Spring 2015 CIS 610

Advanced Geometric Methods in Computer Science
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Homework 5

April 28; Due May 7, 2015

Problem B1 (160). Recall that a nonempty k-dimensional affine subspace $\mathcal{A}$ of $\mathbb{R}^n$ is determined by a pair (a_0, U), where $a_0 \in \mathbb{R}^n$ is any point in $\mathcal{A}$ and U is a k-dimensional subspace of $\mathbb{R}^n$ called the *direction* of $\mathcal{A}$. Two pairs (a_0, U) and (b_0, U) define the same affine subspace $\mathcal{A}$ iff $b_0 - a_0 \in U$ (in fact, U consists of all vectors of the form $b - a$, with $a, b \in \mathcal{A}$).

The subspace U can be represented by any basis $(u_1, \ldots, u_k)$ of vectors $u_i \in U$, and so $\mathcal{A}$ is represented by the *affine frame* $(a_0, (u_1, \ldots, u_k))$.

Two affine frames $(a_0, (u_1, \ldots, u_k))$ and $(b_0, (v_1, \ldots, v_k))$ represent the same affine subspace $\mathcal{A}$ iff there is an invertible $k \times k$ matrix $\Lambda = (\lambda_{ij})$ such that

$$v_j = \sum_{i=1}^{k} \lambda_{ij} u_i, \quad 1 \leq j \leq k,$$

and if there is some vector $c \in \mathbb{R}^k$ such that

$$b_0 = a_0 + \sum_{i=1}^{k} c_i u_i.$$

Note that (Λ, c) defines an invertible affine map of $\mathbb{R}^k$.

A basis $(u_1, \ldots, u_k)$ of U is represented by a $n \times k$ matrix of rank k, say A, so the affine subspace $\mathcal{A}$ is represented by the pair (a_0, A), where $a_0 \in \mathbb{R}^n$ and A is a $n \times k$ matrix of rank k. The equivalence relation on pairs (a_0, A) is given by

$$(a_0, A) \equiv (b_0, B)$$

iff there exists a pair (Λ, c), where Λ is an invertible $k \times k$ matrix ($\Lambda \in \mathbf{GL}(k, \mathbb{R})$) and c is some vector in $\mathbb{R}^k$, such that

$$B = A\Lambda \quad \text{and} \quad b_0 = a_0 + Ac.$$

Using Gram-Schmidt, we may assume that $(u_1, \ldots, u_k)$ is an orthonormal basis, which means that the columns of the matrix A are orthonormal; that is,

$$A^\top A = I_k.$$

Then, in the equivalence relation defined above, the matrix Λ is an orthogonal $k \times k$ matrix ($\Lambda \in \mathbf{O}(k)$).

The (real) *affine Grassmannian* $AG(k, n)$ consists of all k-dimensional affine subspaces of $\mathbb{R}^n$ ($1 \leq k \leq n$).

Recall that the *Euclidean group* $\mathbf{E}(n)$ consists of all invertible affine maps (Q, u), with $Q \in \mathbf{O}(n)$ and $u \in \mathbb{R}^n$, and that the *special Euclidean group* $\mathbf{SE}(n)$ consists of all invertible affine maps (Q, u), with $Q \in \mathbf{SO}(n)$ and $u \in \mathbb{R}^n$. As usual, we represent an element (Q, u) of $\mathbf{E}(n)$ (or $\mathbf{SE}(n)$) by the $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} Q & u \\ 0 & 1 \end{pmatrix},$$

with $\mathbb{R}^n$ embedded in $\mathbb{R}^{n+1}$ by adding 1 as $(n+1)$th coordinate.

Define an action of the group $\mathbf{SE}(n)$ on $AG(k, n)$ as follows: if $\mathcal{A} \in AG(k, n)$, for any affine frame (a_0, A) representing $\mathcal{A}$ (where $A^\top A = I_k$), for any $(Q, u) \in \mathbf{SE}(n)$, then

$$(Q, u) \cdot \mathcal{A} = (Qa_0 + u, QA).$$

(1) Check that the above action does not depend on the affine frame (a_0, A) chosen for $\mathcal{A}$.

(2) Prove the above action is transitive.

