Math 602, Fall 2002, HW 5, due 12/10/2002

Part A

AI) (Vector bundles) As usual, TOP is the category of topological spaces and k will be either the real or complex numbers. All vector spaces are to be finite dimensional. A *vector space family over* X, V, is an object of TOP_X, call p the map $V \to X$, so that

i. $(\forall\, x \in X)(p^{-1}(x)$ (denoted V_x) is a k-vector space)
ii. The induced topology on V_x is the usual topology it has as a vector space over k.

Example: the trivial family: $X \amalg k^n$ (fixed n).
Vector space families over X form a category, $\mathrm{VF}(X)$, if we define the morphisms to be those morphisms, φ, from TOP_X which satisfy:

$$(\forall\, x \in X)(\varphi_x : V_x \to W_x \text{ is a linear map.})$$

(a) Say $Y \xrightarrow{\theta} X$ is a continuous map. Define a functor $\theta^* : \mathrm{VF}(X) \rightsquigarrow \mathrm{VF}(Y)$, called pullback. When Y is a subspace of X, the pullback, $\theta^*(V)$, is called the restriction of V to Y, written $V \restriction Y$.
A vector space family is a *vector bundle* $\iff$ it is *locally trivial*, that is:
$(\forall\, x \in X)(\exists \text{ open } U)(x \in U)$ so that $V \restriction U$ is isomorphic (in $\mathrm{VF}(U)$) to $U \amalg k^n$, some n. Let $\mathrm{Vect}(X)$ denote the *full* subcategory of $\mathrm{VF}(X)$ formed by the objects that are vector bundles.

(b) Say X is an r-dimensional vector space considered in TOP. Write $\mathbb{P}(X)$ for the collection of all hyperplanes through $0 \in X$, then $\mathbb{P}(X)$ is a topological space and is covered by opens each isomorphic to an $(r-1)$-dimensional vector space. On $\mathbb{P}(X)$ we make an element of $\mathrm{VF}\big(\mathbb{P}(X)\big)$: W is the set of pairs $(\xi, \nu) \in \mathbb{P}(X) \amalg X^D$ so that $\xi \subset \ker \nu$. Here, X^D is the dual space of X. Show that W is a line bundle on $\mathbb{P}(X)$.

(c) If $V \in \mathrm{Vect}(X)$ and X is connected, then $\dim(V_x)$ is constant on X. This number is the *rank* of V.

(d) A *section of* V *over* U is a map $\sigma : U \to V \restriction U$ so that $p \circ \sigma = \mathrm{id}_U$. Write $\Gamma(U, V)$ for the collection of sections of V over U. Show: if $V \in \mathrm{Vect}(X)$, each section of V over U is just a compatible family of locally defined vector valued functions on U. Show further that $\Gamma(U, V)$ is a vector space in a natural way.

(e) Say V and W are in $\mathrm{Vect}(X)$, with ranks p and q respectively. Show: $\mathrm{Hom}(V, W)$ is isomorphic to the collection of locally defined "compatible" families of *continuous* functions $U \to \mathrm{Hom}(k^p, k^q)$, *via* the local description

$$\varphi \in \mathrm{Hom}(V, W) \rightsquigarrow \widetilde{\varphi} : U \to \mathrm{Hom}(k^p, k^q),$$

where $\varphi(u, v) = \big(u, \widetilde{\varphi}(u)(v)\big)$.
Now $\mathrm{Iso}(k^p, k^q) = \{\psi \in \mathrm{Hom}(k^p, k^q) \mid \psi \text{ is invertible}\}$ is an open of $\mathrm{Hom}(k^p, k^q)$.

(f) Show: $\varphi \in \mathrm{Hom}(V, W)$ is an isomorphism $\iff$ for a covering family of opens $U (\subseteq X)$, $\widetilde{\varphi}(U) \subseteq \mathrm{Iso}(k^p, k^q)$ $\iff$ $(\forall\, x \in X)(\varphi_x : V_x \to W_x$ is an isomorphism).

(g) Show $\{x \mid \varphi_x \text{ is an isomorphism (here, } \varphi \in \mathrm{Hom}(U, V))\}$ is open in X.

