Homework IV (due March 17), Math 603, Spring 2003. (GJZ)

B II(a). In this problem, we assume that A is a noetherian integral domain. Furthermore, in this question, A is a normal domain, not a field.

In this question, of course, we assume that $a \in A$ is not a unit. Let $\mathfrak{p}$ be a prime ideal in A associated with Aa which is also a maximal ideal in A We want to prove that $\mathfrak{p}$ is a projective locally free module of rank 1, i.e., $\mathfrak{p} \in \mathrm{Pic}(A)$.

Set

$$(\mathfrak{p} \longrightarrow A) = \{\xi \in \mathrm{Frac}(A) \mid \xi\mathfrak{p} \subseteq A\}.$$

Obviously, $A \subseteq (\mathfrak{p} \longrightarrow A)$. First, we show that $(\mathfrak{p} \longrightarrow Aa) \neq Aa$. For this we prove

Proposition 1.1 *Let A be a noetherian domain. For any two ideals, $\mathfrak{A}$ and $\mathfrak{B}$ of A, if $(\mathfrak{A} \longrightarrow \mathfrak{B}) = \mathfrak{B}$, then $\mathfrak{A}$ is not contained in any prime associated with $\mathfrak{B}$.*

Proof. Let

$$\mathfrak{B} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$$

be a reduced primary decomposition of $\mathfrak{B}$. Then,

$$\sqrt{\mathfrak{B}} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t,$$

where the $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ are the primes associated with $\mathfrak{B}$. Since $(\mathfrak{A} \longrightarrow \mathfrak{B}) = \mathfrak{B}$, we can easily prove that $(\mathfrak{A}^s \longrightarrow \mathfrak{B}) = \mathfrak{B}$, for all $s \geq 1$. Assume that $\mathfrak{A} \subseteq \mathfrak{p}_i$, for some i with $1 \leq i \leq t$. Since A is noetherian, $\mathfrak{p}_i$ is finitely generated. Let $\alpha_1, \ldots, \alpha_m$ be generators for $\mathfrak{p}_i$. Since $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$, for each α_i, there is some positive integer, d_i, so that $\alpha_i^{d_i} \in \mathfrak{q}_i$; if we let $d = \sum_{i=1}^n d_i$, we find that $\mathfrak{p}_i^d \subseteq \mathfrak{q}_i$. As $\mathfrak{A} \subseteq \mathfrak{p}_i$, we get $\mathfrak{A}^d \subseteq \mathfrak{q}_i$ and so, $(\mathfrak{A}^d \longrightarrow \mathfrak{q}_i) = A$. It follows that

$$\mathfrak{B} = (\mathfrak{A}^d \longrightarrow \mathfrak{B}) = \bigcap_{j=1}^{t} (\mathfrak{A}^d \longrightarrow \mathfrak{q}_j) = \bigcap_{j \neq i} (\mathfrak{A}^d \longrightarrow \mathfrak{q}_j) \supseteq \bigcap_{j \neq i} \mathfrak{q}_j \supseteq \mathfrak{B},$$

and thus, $\mathfrak{B} = \bigcap_{j \neq i} \mathfrak{q}_j$, contradicting the fact that $\bigcap_{j=1}^t \mathfrak{q}_j$ is a reduced primary decomposition of $\mathfrak{B}$. Therefore, $\mathfrak{A}$ is not contained in any prime associated with $\mathfrak{B}$. $\square$

Remark: The converse of Proposition 1.1 also holds and is easier to prove.

Now, if $\mathfrak{p}$ is a prime associated with Aa, by Proposition 1.1, we must have $(\mathfrak{p} \longrightarrow Aa) > Aa$.

