Math 603, Spring 2003, HW 3, due 2/24/2003

Part A

AI) Consider the two rings $A = \mathbb{R}[T]$ and $B = \mathbb{C}[T]$. Show that $\mathrm{Max}(B)$ is in one-to-one correspondence with the points of the complex plane while $\mathrm{Max}(A)$ is in one-to-one correspondence with the closed upper half plane: $\{\xi \in \mathbb{C} \mid \mathrm{Im}(\xi) \geq 0\}$. Since A is a PID (so is B) we can characterize an ideal by its generator. In these terms, which ideals of $\mathrm{Max}(A)$ correspond to points in $\mathrm{Im}(\xi) > 0$, which to points on the real line? What about $\mathrm{Spec}\, B$, $\mathrm{Spec}\, A$?

AII) When X is compact Hausdorff and $A = \mathbb{C}(X)$, we identified X and $\mathrm{Max}(A)$ in class *via* $x \mapsto \mathfrak{m}_x$. Now $\mathrm{Max}(A)$ has the induced topology from $\mathrm{Spec}\, A$.

(a) Show the induced topology on $\mathrm{Max}(A)$ is compact Hausdorff by proving $x \mapsto \mathfrak{m}_x$ is a homeomorphism.

(b) Prove all finitely generated ideals of A are principal but that no maximal ideal is finitely generated.

AIII) (a) Given $A \to B$ a homomorphism prove that B is faithfully flat over A iff B is flat over A and the map $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is surjective.

(b) Say $A \to B$ is a homomorphism and B is faithfully flat over A. Assume A is noetherian. Show that the topology on $\mathrm{Spec}\, A$ is the quotient topology from $\mathrm{Spec}\, B$.

AIV) Here A is a commutative ring, but *not necessarily with unity.* Let $A^{\#}$ denote $A \times \mathbb{Z}$ (category of sets) and addition componentwise and multiplication by

$$\langle a, n\rangle\langle b, q\rangle = \langle ab + nb + qa, nq\rangle.$$

(a) Clearly, $A^{\#}$ is a commutative ring with unity $\langle 0, 1\rangle$. A is a subring of $A^{\#}$, even an ideal. Suppose A has the ACC on ideals, prove that $A^{\#}$ does too.

(b) If you know all the prime ideals of A, can you find all the prime ideals of $A^{\#}$?

AV) Let B, C be commutative A-algebras, where A is also commutative. Write D for the A-algebra $B \otimes_A C$.

(a) Give an example to show that $\mathrm{Spec}\, D$ is not $\mathrm{Spec}\, B \underset{\mathrm{Spec}\, A}{\times} \mathrm{Spec}\, C$ (category of sets over $\mathrm{Spec}\, A$).

(b) We have A-algebra maps $B \to D$ and $C \to D$ and so we get maps $\mathrm{Spec}\, D \to \mathrm{Spec}\, B$ and $\mathrm{Spec}\, D \to \mathrm{Spec}\, C$ (even maps over $\mathrm{Spec}\, A$), and these are maps of topological spaces (over $\mathrm{Spec}\, A$). Hence, we do get a map

$$\theta : \mathrm{Spec}\, D \to \mathrm{Spec}\, B \underset{\mathrm{Spec}\, A}{\Pi} \mathrm{Spec}\, C \quad \text{(top. spaces)}.$$

Show $\exists$ closed sets in $\mathrm{Spec}\, D$ *not* of the form $\theta^{-1}(Q)$, where Q is a closed set in the product topology of $\mathrm{Spec}\, B \underset{\mathrm{Spec}\, A}{\Pi} \mathrm{Spec}\, C$.

Part B

BI) Let $A = \mathbb{Z}[T]$, we are interested in $\mathrm{Spec}\, A$.

(a) If $\mathfrak{p} \in \mathrm{Spec}\, A$, prove that $\mathrm{ht}(\mathfrak{p}) \leq 2$.

(b) If $\{\mathfrak{p}\}$ is closed in $\mathrm{Spec}\, A$, show that $\mathrm{ht}(\mathfrak{p}) = 2$. Is the converse true?

(c) We have the map $\mathbb{Z} \hookrightarrow \mathbb{Z}[T] = A$, hence the continuous map $\mathrm{Spec}\, A \xrightarrow{\pi} \mathrm{Spec}\, \mathbb{Z}$. Pick a prime number, say p, of $\mathbb{Z}$. Describe $\pi^{-1}(p)$, is it closed?