(h) Show all of (a) to (g) go over when $X \in C^k\!-\!\mathrm{MAN}$ $(0 \le k \le \infty)$ with appropriate modification C^k replacing continuity where it appears.

AII) (Linear algebra for vector bundles). First just look at finite dimensional vector spaces over k and say F is some functor from vector spaces to vector spaces (F might even be a several variable functor). Call F *continuous* $\iff$ the map $\mathrm{Hom}(V, W) \to \mathrm{Hom}\big(F(V), F(W)\big)$ is continuous. (Same definition for C^k, $1 \le k \le \infty$, ω). If we have such an F, extend it to bundles *via* the following steps: first define $F(V)$ (for $V \in \mathrm{Vect}(X)$) as the vector space family $\bigcup_{x \in X} F(V_x)$—we must still topologize this.

(a) V is the trivial bundle: $X \Pi k^p$. As sets, $F(X \Pi k^p)$, *via* the definition above, is just $X \Pi F(k^p)$, so we give $F(X \Pi k^p)$ the product topology. Prove: if $\varphi \in \mathrm{Hom}(V, W)$, then $F(\varphi)$ is continuous, therefore $F(\varphi) \in \mathrm{Hom}\big(F(V), F(U)\big)$. Show, further, φ is an isomorphism $\implies$ $F(\varphi)$ is an isomorphism.

(b) The topology on $F(V)$ when V is trivial appears to depend on the specific trivialization. Show this is not true—it is actually independent of same.

(c) If V is any bundle, then $V \restriction U$ is trivial for small open U, so by (a) and (b), $F(V \restriction U)$ is a trivial bundle. Topologize $F(V)$ by calling a set, Z, open iff $Z \cap \big(F(V \restriction U)\big)$ is open in $F(V \restriction U)$ for all U where $V \restriction U$ is trivial. Show that if $Y \subseteq X$, then the topology on $F(V \restriction Y)$ is just that on $F(V) \restriction Y$, that $\varphi : V \to W$ continuous $\implies$ $F(\varphi)$ is continuous and extend all these things to C^k. Finally prove: $f : Y \to X$ in TOP $\implies$ $f^*\big(F(V)\big) \cong F\big(f^*(V)\big)$ and similarly in C^k-MAN.

(d) If we apply (c) , we get for vector bundles:

   i. $V \amalg W$, more generally finite coproducts

   ii. V^D, the dual bundle

   iii. $V \otimes W$

   iv. $\mathcal{H}om(V, W)$, the vector bundle of (locally defined) homomorphisms.

   Prove: $\Gamma\big(U, \mathcal{H}om(V, W)\big) \cong \mathrm{Hom}(V \restriction U, W \restriction U)$ for *every* open, U, of X. Is this true for the bundles of i, ii and iii?

AIII) Recall that if $R \in$ RNG, $J(R)$—the Jacobson radical of R— is just the intersection of all maximal ideals of R. The ideal, $J(R)$, is actually 2-sided.

(a) Say $J(R) = (0)$ (e.g., $R = \mathbb{Z}$). Show that no non-projective R-module has a projective cover.

(b) Suppose M_i, $i = 1, \dots, t$ are R-modules with projective covers $P_1, \dots, P_t$. Prove that $\coprod_i P_i$ is a projective cover of $\coprod_i M_i$.

(c) Say M and N are R-modules and assume M and $M \amalg N$ have projective covers. Show that N has one.

(d) In M is an R-module, write (as usual) $M^D = \mathrm{Hom}_R(M, R)$. Then M^D is an R^{op}-module. Prove that if M is finitely generated and projective as an R-module, then M^D has the same properties as an R^{op}-module.

AIV) Let $\{M_\alpha\}$ be a given family of R^{op}-modules. Define, for R-modules, two functors:

$$U : N \rightsquigarrow \Big(\Big(\prod_\alpha M_\alpha\Big) \otimes_R N\Big)$$
$$V : N \rightsquigarrow \prod_\alpha (M_\alpha \otimes_R N).$$

(a) Show that V is right-exact and is exact iff each M_α is flat over R.

(b) Show there exists a morphism of functors $\theta : U \to V$. Prove that $\theta_N : U(N) \to V(N)$ is surjective if N is finitely generated, while θ_N is an isomorphism if N is finitely presented.