Since $(\mathfrak{p} \longrightarrow Aa) > Aa$, for any $x \in (\mathfrak{p} \longrightarrow Aa) - Aa$, consider $\xi = x/a \in \mathrm{Frac}(A)$. As $x \in (\mathfrak{p} \longrightarrow Aa)$, we have $x\mathfrak{p} \subseteq Aa$, and so $\xi \in (\mathfrak{p} \longrightarrow A) \cap \mathrm{Frac}(A)$, with $\xi \notin A$. This proves that

$$(\mathfrak{p} \longrightarrow A) > A.$$

Consider and $\xi \in (\mathfrak{p} \longrightarrow A) - A$. Clearly, $\xi\mathfrak{p}$ is an ideal in A. Assume that

$$\xi\mathfrak{p} \subseteq \mathfrak{p}.$$

Since A is noetherian, $\mathfrak{p}$ is finitely generated, and as in a previous homework problem, for every generator, m_i, of $\mathfrak{p}$, we can express each ξm_i as a linear combination of the m_i's. From this, we obtain the fact that ξ is the zero of a monic polynomial equation with coefficients from A (the determinant of a linear system). However, A is integrally closed; so, $\xi \in A$, a contradiction.

Using the above fact, we claim that

$$(\mathfrak{p} \longrightarrow A)\mathfrak{p} = A.$$

Since $A \subseteq (\mathfrak{p} \longrightarrow A)$, we have $\mathfrak{p} \subseteq (\mathfrak{p} \longrightarrow A)\mathfrak{p}$; since $\xi\mathfrak{p} \not\subseteq \mathfrak{p}$ for any $\xi \in (\mathfrak{p} \longrightarrow A) - A$, we must have $(\mathfrak{p} \longrightarrow A)\mathfrak{p} > \mathfrak{p}$. As $\mathfrak{p}$ is maximal, we deduce that

$$(\mathfrak{p} \longrightarrow A)\mathfrak{p} = A.$$

This implies that

$$a_1 b_1 + \cdots + a_n b_n = 1$$

for some $a_i \in \mathfrak{p}$ and some $b_i \in (\mathfrak{p} \longrightarrow A)$, for $i = 1, \ldots, n$. Let $\mathfrak{q}$ be any prime in A. We claim that $a_i b_i$ is a unit of $A_{\mathfrak{q}}$, for some i. Otherwise, as $A_{\mathfrak{q}}$ is a local ring, we would have $a_i b_i \in \mathfrak{q}$ for $i = 1, \ldots, n$, so, $1 \in \mathfrak{q}$, a contradiction. Then, we claim that

$$\mathfrak{p}_{\mathfrak{q}} = a_i A_{\mathfrak{q}}.$$

Since $a_i \in \mathfrak{p}$, it is clear that $a_i A_{\mathfrak{q}} \subseteq \mathfrak{p}_{\mathfrak{q}}$. Conversely, pick $x \in \mathfrak{p}_{\mathfrak{q}}$. Since $b_i \in (\mathfrak{p} \longrightarrow A)$, we have $b_i x \in A_{\mathfrak{q}}$, so, $a_i b_i x \in a_i A_{\mathfrak{q}}$ and since $a_i b_i$ is a unit of $A_{\mathfrak{q}}$, we get $x \in a_i A_{\mathfrak{q}}$. Thus, $\mathfrak{p}_{\mathfrak{q}} \subseteq a_i A_{\mathfrak{q}}$, and our claim is proved.

Therefore, each $\mathfrak{p}_{\mathfrak{q}}$ is a free module of rank 1, and so, $\mathfrak{p}$ is locally free or rank 1. Since A is noetherian, $\mathfrak{p}$ is f.g., and by a result proved in class, $\mathfrak{p}$ is projective and locally free of rank 1, as claimed.

B II(b). The ring A is a noetherian domain satisfying:

(i) For every nonzero minimal prime, $\mathfrak{p}$, of A, the ring $A_{\mathfrak{p}}$ is a P.I.D.

(ii) $A = \bigcap_{\mathrm{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$.

By definition, a nonzero minimal prime (in a noetherian domain) is a prime of height 1. Since every P.I.D. is integrally closed, by (i), for every height 1 prime, $\mathfrak{p}$, the local domain $A_{\mathfrak{p}}$ is integrally closed. Since any intersection of integrally closed domains is an integrally closed domain, by (ii), the ring A is an integrally closed domain, i.e., a normal domain.

