Math 603, Spring 2003, HW 1, due 1/27/2003

Part A

AI) (a) Say $A \xrightarrow{\theta} B$ is a homomorphism of commutative rings. and suppose it makes B a faithfully flat A-module. Show that θ is injective.

(b) Hypotheses as in (a), but also assume B is finitely presented as an A-algebra (e.g., B is finitely generated and A is noetherian). Show that there exists an A-module, M, so that $B \cong A \amalg M$, as A-modules.

(c) Assume A and B are local rings, $\theta : A \to B$ is a ring map (N.B. so that we assume $\theta(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$) and B, as an A-module, is flat. Write $\mathcal{N}(A)$, respectively $\mathcal{N}(B)$ for the nilradicals of A, respectively B. That is,

$$\mathcal{N}(A) = \big\{\xi \in A \mid (\exists n \in \mathbb{N})(\xi^n = 0)\big\}, \text{ etc.}$$

Prove:

i. If $\mathcal{N}(B) = (0)$, then $\mathcal{N}(A) = (0)$.

ii. If B is an integral domain, so is A.

Are the converses of i., ii. true? Proof or counter-example.

AII) Here, I is an index set and $\mathcal{S}(I)$ is the set of all *finite* subsets of I. Partially order $\mathcal{S}(I)$ by inclusion, then it has Moore–Smith. Also, $\mathcal{C}$ is a category having *finite* products or finite coproducts as the case may be below (e.g. groups, Ω-groups, modules). Say for each $\alpha \in I$ we are given an object $M_\alpha \in \mathcal{C}$. For ease of notation below, write $M_S = \coprod_{\alpha \in S} M_\alpha$, $M_S^* = \prod_{\alpha \in S} M_\alpha$, where $S \in \mathcal{S}(I)$ is given. Prove:

If $\mathcal{C}$ has right limits and finite coproducts, then $\mathcal{C}$ has arbitrary coproducts; indeed,

$$\varinjlim_{S \in \mathcal{S}(I)} M_S = \coprod_{\alpha \in I} M_\alpha.$$

Prove a similar statement for left limits and products.

AIII) Recall that a ring, Λ, is *semi-simple* iff every Λ-module, M, has the property:

$$(\forall \text{ submodules, } M', \text{ of } M)(\exists \text{ another submodule, } M'' \text{ of } M)(M \cong M' \amalg M'').$$

There is a condition on the positive integer, n, so that n has the condition $\iff$ $\mathbb{Z}/n\mathbb{Z}$ is semi-simple. Find the condition and prove the theorem.

AIV) In this problem, $A \in \mathrm{CR}$. If $\alpha_1, \dots, \alpha_m$ are in A, write $(\alpha_1, \dots, \alpha_n)$ for the ideal generated by $\alpha_1, \dots, \alpha_n$ in A. Recall that $K_0(A)$, the *Grothendieck group* of A, is the quotient of the free abelian group on the (isomorphism classes of) finitely generated A-modules (as generators) by the subgroup generated by the relations: if $0 \to M' \to M \to M'' \to 0$ is exact in $\mathcal{M}\mathrm{od}(A)$, then $[M] \to [M'] \to [M'']$ is a relation.

(a) If $\alpha \in A$, show that in $K_0(A)$ we have

$$\big[((\alpha) \to 0)\big] = \big[A/(\alpha)\big]$$

(b) If A is a PID and M is a finite length A-module, show that $[M] = 0$ in $K_0(A)$.

(c) Prove: if A is a PID, then for all finitely generated A-modules, M, there exists a unique integer $r = r(M)$, so that $[M] = r[A]$ in $K_0(A)$; hence $K_0(A)$ is $\mathbb{Z}$. Prove further that $r(M) = \dim(M \otimes_A \mathrm{Frac}(A))$.

Part B

BI) Write $\mathcal{LCA}$b for the category of locally compact abelian topological groups, the morphisms being continuous homomorphisms. Examples include: every abelian group with the discrete topology, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}/\mathbb{Z} = \mathbb{T}$, etc. If $G \in \mathcal{LCA}$b, write

$$G^D = \mathrm{Hom}_{\mathrm{cts}}(G, \mathbb{T}),$$

make G^D a group via pointwise operations and topologize G^D via the compact-open topology; that is, take the sets

$$U(C,\epsilon) = \{f \in G^D \mid \mathrm{Im} f \restriction C \subseteq -\epsilon < \arg z < \epsilon\}$$

—where C runs over the compact subsets of G containing 0, ϵ is positive and we identify $\mathbb{T}$ with the unit circle in $\mathbb{C}$—as a fundamental system of neighborhoods at 0 in G^D.

(a) Suppose G is actually compact. Prove G^D is discrete in this topology. Likewise, prove if G is discrete, then G^D is compact in this topology. Finally prove G^D is locally compact in this topology.