Part B

BI) (Continuation of AI and AII). Let V and W be vector bundles and $\varphi : V \to W$ a homomorphism. Call φ a *monomorphism* (respectively *epimorphism*) iff $(\forall\, x \in X)(\varphi_x : V_x \to W_x$ is a monomorphism (respectively epimorphism)). Note: φ is a monomorphism iff $\varphi^D : W^D \to V^D$ is an epimorphism. A *sub-bundle* of V is a subset which is a vector bundle in the induced structure.

(a) Prove: if $\varphi : V \to W$ is a monomorphism, then $\varphi(V)$ is a sub-bundle of W. Moreover, locally on X, we have that there exists a vector bundle, G, an open $U \subseteq X$, so that $(V \restriction U) \amalg (G \restriction U) \cong W \restriction U$ (i.e., every sub-bundle is locally part of a coproduct decomposition of W). Prove also: $\{x \mid \varphi_x$ is a monomorphism$\}$ is open in X. (Suggestion: say $x \in X$, pick a subspace of W_x complementary to $\varphi(V_x)$, call it Z. Form $G = X \amalg Z$. Then there exists a homomorphism $V \amalg G \to W$, look at this homomorphism near the point x.)

(b) Say V is a sub-bundle of W, show that $\bigcup_{x\in X} W_x/V_x$ (with the quotient topology) is actually a vector bundle (not just a vector space family) over X.

(c) Now note we took a full subcategory of VF(X), so for $\varphi \in \mathrm{Hom}(V, W)$ ($V, W \in \mathrm{Vect}(X)$) the dimension of $\ker \varphi_x$ need not be locally constant on X. When it is locally constant call φ a *bundle homomorphism*. Prove that if φ is a bundle homomorphism from V to W, then

   i. $\bigcup_x \ker \varphi_x$ is a sub-bundle of V
   ii. $\bigcup_x \mathrm{Im}\, \varphi_x$ is a sub-bundle of W, hence
   iii. $\bigcup_x \mathrm{coker}\, \varphi_x$ is a vector bundle (quotient topology).

   We refer to these bundles as $\ker \varphi$, $\mathrm{Im}\, \varphi$ and $\mathrm{coker}\, \varphi$ respectively. Deduce from your argument for i that

   iv. Given $x \in X$, there exists an open U, $x \in U$, so that $(\forall\, y \in U)(\mathrm{rank}\, \varphi_y \geq \mathrm{rank}\, \varphi_x)$. Of course, this φ is not necessarily a bundle homomorphism.

   From now on to the end, X is COMPACT HAUSDORFF. We need as well two results from analysis:

   A) (Tietze extension theorem). X is a normal space, Y a closed subspace, V a real vector space. Then every continuous map $Y \to V$ admits an extension to a continuous map $X \to V$. Same result for $X \in C^k-$MAN and C^k maps.

   B) (Partitions of unity). Say X is compact Hausdorff, $\{U_\alpha\}$ a finite open cover of X. There exist continuous maps, f_α, taking X to $\mathbb{R}$ such that

      A. $f_\alpha \geq 0$, (all α)
      B. $\mathrm{supp}(f_\alpha) \subseteq U_\alpha$ (so $f_\alpha \in C_0^0(U_\alpha)$)
      C. $(\forall\, x \in X)(\sum_\alpha f_\alpha(x) = 1)$.

   The same is true for C^k-MAN (X compact!) and C^k functions ($1 \leq k \leq \infty$).

(d) Extend Tietze to vector bundles: if X is compact Hausdorff, $Y \subseteq X$ closed and $V \in \mathrm{Vect}(X)$, then every section $\sigma \in \Gamma(Y, V \restriction Y)$ extends to a section in $\Gamma(X, V)$. (Therefore, there exist *plenty* of continuous or C^∞ global sections of V. FALSE for holomorphic sections). Apply this to the bundle $\mathcal{H}om(V, W)$ and prove: $Y \hookrightarrow X$ closed, X (as usual) compact Hausdorff or compact C^k-manifold, if $\varphi : V \restriction Y \to W \restriction Y$ is an isomorphism of vector bundles, there exists open, U, $Y \subseteq U$, so that φ extends to an isomorphism $V \restriction U \to W \restriction U$.