B II(c). Although this is not needed, since it is an interesting fact, we prove that every commutative ring possesses minimal prime ideals. More precisely, we have the proposition:

Proposition 1.2 *Let A be a commutative ring. Then, every prime ideal, $\mathfrak{P}$, contains some minimal prime ideal, $\mathfrak{p}$.*

Proof. Let $\mathcal{S}$ be the set of primes contained in $\mathfrak{P}$ ordered so that $\mathfrak{p} \leq \mathfrak{q}$ iff $\mathfrak{q} \subseteq \mathfrak{p}$. Obviously, $\mathfrak{P} \in \mathcal{S}$ and $\mathcal{S}$ is nonempty. We claim that $\mathcal{S}$ is inductive. Given any chain, $\{\mathfrak{p}_i\}_{i\in I}$, of primes in $\mathcal{S}$, we contend that $\mathfrak{q} = \bigcap_{i\in I} \mathfrak{p}_i$ is a prime ideal. Let $a, b \in A$ with $a \notin \mathfrak{q}$ and $b \notin \mathfrak{q}$. Then, there is some $i \in I$ with $a \notin \mathfrak{p}_i$ and $b \notin \mathfrak{p}_i$. As $\mathfrak{p}_i$ is prime, we must have $ab \notin \mathfrak{p}_i$. But then, $ab \notin \bigcap_{i\in I} \mathfrak{p}_i = \mathfrak{q}$, which shows that $\mathfrak{q}$ is prime. Since $\mathcal{S}$ is inductive, $\mathcal{S}$ has some maximal element, $\mathfrak{p}$, i.e., there is a minimal prime, $\mathfrak{p}$, contained in $\mathfrak{P}$. $\square$

We claim that in a noetherian domain, every nonzero minimal prime is an isolated prime of some principal ideal, (a).

Let $\mathfrak{p} \neq (0)$ be a minimal prime of A. Since $\mathfrak{p} \neq (0)$, we can pick some $a \in \mathfrak{p}$, and we have $(a) \subseteq \mathfrak{p}$. Let

$$(a) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$$

be a reduced primary decomposition. Then,

$$\sqrt{(a)} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t,$$

where the $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ are the primes associated with (a). Clearly, $\mathfrak{p}_i \neq (0)$. We have $\sqrt{(a)} \subseteq \mathfrak{p}$, since $(a) \subseteq \mathfrak{p}$ and every prime is its own radical. Thus, we have

$$\mathfrak{p}_1 \cdots \mathfrak{p}_t \subseteq \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t \subseteq \mathfrak{p},$$

which implies that $\mathfrak{p}_i \subseteq \mathfrak{p}$, for some i. However, $\mathfrak{p}$ is a minimal prime, and so, $\mathfrak{p}_i = \mathfrak{p}$.

Now, assume that A is a noetherian normal domain. We must prove that properties (i) and (ii) of (b), hold. It turns out that (i) is a consequence of Theorem 1.3 proved in B II(d). Indeed, we just proved that every nonzero minimal prime is an isolated prime of some principal ideal, (a), and Theorem 1.3 finishes the proof.

It remains to prove (ii). For this, consider any $a/b \in \mathrm{Frac}(A)$ and assume that $a/b \in A_{\mathfrak{p}}$ for every height 1 prime, $\mathfrak{p}$; equivalently, we have $a \in bA_{\mathfrak{p}}$ for every height 1 prime, $\mathfrak{p}$. We wish to prove that $a \in bA$. Let

$$bA = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$$

be a reduced primary decomposition of bA, and let $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$, be the corresponding isolated primes. If we can prove that

$$\mathfrak{q}_i \supseteq (bA_{\mathfrak{p}_i}) \cap A,$$

we are done. Indeed, we know that $a \in (bA_{\mathfrak{p}_i}) \cap A$ for $i = 1, \ldots, t$, so $a \in \bigcap_{i=1}^t \mathfrak{q}_i = bA$. Now, every element of $A_{\mathfrak{p}_i}$ is of the form α/β, where $\beta \in A - \mathfrak{p}_i$ and $\alpha \in A$. Thus,

$$a\beta = b\alpha \in \mathfrak{q}_i,$$

for $i = 1, \ldots, t$, since $bA = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$. Since $\mathfrak{q}_i$ is primary and $a\beta \in \mathfrak{q}_i$, if $a \notin \mathfrak{q}_i$, then $\beta^k \in \mathfrak{q}_i$, for some $k \geq 1$. As $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$, we deduce that $\beta \in \mathfrak{p}_i$, a contradiction. Therefore, $a \in \mathfrak{q}_i$, as claimed. This proves that

$$\mathfrak{q}_i \supseteq (bA_{\mathfrak{p}_i}) \cap A$$

for $i = 1, \ldots, t$ and concludes the proof of (ii).

