$\boxed{\textbf{Spring 2015} \quad \textbf{CIS 610}}$

Advanced Geometric Methods in Computer Science
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Homework 4

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Problem B1 (180). The "right way" (meaning convenient and rigorous) to define the *unit quaternions* is to define them are the elements of the unitary group $\mathbf{SU}(2)$, namely the group of 2×2 complex matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \quad \alpha, \beta \in \mathbb{C},\ \alpha\overline{\alpha} + \beta\overline{\beta} = 1.$$

Then, the *quaternions* are the elements of the real vector space $\mathbb{H} = \mathbb{R}\,\mathbf{SU}(2)$. Let $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ be the matrices

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

then $\mathbb{H}$ is the set of all matrices of the form

$$X = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, \quad a, b, c, d \in \mathbb{R}.$$

Indeed, every matrix in $\mathbb{H}$ is of the form

$$X = \begin{pmatrix} a + ib & c + id \\ -(c - id) & a - ib \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}.$$

(1) Prove that the quaternions $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy the famous identities discovered by Hamilton:

$$\begin{aligned} \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} &= -\mathbf{1}, \\ \mathbf{ij} = -\mathbf{ji} &= \mathbf{k}, \\ \mathbf{jk} = -\mathbf{kj} &= \mathbf{i}, \\ \mathbf{ki} = -\mathbf{ik} &= \mathbf{j}. \end{aligned}$$

Prove that $\mathbb{H}$ is a skew field (a noncommutative field) called the *quaternions*, and a real vector space of dimension 4 with basis $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k})$; thus as a vector space, $\mathbb{H}$ is isomorphic to $\mathbb{R}^4$.

A concise notation for the quaternion X defined by $\alpha = a + ib$ and $\beta = c + id$ is

$$X = [a, (b, c, d)].$$

We call a the *scalar part* of X and (b, c, d) the *vector part* of X. With this notation, $X^* = [a, -(b, c, d)]$, which is often denoted by $\overline{X}$. The quaternion $\overline{X}$ is called the *conjugate* of q. If q is a unit quaternion, then $\overline{q}$ is the multiplicative inverse of q. A *pure quaternion* is a quaternion whose scalar part is equal to zero.

(2) Given a unit quaternion

$$q = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathbf{SU}(2),$$

the usual way to define the rotation ρ_q (of $\mathbb{R}^3$) induced by q is to embed $\mathbb{R}^3$ into $\mathbb{H}$ as the pure quaternions, by

$$\psi(x, y, z) = \begin{pmatrix} ix & y + iz \\ -y + iz & -ix \end{pmatrix}, \quad (x, y, z) \in \mathbb{R}^3.$$

Observe that the above matrix is skew-Hermitian ($\psi(x, y, z)^* = -\psi(x, y, z)$). But, the space of skew-Hermitian matrices is the Lie algebra $\mathfrak{su}(2)$ of $\mathbf{SU}(2)$, so $\psi(x, y, z) \in \mathfrak{su}(2)$. Then, q defines the map ρ_q (on $\mathbb{R}^3$) given by

$$\rho_q(x, y, z) = \psi^{-1}(q\psi(x, y, z)q^*),$$

where q^* is the inverse of q (since $\mathbf{SU}(2)$ is a unitary group) and is given by

$$q^* = \begin{pmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{pmatrix}.$$

Actually, the *adjoint representation* of the group $\mathbf{SU}(2)$ is the group homomorphism $\mathrm{Ad}\colon \mathbf{SU}(2) \to \mathbf{GL}(\mathfrak{su}(2))$ defined such that for every $q \in \mathbf{SU}(2)$,

$$\mathrm{Ad}_q(A) = qAq^*, \quad A \in \mathfrak{su}(2).$$

Therefore, modulo the isomorphism ψ, the linear map ρ_q is the linear isomorphism Ad_q. In fact, ρ_q is a rotation (and so is Ad_q), which you will prove shortly.

Since the matrix $\psi(x, y, z)$ is skew-Hermitian, the matrix $-i\psi(x, y, z)$ is Hermitian, and we have

$$-i\psi(x, y, z) = \begin{pmatrix} x & z - iy \\ z + iy & -x \end{pmatrix} = x\sigma_3 + y\sigma_2 + z\sigma_1,$$

where $\sigma_1, \sigma_2, \sigma_3$ are the *Pauli spin matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Check that $\mathbf{i} = i\sigma_3$, $\mathbf{j} = i\sigma_2$, $\mathbf{k} = i\sigma_1$. Prove that matrices of the form $x\sigma_3 + y\sigma_2 + z\sigma_1$ (with $x, y, x \in \mathbb{R}$) are exactly the 2×2 Hermitian matrix with zero trace.

