Math 602, Fall 2002, HW 3, due 10/28/2002

Part A

AI) (a) If G is a group of order n, show that $G \rtimes \mathrm{Aut}(G)$ is isomorphic to a subgroup of $\mathfrak{S}_n$.

(b) Consider the cycle $(1, 2, \dots, n) \in \mathfrak{S}_n$; let H be the subgroup (of $\mathfrak{S}_n$) generated by the cycle. Prove that
$$\mathcal{N}_{\mathfrak{S}_n}(H) \cong (\mathbb{Z}/n\mathbb{Z}) \rtimes \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}).$$

AII) Let TOP denote the category of topological spaces.

(a) Show that TOP possesses finite fibred products and finite fibred coproducts.

(b) Is (a) true without the word "finite"?

(c) Write T2TOP for the full subcategory of TOP consisting of Hausdorff topological spaces. Are (a) and (b) true in T2TOP? If you decide the answer is "no", give reasonable conditions under which a positive result holds. What relation is there between the product (coproduct) you construct in (a) (or (b)) and the corresponding objects in this part of the problem?

AIII) Let R be a ring (not necessarily commutative) and write $\mathcal{M}\mathrm{od}(R)$ for the category of (left) R-modules; i.e., the action of R on a module, M, is on the left. We know $\mathcal{M}\mathrm{od}(R)$ has finite products and finite fibred products.

(a) What is the situation for infinite products and infinite fibred products?

(b) What is the situation for coproducts (finite or infinite) and for fibred coproducts (both finite and infinite)?

AIV) As usual, write $\mathcal{G}\mathrm{r}$ for the category of groups. Say G and G' are groups and $\varphi : G \to G'$ is a homomorphism. Then $(G, \varphi) \in \mathcal{G}\mathrm{r}_{G'}$, the comma category of "groups over G'". The group $\{1\}$ possess a canonical morphism to G', namely the inclusion, i. Thus, $\big(\{1\}, i\big) \in \mathcal{G}\mathrm{r}_{G'}$, as well. We form their product in $\mathcal{G}\mathrm{r}_{G'}$, i.e., we form the fibred product $G \prod_{G'} \{1\}$. Prove that there exists a canonical *monomorphism*
$$G \prod_{G'} \{1\} \to G.$$
Identify its image in G.
Now consider the "dual" situation: G' maps to G, so $G \in \mathcal{G}\mathrm{r}^{G'}$ (*via* φ) the "groups co-over G'". We also have the canonical map $G' \to \{1\}$, killing all the elements of G'; so, as above, we can form the fibred coproduct of G and $\{1\}$: $G \overset{G'}{\amalg} \{1\}$. Prove that there exists a canonical epimorphism
$$G \to G \overset{G'}{\amalg} \{1\},$$
identify its kernel in G.

Part B

BI) Write CR for the category of commutative rings with unity and RNG for the category of rings with unity.

(a) Consider the following two functors from CR to $\mathcal{S}$ets:

i. $|\mathcal{M}_{pq}| : A \rightsquigarrow$ underlying set of $p \times q$ matrices with entries from A
ii. $|\mathrm{GL}_n| : A \rightsquigarrow$ underlying set of all invertible $n \times n$ matrices with entries from A.

Prove the these two functors are representable.

(b) A slight modification of ii. above yields a functor from CR to $\mathcal{G}$r: namely,

$$\mathrm{GL}_n : A \rightsquigarrow \textit{group} \text{ of all invertible } n \times n \text{ matrices with entries from } A.$$

When $n = 1$, we can extend this to a functor from RNG to $\mathcal{G}$r. That is we get the functor

$$\mathbb{G}_m : A \rightsquigarrow \text{group of all invertible elements of } A.$$

Prove that the functor $\mathbb{G}_m$ has a left adjoint, let's temporarily call it (†); that is: $\exists$ a functor (†) from $\mathcal{G}$r to RNG, so that

$$(\forall\, G \in \mathcal{G}\mathrm{r})(\forall\, R \in \mathrm{RNG})(\mathrm{Hom}_{\mathrm{RNG}}((\dagger)(G), R) \cong \mathrm{Hom}_{\mathcal{G}\mathrm{r}}(G, \mathbb{G}_m(R))),$$

via a functorial isomorphism.

(c) Show that without knowing what ring $(\dagger)(G)$ is, namely that is exists and that (†) is left adjoint to $\mathbb{G}_m$, we can prove: the category of $(\dagger)(G)$-modules, $\mathcal{M}\mathrm{od}\big((\dagger)(G)\big)$, is equivalent—in fact isomorphic—to the category of G-modules.