(e) Every vector space possesses a metric (take any of the p-norms, or take the 2-norm if you are fussy). It's easy to see that metrics then exist on trivial bundles. In fact, use the 2-norm, so we can "bundleize" the notion of Hermitian form (AII) and get the bundle $\mathcal{H}erm(V)$. Then a Hermitian metric on V is a global section of $\mathcal{H}erm(V)$ which is positive definite, at each $x \in X$. Show every bundle possesses a Hermitian metric.

(f) Say we're given vector bundles and *bundle homomorphisms*, we say the sequence

$$\cdots \to V_j \to V_{j+1} \to V_{j+2} \to \cdots$$

of such is exact $\iff$ $(\forall\, x \in X)$ the sequence of vector spaces

$$\cdots \to V_{j,x} \to V_{j+1,x} \to V_{j+2,x} \to \cdots$$

is exact. Prove: if $0 \to V' \to V \to V'' \to 0$ is an exact sequence of vector bundles and bundle homomorphisms, then $V \cong V' \amalg V''$.

(g) Consider a vector bundle, V and a subspace, Σ of the vector space $\Gamma(X, V)$. We get the trivial bundle $X \amalg \Sigma$ and a natural homomorphism $X \amalg \Sigma \to V$, *via*

$$(x, \sigma) \to \sigma(x).$$

Prove: if X is compact Hausdorff (or compact C^k−HOM), there exists a *finite dimensional* subspace, Σ, of $\Gamma(X, V)$ so that the map $X \amalg \Sigma \to V$ is *surjective*. Thus there exists a finite dimensional surjective family of C (respectively C^k) sections of V. Use (f) to deduce: usual assumption on X, then for each vector bundle, V, on X, there exists a vector bundle, W, on X, so that $V \amalg W$ is a trivial bundle.

(h) Write $C(X)$ (respectively $C^k(X)$, $1 \leq k \leq \infty$) for the ring of continuous (respectively C^k) functions (values in our field) on X, where X is compact Hausdorff (respectively compact manifold). In a natural way (pointwise multiplication) $\Gamma(X, V)$ is an A-module ($A = C(X)$, $C^k(X)$), and Γ gives a functor from vector bundles, V, to $\mathcal{M}\mathrm{od}(A)$. Trivial bundles go to free modules of finite rank over A (why?) Further, show Γ is an equivalence of categories

$$(\text{Trivial bundles, homs}) \rightsquigarrow (\text{Free } A\text{-modules, f.g.}).$$

Now use your results to prove:

> Γ gives an equivalence of categories: (Vect(X), homs from VF) and (f.g. projective A-modules, module homs).

BII) (a) Say M is a f.g. $\mathbb{Z}$-module, $\neq (0)$. Prove there exists a prime p so that $M \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \neq (0)$. Deduce: no divisible group (abelian) [i.e., one such that $N \xrightarrow{r} N \to 0$ is exact, all $r \geq 1$] can be f.g.

(b) Say M, M'' are $\mathbb{Z}$-modules and M is f.g. while M'' is torsion free. Given $\varphi \in \mathrm{Hom}(M, M'')$ suppose ($\forall$ primes p)(the induced map $M \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \to M'' \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ *is* a monomorphism). Show that φ is a monomorphism and M is free.

BIII) Given Λ, $\Gamma \in$ RNG and a ring homomorphism $\Lambda \to \Gamma$ (thus, Γ is a Λ-algebra), if M is a Λ-module, then $M \otimes_\Lambda \Gamma$ has the natural structure of a Γ^{op}-module. Similarly, if Z is both a Λ^{op}-module and a Γ-module, then $Z \otimes_\Lambda M$ is still a Γ-module. Now let N be a Γ-module,

(a) prove there is a *natural* isomorphism

$$(*) \qquad \mathrm{Hom}_\Gamma(Z \otimes_\Lambda M, N) \xrightarrow{\sim} \mathrm{Hom}_\Lambda(M, \mathrm{Hom}_\Gamma(Z, N)).$$

Prove, in fact, the functors $M \rightsquigarrow M \otimes_\Lambda Z$ and $N \rightsquigarrow \mathrm{Hom}_\Gamma(Z, N)$ are adjoint functors, i.e., $(*)$ is functorial.

