Homework III (due February 24), Math 603, Spring 2003. (GJZ)

B II(a). In this question, k is a field and $A = k[X_1, \ldots, X_n]$. We define a set-theoretic map, $\varphi\colon \Omega^n \to \operatorname{Spec} A$, as follows: For every $\xi = (\xi_1, \ldots, \xi_n) \in \Omega^n$,

$$\varphi(\xi) = \mathfrak{p}(\xi) = \{f \in A \mid f(\xi) = 0\}.$$

It is clear that $\mathfrak{p}(\xi)$ is a prime ideal. The map $X_i \mapsto \xi_i$, for $i = 1, \ldots, n$, extends uniquely to a homomorphism, θ, from $A = k[X_1, \ldots, X_n]$ to Ω, such that $\theta(f) = f(\xi)$ and obviously,

$$\operatorname{Ker} \theta = \mathfrak{p}(\xi).$$

Thus, we obtain an isomorphism, $\overline{\theta}\colon A/\mathfrak{p}(\xi) \to \operatorname{Im} \theta$ between $A/\mathfrak{p}(\xi)$ and a subring of Ω. So, $\operatorname{Frac}(A/\mathfrak{p}(\xi))$ can be viewed as a subfield of Ω. By definition, $f(\mathfrak{p}(\xi))$ is the image, $\overline{f}$, of f in $A/\mathfrak{p}(\xi)$, and this image can be identified with $\overline{\theta}(\overline{f}) = \theta(f) = f(\xi)$ in Ω. Since we will prove shortly that φ is surjective, every prime ideal of $\operatorname{Spec} A$ is of the form $\mathfrak{p}(\xi)$, for some $\xi \in \Omega^n$, and the function induced by f on $\operatorname{Spec} A$ can be identified with the function induced by f on Ω^n, *via* $\overline{\theta}$.

To prove that φ is continuous, it is enough to show that the inverse image of a closed set of $\operatorname{Spec} A$ is a closed set in the k-topology. A closed set, C, in $\operatorname{Spec} A$ is of the form

$$C = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supseteq \mathfrak{A}, \}$$

where $\mathfrak{A}$ is an ideal in A. A closed set in the k-topology is a set of the form

$$V(\mathfrak{A}) = \{\xi \in \Omega^n \mid f(\xi) = 0 \quad \text{for all } f \in \mathfrak{A}\},$$

where $\mathfrak{A}$ is any ideal of $A = k[X_1, \ldots, X_n]$, because any ideal of A is finitely generated (since A is noetherian, as k is a field). Then,

$$\varphi^{-1}(C) = \{\xi \in \Omega^n \mid \{f \in A \mid f(\xi) = 0\} \subseteq \mathfrak{A}\}$$

and so,

$$\varphi^{-1}(C) = \{\xi \in \Omega^n \mid f(\xi) = 0 \quad \text{for all } f \in \mathfrak{A}\},$$

a closed set in the k-topology.

Let us now prove that φ is surjective. Let $\mathfrak{p}$ be any prime ideal of $A = k[X_1, \ldots, X_n]$ and let $x_i = \overline{X_i}$ be the image of X_i in $A/\mathfrak{p}$, for $i = 1, \ldots, n$. Then,

$$k[x_1, \ldots, x_n] \cong A/\mathfrak{p} \quad \text{and} \quad k(x_1, \ldots, x_n) \cong \operatorname{Frac}(A/\mathfrak{p}).$$

We may assume after renumbering $x_1, \ldots, x_n$ that $\{x_1, \ldots, x_r\}$ is a transcendence basis of $k(x_1, \ldots, x_n)$ over k. As Ω is assumed to have infinitely many transcendental elements over

k, we can find $y_1, \ldots, y_r \in \Omega$ algebraically independent over k so that there is a (injective) k-algebra homomorphism

$$\theta\colon k(x_1, \ldots, x_r) \longrightarrow \Omega$$

with $\theta(x_i) = y_i$, for $i = 1, \ldots, r$. Now, as $k(x_1, \ldots, x_n)$ is algebraic over $k(x_1, \ldots, x_r)$ and Ω is algebraically closed, the homomorphism θ can be extended to an injective homomorphism also denoted θ:

$$\theta\colon k(x_1, \ldots, x_n) \longrightarrow \Omega.$$

This is easily proved using Zorn's lemma and the fact that Ω is algebraically closed. If we let $y_i = \theta(x_i)$, for $i = 1, \ldots, n$, then we claim that $y = (y_1, \ldots, y_n)$ is such that $\varphi(y) = \mathfrak{p}$. Indeed, we have $f(y_1, \ldots, y_n) = 0$ iff $f(x_1, \ldots, x_n) = 0$ iff $f \in \mathfrak{p}$.