(3) Prove that for every $q \in \mathbf{SU}(2)$, the map $A \mapsto qAq^*$ preserves the Hermitian matrices with zero trace.

Prove that
$$\det(x\sigma_3 + y\sigma_2 + z\sigma_1) = \det(qAq^*) = -(x^2 + y^2 + z^2).$$

We can embed $\mathbb{R}^3$ into the space of Hermitian matrices with zero trace by
$$\varphi(x, y, z) = x\sigma_3 + y\sigma_2 + z\sigma_1.$$

Check that
$$\varphi = -i\psi$$
and
$$\varphi^{-1} = i\psi^{-1}.$$

Prove that every quaternion q induces a map r_q on $\mathbb{R}^3$ by
$$r_q(x, y, z) = \varphi^{-1}(q\varphi(x, y, z)q^*) = \varphi^{-1}(q(x\sigma_3 + y\sigma_2 + z\sigma_1)q^*)$$
which is clearly linear, and an isometry. Thus, $r_q \in \mathbf{O}(3)$.

(4) Find the fixed points of r_q, where $q = (a, (b, c, d))$. If $(b, c, d) \neq (0, 0, 0)$, then show that the fixed points (x, y, z) of r_q are solutions of the equations
$$\begin{aligned} -dy + cz &= 0 \\ cx - by &= 0 \\ dx - bz &= 0. \end{aligned}$$
This linear system has the nontrivial solution (b, c, d) and the matrix of this system is
$$\begin{pmatrix} 0 & -d & c \\ c & -b & 0 \\ d & 0 & -b \end{pmatrix}.$$
Prove that the above matrix has rank 2, so the fixed points of r_q form the one-dimensional space spanned by (b, c, d). Deduce from this that r_q must be a rotation.

Prove that $r\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$ given by $r(q) = r_q$ is a group homomorphism whose kernel is $\{I, -I\}$.

(5) Find the matrix R_q representing r_q explicitly by computing
$$q(x\sigma_3 + y\sigma_2 + z\sigma_1)q^* = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} x & z - iy \\ z + iy & -x \end{pmatrix} \begin{pmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{pmatrix}.$$

You should find

$$R_q = \begin{pmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2ac+2bd \\ 2bc+2ad & a^2-b^2+c^2-d^2 & -2ab+2cd \\ -2ac+2bd & 2ab+2cd & a^2-b^2-c^2+d^2 \end{pmatrix}.$$

Since $a^2+b^2+c^2+d^2=1$, this matrix can also be written as

$$R_q = \begin{pmatrix} 2a^2+2b^2-1 & 2bc-2ad & 2ac+2bd \\ 2bc+2ad & 2a^2+2c^2-1 & -2ab+2cd \\ -2ac+2bd & 2ab+2cd & 2a^2+2d^2-1 \end{pmatrix}.$$

Prove that $r_q = \rho_q$.

(6) To prove the surjectivity of r algorithmically, proceed as follows.

First, prove that $\mathrm{tr}(R_q) = 4a^2 - 1$, so

$$a^2 = \frac{\mathrm{tr}(R_q)+1}{4}.$$

If $R \in \mathbf{SO}(3)$ is any rotation matrix and if we write

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33}, \end{pmatrix}$$

we are looking for a unit quaternion $q \in \mathbf{SU}(2)$ such that $r_q = R$. Therefore, we must have

$$a^2 = \frac{\mathrm{tr}(R)+1}{4}.$$

We also know that

$$\mathrm{tr}(R) = 1 + 2\cos\theta,$$

where $\theta \in [0,\pi]$ is the angle of the rotation R. Deduce that

$$|a| = \cos\left(\frac{\theta}{2}\right) \quad (0 \le \theta \le \pi).$$

There are two cases.