B II(d).

Theorem 1.3 *If A is a noetherian normal domain and $\mathfrak{A} = (a) = Aa$ is principal ideal (where a is not a unit), then every isolated prime ideal of $\mathfrak{A}$ has height* 1.

Proof. First, we need the fact that every isolated prime, $\mathfrak{p}$, of $\mathfrak{A} = Aa$ is a prime of the form $\mathfrak{p} = (b \longrightarrow Aa)$ and that such a prime is also minimal among the primes containing Aa. This fact has not yet been shown in class, but it is standard; for instance, see Serre, *Local Algebra*.

Let $\mathfrak{m} = \mathfrak{p}^e$ in $A_{\mathfrak{p}}$ be the maximal ideal of $A_{\mathfrak{p}}$. Since $\mathfrak{p} = (b \longrightarrow Aa)$, we get $\mathfrak{m} = (b \longrightarrow A_{\mathfrak{p}}a)$. As in (a), consider

$$(\mathfrak{m} \longrightarrow A_{\mathfrak{p}}) = \{\xi \in \mathrm{Frac}(A) \mid \xi\mathfrak{m} \subseteq A_{\mathfrak{p}}\}.$$

As $\mathfrak{m} = (b \longrightarrow A_{\mathfrak{p}}a)$, we have $b/a \in (\mathfrak{m} \longrightarrow A_{\mathfrak{p}})$, yet $b/a \notin A_{\mathfrak{p}}$ (because if $b/a \in A_{\mathfrak{p}}$, then $b \in aA_{\mathfrak{p}}$ and $\mathfrak{m} = A_{\mathfrak{p}}$, which is absurd). Note that $(b/a)\mathfrak{m}$ is an ideal of $A_{\mathfrak{p}}$. If $(b/a)\mathfrak{m} \subseteq \mathfrak{m}$, then, as in (a), b/a would be integral over $A_{\mathfrak{p}}$. However, A is noetherian and integrally closed. It follows that $A_{\mathfrak{p}}$ is also noetherian and integrally closed. Thus, $b/a \in A_{\mathfrak{p}}$, a contradiction. As $\mathfrak{m}$ is maximal in $A_{\mathfrak{p}}$, we conclude that

$$(b/a)\mathfrak{m} = A_{\mathfrak{p}}.$$

Thus, we have $(b/a)c = 1$, for some $c \in \mathfrak{m}$. We claim that

$$\mathfrak{m} = cA_{\mathfrak{p}}.$$

Since $c \in \mathfrak{m}$, we have $cA_{\mathfrak{p}} \subseteq \mathfrak{m}$. Pick any $x \in \mathfrak{m}$. We have $(b/a)x \in A_{\mathfrak{p}}$, so, $(b/a)cx = x \in cA_{\mathfrak{p}}$. Therefore, $\mathfrak{m} \subseteq cA_{\mathfrak{p}}$, and we are done.

It follows that $\mathfrak{m}$ is a principal ideal in $A_{\mathfrak{p}}$. Therefore, $A_{\mathfrak{p}}$ is a noetherian local domain whose maximal ideal is principal. The following proposition implies that $A_{\mathfrak{p}}$ is a P.I.D.

Proposition 1.4 *If B is a noetherian local domain and its maximal ideal, $\mathfrak{m}$, is principal, then B is a P.I.D.*

Proof. Let $\mathfrak{m} = bB$. First, we claim that $\bigcap_n \mathfrak{m}^n = (0)$. Indeed, if we let $\mathfrak{q} = \bigcap_n \mathfrak{m}^n = (0)$, we have $\mathfrak{q}\mathfrak{m} = \mathfrak{q}$. As A is noetherian, $\mathfrak{q}$ is finitely generated; so, by Nakayama's lemma, $\mathfrak{q} = (0)$.