(d) When exactly is a $\mathfrak{p} \in \operatorname{Spec} A$ the generic point (point whose closure is everything) of $\pi^{-1}(p)$ for some prime number p?

(e) Describe exactly those $\mathfrak{p} \in \operatorname{Spec} A$ whose image, $\pi(\mathfrak{p})$, is dense in $\operatorname{Spec} \mathbb{Z}$. What is $\operatorname{ht}(\mathfrak{p})$ in these cases?

(f) Is there a $\mathfrak{p} \in \operatorname{Spec} A$ so that the closure of $\{\mathfrak{p}\}$ is all of $\operatorname{Spec} A$? What is $\operatorname{ht}(\mathfrak{p})$?

(g) For a general commutative ring, B, if $\mathfrak{p}$ and $\mathfrak{q}$ are elements of $\operatorname{Spec} B$ and if $\mathfrak{q} \in \overline{\{\mathfrak{p}\}}$ show that $\operatorname{ht}(\mathfrak{q}) \geq \operatorname{ht}(\mathfrak{p})$ (assuming finite height). If $\mathfrak{p}$, $\mathfrak{q}$ are as just given and $\operatorname{ht}(\mathfrak{q}) = \operatorname{ht}(\mathfrak{p})$ is $\mathfrak{q}$ necessarily $\mathfrak{p}$? Prove that the following are equivalent:

i. $\operatorname{Spec} B$ is *irreducible* (that is, it is NOT the union of two properly contained closed subsets)

ii. $(\exists\, \mathfrak{p} \in \operatorname{Spec} B)$(closure of $\{\mathfrak{p}\} = \operatorname{Spec} B$)

iii. $(\exists$ *unique* $\mathfrak{p} \in \operatorname{Spec} B)$(closure of $\{\mathfrak{p}\} = \operatorname{Spec} B$)

iv. $\mathcal{N}(B) \in \operatorname{Spec} B$.

(h) Draw a picture of $\operatorname{Spec} \mathbb{Z}[T]$ as a kind of plane over the "line" $\operatorname{Spec} \mathbb{Z}$ and exhibit in your picture all the different kinds of $\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}[T]$.

BII) If A is a commutative ring, we can view $f \in A$ as a "function" on the topological space $\operatorname{Spec} A$ as follows: for each $\mathfrak{p}$ in $\operatorname{Spec} A$, as usual write $\kappa(\mathfrak{p})$ for $\operatorname{Frac}(A/\mathfrak{p})$; [note that $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/$its max. ideal] and set $f(\mathfrak{p})$ = image of f in $A/\mathfrak{p}$ considered in $\kappa(\mathfrak{p})$. Thus, $f : \operatorname{Spec} A \to \bigcup_{\mathfrak{p} \in \operatorname{Spec} A} \kappa(\mathfrak{p})$. Observe that if $f \in \mathcal{N}(A)$, then $f(\mathfrak{p}) = 0$ all $\mathfrak{p}$, *yet* f need not be zero as an element of A.

(a) Let $A = k[X_1, \ldots, X_n]$. We'll prove soon that there are fields, Ω, containing k so that

i. Ω has infinitely many transcendental elements independent of each other and of the X_j over k and

ii. Ω is algebraically closed, i.e., all polynomials with coefficients in Ω have a root in Ω.

An example of this is when $k = \mathbb{Q}$ or some finite extension of $\mathbb{Q}$ and then we can take $\Omega = \mathbb{C}$. In any case, fix such an Ω. Establish a set-theoretic map $\Omega^n \to \operatorname{Spec} A$ so that $f \in A = k[X_1, \ldots, X_n]$ viewed in the usual way as a function on Ω^n agrees with f viewed as a function on $\operatorname{Spec} A$. We can topologize Ω^n as follows: call a subset of Ω^n k-closed iff $\exists$ finitely many polynomials $f_1, \ldots, f_p$ from A so that the subset is exactly the set of common zeros of $f_1, \ldots, f_p$. This gives Ω^n the k-topology (an honest topology, as one checks). Show that your map $\Omega^n \to \operatorname{Spec} A$ is continuous between these topological spaces. Prove, further, that Ω^n maps *onto* $\operatorname{Spec} A$.

(b) Show that Ω^n is irreducible in the k-topology.

(c) Define an equivalence relation on Ω^n: $\xi \sim \eta \iff$ each point lies in the closure (k-topological) of the other. Prove that $\Omega^n/\sim$ is homeomorphic to $\operatorname{Spec} A$ under your map.