Case 1. $\mathrm{tr}(R) \neq -1$, or equivalently $\theta \neq \pi$. In this case $a \neq 0$. Pick

$$a = \frac{\sqrt{\mathrm{tr}(R)+1}}{2}.$$

Then, show that

$$b = \frac{r_{32}-r_{23}}{4a}, \quad c = \frac{r_{13}-r_{31}}{4a}, \quad d = \frac{r_{21}-r_{12}}{4a}.$$

Case 2. $\mathrm{tr}(R) = -1$, or equivalently $\theta = \pi$. In this case $a = 0$. Prove that

$$\begin{aligned} 4bc &= r_{21} + r_{12} \\ 4bd &= r_{13} + r_{31} \\ 4cd &= r_{32} + r_{23} \end{aligned}$$

and

$$\begin{aligned} b^2 &= \frac{1 + r_{11}}{2} \\ c^2 &= \frac{1 + r_{22}}{2} \\ d^2 &= \frac{1 + r_{33}}{2}. \end{aligned}$$

Since $q \neq 0$ and $a = 0$, at least one of b, c, d is nonzero.

If $b \neq 0$, let

$$b = \frac{\sqrt{1 + r_{11}}}{\sqrt{2}},$$

and determine c, d using

$$\begin{aligned} 4bc &= r_{21} + r_{12} \\ 4bd &= r_{13} + r_{31}. \end{aligned}$$

If $c \neq 0$, let

$$c = \frac{\sqrt{1 + r_{22}}}{\sqrt{2}},$$

and determine b, d using

$$\begin{aligned} 4bc &= r_{21} + r_{12} \\ 4cd &= r_{32} + r_{23}. \end{aligned}$$

If $d \neq 0$, let

$$d = \frac{\sqrt{1 + r_{33}}}{\sqrt{2}},$$

and determine b, c using

$$\begin{aligned} 4bd &= r_{13} + r_{31} \\ 4cd &= r_{32} + r_{23}. \end{aligned}$$

(7) Given any matrix $A \in \mathfrak{su}(2)$, with

$$A = \begin{pmatrix} iu_1 & u_2 + iu_3 \\ -u_2 + iu_3 & -iu_1 \end{pmatrix},$$

write $\theta = \sqrt{u_1^2 + u_2^2 + u_3^2}$ and prove that

$$e^A = \cos\theta I + \frac{\sin\theta}{\theta} A, \quad \theta \neq 0,$$

with $e^0 = 0$. Therefore, e^A is a unit quaternion representing the rotation of angle 2θ and axis (u_1, u_2, u_3) (or I when $\theta = k\pi$, $k \in \mathbb{Z}$). The above formula shows that we may assume that $0 \leq \theta \leq \pi$.

An equivalent but often more convenient formula is obtained by assuming that $u = (u_1, u_2, u_3)$ is a unit vector, equivalently $\det(A) = -1$, in which case $A^2 = -I$, so we have

$$e^{\theta A} = \cos\theta I + \sin\theta A.$$

Using the quaternion notation, this read as

$$e^{\theta A} = [\cos\theta, \sin\theta\, u].$$

Prove that the logarithm $A \in \mathfrak{su}(2)$ of a unit quaternion

$$q = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

with $\alpha = a + bi$ and $\beta = c + id$ can be determined as follows:

If $q = I$ (*i.e.* $a = 1$) then $A = 0$. If $q = -I$ (*i.e.* $a = -1$), then

$$A = \pm\pi \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Otherwise, $a \neq \pm 1$ and $(b, c, d) \neq (0, 0, 0)$, and we are seeking some $A = \theta B \in \mathfrak{su}(2)$ with $\det(B) = 1$ and $0 < \theta < \pi$, such that

$$q = e^{\theta B} = \cos\theta I + \sin\theta B.$$

Then,

$$\cos\theta = a \qquad (0 < \theta < \pi)$$
$$(u_1, u_2, u_3) = \frac{1}{\sin\theta}(b, c, d).$$

Since $a^2 + b^2 + c^2 + d^2 = 1$ and $a = \cos\theta$, the vector $(b, c, d)/\sin\theta$ is a unit vector. Furthermore if the quaternion q is of the form $q = [\cos\theta, \sin\theta u]$ where $u = (u_1, u_2, u_3)$ is a unit vector (with $0 < \theta < \pi$), then

$$A = \theta \begin{pmatrix} iu_1 & u_2 + iu_3 \\ -u_2 + iu_3 & -iu_1 \end{pmatrix}$$

is a logarithm of q.

