Math 602, Fall 2002, HW 4, due 11/18/2002

Part A

AI) Recall that for every integral domain, A, there is a field, $\mathrm{Frac}(A)$, containing A minimal among all fields containing A. If B is an A-algebra, an element $b \in B$ is *integral over* $A \iff$ there exists a *monic* polynomial, $f(X) \in A[X]$, so that $f(b) = 0$. The domain, A, is *integrally closed in* B iff every $b \in B$ which is integral over A actually comes from A (*via* the map $A \to B$). The domain, A, is *integrally closed* (also called *normal*) iff it is integrally closed in $\mathrm{Frac}(A)$. Prove:

(a) A is integrally closed $\iff$ $A[X]/\big(f(X)\big)$ is an integral domain for every MONIC irreducible polynomial, $f(X)$.

(b) A is a UFD $\iff$ A possesses the ACC on principal ideals and $A[X]/\big(f(X)\big)$ is an integral domain for every irreducible polynomial $f(X)$. (It follows that every UFD is a normal domain.)

(c) If k is a field and characteristic of k is not 2, show that $A = k[X, Y, Z, W]/(XY - ZW)$ is a normal domain. What happens if $\mathrm{char}(k) = 2$?

AII) If A is a ring, write $\mathrm{End}^*(A)$ for the collection of *surjective* ring endomorphisms of A. Suppose A is commutative and noetherian, prove $\mathrm{End}^*(A) = \mathrm{Aut}(A)$.

AIII) Write $M(n, A)$ for the ring of all $n \times n$ matrices with entries from A (A is a ring). Suppose K and k are fields and $K \supseteq k$.

(a) Show that if $M, N \in M(n, k)$ and if $\exists\, P \in \mathrm{GL}(n, K)$ so that $PMP^{-1} = N$, then $\exists\, Q \in \mathrm{GL}(n, k)$ so that $QMQ^{-1} = N$.

(b) Prove that (a) is false for rings $B \supseteq A$ *via* the following counterexample:
$A = \mathbb{R}[X, Y]/(X^2 + Y^2 - 1)$, $B = \mathbb{C}[X, Y]/(X^2 + Y^2 - 1)$. Find two matrices similar in $M(2, B)$ but NOT similar in $M(2, A)$.

(c) Let S^n be the n-sphere and represent $S^n \subseteq \mathbb{R}^{n+1}$ as $\{(z_0, \ldots, z_n) \in \mathbb{R}^{n+1} \mid \sum_{j=0}^{n} z_j^2 = 1\}$. Show that there is a *natural injection* of $\mathbb{R}[X_0, \ldots, X_n]/(\sum_{j=0}^{n} X_j^2 - 1)$ into $C(S^n)$, the ring of (real valued) continuous functions on S^n. Prove further that the former ring is an integral domain but $C(S^n)$ is not. Find the group of units in the former ring.

AIV) (Rudakov) Say A is a ring and M is a rank 3 free A-module. Write Q for the bilinear form whose matrix (choose some basis for M) is

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, if $v = (x, y, z)$ and $w = (\xi, \eta, \zeta)$, we have

$$Q(v, w) = (x, y, x) \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}.$$

Prove that $Q(w, v) = Q(v, Bw)$ with $B = I + \text{nilpotent} \iff a^2 + b^2 + c^2 = abc$.

AV) Let M be a Λ-module (Λ is not necessarily commutative) and say N and N' are submodules of M.

(a) Suppose $N + N'$ and $N \cap N'$ are f.g. Λ-modules. Prove that both N and N' are then f.g. Λ-modules.

(b) Give a generalization to finitely many submodules, $N_1, \ldots, N_t$ of M.

(c) Can you push part (b) to an infinite number of N_j?

(d) If M is noetherian as a Λ-module, is Λ necessarily noetherian as a ring (left noetherian as M is a left module)? What about $\overline{\Lambda} = \Lambda/\mathrm{Ann}(M)$?

Part B

BI) (Continuation of AI)

(a) Consider the ring $A(n) = \mathbb{C}[X_1, \dots, X_n]/(X_1^2 + \cdots + X_n^2)$. There is a condition on n, call it $C(n)$, so that $A(n)$ is a UFD iff $C(n)$ holds. Find explicitly $C(n)$ and prove that theorem.