(3) Next, we determine the stabilizer of the affine subspace defined by the affine frame $(0, (e_1, \ldots, e_k))$, where $e_1, \ldots, e_k$ are the first k canonical basis vectors of $\mathbb{R}^n$. This affine subspace is also represented by $(0, P_{n,k})$, where $P_{n,k}$ is the $n \times k$ matrix consisting of the first k columns of the identity matrix I_n; namely

$$P_{n,k} = \begin{pmatrix} I_k \\ 0_{n-k,k} \end{pmatrix}.$$

Prove that the stabilizer of the affine subspace defined by $(0, P_{n,k})$ is the group $H = S(\mathbf{E}(k) \times \mathbf{O}(n-k))$ given by the set of matrices

$$H = \left\{ \begin{pmatrix} Q & 0 & u \\ 0 & R & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; Q \in \mathbf{O}(k),\ R \in \mathbf{O}(n-k),\ \det(Q)\det(R) = 1,\ u \in \mathbb{R}^k \right\}.$$

(4) For any k and n such that $1 \leq k \leq n$, let $I_{k,n-k}$ be the matrix

$$I_{k,n-k} = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix}.$$

Note that $I_{k,n-k}^2 = I_n$.

Let $\sigma\colon \mathbf{SE}(n) \to \mathbf{SE}(n)$ be the map given by

$$\sigma \begin{pmatrix} Q & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I_{k,n-k} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Q & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_{k,n-k} & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} Q & z \\ 0 & 1 \end{pmatrix} \in \mathbf{SE}(n).$$

Prove that $\sigma^2 = \mathrm{id}$, and that σ is a group homomorphism (that is, $\sigma((Q,u)(R,v)) = \sigma(Q,u)\sigma(R,v)$, for all $(Q,u),(R,v) \in \mathbf{SE}(n)$).

(5) The *subgroup* $\mathbf{SE}(n)^\sigma$ *fixed by* σ is defined by

$$\mathbf{SE}(n)^\sigma = \{P \in \mathbf{SE}(n) \mid \sigma(P) = P\}.$$

Prove that

$$\mathbf{SE}(n)^\sigma = \left\{ \begin{pmatrix} Q & 0 & u \\ 0 & R & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; Q \in \mathbf{O}(k),\ R \in \mathbf{O}(n-k),\ \det(Q)\det(R) = 1,\ u \in \mathbb{R}^k \right\}.$$

(6) Let $\mathfrak{se}(n)$ be the following vector space

$$\mathfrak{se}(n) = \left\{ \begin{pmatrix} S & -A^\top & u \\ A & T & v \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; S \in \mathfrak{so}(k),\ T \in \mathfrak{so}(n-k),\ A \in \mathrm{M}_{n-k,k},\ u \in \mathbb{R}^k,\ v \in \mathbb{R}^{n-k} \right\}.$$

Are the matrices in $\mathfrak{se}(n)$ skew-symmetric? If not, give a necessary and sufficient condition for such matrices to be skew-symmetric.

Check that the map $\theta\colon \mathfrak{se}(n) \to \mathfrak{se}(n)$ given by

$$\theta(X) = \begin{pmatrix} I_{k,n-k} & 0 \\ 0 & 1 \end{pmatrix} X \begin{pmatrix} I_{k,n-k} & 0 \\ 0 & 1 \end{pmatrix}, \quad X \in \mathfrak{se}(n)$$

is the derivative $d\sigma_I$.

Prove that θ is a linear involution of $\mathfrak{se}(n)$. Prove that the subspaces

$$\begin{aligned} \mathfrak{h} &= \{X \in \mathfrak{se}(n) \mid \theta(X) = X\} \\ \mathfrak{m} &= \{X \in \mathfrak{se}(n) \mid \theta(X) = -X\} \end{aligned}$$

are given by

$$\mathfrak{h} = \left\{ \begin{pmatrix} S & 0 & u \\ 0 & T & 0 \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; S \in \mathfrak{so}(k),\ T \in \mathfrak{so}(n-k),\ u \in \mathbb{R}^k \right\}$$

and

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & -A^\top & 0 \\ A & 0 & v \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; A \in \mathrm{M}_{n-k,k},\ v \in \mathbb{R}^{n-k} \right\}.$$

(7) Prove (very quickly) that

$$\mathfrak{se}(n) = \mathfrak{h} \oplus \mathfrak{m},$$

and that $\dim(\mathfrak{m}) = (k+1)(n-k)$.