Show that the exponential map $\exp\colon \mathfrak{su}(2) \to \mathbf{SU}(2)$ is surjective, and injective on the open ball

$$\{\theta B \in \mathfrak{su}(2) \mid \det(B) = 1, 0 \leq \theta < \pi\}.$$

(8) You are now going to derive a formula for interpolating between two quaternions. This formula is due to Ken Shoemake, once a Penn student and my TA! Since rotations in $\mathbf{SO}(3)$ can be defined by quaternions, this has applications to computer graphics, robotics, and computer vision.

First, we observe that multiplication of quaternions can be expressed in terms of the inner product and the cross-product in $\mathbb{R}^3$. Indeed, if $q_1 = [a, u_1]$ and $q_2 = [a_2, u_2]$, then check that

$$q_1 q_2 = [a_1, u_1][a_2, u_2] = [a_1 a_2 - u_1 \cdot u_2,\ a_1 u_2 + a_2 u_1 + u_1 \times u_2].$$

We will also need the identity

$$u \times (u \times v) = (u \cdot v)u - (u \cdot u)v.$$

Given a quaternion q expressed as $q = [\cos\theta, \sin\theta\, u]$, where u is a unit vector, we can interpolate between I and q by finding the logs of I and q, interpolating in $\mathfrak{su}(2)$, and then exponentiating. We have

$$A = \log(I) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \log(q) = \theta \begin{pmatrix} i u_1 & u_2 + i u_3 \\ -u_2 + i u_3 & -i u_1 \end{pmatrix}.$$

Since $\mathbf{SU}(2)$ is a compact Lie group and since the inner product on $\mathfrak{su}(2)$ given by

$$\langle X, Y \rangle = \mathrm{tr}(X^\top Y)$$

is $\mathrm{Ad}(\mathbf{SU}(2))$-invariant, it induces a biinvariant Riemannian metric on $\mathbf{SU}(2)$, and the curve

$$\lambda \mapsto e^{\lambda B}, \quad \lambda \in [0, 1]$$

is a geodesic from I to q in $\mathbf{SU}(2)$. We write $q^\lambda = e^{\lambda B}$. Given two quaternions q_1 and q_2, because the metric is left invariant, the curve

$$\lambda \mapsto Z(\lambda) = q_1 (q_1^{-1} q_2)^\lambda, \quad \lambda \in [0, 1]$$

is a geodesic from q_1 to q_2. Remarkably, there is a closed-form formula for the interpolant $Z(\lambda)$. Say $q_1 = [\cos\theta, \sin\theta\, u]$ and $q_2 = [\cos\varphi, \sin\varphi\, v]$, and assume that $q_1 \neq q_2$ and $q_1 \neq -q_2$.

Define Ω by

$$\cos\Omega = \cos\theta\cos\varphi + \sin\theta\sin\varphi(u \cdot v).$$

Since $q_1 \neq q_2$ and $q_1 \neq -q_2$, we have $0 < \Omega < \pi$. Prove that

$$Z(\lambda) = q_1(q_1^{-1}q_2)^\lambda = \frac{\sin(1-\lambda)\Omega}{\sin\Omega} q_1 + \frac{\sin\lambda\Omega}{\sin\Omega} q_2.$$

(9) We conclude by discussing the problem of a consistent choice of sign for the quaternion q representing a rotation $R = \rho_q \in \mathbf{SO}(3)$. We are looking for a "nice" section $s\colon \mathbf{SO}(3) \to \mathbf{SU}(2)$, that is, a function s satisfying the condition

$$\rho \circ s = \mathrm{id},$$

where ρ is the surjective homomorphism $\rho\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$.

I claim that any section $s\colon \mathbf{SO}(3) \to \mathbf{SU}(2)$ of ρ is neither a homomorphism nor continuous. Intuitively, this means that there is no "nice and simple " way to pick the sign of the quaternion representing a rotation.

To prove the above claims, let Γ be the subgroup of $\mathbf{SU}(2)$ consisting of all quaternions of the form $q = [a, (b, 0, 0)]$. Then, using the formula for the rotation matrix R_q corresponding to q (and the fact that $a^2 + b^2 = 1$), show that

$$R_q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2a^2 - 1 & -2ab \\ 0 & 2ab & 2a^2 - 1 \end{pmatrix}.$$

Since $a^2 + b^2 = 1$, we may write $a = \cos\theta, b = \sin\theta$, and we see that

$$R_q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix},$$

a rotation of angle 2θ around the x-axis. Thus, both Γ and its image are isomorphic to $\mathbf{SO}(2)$, which is also isomorphic to $\mathbf{U}(1) = \{w \in \mathbb{C} \mid |w| = 1\}$. By identifying $\mathbf{i}$ and i and identifying Γ and its image to $\mathbf{U}(1)$, if we write $w = \cos\theta + i\sin\theta \in \Gamma$, show that the restriction of the map ρ to Γ is given by $\rho(w) = w^2$.