(b) Consider the ring $B(n) = \mathbb{C}[X_1, \dots, X_n]/(X_1^2 + X_2^2 + X_3^3 + \cdots + X_n^3)$. There is a condition on n, call it $D(n)$, so that $B(n)$ is a UFD iff $D(n)$ holds. Find explicitly $D(n)$ and prove the theorem.

(c) Investigate exactly what you can say if $C(n)$ (respectively $D(n)$) does not hold.

(d) Replace $\mathbb{C}$ by $\mathbb{R}$ and answer (a) and (b).

(e) Can you formulate a theorem about the ring $A[X,Y]/\big(f(X,Y)\big)$, where A is a given UFD and f is a polynomial in $A[X,Y]$, of the form $A[X,Y]/\big(f(X,Y)\big)$ is a UFD provided $f(X,Y)\cdots$? Your theorem must be general enough to yield (a) and (b) as easy consequences. (You must prove it too.)

BII) (Exercise on projective modules) In this problem, $A \in \mathcal{O}\mathrm{b}(\mathrm{CR})$.

(a) Suppose P and P' are projective A-modules, and M is an A-module. If

$$0 \to K \to P \to M \to 0 \qquad \text{and}$$
$$0 \to K' \to P' \to M \to 0$$

are exact, prove that $K' \amalg P \cong K \amalg P'$.

(b) If P is a f.g. projective A-module, write P^D for the A-module $\mathrm{Hom}_A(P, A)$. We have a canonical map $P \to P^{DD}$. Prove this is an isomorphism.

(c) Again, P is f.g. projective; suppose we're given an A-linear map $\mu : \mathrm{End}_A(P) \to A$. Prove: there exists a unique element $f \in \mathrm{End}_A(P)$ so that $(\forall\, h \in \mathrm{End}_A(P))(\mu(h) = \mathrm{tr}(hf))$. Here, you must define the trace, tr, for f.g. projectives, P, as a well-defined map, then prove the result.

(d) Again, P is f.g. projective; μ is as in (c). Show that $\mu(gh) = \mu(hg) \iff \mu = a\,\mathrm{tr}$ for some $a \in A$.

(e) Situation as in (b), then each $f \in \mathrm{End}_A(P)$ gives rise to $f^D \in \mathrm{End}_A(P^D)$. Show that $\mathrm{tr}(f) = \mathrm{tr}(f^D)$.

(f) Using categorical principles, reformulate (a) for injective modules and prove your reformulation.

BIII) Suppose K is a commutative ring and $a, b \in K$. Write $A = K[T]/(T^2 - a)$; there is an automorphism of A (the identity on K) which sends t to $-t$, where t is the image of T in A. If $\xi \in A$, we write $\overline{\xi}$ for the image of ξ under this automorphism. Let $\mathbb{H}(K; a, b)$ denote the set

$$\mathbb{H}(K; a, b) = \left\{ \begin{pmatrix} \xi & b\eta \\ \overline{\eta} & \overline{\xi} \end{pmatrix} \;\middle|\; \xi, \eta \in A \right\},$$

this is a subring of the 2×2 matrices over A. Observe that $q \in \mathbb{H}(K; a, b)$ is a unit there iff q is a unit of the 2×2 matrices over A.

(a) Consider the non-commutative polynomial ring $K\langle X, Y\rangle$. There is a 2-sided ideal, $\mathcal{I}$, in $K\langle X, Y\rangle$ so that $\mathcal{I}$ is symmetrically generated *vis a vis* a and b and $K\langle X, Y\rangle/\mathcal{I}$ is naturally isomorphic to $\mathbb{H}(K; a, b)$. Find the generators of $\mathcal{I}$ and establish the explicit isomorphism.

(b) For pairs (a,b) and (α,β) decide exactly when $\mathbb{H}(K;a,b)$ is isomorphic to $\mathbb{H}(K;\alpha,\beta)$ as objects of the comma category RNG^K.

