \to \mathrm{Pic}(A)$?

(h) Lastly, assume A is actually a PID. Say $M = A^n$ is a free A-module of rank n and let $u \in \mathrm{End}_A M$. Assume $\det(u) \neq 0$ and show

$$\det(u) \cdot A = \chi_A(\operatorname{coker} u).$$

BIV) More examples.

(a) Let $A = k[X, Y, Z, W]/(XY - ZW)$, where k is a field and $\mathrm{char}(k) \neq 2$. Prove that A is a normal domain and compute $\mathrm{Pic}(A)$.

(b) If $A = \mathbb{C}[t^3, t^7, t^8] \subseteq \mathbb{C}[t]$, compute $\mathrm{Pic}(A)$. If $A = \{f \in \mathbb{C}[T] \mid f'(0) = f''(0) \text{ and } f(1) = f(-1)\}$ compute $\mathrm{Pic}(A)$.

(c) If $A = \mathbb{C}[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$, show $\mathrm{Pic}(A) \neq (0)$.

BV) (a) $A = K[X, Y, Z]$, K a field. Set $\mathfrak{a} = (X, Y)(X, Z)$. Find a primary decomposition of $\mathfrak{a}$.

(b) Let $A = K[X, XY, Y^2, Y^3] \subseteq K[X, Y] = B$, here K is a field. Write $\mathfrak{p} = YB \cap A = (XY, Y^2, Y^3)$. Prove that $\mathfrak{p}^2 = (X^2Y^2, XY^3, Y^4, Y^5)$ and is not primary. Find a primary decomposition of $\mathfrak{p}^2$ involving (Y^2, Y^3). All ideals are ideals of A.

BVI) (a) Say A is an integral domain. Prove

$$A = \bigcap_{\mathfrak{p} \in \mathrm{Spec}\, A} A_{\mathfrak{p}} = \bigcap_{\mathfrak{m} \in \mathrm{Max}(A)} A_{\mathfrak{m}}.$$

(b) Now let A be a commutative ring and let $f(T)$ be a polynomial of degree d in $A[T]$. Prove that $A[T]/\big(f(T)\big)$ is an A-projective module of rank d iff the coefficient of T^d in $f(T)$ is a unit of A.