Math 602, Fall 2002, HW 1, due 9/30/2002

Part A

AI) (a) Suppose G is a finite group and that $\mathrm{Aut}_{\mathrm{Gr}}(G) = \{1\}$. (Here, $\mathrm{Aut}_{\mathrm{Gr}}(G)$ is the group of all bijections, $G \to G$, which are also group homomorphisms.) Find *all* such groups G.

(b) Write $\mathbb{Z}/2\mathbb{Z}$ for the cyclic group of order 2. If $G = \mathbb{Z}/2\mathbb{Z} \prod \cdots \prod \mathbb{Z}/2\mathbb{Z}$, t-times, compute $\#\big(\mathrm{Aut}_{\mathrm{Gr}}(G)\big)$. When $t = 2$, determine the group $\mathrm{Aut}_{\mathrm{Gr}}(G)$. When $t = 3$, determine the structure of the odd prime Sylows. Can you decide whether $\mathrm{Aut}_{\mathrm{Gr}}(G)$ has any normal subgroups in the case $t = 3$?

AII) (a) (Poincaré). In an infinite group, prove that the intersection of two subgroups of finite index has finite index itself.

(b) Show that if a group, G, has a subgroup of finite index, then it possesses a normal subgroup of finite index. Hence, an infinite simple group has no subgroups of finite index.

(c) Sharpen (b) by proving: if $(G : H) = r$, then G possesses a normal subgroup, N, with $(G : N) \le r!$. Conclude immediately that a group of order 36 cannot be simple.

AIII) Let $G = \mathrm{GL}(n, \mathbb{C})$ and Δ_n be the subgroup of matrices with entries only along the diagonal. Describe precisely $N_G(\Delta_n)$ in terms of what the matrices look like.

AIV) Say G is a group and $\#(G) = p^n g_0$, where p is a prime and $(p, g_0) = 1$. Assume

$$r > \sum_{j=1}^{g_0-1} \sum_{k>0} [j/p^k]$$

$\big([x]$ = largest integer $\le x\big)$. Prove that G is not simple. Show that this governs all groups of order < 60, except for $\#(G) = 30, 40, 56$. We know, from class, that $\#(G) = 30 \implies G$ not simple. Show by explicit argument that groups of orders $40, 56$ are not simple. (Here, of course, by simple we mean non-abelian and simple.)

AV) In a p-group, G, we must have

$$(G : Z(G)) \ge p^2$$

(provided, of course, G is non-abelian). Show that for non-abelian groups of order p^3, $Z(G) \cong \mathbb{Z}/p\mathbb{Z}$ and $G/Z(G) \cong \mathbb{Z}/p\mathbb{Z} \prod \mathbb{Z}/p\mathbb{Z}$.

AVI) Let G be the group of automorphisms of a regular polyhedron with v vertices, e edges, and f faces. Show that G has order $g = fs = vr = 2e$, where s is the number of sides to a face and r is the number of edges emanating from a vertex. From the topology, one knows Euler's formula

$$v - e + f = 2.$$

Find the only possible values for v, e, f, r, s, g. Make a table.

Part B

BI) Let p be a prime number. Find all non-abelian groups of order p^3. Get started with the Burnside basis theorem, but be careful to check that the groups on your list are non-isomorphic. Also make sure your list is exhaustive. Your list should be a description of the generators of your groups and the relations they satisfy.

BII) Let G be a finite group and write $c(G)$ for the number of distinct conjugacy classes in G. This number will increase (in general) as $\#(G) \to \infty$; so, look at

$$\overline{c}(G) = \frac{c(G)}{\#(G)}.$$

The number $\overline{c}(G)$ measures the "average number of conjugacy classes per element of G" and is 1 if G is abelian. Assume G is *non-abelian* from now on. Then $0 < \overline{c}(G) < 1$.

(a) Prove for all such G, $\overline{c}(G) \leq 5/8$.

(b) Suppose p is the smallest prime with $p \,|\, \#(G)$. Prove that

$$\overline{c}(G) \leq \frac{1}{p} + \frac{1}{p^2} - \frac{1}{p^3}.$$

Is the bound of (a) sharp; that is, does there exist G with $\overline{c}(G) = 5/8$? How about the bound of (b)?

BIII) If G is a finite group and H a normal subgroup of G, write P for a p-Sylow subgroup *of H*.

(a) Show that the natural injection

$$N_G(P)/N_H(P) \to G/H$$

(why does it exist, why injective?) is actually an isomorphism.

(b) Prove that the Frattini subgroup, $\varphi(G)$, of ANY finite group, G, has property N.

BIV) We've remarked that $\varphi(G)$ is a kind of "radical" in the group-theoretic setting. In this problem we study various types of radicals.
A *normal* subgroup, H, of G is called *small* iff for every $X \lhd G$, the equality $H \cdot X = G$ implies that $X = G$. (Note: $\{1\}$ is small, $\varphi(G)$ is small; so they exist.) Check that if H and L are small, so is HL, and if H is small and $K \lhd G$, then $K \subseteq H \implies K$ is small.