B II(b). Observe that $\Omega^n = V((0))$. The fact that Ω^n is irreducible will follow from the more general fact that if $\mathfrak{A}$ is a prime ideal, then $V(\mathfrak{A})$ is irreducible, as (0) is a prime ideal.

Assume that $V(\mathfrak{A}) = V(\mathfrak{A}_1) \cup V(\mathfrak{A}_2)$, where $V(\mathfrak{A}_1)$ and $V(\mathfrak{A}_2)$ are proper subsets of $V(\mathfrak{A})$. Then, there is some $f \in A$ with $f \in \mathfrak{A}_1$ and $f \notin \mathfrak{A}$ and there is some $g \in A$ with $g \in \mathfrak{A}_2$ and $g \notin \mathfrak{A}$. But, fg vanishes on $V(\mathfrak{A})$, so $fg \in \mathfrak{A}$, contradicting the fact that $\mathfrak{A}$ is prime.

B II(c). The closure of a point $\xi \in \Omega^n$ is the set $V(\mathfrak{p}(\xi))$. Thus, $\xi \sim \eta$ iff $\overline{\{\xi\}} = \overline{\{\eta\}}$ iff $\mathfrak{p}(\xi) = \mathfrak{p}(\eta)$ iff $\varphi(\xi) = \varphi(\eta)$. It follows that φ yields a bijection, $\overline{\varphi}$, between $\Omega^n/\sim$ and $\operatorname{Spec} A$. We already know from (a) that the map φ is continuous, and clearly, it induces a continuous map on $\Omega^n/\sim$. It remains to prove that $\overline{\varphi}$ maps closed sets to closed sets. Now, the closed sets $V(f) = \{\xi \in \Omega^n \mid f(\xi) = 0\}$ generate the closed sets in the k-topology (every closed set is some intersection of sets of the form $V(f)$), so it is enough to consider the image of a closed set of the form $\{\overline{\xi} \mid f(\xi) = 0\}$ under $\overline{\varphi}$ (here, $\overline{\xi}$ denotes the equivalence class of ξ modulo $\sim$). The image of this closed set is clearly $\{\mathfrak{p} \in \operatorname{Spec} A \mid f \in \mathfrak{p}\}$ (since every prime ideal of A is of the form $\mathfrak{p}(\xi)$, as we just showed), a closed set in $\operatorname{Spec} A$.

B V(a). Let A be an integral domain and let $K = \mathrm{Frac}(A)$. If $\mathfrak{p} \in \operatorname{Spec}(A[X])$ is a prime ideal such that $\mathfrak{p} \cap A = (0)$, then we claim that $\mathfrak{p}$ is a principal ideal.

Consider the multiplicative subset $S = A - \{0\}$. It is immediate by definition that $S^{-1}A[X] \cong K[X]$. Furthermore, there is a one-to-one inclusion preserving correspondence between the prime ideals, $\mathfrak{p}$, of $A[X]$ such that $\mathfrak{p} \cap S = \emptyset$ and the extended ideals, $\mathfrak{p}^e$, in $K[X]$. As $S = A - \{0\}$, we have $\mathfrak{p} \cap S = \emptyset$ iff $\mathfrak{p} \cap A = (0)$. Furthermore, since K is a field, every ideal of $K[X]$ is a principal ideal, $f(X)K[X]$, and we may assume by multiplying by a common denominator that $f(X) \in A[X]$. But, then $\mathfrak{p}^e = f(X)K[X]$ for some $f(X) \in A[X]$ and so, $\mathfrak{p}$ is the principal ideal $f(X)A[X]$.