Prove that any section s of ρ is not a homomorphism. (Consider the restriction of s to the image $\rho(\Gamma)$).

Prove that any section s of ρ is not continuous.

Problem B2 (120 pts). (a) Let $f\colon M \to N$ be a map of smooth manifolds. A point, $p \in M$, is called a *critical point (of f)* iff df_p is *not* surjective and a point $q \in N$ is called a *critical value (of f)* iff $q = f(p)$, for some critical point, $p \in M$. A point $p \in M$ is a *regular point (of f)* iff p is not critical, i.e., df_p is surjective, and a point $q \in N$ is a *regular value (of f)* iff it is not a critical value. In particular, any $q \in N - f(M)$ is a regular value and

$q \in f(M)$ is a regular value iff *every* $p \in f^{-1}(q)$ is a regular point (but, in contrast, q is a critical value iff *some* $p \in f^{-1}(q)$ is critical).

Prove that for every regular value, $q \in f(M)$, the preimage $Z = f^{-1}(q)$ is a manifold of dimension $\dim(M) - \dim(N)$.

Hint. Pick any $p \in f^{-1}(q)$ and some parametrizations φ at p and ψ at q, with $\varphi(0) = p$ and $\psi(0) = q$, and consider $h = \psi^{-1} \circ f \circ \varphi$. Prove that dh_0 is surjective and then apply Lemma 2.29.

(b) Under the same assumptions as (a), prove that for every point $p \in Z = f^{-1}(q)$, the tangent space, T_pZ, is the kernel of $df_p \colon T_pM \to T_qN$.

(c) If $X, Z \subseteq \mathbb{R}^N$ are manifolds and $Z \subseteq X$, we say that Z *is a submanifold of* X. Assume there is a smooth function, $g \colon X \to \mathbb{R}^k$, and that $0 \in \mathbb{R}^k$ is a regular value of g. Then, by (a), $Z = g^{-1}(0)$ is a submanifold of X of dimension $\dim(X) - k$. Let $g = (g_1, \ldots, g_k)$, with each g_i a function, $g_i \colon X \to \mathbb{R}$. Prove that for any $p \in X$, dg_p is surjective iff the linear forms, $(dg_i)_p \colon T_pX \to \mathbb{R}$, are linearly independent. In this case, we say that $g_1, \ldots, g_k$ are *independent at* p. We also say that Z is *cut out* by $g_1, \ldots, g_k$ when

$$Z = \{p \in X \mid g_1(p) = 0, \ldots, g_k(p) = 0\}$$

with $g_1, \ldots, g_k$ independent for all $p \in Z$.

Let $f \colon X \to Y$ be a smooth maps of manifolds and let $q \in f(X)$ be a regular value. Prove that $Z = f^{-1}(q)$ is a submanifold of X cut out by $k = \dim(X) - \dim(Y)$ independent functions.

Hint. Pick some parametrization, ψ, at q, so that $\psi(0) = q$ and check that 0 is a regular value of $g = \psi^{-1} \circ f$, so that $g_1, \ldots, g_k$ work.

(d) Now, assume Z is a submanifold of X. Prove that locally, Z is cut out by independent functions. This means that if $k = \dim(X) - \dim(Z)$, the *codimension* of Z in X, then for every $z \in Z$, there are k independent functions, $g_1, \ldots, g_k$, defined on some open subset, $W \subseteq X$, with $z \in W$, so that $Z \cap W$ is the common zero set of the g_i's.

Hint. Apply Lemma 2.28 to the immersion $Z \longrightarrow X$.

(e) We would like to generalize our result in (a) to the more general situation where we have a smooth map, $f \colon X \to Y$, but this time, we have a submanifold, $Z \subseteq Y$ and we are investigating whether $f^{-1}(Z)$ is a submanifold of X. In particular, if X is also a submanifold of Y and f is the inclusion of X into Y, then $f^{-1}(Z) = X \cap Z$.

