Math 602, Fall 2002, HW 2, due 10/14/2002

Part A

AI) A *Fermat prime*, p, is a prime number of the form $2^\alpha + 1$. E.g., $2, 3, 5, 17, 257, \ldots$.

(a) Show if $2^\alpha + 1$ is prime then $\alpha = 2^\beta$.

(b) Say p is a Fermat prime (they are quite big) and g_0 is an *odd* number with $g_0 < p$. Prove that any group of order $g_0 p$ is isomorphic to a product $G_0 \prod (\mathbb{Z}/p\mathbb{Z})$, where $\#(G_0) = g_0$. Hence, for example, the groups of orders $51 (= 3 \cdot 17)$, $85 (= 5 \cdot 17)$, $119 (= 7 \cdot 17)$, $153 (= 9 \cdot 17)$, $187 (= 11 \cdot 17)$, $221 (= 13 \cdot 17)$, $255 (= 3 \cdot 5 \cdot 17)$ are all abelian. Most we knew already, but $153 = 3^2 \cdot 17$ and $255 = 3 \cdot 5 \cdot 17$ are new.

(c) Generalize to any prime, p, and $g_0 < p$, with $p \not\equiv 1 \bmod g_0$. For example, find all groups of order 130.

AII) Recall that a group, G, is *finitely generated* (f.g.) $\iff (\exists\, \sigma_1, \ldots, \sigma_n \in G)(G = \mathrm{Gp}\{\sigma_1, \ldots, \sigma_n\})$.

(a) If G is an *abelian* f.g. group, prove each of its subgroups is f.g.

(b) In an arbitrary group, G, an element $\sigma \in G$ is called *n-torsion* ($n \in \mathbb{N}$) $\iff \sigma^n = 1$; σ is torsion iff it is n-torsion for some $n \in \mathbb{N}$. The element $\sigma \in G$ is *torsion free* $\iff$ it is NOT torsion. Show that in an abelian group, the set

$$t(G) = \{\sigma \in G \mid \sigma \text{ is torsion}\}$$

is a subgroup and that $G/t(G)$ is torsion free (i.e., all its non-identity elements are torsion free).

(c) In one of the groups discussed in class, namely the solvable group $0 \to \mathbb{Z} \to G \to \mathbb{Z}/2\mathbb{Z} \to 0$ (split extension, non-trivial action) find two elements x, y satisfying: $x^2 = y^2 = 1$ and xy is torsion free. Can you construct a group, $\widetilde{G}$, possessing elements x, y of order 2, so that xy has order n, where n is predetermined in $\mathbb{N}$? Can you construct $\widetilde{G}$ solvable with these properties?

(d) Back to the abelian case. If G is abelian and finitely generated show that $t(G)$ is a finite group.

(e) Say G is abelian, f.g., and torsion-free. Write d for the minimal number of generators of G. Prove that G is isomorphic to a product of d copies of $\mathbb{Z}$.

(f) If G is abelian and f.g., prove that

$$G \cong t(G) \prod \big(G/t(G)\big).$$

AIII) Let (P) be a property of groups. We say a group, G, is *locally (P)* $\iff$ each f.g. subgroup of G has (P). Usually, one says a locally cyclic group is a *rank one group.*

(a) Prove that a rank one group is abelian.

(b) Show that the additive group of rational numbers, $\mathbb{Q}^+$, is a rank one group.

(c) Show that every torsion-free, rank one group is isomorphic to a subgroup of $\mathbb{Q}^+$.