We define the function $v\colon B \to \mathbb{N} \cup \{-\infty, +\infty\}$ as follows: For any $b \in B$,

$$v(b) = \begin{cases} -\infty & \text{if } b \notin \mathfrak{m} \\ n & \text{if } b \in \mathfrak{m}^n \text{ and } b \notin \mathfrak{m}^{n+1} \\ +\infty & \text{if } b = 0. \end{cases}$$

Let $\mathfrak{A}$ be any nonzero ideal in B. Since B is local, $\mathfrak{A} \subseteq \mathfrak{m}$. Thus, there is some $a \in \mathfrak{A}$ for which v is minimal when a ranges over $\mathfrak{A}$. Say $v(a) = n$. Then, $\mathfrak{A} \subseteq \mathfrak{m}^n = b^n B$. In particular, we have $a = b^n c$, for some $c \in B$. Since $a \in \mathfrak{m}^n$ and $a \notin \mathfrak{m}^{n+1}$, we must have $c \notin \mathfrak{m}$. However, as B is a local ring, this implies that c is a unit, and so, $b^n \in aB \subseteq \mathfrak{A}$. Since $\mathfrak{A} \subseteq \mathfrak{m}^n = b^n B$, we deduce that $\mathfrak{A} = b^n B$, i.e., $\mathfrak{A}$ is a principal ideal. Therefore, B is a P.I.D. $\square$

Now, as $A_\mathfrak{p}$ is a P.I.D., it has Krull dimension 1; thus, $\mathfrak{m}$ has height 1, and so, $\mathfrak{p}$ also has height 1. $\square$

B VI(a). Let A be an integral domain. We want to prove that

$$A = \bigcap_{\mathfrak{p}\in \operatorname{Spec} A} A_{\mathfrak{p}} = \bigcap_{\mathfrak{m}\in \operatorname{Max} A} A_{\mathfrak{m}}.$$

Since A is an integral domain, there is an natural inclusion $A \hookrightarrow \operatorname{Frac}(A)$ of A into its total fraction field, $\operatorname{Frac}(A)$. Also, for every prime ideal, $\mathfrak{p} \in \operatorname{Spec} A$, we have a natural inclusion $A \hookrightarrow A_{\mathfrak{p}}$. It follows that

$$A \subseteq \bigcap_{\mathfrak{p}\in \operatorname{Spec} A} A_{\mathfrak{p}} \subseteq \bigcap_{\mathfrak{m}\in \operatorname{Max} A} A_{\mathfrak{m}},$$

since every maximal ideal is a prime ideal. Thus, it suffices to prove that

$$\bigcap_{\mathfrak{m}\in \operatorname{Max} A} A_{\mathfrak{m}} \subseteq A.$$

Let $B = \bigcap_{\mathfrak{m}\in \operatorname{Max} A} A_{\mathfrak{m}}$; we need to prove that $B \subseteq A$. We will use the proposition used in class that says that for every A-module, M, if $M_{\mathfrak{m}} = (0)$ for every maximal ideal, $\mathfrak{m} \in \operatorname{Max} A$, then $M = (0)$. Here, $M = B/A$.

For every $\mathfrak{m} \in \operatorname{Max} A$, it is clear that the multiplicative set $S = A - \mathfrak{m}$ is also a multiplicative set in B, and since $A \subseteq B \subseteq A_{\mathfrak{m}}$, we get

$$A_{\mathfrak{m}} = S^{-1}A \subseteq S^{-1}B \subseteq S^{-1}A_{\mathfrak{m}} = A_{\mathfrak{m}}.$$

Therefore, $A_{\mathfrak{m}} = S^{-1}A = S^{-1}B$. Moreover, since we have an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$$

and since $S^{-1}A$ is flat over A, by tensoring with $S^{-1}A$ over A (using the fact that $S^{-1}M \cong M \otimes_A S^{-1}A$ for any A-module, M), we get

$$0 \longrightarrow S^{-1}A \longrightarrow S^{-1}B \longrightarrow S^{-1}(B/A) \longrightarrow 0 \quad \text{is exact;}$$

we deduce that

$$S^{-1}(B/A) \cong S^{-1}B/S^{-1}A.$$

However, we just proved that $S^{-1}A = S^{-1}B$, so, $S^{-1}(B/A) = (0)$, i.e., $(B/A)_{\mathfrak{m}} = (0)$ for all $\mathfrak{m} \in \operatorname{Max} A$, which implies that $B/A = (0)$, i.e., $B = A$, as required.

