Homework II (due February 10), Math 603, Spring 2003. (GJZ)

B III(a). First, we need a few preliminary results.

Proposition 1.1 *Let A and B be integral domains, with A a subring of B, and assume that B is a finitely generated A-module. Then, B is integral over A.*

Proof. Recall that B is integral over A iff every $b \in B$ is the zero some *monic* polynomial,

$$b^n + a_1 b^{n-1} + \cdots + a_n = 0,$$

with $a_1, \ldots, a_n \in A$. Since B is f.g. over A, let $g_1, \ldots, g_m$ be a finite set of generators of B as an A-module. Pick any $b \in B$. Then, each $bg_i \in B$, and so, we can write

$$bg_i = \sum_{j=1}^{m} a_{ij} g_j, \quad \text{for } i = 1, \ldots, m,$$

for some $a_{ij} \in A$. Consequently, we have the linear system

$$\begin{aligned}
(a_{11} - b)g_1 + a_{12}g_2 + \cdots + a_{1m}g_m &= 0 \\
a_{21}g_1 + (a_{22} - b)g_2 + \cdots + a_{2m}g_m &= 0 \\
\vdots \qquad\qquad &= \vdots \\
a_{m1}g_1 + a_{m2}g_2 + \cdots + (a_{mm} - b)g_m &= 0.
\end{aligned}$$

Then, it is well-known by linear algebra that if C is the matrix of the above system and if $\widetilde{C}$ is the transpose of the matrix of cofactors of C, then

$$\widetilde{C}C = \det(C) I_m.$$

Since the above system is $Cg = 0$ (where g is the vector (g_i)), we get

$$\det(C) g_j = \det(a_{ij} - \delta_{ij} b) g_j = 0, \quad \text{for } j = 1, \ldots, m.$$

Since the g_j generate B, we see that

$$\det(a_{ij} - \delta_{ij} b) y = 0 \quad \text{for all } y \in B;$$

in particular, this holds for $y = 1$, and so,

$$\det(a_{ij} - \delta_{ij} b) = 0$$

is a monic polynomial satisfied by b. $\square$

Remark: The converse of Proposition 1.1 also holds (and is easier to prove).

Proposition 1.2 *Let A and B be integral domains, with A a subring of B, and assume that B is integral over A. Then, A is a field iff B is a field.*

Proof. First, assume that B is a field. Pick $a \neq 0$ in A. Since $A \subseteq B$ and B is a field, $a^{-1} \in B$. Since B is integral over A, we have

$$a^{-n} + a_1 a^{-n+1} + \cdots + a_n = 0,$$

for some $a_1, \ldots, a_n \in A$. But then, multiplying by a^{n-1}, we get

$$a^{-1} = -(a_1 + \cdots + a_n a^{n-1}),$$

where the righthand side is in A, and so, $a^{-1} \in A$ and A is a field.

Conversely, assume that A is a field. Pick $b \neq 0$ in B. Since B is integral over A, we have

$$b^n + a_1 b^{n-1} + \cdots + a_{n-1} b + a_n = 0,$$

for some $a_1, \ldots, a_n \in A$, and we may assume that n is minimal, so that $a_n \neq 0$, since $b \neq 0$ and B is an integral domain. We know that $b^{-1} \in \mathrm{Frac}(B)$, and in $\mathrm{Frac}(B)$, if we divide by b and by a_n, with $a_n^{-1} \in A$, since A is a field, we can write

$$b^{-1} = -(a_n^{-1} b^{n-1} + \cdots + a_n^{-1} a_{n-1}),$$

where the righthand side is in B. Thus, $b^{-1} \in B$ and B is a field. $\square$

We can now give the solution to BIII(a). Assume that K is a field, A is a subring of K, and the ring K is a f.g. A-module. Write $k = \mathrm{Frac}(A)$. By Proposition 1.1, the ring K is integral over A. By Proposition 1.2, since K is a field, so is A. But then, $k = \mathrm{Frac}(A) = A$, as claimed.

B III(b) This time, we are assuming that K is a field and that there are elements $\alpha_1, \ldots, \alpha_m \in K$ algebraic over $k = \mathrm{Frac}(A)$, so that

$$K = A[\alpha_1, \ldots, \alpha_m].$$

To say that α_i is algebraic over k means that α_i satisfies some polynomial equation

$$c_0^{(i)} \alpha_i^{n_i} + c_1^{(i)} \alpha_i^{n_i - 1} + \cdots + c_{n_i}^{(i)} = 0,$$

where $c_j^{(i)} \in k$ for $j = 0, \ldots, n_j$. Since $k = \mathrm{Frac}(A)$ is a field, we may assume that $c_0^{(i)} = 1$, i.e., the polynomial is monic. Now, as $k = \mathrm{Frac}(A)$, each $c_j^{(i)}$ is of the form

$$c_j^{(i)} = \frac{a_j^{(i)}}{b_j^{(i)}},$$

for some $a_j^{(i)}, b_j^{(i)} \in A$, with $b_j^{(i)} \neq 0$. We can form a unique common denominator, $b \neq 0$, by multiplying all these nonzero denominators, $b_j^{(i)}$, and we see that the α_i's satisfy monic polynomials

