Lecture 3 — January 22, 2002

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3.1 Helly Theorem

Recall from last time

Definition 3.1 (Convex Sets). *Ω is a convex set in $\mathbb{R}^d$ if $\forall$ p,q $\in \Omega$, segment(p,q) $\in \Omega$.*

Theorem 3.1 (Helly). *If $C = \{c_1, c_2, \ldots, c_n\}$ where c_i is convex in $\mathbb{R}^d$, and if $\forall$ subsets $c_{i1}, c_{i2}, \ldots, c_{id+1}$, $\bigcap_{j=0}^{d+1} c_{ij} \neq 0$ then $\bigcap_{i=1}^{n} c_i \neq 0$*

That is given $n \geq d+1$ convex subsets in $\mathbb{R}^d$,
If any d+1 of the subsets have a common point or points then all n subsets have a common point or points.

3.2 Centerpoints

Definition 3.2 (Centerpoint). *Given $P = \{\ p_1, p_2, \ldots, p_n\} \in \mathbb{R}^d$, $c \in \mathbb{R}^d$ is a δ-centerpoint of P if for every hyperplane h passing through c, $\mid h^+ \cap P \mid \leq (1-\delta)n$*

In d dimensions, a $\frac{1}{d+1}$-centerpoint always exists, this is a corollary of Helly's Theorem.

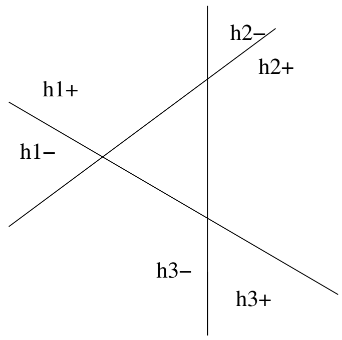
Let $\mathcal{H}$ be a collection of half spaces, that is

$$\mathcal{H} = \left\{ h^+ : \mid h^+ \cap P \mid \geq \left\lfloor \left(\frac{d}{d+1} \right) n \right\rfloor + 1 \right\}$$

$\mathcal{H}$ satisfies the Helly condition, consider $h_1^+, h_2^+, \ldots, h_{d+1}^+$

$$\left| \bigcap_{i=1}^{d+1} h_i^+ \cap P \right| \neq 0 \Leftrightarrow \bigcup_{i=1}^{d+1} \mid h_i^- \cap P \mid < n$$

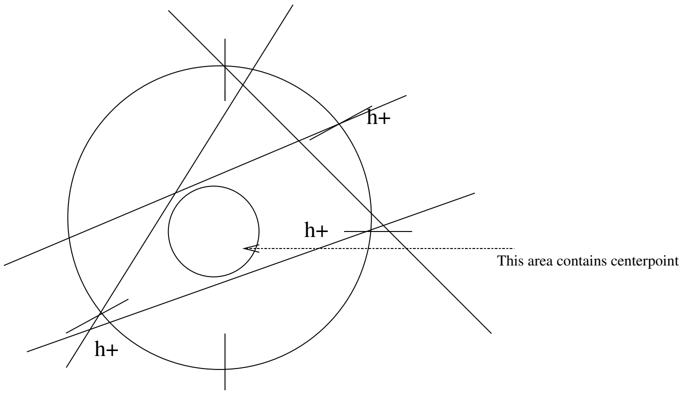
Pictorally,



By Helly theorem we know

$$\mathcal{H} = \left\{ h^+ : \mid h^+ \cap P \mid \geq \left\lfloor \left(\frac{d}{d+1} \right) n \right\rfloor + 1 \right\} \iff \left| \bigcap_{i=1}^{d+1} h_i^+ \cap P \right| \neq 0$$

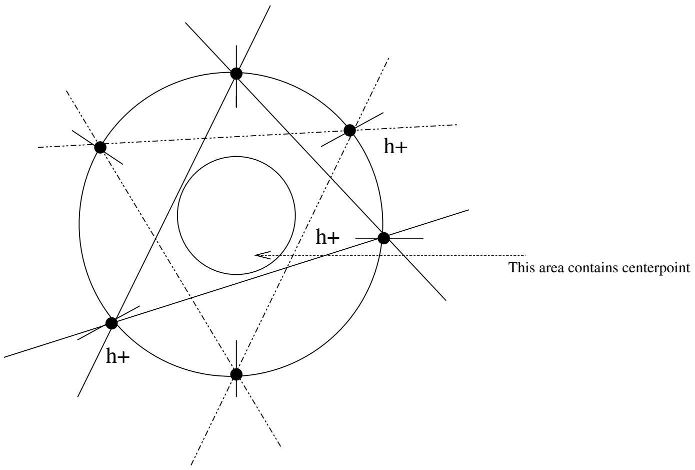
Consider,



If we choose c $\in \bigcap_{h^+ \in \mathcal{H}} h^+ \neq 0$ then c is a $\frac{1}{d+1}$-centerpoint

3.3 Combinatorially Interesting Events

Example of 6 points in 2 dimensions, if we draw h^+ to leave out 5 points this is combinatorally interesting, now we need only to compute $\binom{n}{d}$ half spaces to find centerpoints.



