Homework I (due January 27), Math 603, Spring 2003. (GJZ)

B III(a). First, we need a few preliminary results. Assume that N is a flat R^{op}-module and let $\theta\colon M \to Z$ be a linear map of R-modules. We have the map, $1\otimes\theta\colon N\otimes_R M \to N\otimes_R Z$, and we claim that

$$\mathrm{Ker}\,(1\otimes\theta) \cong N\otimes_R \mathrm{Ker}\,(\theta) \quad \text{and} \quad \mathrm{Im}\,(1\otimes\theta) \cong N\otimes_R \mathrm{Im}\,(\theta).$$

Indeed, since N is flat, from the exact sequence

$$0 \longrightarrow \mathrm{Ker}\,(\theta) \longrightarrow M \xrightarrow{\theta} Z,$$

we get the exact sequence

$$0 \longrightarrow N\otimes_R \mathrm{Ker}\,(\theta) \longrightarrow N\otimes_R M \xrightarrow{1\otimes\theta} N\otimes_R Z,$$

which shows that $\mathrm{Ker}\,(1\otimes\theta) \cong N\otimes_R \mathrm{Ker}\,(\theta)$. We also have the exact sequences

$$M \xrightarrow{\theta'} \mathrm{Im}\,(\theta) \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \mathrm{Im}\,(\theta) \xrightarrow{i} Z,$$

where is θ' is the corestriction of θ to $\mathrm{Im}\,(\theta)$ and i is the inclusion of $\mathrm{Im}\,(\theta)$ into Z, with $\theta = i\circ\theta'$; since N is flat, we get the exact sequences

$$N\otimes_R M \xrightarrow{1\otimes\theta'} N\otimes_R \mathrm{Im}\,(\theta) \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow N\otimes_R \mathrm{Im}\,(\theta) \xrightarrow{1\otimes i} N\otimes_R Z,$$

and we have $(1\otimes i)\circ(1\otimes\theta') = 1\otimes(i\circ\theta') = 1\otimes\theta$, with $1\otimes\theta'$ surjective and $1\otimes i$ injective, which shows that $\mathrm{Im}\,(1\otimes\theta) \cong N\otimes_R \mathrm{Im}\,(\theta)$.

We also need the following propositions.

Proposition 1.1 *Let M be a faithfully flat R^{op}-module. For any linear map, $\theta\colon N' \to N$, of R-modules, if $1\otimes\theta = 0$, then $\theta = 0$.*

Proof. As M is faithfully flat, it is flat, and we observed that $\mathrm{Im}\,(1\otimes\theta) \cong M\otimes_R \mathrm{Im}\,(\theta)$. Thus, if $1\otimes\theta = 0$, we have $\mathrm{Im}\,(1\otimes\theta) = (0)$, i.e., $M\otimes_R \mathrm{Im}\,(\theta) = (0)$; since M is faithfully flat, we must have $\mathrm{Im}\,(\theta) = (0)$, i.e., $\theta = 0$. $\square$

Proposition 1.2 *Let M be a faithfully flat R^{op}-module. Then, any sequence*

$$N' \xrightarrow{\varphi} N \xrightarrow{\psi} N'' \quad \textit{is exact}$$

iff the sequence

$$M\otimes_R N' \xrightarrow{1\otimes\varphi} M\otimes_R N \xrightarrow{1\otimes\psi} M\otimes_R N'' \quad \textit{is exact.}$$

Proof. One direction is obvious, since M is flat (namely, if the first sequence is exact, then the tensored one is exact).

Conversely, assume that the sequence

$$M \otimes_R N' \xrightarrow{1\otimes\varphi} M \otimes_R N \xrightarrow{1\otimes\psi} M \otimes_R N'' \quad \text{is exact.} \tag{$*$}$$

Exactness implies that $(1 \otimes \psi) \circ (1 \otimes \varphi) = 0$, i.e., $1 \otimes (\psi \circ \varphi) = 0$. As M is faithfully flat, by Proposition 1.1, $\psi \circ \varphi = 0$. Let $K = \mathrm{Ker}\, \psi$ and $I = \mathrm{Im}\, \varphi$; we just proved that $I \subseteq K$. Consider the exact sequence

$$0 \longrightarrow I \xrightarrow{i} K \xrightarrow{\pi} K/I \longrightarrow 0.$$

We would like to prove that $K/I = (0)$. Since M is flat, we get the exact sequence

$$0 \longrightarrow M \otimes_R I \xrightarrow{1\otimes i} M \otimes_R K \xrightarrow{1\otimes\pi} M \otimes_R (K/I) \longrightarrow 0. \tag{$\dagger$}$$