(d) There is a functor from $\mathcal{G}$r to Ab, namely send G to $G^{\mathrm{ab}} = G/[G, G]$. Show this functor has a right adjoint, call it I. Namely, there exists a functor $I : \mathrm{Ab} \to \mathcal{G}\mathrm{r}$, so that

$$(\forall\, G \in \mathcal{G}\mathrm{r})(\forall\, H \in \mathrm{Ab})(\mathrm{Hom}_{\mathcal{G}\mathrm{r}}(G, I(H)) \cong \mathrm{Hom}_{\mathrm{Ab}}(G^{\mathrm{ab}}, H)).$$

Does $G \rightsquigarrow G^{\mathrm{ab}}$ have a left adjoint?

BII) We fix a commutative ring with unity, A, and write $\mathcal{M}$ for $\mathcal{M}_{pq}(A)$, the $p \times q$ matrices with entries in A. Choose a $q \times p$ matrix, Γ, and make $\mathcal{M}$ a ring *via*:

Addition: as usual among $p \times q$ matrices
Multiplication: if $R, S \in \mathcal{M}$, set $R * S = R\Gamma S$, where $R\Gamma S$ is the ordinary product of matrices.

Write $\mathcal{M}(\Gamma)$ for $\mathcal{M}$ with these operations, then $\mathcal{M}(\Gamma)$ is an A-algebra (a ring which is an A-module).

(a) Suppose that A is a field. Prove that the isomorphism classes of $\mathcal{M}(\Gamma)$'s are finite in number (here p and q are fixed while Γ varies); in fact, are in one-to-one correspondence with the integers $0, 1, 2, \ldots, B$ where B is to be determined by you.

(b) Given two $q \times p$ matrices Γ and $\widetilde{\Gamma}$ we call them equivalent iff $\widetilde{\Gamma} = W\Gamma Z$, where $W \in \mathrm{GL}(q, A)$ and $Z \in \mathrm{GL}(p, A)$. Prove: each Γ is equivalent to a matrix

$$\begin{pmatrix} I_r & 0 \\ 0 & H \end{pmatrix}$$

where $I_r = r \times r$ identity matrix and the entries of H are non-units of A. Is r uniquely determined by Γ? How about the matrix H?

(c) Call the commutative ring, A, a *local ring* provided it possesses exactly one maximal ideal, $\mathfrak{m}_A$. For example, any field is a local ring; the ring $\mathbb{Z}/p^n\mathbb{Z}$ is local if p is a prime; other examples of this large, important class of rings will appear below. We have the descending chain of ideals

$$A \supseteq \mathfrak{m}_A \supseteq \mathfrak{m}_A^2 \supseteq \cdots .$$

For some local rings one knows that $\bigcup_{t \geq 0} \mathfrak{m}_A^t = (0)$; let's call such local rings "good local rings" for temporary nomenclature. If A is a good local ring, we can define a function on A to $\mathbb{Z} \cup \{\infty\}$, call it ord, as follows:

$\mathrm{ord}(\xi) = 0$ if $\xi \notin \mathfrak{m}_A$
$\mathrm{ord}(\xi) = n$ if $\xi \in \mathfrak{m}_A^n$ but $\xi \notin \mathfrak{m}_A^{n+1}$
$\mathrm{ord}(0) = \infty$.

The following properties are simple to prove:

$\mathrm{ord}(\xi \pm \eta) \geq \min\{\mathrm{ord}(\xi), \mathrm{ord}(\eta)\}$
$\mathrm{ord}(\xi\eta) \geq \mathrm{ord}(\xi) + \mathrm{ord}(\eta)$.

Consider the $q \times p$ matrices under equivalence and look at the following three conditions:

i. Γ is equivalent to $\begin{pmatrix} I_r & 0 \\ 0 & H \end{pmatrix}$, with $H = (0)$

ii. Γ is equivalent to $\begin{pmatrix} I_r & 0 \\ 0 & H \end{pmatrix}$ with H having non-unit entries and $r \geq 1$

iii. $(\exists\, Q \in \mathcal{M})(\Gamma Q \Gamma = \Gamma)$.

Of course, i. $\Longrightarrow$ ii. if $\Gamma \neq (0)$, A any ring. Prove: if A is any (commutative) ring then i. $\Longrightarrow$ iii., and if A is good local i. and iii. are equivalent. Show further that if A is good local then $\mathcal{M}(\Gamma)$ possesses a non-trivial idempotent, P, (an element such that $P * P = P$, $P \neq 0, \neq 1$) if and only if Γ has ii.

(d) Write $\mathcal{I} = \{U \in \mathcal{M}(\Gamma) \mid \Gamma U \Gamma = 0\}$ and given $P \in \mathcal{M}(\Gamma)$, set

$$B(P) = \{V \in \mathcal{M}(\Gamma) \mid (\exists\, Z \in \mathcal{M}(\Gamma))(V = P * Z * P)\}.$$

If iii. above holds, show there exists $P \in \mathcal{M}(\Gamma)$ so that $P * P = P$ and $\Gamma P \Gamma = \Gamma$. For such a P, prove that $B(P)$ is a subring of $\mathcal{M}(\Gamma)$, that $\mathcal{M}(\Gamma) \cong B(P) \amalg \mathcal{I}$ in the category of A-modules, and that $\mathcal{I}$ is a two-sided ideal of $\mathcal{M}(\Gamma)$ (by exhibiting $\mathcal{I}$ as the kernel of a surjective ring homomorphism whose image you should find). Further show if i. holds, then $B(P)$ is isomorphic to the ring of $r \times r$ matrices with entries from A. When A is a field show $\mathcal{I}$ is a maximal 2-sided ideal of $\mathcal{M}(\Gamma)$, here $\Gamma \neq (0)$. Is $\mathcal{I}$ the unique maximal (2-sided) ideal in this case?