Apply Proposition 18.22 to conclude that the affine Grassmannian $AG(k,n)$ is a reductive homogeneous space with a simple reductive decomposition $\mathfrak{se}(n) = \mathfrak{h} \oplus \mathfrak{m}$. In fact, except for the fact that there is no $\mathrm{Ad}(H)$-invariant metric on $\mathfrak{m}$ (because H is not compact), all the other properties of a symmetric space are satisfied.

Problem B2 (60). Consider the Lie group $\mathbf{SO}(n)$ with the bi-invariant metric induced by the inner product on $\mathfrak{so}(n)$ given by

$$\langle B_1, B_2 \rangle = \frac{1}{2}\mathrm{tr}(B_1^\top B_2).$$

For any two matrices $B_1, B_2 \in \mathfrak{so}(n)$, let γ be the curve given by

$$\gamma(t) = e^{(1-t)B_1 + tB_2}, \quad 0 \leq t \leq 1.$$

This is a curve "interpolating" between the two rotations $R_1 = e^{B_1}$ and $R_2 = e^{B_2}$.

(1) Prove that the length $L(\gamma)$ of the curve γ is given by

$$L(\gamma) = \left(-\frac{1}{2}\mathrm{tr}((B_2 - B_1)^2)\right)^{\frac{1}{2}}.$$

(2) We know that the geodesic from R_1 to R_2 is given by

$$\gamma_g(t) = R_1 e^{tB}, \quad 0 \leq t \leq 1,$$

where $B \in \mathfrak{so}(n)$ is the principal log of $R_1^\top R_2$ (if we assume that $R_1^\top R_2$ is not a rotation by π, i.e, does not admit -1 as an eigenvalue).

Conduct numerical experiments to verify that in general, $\gamma(1/2) \neq \gamma_g(1/2)$.

Problem B3 (40). Prove that for any matrix

$$X = \begin{pmatrix} 0 & -u^\top \\ u & 0 \end{pmatrix},$$

where $u \in \mathbb{R}^n$ (a column vector), we have

$$e^{tX} = \begin{pmatrix} \cos(\|u\|\, t) & -\sin(\|u\|\, t)\frac{u^\top}{\|u\|} \\ \sin(\|u\|\, t)\frac{u}{\|u\|} & I + (\cos(\|u\|\, t) - 1)\frac{uu^\top}{\|u\|^2} \end{pmatrix}.$$

Problem B4 ((Extra Credit) (50). Review Gauss' Lemma (Proposition 13.7) and its proof. Do Carmo states Gauss' lemma in the following form:

Let $(M, \langle -, - \rangle)$ be a Riemannian manifold. For any $p \in M$ and for any $v \in T_pM$ such that $\|v\| < i(p)$ (so that $\exp_p(v)$ is defined), for every $w \in T_pM \approx T_v(T_pM)$, we have

$$\langle (d \exp_p)_v(v), (d \exp_p)_v(w) \rangle = \langle v, w \rangle.$$

Do Carmo begins his proof by saying, let

$$w = w_T + w_N,$$

where w_T is the projection of w on the subspace spanned by v and w_N is the projection of w onto the orthogonal complement of that subspace (in T_pM). He then says "since $(d \exp_p)_v$ is linear and by definition of $\exp_p$, we have

$$\langle (d \exp_p)_v(v), (d \exp_p)_v(w_T) \rangle = \langle v, w_T \rangle,$$

it suffices to prove that

$$\langle (d \exp_p)_v(v), (d \exp_p)_v(w_N) \rangle = \langle v, w_N \rangle,$$

and we can assume that $w_N \neq 0$."

Explain why what Do carmo says is correct, and why his version of Gauss Lemma reduces to the version in Proposition 13.7.