AIV) Fix a group, G, and consider the set, $\mathcal{M}_n(G)$, of $n \times n$ matrices with entries from G or so that $\alpha_{ij} = 0$ (i.e., entries are 0 or from G). Assume for each row and each column there is one and only one non-zero entry. These matrices form a group under ordinary "matrix multiplication" if we define $0 \cdot$ group element $=$ group element $\cdot\, 0 = 0$. Establish an isomorphism of this group with the wreath product $G^n \wr \mathfrak{S}_n$. As an application, for the subgroup of $\mathrm{GL}(n, \mathbb{C})$ consisting of diagonal matrices, call it Δ_n, show that

$$N_G(\Delta_n) \cong \mathbb{C}^n \wr \mathfrak{S}_n, \quad \text{here } G = \mathrm{GL}(n, \mathbb{C}).$$

AV) (a) Say G is a simple group of order n and say p is a prime number dividing n. If $\sigma_1, \ldots, \sigma_t$ is a listing of the elements of G of exact order p, prove that $G = \mathrm{Gp}\{\sigma_1, \ldots, \sigma_t\}$.

(b) Suppose G is any finite group of order n and that d is a positive integer relatively prime to n. Show that every element of G is a dth power.

Part B

BI) In class, we stated that when G is a (finite) cyclic group, and A is any G-module, we have an isomorphism

$$A^G/\mathcal{N}(A) \xrightarrow{\sim} H^2(G, A).$$

This problem is designed to lead you to a proof. I am quite aware of other proofs which you might dig out of books (after some effort), but I want you to do *this* proof.

(a) Suppose G is any group and A, B, C are G-modules. Suppose further, we are given a *G-pairing of* $A \prod B \to C$ i.e., a map

$$\theta : A \prod B \to C$$

which is bi-additive and "G-linear":

$$\sigma\theta(a, b) = \theta(\sigma a, \sigma b).$$

If f, g are r-, s-cochains of G with values in A, B (respectively) we can define an $(r+s)$-cochain of G with values in C *via* the formula:

$$(f \smile_\theta g)(\sigma_1, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_{r+s}) = \theta\big(f(\sigma_1, \ldots, \sigma_r), \sigma_1 \ldots \sigma_r g(\sigma_{r+1}, \ldots, \sigma_{r+s})\big).$$

Prove that $\delta(f \smile_\theta g) = \delta f \smile_\theta g + (-1)^r f \smile_\theta \delta g$. (It may be that the sign should be $(-1)^{rs}$, I'm not sure, but for what follows the "correct" sign will be irrelevant. Also, you will figure it out!) Show how you conclude from this that we have a pairing of abelian groups

$$\smile_\theta\colon H^r(G, A) \prod H^s(G, B) \to H^{r+s}(G, C).$$

(Notation and nomenclature: $\alpha \smile_\theta \beta$, *cup-product.*)

(b) Again G is any group, this time finite. Let $\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}$ be G-modules with trivial action. Consider the abelian group $\mathrm{Hom}_{\mathrm{gr}}(G, \mathbb{Q}/\mathbb{Z}) = \widetilde{G}$, where addition in $\widetilde{G}$ is by pointwise operation on functions. If $\chi \in \widetilde{G}$, then $\chi(\sigma) \in \mathbb{Q}/\mathbb{Z}$, all $\sigma \in G$. Show that the function

$$f_\chi(\sigma, \tau) = \delta\chi(\sigma, \tau) = \sigma\chi(\tau) - \chi(\sigma\tau) + \chi(\sigma)$$

has values in $\mathbb{Z}$ and actually is a 2-cocycle with values in $\mathbb{Z}$. (This is an example of the "principle": "if it looks like a coboundary, it is certainly a cocycle.") The map

$$\chi \in \widetilde{G} \mapsto \text{cohomology class of } f_\chi(\sigma, \tau) \tag{†}$$

gives a homomorphism $\widetilde{G} \to H^2(G, \mathbb{Z})$.
Now *any* 2-cocycle $g(\sigma, \tau)$ with values in $\mathbb{Z}$ can be regarded as a 2-cocycle with values in $\mathbb{Q}$ (corresponding to the injection $\mathbb{Z} \hookrightarrow \mathbb{Q}$). Show that *as a 2-cocycle in* $\mathbb{Q}$ it is a coboundary (of some $h(\sigma)$, *values in* $\mathbb{Q}$). So, $g(\sigma, \tau) = \delta h(\sigma, \tau)$, some h. Use this construction to prove:

For any finite group, G, the map (†) above gives an *isomorphism* of $\widetilde{G}$ with $H^2(G, \mathbb{Z})$.