However, we showed earlier that

$$M \otimes_R K \cong \mathrm{Ker}\,(1 \otimes \psi) \quad \text{and} \quad M \otimes_R I \cong \mathrm{Im}\,(1 \otimes \varphi).$$

As exactness of the sequence $(*)$ means that

$$\mathrm{Im}\,(1 \otimes \varphi) = \mathrm{Ker}\,(1 \otimes \psi),$$

we get $M \otimes_R I \cong M \otimes_R K$; exactness of the sequence $(\dagger)$ implies that

$$M \otimes_R (K/I) \cong (M \otimes_R K)/(M \otimes_R I) = (0).$$

But then, $1 \otimes \pi = 0$, and since M is faithfully flat, by Proposition 1.1, we get $\pi = 0$. Therefore, $K/I = (0)$, i.e., $K = I$, and the sequence

$$N' \xrightarrow{\varphi} N \xrightarrow{\psi} N'' \quad \text{is exact.} \quad \square$$

As a corollary of Proposition 1.2, we get

Corollary 1.3 *Let M be a faithfully flat R^{op}-module (resp. R-module). For any linear map, $\theta\colon N' \to N$, of R-modules (resp. of R^{op}-modules), $1 \otimes \theta$ (resp. $\theta \otimes 1$) is injective (resp. surjective) iff θ is injective (surjective).*

We are now ready to prove B III(a). Assume that $\theta\colon A \to B$ is a homomorphism of rings and that B is faithfully flat over A *via* θ. First, assume that M is a finitely generated A-module, and let $e_1, \ldots, e_s$ be a set of generators. We know that if M and N are two modules and M is generated by $e_1, \ldots, e_s$ and N is generated by $f_1, \ldots, f_t$, then $M \otimes_A N$ is generated by the $e_i \otimes f_j$ (this also holds for infinite sets of generators). As B is generated by 1 (over A), we see that $e_1 \otimes 1, \ldots, e_s \otimes 1$ generate $M \otimes_A B$.

Conversely, assume that $M \otimes_A B$ is finitely generated. As $M \otimes_A B$ is generated by vectors of the form $e_i \otimes 1$, where $e_i \in M$, there is a finite number of vectors, $e_1, \ldots, e_s$, such that $e_1 \otimes 1, \ldots, e_s \otimes 1$ generate $M \otimes_A B$. Let N be the submodule of M generated by $e_1, \ldots, e_s$. We have an exact sequence $0 \longrightarrow N \xrightarrow{i} M$, where i is injective, and since B is faithfully flat over A, it is flat, and so, we get the exact sequence

$$0 \longrightarrow N \otimes_A B \xrightarrow{i \otimes 1} M \otimes_A B.$$

However, since $M \otimes_A B$ is generated by $e_1 \otimes 1, \ldots, e_s \otimes 1$ and N is generated by $e_1, \ldots, e_s$, the map $i \otimes 1$ is surjective. Since M is faithfully flat, by Corollary 1.3, the map i is surjective. But now, i is bijective, so $M \cong M'$ is finitely generated.

B III(b). We also need a preliminary proposition.

Proposition 1.4 *Let*

$$0 \longrightarrow M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \longrightarrow 0 \tag{$*$}$$

be an exact sequence. If M is f.g. and M'' is f.p., then, M' is f.g.

Proof. Let

$$F_1 \xrightarrow{\alpha} F_0 \xrightarrow{\beta} M'' \longrightarrow 0 \tag{$\dagger$}$$

be a finite presentation of M'', which means that ($\dagger$) is exact, with F_0, F_1 some f.g. free modules. Say $e_1, \ldots, e_s$ is a basis of F_0. Since ($*$) is exact, the map ψ is surjective, and so, for $i = 1, \ldots, s$, there is some $g_i \in M$ so that $\psi(g_i) = \beta(e_i)$. If we define the linear map, $\theta \colon F_0 \to M$, by

$$\theta(e_i) = g_i,\ i = 1, \ldots, s,$$

we see that $\beta = \psi \circ \theta$. Now, as ($\dagger$) is exact, $\beta \circ \alpha = 0$, so $\psi \circ \theta \circ \alpha = 0$; thus,

$$\theta \circ \alpha(F_1) \subseteq \mathrm{Ker}\ \psi = \mathrm{Im}\ \varphi.$$