B V(b). In this question, let A be a UFD, let $K = \mathrm{Frac}(A)$, and let $\eta = a/b \in K$, where a and b are relatively prime in A. Consider the homomorphism $\theta\colon A[X] \to K$ given by

$$\theta(f(X)) = f(\eta).$$

We claim that Ker $\theta = (bX - a)$.

We need to prove that any polynomial, $f(X) \in A[X]$ which vanishes on $\eta = a/b$ in K is divisible by $bX - a$. Since A is a UFD, so is $A[X]$. So, we can write

$$f(X) = \alpha p_1(X) \cdots p_s(X)$$

where $\alpha \in A$ is a unit and $p_1(X), \ldots, p_s(X)$ are irreducible polynomials in $A[X]$. If $f(\eta) = 0$, then $\alpha p_1(\eta) \cdots p_s(\eta) = 0$, and since K is a field, we have $p_i(\eta) = 0$, for some i with $1 \leq i \leq s$. Viewing $p_i(X)$ as a polynomial in $K[X]$, this implies that

$$p_i(X) = \left(X - \frac{a}{b}\right) q(X),$$

i.e., $p_i(X) = (bX - a)r(X)$, with $r(X) \in K(X)$. Thus, $p_i(X)$ is reducible over $K(X)$. However, by Gauss' lemma, if a polynomial is reducible over $K[X]$ then it is already reducible over $A[X]$. As $p_i(X)$ is irreducible in $A[X]$ and a and b are relatively prime we must have $r(X) = 1$ and $q_i(X) = bX - a$; so, $f(X)$ is divisible by $bX - a$.

It follows that Ker $\theta = (bX - a)$, and by the first homomorphism theorem,
$A[\eta] \cong A[X]/(bX - a)$.

If b is a unit, then we can divide any polynomial in $A[X]$ by $bX - a$, and we get $A[\eta] \cong A[X]/(bX - a) \cong A$. In this case, $A[\eta]$ is, of course, flat. The converse is true (i.e., if $A[\eta]$ is flat over A, then b is a unit). In fact, this holds for any ring, see HW IV, Problem B VI(b).

B V(c). In this question, k is a field and $\xi = f(X)/g(X) \in k(X)$ is a non-constant rational function with f and g relatively prime, which implies that f and g are not both constant. First, we claim that ξ is transcendental over k. If not, ξ would be the zero of some polynomial over k and after clearing denominators, we would obtain an equation

$$a_0 f^d + a_1 f^{d-1} g + \cdots + a_{d-1} f g^{d-1} + a_d g^d = 0,$$

and so, we would have

$$a_d g^d = -f(a_0 f^{d-1} + a_1 f^{d-2} g + \cdots + a_{d-1} g^{d-1}).$$

Since g is relatively prime to f, it should divide $a_0 f^{d-1} + a_1 f^{d-2} g + \cdots + a_{d-1} g^{d-1}$, and thus, g would divide $a_0 f^{d-1}$. Since g is relatively prime to f, it would have to be a constant. But then, f would have to be a constant too, a contradiction.

Note that X is a zero of the polynomial

$$f(Y) - \xi g(Y) \in k(\xi)[Y],$$

and thus, X is algebraic over $k(\xi)$, which shows that $\dim_{k(\xi)} k(X)$ is finite.

We claim that $f(Y) - \xi g(Y)$ is irreducible over $k(\xi)$. Since ξ is transcendental over k, this is equivalent to proving that $f(Y) - Z g(Y)$ is irreducible over $k(Z)$, for some new indeterminate, Z. However, by Gauss's lemma, as $k[Z]$ is a UFD, $f(Y) - Zg(Y)$ is irreducible in $k(Z)[Y]$ iff it is irreducible in $k[Z][Y]$. Moreover, $k[Z][Y] \cong k[Y][Z]$, and $f(Y) - Zg(Y)$ is a polynomial of degree 1 in Z and $f(Y)$ and $g(Y)$ are relatively prime, so $f(Y) - Zg(Y)$ is irreducible in $k[Y][Z]$.

It follows that $\dim_{k(\xi)} k(X)$ is equal to the degree (in Y) of $f(Y) - \xi g(Y)$. But, this degree is clearly $\max\{\deg(f(X)), \deg(g(X))\}$ and so,

$$\dim_{k(\xi)} k(X) = \max\{\deg(f(X)), \deg(g(X))\}.$$