(c) Now let G be finite, A be any G-module, and $\mathbb{Z}$ have the trivial G-action. We have an obvious G-pairing $\mathbb{Z} \prod A \to A$, namely $(n, a) \mapsto na$, hence by (a) and (b) we obtain a pairing

$$\widetilde{G}(= H^2(G, \mathbb{Z})) \prod A^G \to H^2(G, A).$$

Show that if $\xi = \mathcal{N}\alpha$, $\alpha \in A$, then (χ, ξ) goes to 0 in $H^2(G, A)$; hence, we obtain a pairing:

$$\widetilde{G} \prod (A^G/\mathcal{N}A) \to H^2(G, A).$$

(Hint: if $f(\sigma, \tau)$ is a 2-cocycle of G in A, consider the 1-cochain $u_f(\tau) = \sum_{\sigma \in G} f(\sigma, \tau)$. Using the cocycle condition and suitable choices of the variables, show the values of u_f are in A^G and that u_f is related to $\mathcal{N}f$, i.e., $\mathcal{N}f(\tau, \rho) = \sum_\sigma \sigma f(\tau, \rho)$ can be expressed by u_f.)

(d) Finally, when G is cyclic, we pick a generator σ_0. There exists a distinguished element, χ_0, of $\widetilde{G}$ corresponding to σ_0, namely χ_0 is that homomorphism $G \to \mathbb{Q}/\mathbb{Z}$ whose value at σ_0, $\chi_0(\sigma_0)$, is $\frac{1}{n} \bmod \mathbb{Z}$, where $n = \#(G)$. Show that the map

$$A^G/\mathcal{N}A \to H^2(G, A)$$

via

$$\alpha \mapsto (\chi_0, \alpha) \mapsto \delta\chi_0 \smile \alpha \in H^2(G, A)$$

is the required isomorphism. For surjectivity, I suggest you consider the construction of u_f in part (c) above.

BII) Let $G = \mathrm{SL}(2, \mathbb{Z})$ be the group of all 2×2 integral matrices of determinant 1; pick a prime, p, and write U for the set of 2×2 integral matrices having determinant p. G acts on U *via* $u(\in U) \mapsto \sigma u$, where $\sigma \in G$.

(a) Show that the orbit space has $p+1$ elements: $0, 1, \ldots, p-1, \infty$, where j corresponds to the matrix

$$w_j = \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}$$

and ∞ corresponds to the matrix $w_\infty = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.

(b) If $\tau \in G$ and $r \in S = \{0, 1, \ldots, p-1, \infty\} = G \backslash U$, show there exists a unique $r' \in S$ with $w_n \tau^{-1}$ in the orbit of $w_{r'}$. Write $\tau \cdot r = r'$ and prove this gives an action of G on S. Hence, we have a group homomorphism $P : G \to \mathrm{Aut}(S) = \mathfrak{S}_{p+1}$.

(c) If $N = \ker P$, prove that G/N is isomorphic to the group $\mathrm{PSL}(2, \mathbb{F}_p)$ consisting of all "fractional linear transformations"

$$x \mapsto x' = \frac{ax+b}{cx+d}, \quad a, b, c, d \in \mathbb{F}_p,\ ad - bc = 1.$$

Show further that

i. $\#\big(\mathrm{PSL}(2, \mathbb{F}_p)\big) = \begin{cases} \dfrac{p(p+1)(p-1)}{2} & \text{if } p \neq 2 \\ 6 & \text{if } p = 2 \end{cases}$

and

ii. $\mathrm{PSL}(2, \mathbb{F}_p)$ act transitively on S under the action of (b).

(d) Now prove: $\mathrm{PSL}(2, \mathbb{F}_p)$ is simple if $p \geq 5$. (Note: $\mathrm{PSL}(2, \mathbb{F}_3)$ is A_4, $\mathrm{PSL}(2, \mathbb{F}_5)$ is A_5, but $\mathrm{PSL}(2, \mathbb{F}_p)$ is not A_n if $p \geq 7$. So, you now have a second infinite collection of simple finite groups—these are the finite group analogs of the Lie groups $\mathrm{PSL}(2, \mathbb{C})$).