Since F_1 is free (and so, projective), the above implies that there is a linear map, $\gamma \colon F_1 \to M'$, so that

$$\varphi \circ \gamma = \theta \circ \alpha.$$

Therefore, we get the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & \mathrm{Ker}\ 1_{M''} & & \\
 & & & & & & \downarrow & & \\
 & & F_1 & \xrightarrow{\alpha} & F_0 & \xrightarrow{\beta} & M'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle 1_{M''}} & & \\
0 & \longrightarrow & M' & \xrightarrow[\varphi]{} & M & \xrightarrow[\psi]{} & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{Coker}\ \gamma & \longrightarrow & \mathrm{Coker}\ \theta & \longrightarrow & \mathrm{Coker}\ 1_{M''} & \longrightarrow & 0
\end{array}$$

in which the second row is exact because (†) is exact and the thid row is exact because (∗) is exact. Thus, we can apply the snake lemma, and we get the exact sequence

$$0 = \mathrm{Ker}\ 1_{M''} \longrightarrow \mathrm{Coker}\ \gamma \longrightarrow \mathrm{Coker}\ \theta \longrightarrow \mathrm{Coker}\ 1_{M''} = 0.$$

Consequently,

$$\mathrm{Coker}\ \gamma \cong \mathrm{Coker}\ \theta = M/\theta(F_0).$$

Now, F_0 and M are finitely generated, and so, $\mathrm{Coker}\ \gamma \cong M/\theta(F_0)$ is f.g. We also have the exact sequence

$$0 \longrightarrow \gamma(F_1) \longrightarrow M' \longrightarrow \mathrm{Coker}\ \gamma \longrightarrow 0.$$

As F_1 is f.g., so if $\gamma(F_1)$, and we just proved that $\mathrm{Coker}\ \gamma$ is f.g. By a proposition proved in class, M' is also f.g., as desired. $\square$

Now, for the proof of B III(b). Assume that M is f.p., and let

$$F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0 \tag{†}$$

be a finite presentation of M, with F_0, F_1 some f.g. free modules. Since B is faithfully flat, we get the exact sequence

$$F_1 \otimes_A B \longrightarrow F_0 \otimes_A B \longrightarrow M \otimes_A B \longrightarrow 0.$$

However, $F_1 = \coprod_S A$ and $F_0 = \coprod_T A$, for some finite sets, S, T, and $F_1 \otimes_A B \cong \coprod_S (A \otimes_A B) \cong \coprod_S B$ and similarly $F_2 \cong \coprod_T B$, which shows that $F_1 \otimes_A B$ and $F_0 \otimes_A B$ are still f.g. free modules. Therefore, $M \otimes_A B$ is still f.p.

Conversely, assume that $M \otimes_A B$ is f.p. In particular, $M \otimes_A B$ is f.g., and by B III(a), M is f.g. Thus, there is an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0,$$

where F is a f.g, free module and $K = \mathrm{Ker}\,(F \longrightarrow M)$. As B is faithfully flat, we get the exact sequence

$$0 \longrightarrow K \otimes_A B \longrightarrow F \otimes_A B \longrightarrow M \otimes_A B \longrightarrow 0.$$

In the above sequence, $F \otimes_A B$ is f.g. because F is free and f.g., and by hypothesis, $M \otimes_A B$ is f.p. By Proposition 1.4, the module $K \otimes_A B$ is f.g. By B III(a), again, we see that K is f.g. Now, since K is a f.g. submodule, there is a f.g. free module, F_0, and a surjection $F_0 \longrightarrow K$, and

$$F_0 \longrightarrow F \longrightarrow M \longrightarrow 0,$$

is a finite presentation of M.

B III(c). By definition, an A-module, M, is locally free iff $M_{\mathfrak{p}} \cong M \otimes_A A_{\mathfrak{p}}$ is free over $A_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p} \subseteq A$.

Assume that B is faithfully flat over A *via* $\theta\colon A \to B$ and that M is an A-module that is locally free. Let $\mathfrak{q} \subseteq B$ be any prime ideal in B, and let $\mathfrak{p} = \mathfrak{q}^c = \theta^{-1}(\mathfrak{q})$ be the contraction of $\mathfrak{q}$ in A. From the definition of $B_\mathfrak{q}$, it is obvious that $B_\mathfrak{q}$ can be viewed as an $A_\mathfrak{p}$-module and as an A-module. Then, we have

$$\begin{aligned}
(M \otimes_A B)_\mathfrak{q} &\cong (M \otimes_A B) \otimes_B B_\mathfrak{q} \\
&\cong M \otimes_A (B \otimes_B B_\mathfrak{q}) \\
&\cong M \otimes_A B_\mathfrak{q} \\
&\cong M \otimes_A (A_\mathfrak{p} \otimes_{A_\mathfrak{p}} B_\mathfrak{q}) \\
&\cong (M \otimes_A A_\mathfrak{p}) \otimes_{A_\mathfrak{p}} B_\mathfrak{q} \\
&\cong M_\mathfrak{p} \otimes_{A_\mathfrak{p}} B_\mathfrak{q}.
\end{aligned}$$