The *linear programming algorithm* uses linear programming to calculate these intersections of $\binom{n}{d}$ half spaces in $O(n^d)$ time. Notice that this algorithm is polnomial, but for $n \geq 4$ it is not very fast.

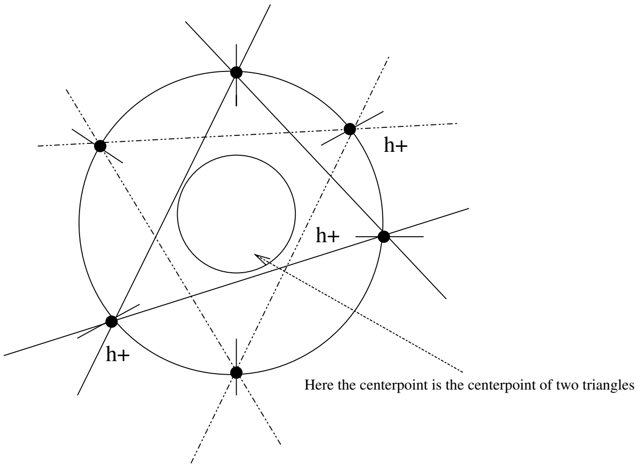
3.4 Geometric Sampling

Paradigms for geometric approximation

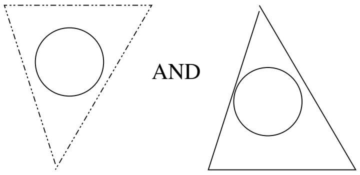
1. Find a somewhat polynomial solution/algorithm
2. Take a small random sample of the input, $\mathcal{S}$
3. Apply algorithm in 1. to $\mathcal{S}$

Now finding centerpoints can be sped up to $\frac{1}{d+1}+\epsilon$, constant time.

Interesting fact about two dimensions, suppose we have n=3m points - recall the previous example with 6 points.



But notice the two traingles above both include the centerpoint



So given $n = 3m = P = \{p_1, p_2, \ldots, p_n\}$, c is a $\frac{1}{3}$-centerpoint.

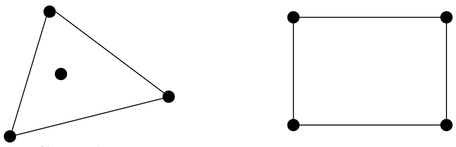
P can be divided into m subsets of 3 each $\bar{Q_1}, \bar{Q_2}, \ldots, \bar{Q_m}$ with $c \in \bigcap_{i=1}^{m} \Delta(Q_i)$

Generically if we have $(d+1)m$ points in $\mathbb{R}^d$, c is a $\frac{1}{d+1}$ centerpoint.

3.5 Finding centerpoints using Radon points

We would like to generalize the 1-dimensional case where given 3 elements we simply pick the middle one, and to do so without using linear programming.

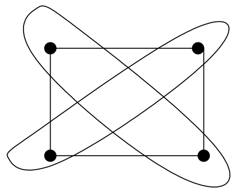
Consider case in 2-dimensions with 4 points



Case 1
1 point in a triangle

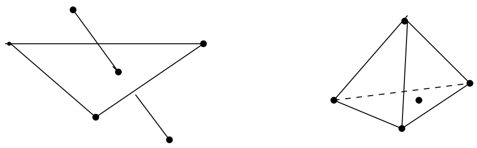
Case 2
No point in a triangle

To find the centerpoint in Case 1 it is easy, it is the point in the middle, for Case 2 consider



The center is located in the intersection of the ellipses.

Consider 3 dimensions with 5 points.



Definition 3.3 (Convex Hull). *Given a set of points $P = \{p_1, p_2, \ldots, p_n\}$, a convex hull is the smallest convex set that contains P.*

Theorem 3.2 (Radon). *If $P \subset \mathbb{R}^d$ with $\mid P \mid \geq d+2$ then there is a partition(P_1, P_2) of P such that the convex hull of P_1 has a common point with the convex hull of P_2.*

Definition 3.4 (Radon Point). *Let P be a set of points in $\mathbb{R}^d$. A point $q \in \mathbb{R}^d$ is a Radon point if P can be partioned into 2 disjoint subsets P_1 and P_2 such that q is a common point of the convex hull of P_1 and the convex hull of P_2.*

Before giving a formal proof of Radon's theorem consider example, this will also show us how an algorithm can be created from the proof.

Ex. 3 dimensions, 5 points, call them

$$\begin{array}{ccccc} P_1 & P_2 & P_3 & P_4 & P_5 \\ \begin{pmatrix} x_1 \\ y_1 \\ z_3 \end{pmatrix} & \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} & \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} & \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} & \begin{pmatrix} x_5 \\ y_5 \\ z_5 \end{pmatrix} \end{array}$$

Now consider the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

This is equivalent to solving

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 0$$
$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 + \alpha_5 x_5 = 0$$
$$\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 + \alpha_4 y_4 + \alpha_5 y_5 = 0$$
$$\alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3 + \alpha_4 z_4 + \alpha_5 z_5 = 0$$

Here we have 5 variables and 4 equations, so there is a solution, indeed there are infinitely many.