B VI(b). Now, A is any commutative ring and $f(T)$ is a polynomial of degree d in $A[T]$.

First, assume that the coefficient, a_0, of T^d in $f(T)$ is a unit. If so, the ideal $(f(T))$ is also generated by the monic polynomial $g(T) = a_0^{-1}f(T)$ of degree d. Now, since $g(T)$ is monic, we can divide any polynomial, $p(T) \in A[T]$ by $g(T)$. So, we can write

$$p(T) = g(T)q(T) + r(T), \quad \text{with} \quad \deg(r(T)) \leq d - 1.$$

It follows that $A[T]/(f(T))$ is isomorphic to the A-module of polynomials of degree at most $r-1$, modulo $g(T)$ (this means that $p_1(T)p_2(T) =$ the remainder of the division of $p_1(T)p_2(T)$ modulo $g(T)$). However, this module is generated by $1, T, \ldots, T^{d-1}$. Furthermore, these monomials are linearly independent, because if

$$a_0 T^{d-1} + \cdots + a_{d-2}T + a_{d-1} = 0 \pmod{g(T)},$$

as $\deg(g(T)) = d$, we must have

$$a_0 T^{d-1} + \cdots + a_{d-2}T + a_{d-1} \equiv 0 \quad \text{in } A[T].$$

(i.e., it is the zero polynomial.) Therefore, $A[T]/(f(T))$ is isomorphic to a free A-module of rank d; hence, $A[T]/(f(T))$ is isomorphic to a projective A-module of rank d.

Now, assume that $A[T]/(f(T))$ is isomorphic to a projective A-module of rank d. This means that for every prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, the A-module $(A[T]/(f(T)))_{\mathfrak{p}}$ is free of rank d. Assume that that the coefficient, a_0, of T^d in $f(T)$ is not a unit; we are going to derive a contradiction. If a_0 is not a unit, then the ideal $(a_0) = Aa_0$ is properly contained in A, so, there is some maximal (thus, prime) ideal, $\mathfrak{p}$, with $(a_0) \subseteq \mathfrak{p}$. For this $\mathfrak{p}$, by hypothesis, we have an exact sequence

$$0 \longrightarrow (f(T))_{\mathfrak{p}} \longrightarrow A[T]_{\mathfrak{p}} \longrightarrow A_{\mathfrak{p}}^d \longrightarrow 0.$$

However, going back to the defintion of localization, it is clear that $A[T]_{\mathfrak{p}} \cong A_{\mathfrak{p}}[T]$ and $(f(T))_{\mathfrak{p}} \cong f(T)A_{\mathfrak{p}}[T]$. Now, $A_{\mathfrak{p}}$ is a local ring, and since $A_{\mathfrak{p}}^d$ is free, $A_{\mathfrak{p}}^d$ is flat. Therefore, if we tensor with $\kappa(A_{\mathfrak{p}})$, by a proposition proved in a previous homework and in class, we get an exact sequence:

$$0 \longrightarrow \overline{f}(T)\kappa(A_{\mathfrak{p}})[T] \longrightarrow \kappa(A_{\mathfrak{p}})[T] \longrightarrow \kappa(A_{\mathfrak{p}})^d \longrightarrow 0.$$

In this sequence, all modules involved are vector spaces over $\kappa(A_{\mathfrak{p}})$ and $\overline{f}(T)$ denotes the polynomial obtained from $f(T)$ by reducing the coefficients of $f(T)$ modulo $\mathfrak{p}^e$, the maximal ideal of $A_{\mathfrak{p}}$. But, $a_0 \in \mathfrak{p}$, so $\overline{a_0} = 0$ and $\overline{f}(T)$ is a polynomial of degree at most $d-1$. By the first part of the proof, this would imply that $\kappa(A_{\mathfrak{p}})[T]/(\overline{f}(T)\kappa(A_{\mathfrak{p}})[T])$ has dimension at most $d-1$, a contradiction. Therefore, a_0 must be a unit.