Since $M_\mathfrak{p}$ is a free $\mathfrak{p}$-module,

$$M_\mathfrak{p} \cong \coprod_T A_\mathfrak{p},$$

for some set, T; so,

$$M_\mathfrak{p} \otimes_{A_\mathfrak{p}} B_\mathfrak{q} \cong (\coprod_T A_\mathfrak{p}) \otimes_{A_\mathfrak{p}} B_\mathfrak{q} \cong \coprod_T B_\mathfrak{q},$$

and we see that $(M \otimes_A B)_\mathfrak{q}$ is a free $B_\mathfrak{q}$-module.

Conversely, assume that $M \otimes_A B$ is locally free over B and that M is f.p. (we don't know how to prove the required statement in general, and we suspect that it is false, although we don't have a counterexample either). By a theorem proved in class, as M is f.p., M is locally free (over A) iff M is flat over A. Since M is f.p., so is $M \otimes_A B$ (we proved that in (b)), and so, $M \otimes_A B$ is locally free over B iff it is flat over B. Consider any exact sequence

$$0 \longrightarrow N \longrightarrow N'.$$

We need to prove that

$$0 \longrightarrow N \otimes_A M \longrightarrow N' \otimes_A M \quad \text{is still exact.}$$

If not, let K be the kernel of the map $N \otimes_A M \longrightarrow N' \otimes_A M$; we have an exact sequence

$$0 \longrightarrow K \longrightarrow N \otimes_A M \longrightarrow N' \otimes_A M.$$

Since B is flat over A (in fact, faithfully flat), the sequence

$$0 \longrightarrow K \otimes_A B \longrightarrow (N \otimes_A M) \otimes_A B \longrightarrow (N' \otimes_A M) \otimes_A B \quad \text{is exact,}$$

that is, the sequence

$$0 \longrightarrow K \otimes_A B \longrightarrow N \otimes_A (M \otimes_A B) \longrightarrow N' \otimes_A (M \otimes_A B) \quad \text{is exact.}$$

However, as $M \otimes_A B$ is flat over B, by hypothesis, we must have $K \otimes_A B = (0)$ and since B is faithfully flat over A, we get $K = (0)$. Therefore, M is indeed flat over A. In conclusion, under the hypothesis that M is f.p., we proved that if $M \otimes_A B$ is locally free over B, then M is locally free over A (of course, B is faithfully flat over A). $\square$

B IV(a). Let Λ be a ring and consider the exact sequence of Λ-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0. \tag{$*$}$$

Assume that M'' is flat and let N be any Λ^{op}-module. We need to prove that the sequence

$$0 \longrightarrow N \otimes_\Lambda M' \longrightarrow N \otimes_\Lambda M \longrightarrow N \otimes_\Lambda M'' \longrightarrow 0 \quad \text{is still exact.}$$

We can write N as a factor of some free Λ^{op}-module, F:

$$0 \longrightarrow K \longrightarrow F \longrightarrow N \longrightarrow 0. \tag{$**$}$$

Then, by tensoring $(*)$ with K, F and N by and tensoring $(**)$ with M', M and M'' we obtain the following commutative diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & K \otimes_\Lambda M' \arrow[r] \arrow[d, "\delta_1"] & K \otimes_\Lambda M \arrow[r] \arrow[d, "\delta_2"] & K \otimes_\Lambda M'' \arrow[r] \arrow[d, "\delta_3"] & 0 \\
0 \arrow[r] & F \otimes_\Lambda M' \arrow[r] \arrow[d] & F \otimes_\Lambda M \arrow[r] \arrow[d] & F \otimes_\Lambda M'' \arrow[r] \arrow[d] & 0 \\
 & N \otimes_\Lambda M' \arrow[r, "\theta"] \arrow[d] & N \otimes_\Lambda M \arrow[r] \arrow[d] & N \otimes_\Lambda M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & .
\end{tikzcd}$$

The second row is exact because F is free, and thus flat; the third column is exact because M'' is flat, and the other rows and columns are exact because tensor is right-exact. We need to prove that $\theta\colon N \otimes_\Lambda M' \to N \otimes_\Lambda M$ is injective.