(b) If $\{G_\alpha, \varphi_\alpha^\beta\}$ is a right (respectively left) mapping family of *finite* abelian groups, then $\left\{G_\alpha^D, (\varphi_\alpha^\beta)^D\right\}$ becomes a left (respectively right) mapping family, again of *finite* abelian groups (how, why?). Prove that

$$\left(\varinjlim_\alpha G_\alpha\right)^D \cong \varprojlim_\alpha G_\alpha^D$$

and

$$\left(\varprojlim_\alpha G_\alpha\right)^D \cong \varinjlim_\alpha G_\alpha^D$$

as *topological* groups. We call a group *profinite* $\iff$ it is isomorphic, as a *topological* group, to a left limit of finite groups.

(c) Prove the following three conditions are equivalent for an abelian topological group, G:

i. G is profinite

ii. G is a compact, Hausdorff, totally disconnected group

iii. G^D is a discrete torsion group.

(d) For this part, $\{G_\alpha\}$ is a family of *compact* groups, not necessarily abelian, and the index set has Moore–Smith. Assume $(\forall\alpha)(\exists S_\alpha)(S_\alpha = \text{closed, normal subgroup of } G_\alpha)$ and that $b \geq \alpha \implies G_\beta \subseteq G_\alpha$ and $S_\beta \subseteq S_\alpha$. Show that the family $\{H_\alpha = G_\alpha/S_\alpha\}_\alpha$ can be made into a left mapping family, in a natural way, and that

$$\varprojlim_\alpha H_\alpha \cong \bigcap_\alpha G_\alpha / \bigcap_\alpha S_\alpha \quad \text{(as topological groups.)}$$

(e) If G is a compact topological group, write $\{U_\alpha | \alpha \in I\}$ for the family of *all open, normal* subgroups of G. Continue (c) by proving:

$$G \text{ is profinite} \iff G \text{ is compact and } \bigcap_\alpha U_\alpha = \{1\}.$$

(f) In class, we defined $\mathbb{Z}_p$ as $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$ and $\hat{\mathbb{Z}}$ as $\varprojlim_n \mathbb{Z}/n\mathbb{Z}$ (Artin ordering for the n's). Quickly use (b) to compute $\mathbb{Z}_p^D$ and $(\hat{\mathbb{Z}})^D$. Now consider the following mathematical statements:

i. $\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$

ii. $\mathbb{Q}^* \cong \mathbb{Z}/2\mathbb{Z}\ \Pi\ \prod_p \mathbb{Z}$

iii. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-1/p^s}$, if Re $s > 1$

iv. A statement you know well and are to fill in here concerning arithmetic in $\mathbb{Z}$.

Show i-iv are mutually equivalent.

BII) Fix an abelian group, A, for what follows. Write $A_n = A$, all $n \in \mathbb{N}$ and give $\mathbb{N}$ the Artin ordering. If $n \leq m$ (i.e. $n|m$) define $\varphi_n^m : A_n \to A_m$ by $\varphi_n^m(\xi) = m\xi/n$, and define $\psi_m^n : A_m \to A_n$ by $\psi_m^n(\xi) = m\xi/n$, too. Let

$$\tilde{A} = \varinjlim \{A_n, \varphi_n^m\} \quad \text{and} \quad T(A) = \varprojlim \{A_m, \psi_m^n\}.$$

($T(A) =$ *full Tate group of* A).

(a) Prove that both $\tilde{A}$ and $T(A)$ are divisible groups.

(b) Show that if $A = A_1 \xrightarrow{\varphi} \tilde{A}$ is the canonical map into the direct limit, then $\ker(\varphi) = t(A)$, the torsion subgroup of A. Hence, every torsion free abelian group is a subgroup of a divisible group. Given any abelian group , A, write

$$0 \to K \to F \to A \to 0,$$

for some free abelian group F. Show that A may be embedded in $\tilde{F}/K$; hence deduce anew that every abelian group embeds in a divisible abelian group.

(c) If A is a free $\mathbb{Z}$-module, what is $T(A)$?

(d) If $A \to B \to 0$ is exact, need $T(A) \to T(B) \to 0$ also be exact? Proof or counterexample.

(e) Show that if $T(A) \neq (0)$, then A is not finitely generated.

BIII) Again, as in AI), let $\theta : A \to B$ be a homomorphism of commutative rings and assume B is faithfully flat over A *via* θ. If M is an A-module, write M_B for $M \otimes_A B$.

(a) Prove: M is finitely generated as an A-module iff M_B is finitely generated as a B-module.

(b) Same as (a) but for finite presentation instead of finite generation.

(c) Show: M is locally free over A iff M_B is locally free over B.

(d) When, if ever, is $S^{-1}A$ faithfully flat over A?

BIV) Here, $\Lambda \in$ RNG. Say

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of Λ-modules.

(a) Assume M'' is a flat Λ-module. Prove: for all Λ^{op}-modules, N, the sequence

$$0 \to N \otimes_\Lambda M' \to N \otimes_\Lambda M \to N \otimes_\Lambda M'' \to 0$$

is again exact. (You might look at the special case when M is free first.)

(b) Again assume M'' is flat; prove M and M' are flat $\iff$ either is flat. Give an example of Λ, M', M, M'' in which both M and M' are flat but M'' is NOT flat.