(b) Establish an analog of $(*)$:

$$(**) \qquad \mathrm{Hom}_\Gamma(M, \mathrm{Hom}_\Lambda(Z, N)) \cong \mathrm{Hom}_\Lambda(Z \otimes_\Gamma M, N)$$

under appropriate conditions on Z, M and N (what are they?)

(c) Show: M projective as a Λ^{op}-module, Z projective as a Γ^{op}-module $\implies$ $M \otimes_\Lambda Z$ is projective as a Γ^{op}-module. In particular, M projective as a Λ^{op}-module $\implies$ $M \otimes_\Lambda \Gamma$ is projective as a Γ^{op}-module and of course, the same statement with the op for $Z \otimes_\Lambda M$, $\Gamma \otimes_\Lambda M$. Show further, that if N is Λ-injective, then $\mathrm{Hom}_\Lambda(\Gamma, N)$ is Γ-injective.

(d) For abelian groups, M, write $M^D = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. Then, if M is free, M^D is injective as a $\mathbb{Z}$-module (why?). From this deduce: every abelian group is a subgroup of an injective abelian group.

(e) Give Eckmann's proof of the Baer Embedding Theorem: for every ring, Γ, each Γ-module is a submodule of an injective Γ-module. (Eckmann's proof uses (c) and (d)).

BIV) Here, A and B are commutative rings and $\varphi : A \to B$ a ring homomorphism so that B is an A-algebra. Assume B is flat (i.e., as an A-module, it's flat). Define a homomorphism

$$\theta : \mathrm{Hom}_A(M, N) \otimes_A B \to \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B)$$

(functorial in M and N)—how?

(a) If M is f.g. as an A-module, θ is injective.

(b) If M is f.p. as an A-module, θ is an isomorphism.

(c) Assume M is f.p. as an A-module, write $\mathfrak{a}$ for $\mathrm{Ann}(M) = \big(M \to (0)\big)$. Prove that $\mathfrak{a} \otimes_A B$ is the annihilator of $M \otimes_A B$ in B.

BV) Here, k is a field and f is a *monic* polynomial of *even* degree in $k[X]$.

(a) Prove there exist g, $r \in k[X]$ such that $f = g^2 + r$ and $\deg r < \frac{1}{2} \deg f$. Moreover, g and r are unique.

Now specialize to the case $k = \mathbb{Q}$, and suppose f has *integer* coefficients. Assume $f(X)$ is NOT the square of a polynomial with rational coefficients.

(b) Prove there exist only *finitely* many integers, x, such that the value $f(x)$ is a square, say y^2, where $y \in \mathbb{Z}$. In which ways can you get the square of an integer, y, by adding 1 to third and fourth powers of an integer, x?

(c) Show there exists a constant, K_N, depending ONLY on the degree, N, of f so that:

if all coefficients of f are bounded in absolute value by C (≥ 1) then whenever $\langle x, y\rangle$ is a solution of $y^2 = f(x)$ (with x, $y \in \mathbb{Z}$) we have $|x| \leq K_N C^N$.

(d) What can you say about the number of points $\langle x, y\rangle$ with rational coordinates which lie on the (hyper-elliptic) curve $Y^2 = f(X)$?

BVI) Consider $\mathcal{M}\mathrm{od}(\mathbb{Z})$ and copies of $\mathbb{Z}$ indexed by $\mathbb{N} = \{1, 2, \ldots\}$. Form the module $\prod_{\mathbb{Z}} \mathbb{Z}$. It is a product of $\aleph_0$ projective modules. Show $M = \prod_{\mathbb{Z}} \mathbb{Z}$ is *not* projective as a $\mathbb{Z}$-module. (Suggestions: establish that each submodule of a free module over a PID is again free, therefore need to show M is not free. Look at

$$K = \{\xi = (\xi_j) \in M \mid (\forall\, n)(\exists\, k = k(n))(2^n \mid \xi_j \text{ if } j > k(n))\}.$$

This is a submodule of M; show $K/2K$ is a vector space over $\mathbb{Z}/2\mathbb{Z}$ of the same dimension as K and finish up. Of course, 2 could be replaced by any prime). So, products of projectives need not be projective.