$$\alpha_i^{n_i} + \frac{\widetilde{a}_1^{(i)}}{b}\alpha_i^{n_i-1} + \cdots + \frac{\widetilde{a}_{n_i}^{(i)}}{b} = 0,$$

where $b \in B$ and $\widetilde{a}_j^{(i)} \in A$, for $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$. These monic equations show that each α_i is integral over $A[1/b]$. So, we have $K = A[\alpha_1, \ldots, \alpha_m]$, with the α_i integral over $A[1/b]$. This implies that K is f.g. over $A[1/b]$. Indeed, every element of $K = A[\alpha_1, \ldots, \alpha_m]$ is a polynomial expression with coefficients in A and monomials of the form

$$\alpha_1^{k_1} \cdots \alpha_m^{k_m}.$$

Using the monic equations sastisfied by the α_i's, we can express any power α_i^k for $k \geq n_i$ in terms of $1, \alpha_1, \ldots, \alpha_i^{n_i-1}$, with coefficents in $A[1/b]$, and thus, every monomial $\alpha_1^{k_1} \cdots \alpha_m^{k_m}$ can be expressed as a linear combination of the monomials $\alpha_1^{h_1} \cdots \alpha_m^{h_m}$, with $h_i \leq n_i - 1$, for $i = 1, \ldots, m$. So, K is indeed f.g. over $A[1/b]$. Now, we can apply B III(a) to K and $A[1/b]$, and we get that

$$\mathrm{Frac}(A[1/b]) = A[1/b].$$

However, $\mathrm{Frac}(A)$ contains $1/b$, and so, $\mathrm{Frac}(A) = \mathrm{Frac}(A[1/b])$; it follows that

$$k = \mathrm{Frac}(A) = \mathrm{Frac}\left(A\left[\frac{1}{b}\right]\right),$$

for some nonzero element $b \in A$.

Now, assume that A is not a field, so that $A \neq \mathrm{Frac}(A)$. We claim that there is an isomorphism, $\varphi\colon A_b \longrightarrow A[1/b]$, where $A_b = S^{-1}A$ is the localization of A w.r.t. the multiplicative set $S = \{b^k \mid k \geq 0\}$.

We have the inclusion map, $i\colon A \to A[1/b]$ and since every b^k becomes invertible in $A[1/b]$, by the universal mapping property of A_b, there is a unique homomorphism, $\varphi\colon A_b \longrightarrow A[1/b]$, so that

$$i = \varphi \circ h,$$

where $h\colon A \to A_b$ is the canonical map. Now, since A is an integral domain, the map $h\colon A \to A_b$ is injective (because $h(a) = a/1$, and $a/1 = a'/1$ iff $b^k(a - a') = 0$ iff $a = a'$, since $b \neq 0$ and A is an integral domain). Now, every element of A_b is of the form a/b^k, for some $a \in A$ and some $k \geq 0$, and

$$\varphi\left(\frac{a}{b^n}\right) = \varphi\left(\frac{a}{1}\frac{1}{b^n}\right) = \varphi\left(\frac{a}{1}\right)\varphi\left(\frac{1}{b}\right)^n = \varphi\left(\frac{a}{1}\right)\varphi\left(\frac{b}{1}\right)^{-n} = \varphi(h(a))\varphi(h(b))^{-n}.$$

However, $\varphi \circ h = i$, and so,

$$\varphi\left(\frac{a}{b^n}\right) = \varphi(h(a))\varphi(h(b))^{-n} = i(a)(i(b))^{-n} = \frac{a}{b^n} \in A[1/b].$$

Therefore, it is clear that φ is an injection. Since every element in $A[1/b]$ is also of the form a/b^k, with $a \in A$, the morphism φ is also surjective and thus, it is an isomorphism.

Now, we proved in class that there is a one-to-one, inclusion-preserving, correspondence between the prime ideals in $S^{-1}A = A_b$ and the prime ideals, $\mathfrak{p}$, in A, for which $\mathfrak{p} \cap S = \emptyset$. However, since A_b is a field, its only prime ideal is (0). Since A is an integral domain, (0) is a prime ideal, so it corresponds to (0) in A_b; consequently, every prime $\mathfrak{p} \neq (0)$ in A must intersect S, which means that $b \in \mathfrak{p}$ for every prime $\mathfrak{p} \neq (0)$, and in particular, for maximal ideals (since A is not a field, there are maximal ideals distinct from (0)). Therefore, b belongs to every maximal ideal in A.