BIII) Let G be a finite group in this problem.

(a) Classify all group extensions

$$0 \to \mathbb{Q} \to \mathcal{G} \to G \to 0 \qquad (E).$$

Your answer should be in terms of the collection of all subgroups of G, say H, with $(G : H) \leq 2$, plus, perhaps, other data.

(b) Same question as (a) for group extensions

$$0 \to \mathbb{Z} \to \mathcal{G} \to G \to 0 \qquad (E),$$

same kind of answer.

(c) Write V for the "four-group" $\mathbb{Z}/2\mathbb{Z} \prod \mathbb{Z}/2\mathbb{Z}$. There are two actions of $\mathbb{Z}/2\mathbb{Z}$ on V: i. flip the factors, ii. take each element to its inverse. Are these the only actions? Find all group extensions

$$0 \to V \to \mathcal{G} \to \mathbb{Z}/p\mathbb{Z} \to 0 \qquad (E).$$

The group $\mathcal{G}$ is a group of order 8; compare your results with what you know from Assignment 1.

(d) Say H is any other group, G need no longer be finite and A, B are abelian groups. Suppose $\varphi : H \to G$ is a homomorphism and we are given a group extension

$$0 \to A \to \mathcal{G} \to G \to 0, \qquad (E).$$

Show that, in a canonical way, we can make a group extension

$$0 \to A \to \widetilde{\mathcal{G}} \to H \to 0 \qquad (\varphi^* E).$$

(Note: your answer has to be in terms of G, H, $\mathcal{G}$ and any homomorphisms between them as these are the only "variables" present. You'll get the idea if you view an extension as a fibre space as remarked in class.)
Now say $\psi : A \to B$ is a group homomorphism and we are given an extension

$$0 \to A \to \mathcal{G} \to G \to 0 \qquad (E).$$

Construct, in a canonical way, an extension

$$0 \to B \to \widetilde{\mathcal{G}} \to G \to 0 \qquad (\psi_* E).$$

(e) Explain, carefully, the relevance of these two constructions to parts (a) and (b) of this problem.

BIV) Say A is any abelian group, and write G for the *wreath product* $A \wr \mathfrak{S}_n$, as in class. Show:

(a) $[G, G] \neq G$

(b) $\big(G : [G, G]\big) = \infty \iff A$ is infinite

(c) If $n \geq 2$, then $[G, G] \neq \{1\}$.

(d) Give a restriction on n which prevents G from being solvable.

BV) If $\{G_\alpha\}_{\alpha \in \Lambda}$ is a family of *abelian* groups, write $\coprod_\alpha G_\alpha$ for

$$\coprod_\alpha G_\alpha = \Big\{ (\xi_\alpha) \in \prod_\alpha G_\alpha \mid \text{ for all but finitely many } \alpha, \text{ we have } \xi_\alpha = 0 \Big\}.$$

Let's refer to $\coprod_\alpha G_\alpha$ as the *coproduct* of the G_α. Write as well

$$(\mathbb{Q}/\mathbb{Z})_p = \{\xi \in \mathbb{Q}/\mathbb{Z} \mid p^r\xi = 0, \text{ some } r > 0\};$$

here, p is a prime. Further, call an *abelian* group A *divisible* iff

$$(\forall\, n)(A \xrightarrow{n} A \to 0 \text{ is exact}).$$

Prove: Theorem Every divisible (abelian) group is a coproduct of copies of $\mathbb{Q}$ and $(\mathbb{Q}/\mathbb{Z})_p$ for various primes p. The group is torsion iff no copies of $\mathbb{Q}$ appear, it is torsion-free iff no copies of $(\mathbb{Q}/\mathbb{Z})_p$ appear (any p). Every torsion-free, divisible, abelian group is naturally a vector space over $\mathbb{Q}$.