Problem B5 (100). Let E be a real vector space of dimension $n \geq 1$, and let $\langle -, - \rangle_1$ and $\langle -, - \rangle_2$ be two inner products on E. Let $\varphi_k \colon E \to E^*$ be the linear map given by

$$\varphi_k(u)(v) = \langle u, v \rangle_k, \quad u, v \in E, k = 1, 2.$$

(1) Prove that if $(u_1, \ldots, u_n)$ is an orthonormal basis for $(E, \langle -, - \rangle_1)$, then

$$\varphi_1(u_i) = u_i^*, \quad i = 1, \ldots, n,$$

where $(u_1^*, \ldots, u_n^*)$ is the dual basis in E^* of $(u_1, \ldots, u_n)$ (recall that $u_i^*(u_j) = \delta_{ij}$).

Prove that for any basis $(u_1, \ldots, u_n)$ in E and its dual basis $(u_1^*, \ldots, u_n^*)$ in E^*, the matrix A_k representing φ_k $(k = 1, 2)$ is given by

$$(A_k)_{ij} = \varphi_k(u_j)(u_i) = \langle u_j, u_i \rangle_k, \quad 1 \leq i, j, \leq n$$

Conclude that A_k is symmetric positive definite $(k = 1, 2)$.

(2) Consider the linear map $f \colon E \to E$ defined by

$$f = \varphi_1^{-1} \circ \varphi_2.$$

Check that

$$\langle u, v\rangle_2 = \langle f(u), v\rangle_1, \quad \text{for all } u, v \in E,$$

and deduce from the above that f is self-adjoint with respect to $\langle -, -\rangle_1$.

(3) Prove that there is some orthonormal basis $(u_1, \ldots, u_n)$ for $(E, \langle -, -\rangle_1)$ which is also an orthogonal basis for $(E, \langle -, -\rangle_2)$. Prove that this result still holds if $\langle -, -\rangle_1$ is an inner product and $\langle -, -\rangle_2$ is any symmetric bilinear form. We say that $\langle -, -\rangle_2$ is *diagonalized* by $\langle -, -\rangle_1$.

Hint. Use Theorem 12.7 from my notes linalg.pdf; see the notes for CIS515.

Assume that $\langle -, -\rangle_1$ is a symmetric, nondegenerate, bilinear form and that $\langle -, -\rangle_2$ is any symmetric bilinear form. Prove that for any basis $(e_1, \ldots, e_n)$ of E, if $(e_1, \ldots, e_n)$ is orthogonal for $\langle -, -\rangle_1$ implies that it is also orthogonal for $\langle -, -\rangle_2$, which means that

$$\text{if} \quad \langle e_j, e_j\rangle_1 = 0 \quad \text{then} \quad \langle e_j, e_j\rangle_2 = 0, \quad \text{for all } i \neq j,$$

then $f = \varphi_1^{-1} \circ \varphi_2$ has $(e_1, \ldots, e_n)$ as a basis of eigenvectors.

Find an example of two symmetric, nondegenerate bilinear forms that do not admit a common orthogonal basis.

(4) Given a group G and a real finite dimensional vector space E, a *representation* of G is any homomorphism $\rho\colon G \to \mathbf{GL}(E)$. A subspace $U \subseteq E$ is *invariant* under ρ if for every $g \in G$, we have $\rho(g)(u) \in U$ for all $u \in U$. A representation is said to be *irreducible* if its only invariant subspaces are (0) and E.

For any two inner products $\langle -, -\rangle_1$ and $\langle -, -\rangle_2$ on E, if $\rho(g)$ is an isometry for both $\langle -, -\rangle_1$ and $\langle -, -\rangle_2$ for all $g \in G$ (which means that $\langle \rho(g)(u), \rho(g)(v)\rangle_k = \langle u, v\rangle_k$ for all $u, v \in E$, $k = 1, 2$) and if ρ is irreducible, then prove that $\langle -, -\rangle_2 = \lambda\langle -, -\rangle_1$, for some nonzero $\lambda \in \mathbb{R}$.

Hint. Compare $\rho(g) \circ f$ and $f \circ \rho(g)$ and show that the eigenspaces of f (as defined in (2)) are invariant under each $\rho(g)$.

In the situation of Proposition 18.16, where we have a homogeneous reductive space G/H with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, prove that if the representation $\mathrm{Ad}^{\mathrm{G}}\colon H \to \mathbf{GL}(\mathfrak{m})$ is irreducible (where Ad_h is restricted to $\mathfrak{m}$ for all $h \in H$), then any two $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{m}$ are proportional to each other.

TOTAL: 360 + 50 points.