If we look at the first two rows, we see that the snake lemma applies, and we get the exact sequence

$$\mathrm{Ker}\ \delta_3 \xrightarrow{\delta} \mathrm{Coker}\ \delta_1 \longrightarrow \mathrm{Coker}\ \delta_2.$$

However, the right-exactness of the first two rows implies $\mathrm{Coker}\ \delta_1 = N \otimes_\Lambda M'$ and $\mathrm{Coker}\ \delta_2 = N \otimes_\Lambda M$; so, we have the exact sequence

$$\mathrm{Ker}\ \delta_3 \xrightarrow{\delta} N \otimes_\Lambda M' \xrightarrow{\theta} N \otimes_\Lambda M.$$

Since the third column is exact (because M'') is flat, we have $\mathrm{Ker}\ \delta_3 = 0$, and so, we have the exact sequence

$$0 \longrightarrow N \otimes_\Lambda M' \xrightarrow{\theta} N \otimes_\Lambda M,$$

which proves that θ is injective, as required. Therefore, the sequence

$$0 \longrightarrow N \otimes_\Lambda M' \longrightarrow N \otimes_\Lambda M \longrightarrow N \otimes_\Lambda M'' \longrightarrow 0 \quad \text{is indeed exact.}$$

B IV(b). Again, we have an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0, \tag{$*$}$$

where M'' is flat. Further, assume that M is also flat. Consider any exact sequence, $0 \longrightarrow N' \longrightarrow N$, of Λ^{op}-modules and tensor $(*)$ with N' and N. We get the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N' \otimes_\Lambda M' \arrow[r, "\alpha'"] \arrow[d, "\theta"] & N' \otimes_\Lambda M \arrow[r] \arrow[d, "\psi"] & N' \otimes_\Lambda M'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \otimes_\Lambda M' \arrow[r, "\alpha"] & N \otimes_\Lambda M \arrow[r] & N \otimes_\Lambda M'' \arrow[r] & 0.
\end{tikzcd}$$

The second column is exact because M is flat and the third column is exact because M'' is flat; the rows are exact, by (a), since M'' is flat. We need to prove that $\theta\colon N' \otimes_\Lambda M' \to N \otimes_\Lambda M'$ is injective.

As the rows are exact, both α and α' are injective, and as the middle column is exact, ψ is also injective. However, from the commutative diagram, we have

$$\psi \circ \alpha' = \alpha \circ \theta,$$

and since $\psi \circ \alpha'$ is injective, it follows that θ is injective. So, we proved that if M is flat, then M' is flat.

Now, assume that M' is flat. This time, we have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & N' \otimes_\Lambda M' \arrow[r, "\alpha'"] \arrow[d, "\varphi"] & N' \otimes_\Lambda M \arrow[r, "\beta'"] \arrow[d, "\theta"] & N' \otimes_\Lambda M'' \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & N \otimes_\Lambda M' \arrow[r, "\alpha"] & N \otimes_\Lambda M \arrow[r, "\beta"] & N \otimes_\Lambda M'' \arrow[r] & 0.
\end{tikzcd}$$

The rows are exact and the first and third column are exact. We need to prove that $\theta\colon N' \otimes_\Lambda M \to N \otimes_\Lambda M$ is injective. This time, φ and γ are injective, since the first and the third columns are exact. We can apply the five lemma (since the map $0 \longrightarrow 0$ is surjective), and we deduce that θ is injective. A direct diagram chase goes as follows. Pick $x \in N' \otimes_\Lambda M$ and assume that $\theta(x) = 0$. Then,

$$\beta \circ \theta(x) = \gamma \circ \beta'(x) = 0.$$

However, γ is injective, which implies that $\beta'(x) = 0$. Since $\mathrm{Im}\ \alpha' = \mathrm{Ker}\ \beta'$, there is some $y \in N' \otimes_\Lambda M'$ so that $\alpha'(y) = x$. But, $\theta \circ \alpha'(y) = \theta(x) = 0$ and

$$\theta \circ \alpha'(y) = \alpha \circ \varphi(y),$$

where both φ and α are injective. Thus, $y = 0$, and so $x = 0$.

Therefore, assuming that M'' is flat, we proved that M is flat iff M' is flat.

The modules M and M' may both be flat with M'' not flat. Let $\Lambda = \mathbb{Z}$, $M' = n\mathbb{Z}$, $M = \mathbb{Z}$ and $M'' = \mathbb{Z}/n\mathbb{Z}$, where $n \geq 2$. The module M'' is not flat since it is torsion, the sequence

$$0 \longrightarrow n\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0 \quad \text{is exact,}$$

and M and M' are flat over $\mathbb{Z}$, as free modules.