(e) Call an idempotent, P, of a ring *maximal* (also called *principal*) iff when L is another idempotent, then $PL = 0 \implies L = 0$. Suppose Γ satisfies condition iii. above, prove that an idempotent, P, of $\mathcal{M}(\Gamma)$ is maximal iff $\Gamma P \Gamma = \Gamma$.

BIII) Let A be the field of real numbers $\mathbb{R}$ and conserve the notations of problem BII. Write X for a $p \times q$ matrix of functions of one variable, t, and consider the Γ-*Riccati Equation*

$$\frac{dX}{dt} = X\Gamma X \qquad (*)_\Gamma.$$

(a) If $q = p$ and Γ is invertible, show that either the solution, $X(t)$, blows up at some finite t, or else $X(t)$ is equivalent to a matrix

$$\widetilde{X}(t) = \begin{pmatrix} 0 & O(1) & O(t) & \dots & O(t^{p-1}) \\ 0 & 0 & O(1) & \dots & O(t^{p-2}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

where $O(t^s)$ means a polynomial of degree $\leq s$. Hence, in this case, $X(t)$ must be nilpotent.

(b) Suppose $q \neq p$ and Γ has rank r. Let P be an idempotent of $\mathcal{M}(\Gamma)$ with $\Gamma P \Gamma = \Gamma$. If $Z \in \mathcal{M}(\Gamma)$, write $Z^\flat$ for $Z - P * Z * P$; so $Z^\flat \in \mathcal{I}$. Observe that $\mathcal{I}$ has dimension $pq - r^2$ as an $\mathbb{R}$-vector space. Now assume that for a solution, $X(t)$, of $(*)_\Gamma$, we have $X(0) \in \mathcal{I}$. Prove that $X(t)$ exists for all t. Can you give necessary and sufficient conditions for $X(t)$ to exist for all t?

(c) Apply the methods of (b) to the case $p = q$ but $r = \mathrm{rank}\,\Gamma < p$. Give a similar discussion.

BIV) A module, M, over a ring, R, is called *indecomposable* iff we *cannot* find two submodules M_1 and M_2 of M so that $M \xrightarrow{\sim} M_1 \amalg M_2$ in the category of R-modules.

(a) Every ring is a module over itself. Show that if R is a local ring, then R is indecomposable as an R-module.

(b) Every ring, R, with unity admits a homomorphism $\mathbb{Z} \to R$ (i.e., $\mathbb{Z}$ is an *initial object* in the category RNG). The kernel of $\mathbb{Z} \to R$ is the principal ideal $n\mathbb{Z}$ for some $n \geq 0$; this n is the *characteristic of* R. Show that the characteristic of a local ring must be 0 or a prime power. Show by example that every possibility occurs as a characteristic of some local ring.

(c) Pick a point in $\mathbb{R}$ or $\mathbb{C}$; without loss of generality, we may assume this point is 0. If f is a function we say f is locally defined at 0 iff f has a domain containing some (small) open set, U, about 0 (in either $\mathbb{R}$ or $\mathbb{C}$). Here, f is $\mathbb{R}$- or $\mathbb{C}$-valued, independent of where its domain is. When f and g are locally defined at 0, say f makes sense on U and g on V, we'll call f and g *equivalent at* 0 $\Longleftrightarrow$ there exists open W, $0 \in W$, $W \subseteq U \cap V$ and $f \restriction W = g \restriction W$. A *germ of a function at* 0 is an equivalence class of a function. If we consider germs of functions that are at least continuous near 0, then when they form a ring they form a local ring.

Consider the case $\mathbb{C}$ and complex valued germs of holomorphic functions at 0. This is a local ring. Show it is a good local ring.

In the case $\mathbb{R}$, consider the germs of real valued C^k functions at 0, for some k with $0 \leq k \leq \infty$. Again, this is a local ring; however, show it is NOT a good local ring.

Back to the case $\mathbb{C}$ and the good local ring of germs of complex valued holomorphic functions at 0. Show that this local ring is also a principal ideal domain.

In the case of real valued C^∞ germs at $0 \in \mathbb{R}$, exhibit an infinite set of germs, each in the maximal ideal, no finite subset of which generates the maximal ideal (in the sense of ideals). These germs are NOT to belong to $\mathfrak{m}^2$.