BIII) Let A be an integral domain and write K for $\operatorname{Frac}(A)$. For each $\xi \in K$, we set

$$\operatorname{dom}(\xi) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \xi \text{ can be written } \xi = a/b, \text{ with } a, b \in A \text{ and } b(\mathfrak{p}) \neq 0\}.$$

(a) Show $\operatorname{dom}(\xi)$ is open in $\operatorname{Spec} A$.

(b) If $A = \mathbb{R}[X, Y]/(X^2 + Y^2 - 1)$, set $\xi = (1 - y)/x$ (where $x = \overline{X}$ and $y = \overline{Y}$). What is $\operatorname{dom}(\xi)$?

(c) Set $A = \mathbb{C}[X, Y]/(Y^2 - X^2 - X^3)$ and let $\xi = y/x$. What is $\operatorname{dom}(\xi)$?

(d) Note that as ideals of A (any commutative ring) are A-modules, we can ask if they are free or locally free. Check that the non-zero ideal, $\mathfrak{a}$, of A is free $\iff$ it is principal and $\big(\mathfrak{a} \to (0)\big) = (0)$. The second condition is automatic in a domain. Now look again at $A = \mathbb{R}[X, Y]/(X^2 + Y^2 - 1)$, you should see easily that this is a domain. Characterize as precisely as you can the elements $\mathfrak{m} \in \operatorname{Max}(A)$ which are free as A-modules. If there are other elements of $\operatorname{Max}(A)$, are these locally free? What is the complement of $\operatorname{Max}(A)$ in $\operatorname{Spec} A$? Prove that $A \otimes_{\mathbb{R}} \mathbb{C}$ is a PID.

(e) Consider the descent question for PIDs: given rings S and T with $S \to T$ a homomorphism, suppose A is an S-algebra and T is faithfully flat over S. If $A \otimes_S T$ is a PID, is A necessarily a PID?

BIV) Let p be an odd prime number, set $m = 2p - 1$ and write $A = \mathbb{Z}[\sqrt{-m}] \cong \mathbb{Z}[T]/(T^2 + m)$. Assume m is square free.

(a) Let $\mathfrak{a}$ be the ideal $(p, 1 + \sqrt{-m})$ of A. Prove that $\mathfrak{a}$ is not principal, yet that $\mathfrak{a}$, as a module, is locally free (necessarily of rank one). Prove further that A is *not* a UFD.

(b) For $p = 3$ and 7, find all the ideals, $\mathfrak{a}$, which are not free, yet are locally free.

N.B. By our results you have non-free projectives, here.

BV) In this problem A is an integral domain and $K = \mathrm{Frac}(A)$.

(a) Is it true that if $\mathfrak{p} \in \mathrm{Spec}\,\big(A[X]\big)$ and if $\mathfrak{p} \cap A = (0)$, then $\mathfrak{p}$ is a principal ideal? Proof or counterexample.

(b) Say A is a UFD and $\eta \in K$, with $\eta \neq 0$. Write $\eta = a/b$, where a and b are relatively prime. Prove that $A[\eta] \cong A[X]/(bX - a)$. When is $A[\eta]$ a flat A-module?

(c) If k is a field and $\xi \in k(X)$ is a non-constant rational function, write $\xi = f(X)/g(X)$ where f and g are relatively prime polynomials. Of course, $k(\xi)$ is a subfield of $k(X)$, so $k(X)$ is a $k(\xi)$ vector space (and a $k(\xi)$-algebra). Prove that $\dim_{k(\xi)}\big(k(X)\big) < \infty$ and compute this dimension in terms of f and g.

BVI) If A is a commutative ring, $B = A[[X_1, \ldots, X_n]]$ denotes the ring of formal power series in the variables $X_1, \ldots, X_n$ (the case $n = 1$ was discussed in assignment 2).

(a) Prove:

$$\begin{aligned} A \text{ is noetherian} &\iff B \text{ is noetherian} \\ A \text{ is an integral domain} &\iff B \text{ is an integral domain} \\ A \text{ is a local ring} &\iff B \text{ is a local ring.} \end{aligned}$$

(b) Write $K((X_1, \ldots, X_n))$ for $\mathrm{Frac}\, B$, where $K = \mathrm{Frac}\, A$. Say $A = K = \mathbb{C}$, $n = 2$. Is $\mathbb{C}((X, Y))$ equal to $\mathbb{C}((X))((Y))$? If not, does one contain the other; which?