(c) Find all isomorphism classes of $\mathbb{H}(K;a,b)$ when $K=\mathbb{R}$ and when $K=\mathbb{C}$. If $K=\mathbb{F}_p$, $p\neq 2$ answer the same question and then so do for $\mathbb{F}_2$.

(d) When K is just some field, show $\mathbb{H}(K;a,b)$ is a "division ring" (all non-zero elements are units) $\iff$ the equation $X^2-aY^2=b$ has no solution in K (here we assume a is not a square in K). What is the case if a *is* a square in K?

(e) What is the center of $\mathbb{H}(K;a,b)$?

(f) For the field $K=\mathbb{Q}$, prove that $\mathbb{H}(\mathbb{Q};a,b)$ is a division ring $\iff$ the surface $aX^2+bY^2=Z^2$ has no points whose coordinates are integers except 0.

BIV) (a) If A is a commutative ring and $f(X)\in A[X]$, suppose $(\exists\, g(X)\neq 0)(g(X)\in A[X]$ and $g(X)f(X)=0)$. Show: $(\exists\,\alpha\in A)(\alpha\neq 0$ and $\alpha f(X)=0)$. *Caution*: A may possess non-trivial nilpotent elements.

(b) Say K is a field and $A=K[X_{ij},\, 1\le i,j\le n]$. The matrix

$$M=\begin{pmatrix} X_{11} & \dots & X_{1n}\\ \dots & \dots & \dots\\ X_{n1} & \dots & X_{nn}\end{pmatrix}$$

has entries in A and $\det(M)\in A$. Prove that $\det(M)$ is an irreducible polynomial of A.

BV) Let A be a commutative noetherian ring and suppose B is a commutative A-algebra which is f.g. as an A-algebra. If $G\subseteq \mathrm{Aut}_{A-\mathrm{alg}}(B)$ is a *finite* subgroup, write

$$B^G=\{b\in B\mid \sigma(b)=b,\ \text{all } \sigma\in G\}.$$

Prove that B^G is also f.g. as an A-algebra; hence B^G is noetherian.

BVI) Again, A is a commutative ring. Write $\mathrm{RCF}(A)$ for the ring of $\infty\times\infty$ matrices all of whose rows and all of whose columns possess but finitely many (*not* bounded) non-zero entries. This *is* a ring under ordinary matrix multiplication (as you see easily).

(a) Specialize to the case $A=\mathbb{C}$; find a *maximal* two-sided ideal, $\mathcal{E}$, of $\mathrm{RCF}(\mathbb{C})$. Prove it is such and is the only such. You are to find $\mathcal{E}$ explicitly. Write $E(\mathbb{C})$ for the ring $\mathrm{RCF}(\mathbb{C})/\mathcal{E}$.

(b) Show that there exists a natural injection of rings $M_n=n\times n$ complex matrices $\hookrightarrow\mathrm{RCF}(\mathbb{C})$ so that the composition $M_n\to E(\mathbb{C})$ is *still* injective: show further that if $p\mid q$ we have a commutative diagram

$$\begin{tikzcd} M_p \arrow[rr, hook] \arrow[dr] & & M_q \arrow[dl] \\ & E(\mathbb{C}) & \end{tikzcd}$$

BVII) (Left and right noetherian) For parts (a) and (b), let $A=\mathbb{Z}\langle X,Y\rangle/(YX,Y^2)$—a non-commutative ring.

(a) Prove that

$$\mathbb{Z}[X]\hookrightarrow\mathbb{Z}\langle X,Y\rangle\to A$$

is an injection and that $A=\mathbb{Z}[X]\amalg\big(\mathbb{Z}[X]y\big)$ as a left $\mathbb{Z}[X]$-module (y is the image of Y in A); hence A is a left noetherian ring.

(b) However, the right ideal generated by $\{X^ny\mid n\ge 0\}$ is NOT f.g. (prove!); so, A is not right noetherian.

(c) Another example. Let

$$C = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \;\middle|\; a \in \mathbb{Z};\, b, c, \in \mathbb{Q} \right\}.$$

Then C is right noetherian but NOT left noetherian (prove!).