Now let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be any non zero solution.

Consider 1st equation $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 0$
Since $\alpha_j \neq 0 \; \forall j$ there are negative and positive values for α.
Here there is a 1:4 or 2:3 ratio positive to negative, without loss of generality consider the 2:3 case.
Say α_1 and α_2 are positive then
$\alpha_1 + \alpha_2 = (-\alpha_3) + (-\alpha_4) + (\alpha_5) = K$
Now divide by K, yielding

$$\frac{\alpha_1}{K} + \frac{\alpha_2}{K} = 1 \quad AND \quad \frac{-\alpha_3 - \alpha_4 - \alpha_5}{K} = 1$$

Now considering the points in the equation

$$\left(\frac{\alpha_1}{K}\right) P_1 + \left(\frac{\alpha_2}{K}\right) P_2 = \left(-\frac{\alpha_3}{K}\right) P_3 + \left(-\frac{\alpha_4}{K}\right) P_4 + \left(-\frac{\alpha_5}{K}\right) P_5$$

And now
$\left(\frac{\alpha_1}{K}\right) P_1 + \left(\frac{\alpha_2}{K}\right) P_2$ is a point in convex hull.

This can easily be formalized to become an algorithm to compute Radon points

Formal proof of Radon Theorem. [1] As we did above consider
the general case that is P is a set of n points, where $n \geq d+2$.
Now consider a system of d+1 homogenous linear equations just as above

$$\begin{aligned}
&\alpha_1 + \alpha_2 + \cdots + \alpha_n = 0 \\
&\alpha_1 x_{11} + \alpha_2 x_{12} + \cdots + \alpha_d x_{1n} = 0 \\
&\alpha_1 x_{21} + \alpha_2 x_{22} + \cdots + \alpha_d x_{2n} = 0 \\
&\vdots \\
&\alpha_1 x_{d1} + \alpha_2 x_{d2} + \cdots + \alpha_d x_{dn} = 0
\end{aligned}$$

As before this system has a nontrivial solution call it $(\vec{\alpha} n)$.
Now collect the positive and negative α's let $\mathcal{U}$ be the set of all i
where $\alpha_i \geq 0$ and let $\mathcal{V}$ be the set of all j where $\alpha_j \leq 0$ and $K = \sum_{i \in \mathcal{U}} \alpha_i \geq 0$. Then $\sum_{j \in \mathcal{V}} \alpha_j = -K$ and just as before $\sum_{i \in \mathcal{U}} \left(\frac{\alpha_i}{K}\right) p_i = \sum_{j \in \mathcal{V}} \left(\frac{\alpha_j}{K}\right) p_j$. Now let $P_1 = \{p_i \mid i \in \mathcal{U}\}, and P_2 = \{p_i \mid i \in \mathcal{V}\}$.
Then the partition (P_1, P_2) has the desired property, the covnex hull of P_1 has a common point with convex hull of P_2

Now that we have proved Radon's theorem it is very easy to formally prove Helly's theroem.

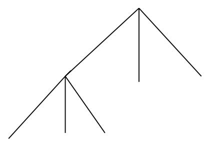
Recall if C $= \{c_1, c_2, \ldots, c_n\}$ where c_i is convex in $\mathbb{R}^d$, and if $\forall$ subsets $c_{i1}, c_{i2}, \ldots, c_{id+1}$, $\bigcap_{j=0}^{d+1} c_{ij} \neq 0$ then $\bigcap_{i=1}^{n} c_i \neq 0$.

Proof by induction.[2] Base case, n=d+1, this is trivial as n=d+1 has a common point by assumption. Now assume that this is true for all cases $d+1 \leq n \leq N-1$. Then the sets $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_i, \mathcal{S}_{i+1}, \ldots, \mathcal{S}_N$ have a common point call it p_i, for each $1 \leq i \leq N$. Let $P = \{p_i \mid 1 \leq i \leq n\}$ But since $N \geq d+2$,

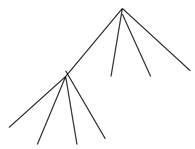
Radon's theorem holds, so there exist P_1 and P_2 such that the convex hull of P_1 and the convex hull of P_2 have a common point q. By the definiton of P, the convex hull of P_j belongs to set $\mathcal{S}_i$ whenever p_i is not in P_j. for $j = 1, 2$.

So q is contained in set $\mathcal{S}_i$ $\forall i, 1 \leq i \leq N$.

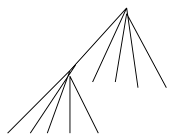
Now recall that to find the median in 1 dimension we need only take the middle of three



Now generalizing to 2 dimensions we take the Radon points of 4 branches (4 points)



In 3 dimensions we take Radon points of 5 branches (5 points)



Bibliography

[1] K. Clarkson, D. Eppstein, G. Miller, C. Sturtivant, and S.-H. Teng, "Approximating Center Points with Iterative Radon Points" *9th ACM Symp Computational Geometry*(1993).

[2] H. Edelsbrunner, *Algorithms in Combinatorial Geometry*, Springer-Verlang, 1987.