B IV(a). Let $\mathcal{C} = \mathcal{P}(X, \mathcal{A}\mathrm{b})$ be the category of presheaves of abelian groups on the topological space, X. For any open subset, U, in X, we have the functor $\mathcal{S}_U : \mathcal{F} \rightsquigarrow \mathcal{F}(U)$, from $\mathcal{P}(X, \mathcal{A}\mathrm{b})$ to $\mathcal{A}\mathrm{b}$. We are seeking a presheaf, $\mathcal{F} \in \mathcal{P}(X, \mathcal{A}\mathrm{b})$ and an object, $\xi \in \mathcal{S}_U(\mathcal{F}) = \mathcal{F}(U)$, so that we have an isomorphism

$$\widetilde{\xi} : \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, -) \longrightarrow \mathcal{S}_U,$$

given by the consistent family of morphisms

$$\widetilde{\xi}_{\mathcal{G}} : \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{S}_U(\mathcal{G}) = \mathcal{G}(U),$$

defined *via* $\widetilde{\xi}_{\mathcal{G}}(\theta) = \mathcal{S}_U(\theta)(\xi)$, for every presheaf, $\mathcal{G} \in \mathcal{P}(X, \mathcal{A}\mathrm{b})$, and every morphism of presheaves, $\theta : \mathcal{F} \to \mathcal{G}$. Now, $\theta : \mathcal{F} \to \mathcal{G}$ is given by a consistent family of morphisms $\theta_V : \mathcal{F}(V) \to \mathcal{G}(V)$, for every open subset, V, of X and $\mathcal{S}_U(\theta) = \theta_U$. Therefore, we need to find a pair $(\mathcal{F}, \xi)$, as above, so that we have an isomorphism

$$\widetilde{\xi}_{\mathcal{G}} : \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{G}(U),$$

defined *via* $\widetilde{\xi}_{\mathcal{G}}(\theta) = \theta_U(\xi)$. We claim that the pair $(\mathcal{F}, \xi) = (\mathbb{Z}_U, 1)$ works, where $\mathbb{Z}_U$ is the presheaf defined in Problem AI(a), i.e.,

$$\mathbb{Z}_U(W) = \begin{cases} \mathbb{Z} & \text{if } W \subseteq U \\ (0) & \text{if } W \not\subseteq U, \end{cases}$$

and with $\rho_U^W = \mathrm{id}$. Indeed, a morphism $\theta \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{Z}_U, \mathcal{G})$ is uniquely determined by the consistent family of homomorphisms of abelian groups, $\theta_W : \mathbb{Z}_U(W) \to \mathcal{G}(W)$. This means that we have the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}_U(V) & \xrightarrow{\theta_V} & \mathcal{G}(V) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \rho_V^W} \\ \mathbb{Z}_U(W) & \xrightarrow{\theta_W} & \mathcal{G}(W) \end{array}$$

whenever $W \subseteq V$. If $W \not\subseteq U$, we have $\mathbb{Z}_U(W) = (0)$ and θ_W is the zero morphism. If $W \subseteq U$, we have $\mathbb{Z}_U(W) = \mathbb{Z}$ and we have the commutative diagram

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\theta_U} & \mathcal{G}(U) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \rho_V^W} \\ \mathbb{Z} & \xrightarrow{\theta_W} & \mathcal{G}(W) \end{array}$$

which implies that $\theta_W = \rho_V^W(\theta_U)$. Thus, θ is uniquely determined by θ_U. But, the homomorphism $\theta_U\colon \mathbb{Z} \to \mathcal{G}(U)$ is uniquely determined by $\theta_U(1)$, and so, θ is uniquely determined by $\theta_U(1)$. Therefore, the maps $\theta \mapsto \theta_U(1)$ is indeed an isomophism between $\mathrm{Hom}_{\mathcal{C}}(\mathbb{Z}_U, \mathcal{G})$ and $\mathcal{G}(U)$.

B IV(b). We need to check conditions (1)(2)(3) of B IV(b) for being a cover. Condition (1) is obvious and condition (3) is condition (b) of B IV(b). We need to check condition (2). This amounts to showing that given morphisms

$$V_\gamma^{(\alpha)} \longrightarrow U_\alpha \longrightarrow U \quad \text{and} \quad V_\delta^{(\beta)} \longrightarrow U_\beta \longrightarrow U,$$

the fibred coproduct $V_\gamma^{(\alpha)} \prod_U V_\delta^{(\beta)}$ exists. This can be demonstrated by constructing a commutative diagram. Sorry, we ran out of time!

B IV(c). By definition, a representable cofunctor, F, on $\mathcal{T}$ is a cofunctor such that there is an isomorphism $\widetilde{\xi}\colon \mathrm{Hom}_{\mathcal{T}}(-, A) \to F$, for some object $A \in \mathcal{C}$ and some $\xi \in F(A)$, given by the consistent family of morphisms, $\widetilde{\xi}_B\colon \mathrm{Hom}_{\mathcal{T}}(B, A) \to F(B)$, *via* $\widetilde{\xi}_B(u) = F(u)(\xi)$. But, the conditions of B IV(b) for being a canonical site are precisely the sheaf conditions for presheaves of the form $\mathrm{Hom}_{\mathcal{T}}(-, A)$. So, a representable cofunctor, F, on $\mathcal{T}$ is a sheaf w.r.t. $\mathcal{T}_{\mathrm{can}}$.