(a) The *small radical* of G, $\mathcal{J}^{**}(G)$, is

$$\mathcal{J}^{**}(G) = \big\{x \in G \,\big|\, \mathrm{Gp}\{\mathrm{Cl}(x)\} \text{ is small}\big\}.$$

(Here, $\mathrm{Cl}(x)$ is the conjugacy class of x in G, and $\mathrm{Gp}\{S\}$ is the group generated by S.) Prove that $\mathcal{J}^{**}(G)$ is a subgroup of G.

(b) The *Jacobson radical* of G, $\mathcal{J}^{*}(G)$, is the intersection of all maximal, normal subgroups of G; while the *Baer radical* of G, $\mathcal{J}(G)$, is the product (inside G) of *all* the small subgroups of G. Prove

$$\mathcal{J}^{**}(G) \subseteq \mathcal{J}(G) \subseteq \mathcal{J}^{*}(G).$$

(c) Prove *Baer's Theorem*: $\mathcal{J}^{**}(G) = \mathcal{J}(G) = \mathcal{J}^{*}(G)$. (Suggestion: if $x \notin \mathcal{J}^{**}(G)$, find $N \lhd G$ ($\neq G$) so that $\mathrm{Gp}\{\mathrm{Cl}(x)\}N = G$. Now construct an appropriate maximal normal subgroup not containing x.)

BV) Recall that a *characteristic* subgroup is one taken into itself by *all* automorphisms of the group.

(a) Prove that a group possessing no proper characteristic subgroups is isomorphic to a product of isomorphic simple groups. (Hints: choose $\widetilde{G}$ of smallest possible order (> 1) normal in G. Consider all subgroups, H, for which $H \cong G_1 \prod \cdots \prod G_t$, where each $G_j \lhd G$ and each $G_j \cong \widetilde{G}$. Pick t so that $\#(H)$ is maximal. Prove that H is characteristic. Show $K \lhd G_1$ (say) $\implies K \lhd G$.)

(b) Prove: in every finite group, G, a minimal normal subgroup, H, is either an elementary abelian p-group or is isomorphic to a product of mutually isomorphic, non-abelian, simple groups.

(c) Show that in a solvable group, G, only the first case in (b) occurs.

BVI) Let G be a finite p-group and suppose $\varphi \in \mathrm{Aut}(G)$ has order n (i.e., $\varphi\big(\varphi(\cdots(\varphi(x))\cdots)\big) = \mathrm{Id}$, all $x \in G$: we do φ n-times in succession and n is minimal). Suppose $(n,p) = 1$. Now φ induces an automorphism of $G/\varphi(G)$, call it $\overline{\varphi}$, as $\varphi(G)$ is characteristic. Remember that $G/\varphi(G)$ is a vector space over $\mathbb{F}_p$; so, $\overline{\varphi} \in \mathrm{GL}\big(G/\varphi(G)\big)$.

(a) Prove $\overline{\varphi} = \text{identity} \iff \varphi = \text{identity}$.

(b) Show that if d is the Burnside dimension of G, then

$$\#\big(\mathrm{GL}(G/\varphi(G))\big) = p^{\frac{d(d-1)}{2}} \prod_{k=1}^{d} (p^k - 1),$$

and that if P is a p-Sylow subgroup of $\mathrm{GL}\big(G/\varphi(G)\big)$, then $P \subseteq \mathrm{SL}\big(G/\varphi(G)\big)$; i.e., $\sigma \in P \implies \det(\sigma) = 1$.

(c) Let $\mathcal{P} = \{\varphi \in \mathrm{Aut}(G) \mid \overline{\varphi} \in P,$ no restriction on order of $\varphi\}$. Show that $\mathcal{P}$ is a p-subgroup of $\mathrm{Aut}(G)$.

(d) Call an element $\sigma \in \mathrm{GL}\big(G/\varphi(G)\big)$ *liftable* if it is $\overline{\varphi}$ for some $\varphi \in \mathrm{Aut}(G)$. Examine all G of order p, p^2, p^3 to help answer the following: is every σ liftable? If not, how can you tell (given σ) if σ is liftable?

BVII) Let p be a prime number and consider a set, S, of p objects: $S = \{\alpha_1, \ldots, \alpha_p\}$. Assume G is a *transitive* group of permutations of S (i.e., the elements of S form an orbit under G); further assume $(\alpha_1\alpha_2) \in G$ (here $(\alpha_1\alpha_2)$ is the transposition). Prove: $G = \mathfrak{S}_p$. (Suggestion: let $M = \{\alpha_j | (\alpha_1\alpha_j) \in G\}$, show that $\sigma \in \mathfrak{S}_p$ and $\sigma = 1$ outside $M \implies \sigma \in G$. Now prove $\#(M) | p$.)