Convince yourself that, in general, the intersection of two submanifolds is *not* a submanifold. Try examples involving curves and surfaces and you will see how bad the situation can be. What is needed is a notion generalizing that of a regular value, and this turns out to be the notion of transversality.

We say that f *is transversal to* Z iff

$$df_p(T_pX) + T_{f(p)}Z = T_{f(p)}Y,$$

for all $p \in f^{-1}(Z)$. (Recall, if U and V are subspaces of a vector space, E, then $U+V$ is the subspace $U+V = \{u+v \in E \mid u \in U,\, v \in V\}$). In particular, if f is the inclusion of X into Y, the transversality condition is

$$T_pX + T_pZ = T_pY,$$

for all $p \in X \cap Z$.

Draw several examples of transversal intersections to understand better this concept. Prove that if f is transversal to Z, then $f^{-1}(Z)$ is a submanifold of X of codimension equal to $\dim(Y) - \dim(Z)$.

Hint. The set $f^{-1}(Z)$ is a manifold iff for every $p \in f^{-1}(Z)$, there is some open subset, $U \subseteq X$, with $p \in U$, and $f^{-1}(Z) \cap U$ is a manifold. First, use (d) to assert that locally near $q = f(p)$, Z is cut out by $k = \dim(Y) - \dim(Z)$ independent functions, $g_1, \ldots, g_k$, so that locally near p, the preimage $f^{-1}(Z)$ is cut out by $g_1 \circ f, \ldots, g_k \circ f$. If we let $g = (g_1, \ldots, g_k)$, it is a submersion and the issue is to prove that 0 is a regular value of $g \circ f$ in order to apply (a). Show that transversality is just what's needed to show that 0 is a regular value of $g \circ f$.

(f) With the same assumptions as in (g) (f is transversal to Z), if $W = f^{-1}(Z)$, prove that for every $p \in W$,

$$T_pW = (df_p)^{-1}(T_{f(p)}Z),$$

the preimage of $T_{f(p)}Z$ by $df_p\colon T_pX \to T_{f(p)}Y$. In particular, if f is the inclusion of X into Y, then

$$T_p(X \cap Z) = T_pX \cap T_pZ.$$

(g) Let $X, Z \subseteq Y$ be two submanifolds of Y, with X compact, Z closed, $\dim(X) + \dim(Z) = \dim(Y)$ and X transversal to Z. Prove that $X \cap Z$ consists of a finite set of points.

Problem B3 (120). (1) All Lie algebras in this problem are finite-dimensional. Let $\mathfrak{g}$ be a Lie algebra (over $\mathbb{R}$ or $\mathbb{C}$). Given two subsets $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathfrak{g}$, we let $[\mathfrak{a}, \mathfrak{b}]$ be the subspace of $\mathfrak{g}$ consisting of all linear combinations of elements of the form $[a, b]$ with $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. Check that if $\mathfrak{a}$ and $\mathfrak{b}$ are ideals, then $[\mathfrak{a}, \mathfrak{b}]$ is an ideal.

(2) The *lower central series* $(C^k\mathfrak{g})$ of $\mathfrak{g}$ is defined as follows:

$$\begin{aligned} C^0\mathfrak{g} &= \mathfrak{g} \\ C^{k+1}\mathfrak{g} &= [\mathfrak{g}, C^k\mathfrak{g}], \quad k \geq 0. \end{aligned}$$

We have a decreasing sequence

$$\mathfrak{g} = C^0\mathfrak{g} \supseteq C^1\mathfrak{g} \supseteq C^2\mathfrak{g} \supseteq \cdots.$$

We say that $\mathfrak{g}$ is *nilpotent* iff $C^k\mathfrak{g} = (0)$ for some $k \geq 1$.

Prove that the following statements are equivalent:

1. The algebra $\mathfrak{g}$ is nilpotent.

2. There is some $n \geq 1$ such that

$$[x_1, [x_2, [x_3, \cdots, [x_n, x_{n+1}] \cdots]]] = 0$$

   for all $x_1, \ldots, x_{n+1} \in \mathfrak{g}$.

3. There is a chain of ideals

$$\mathfrak{g} = \mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \cdots \supseteq \mathfrak{a}_n = (0)$$

   such that $[\mathfrak{g}, \mathfrak{a}_i] \subseteq \mathfrak{a}_{i+1}$ for $i = 0, \ldots, n-1$ $(n \geq 1)$.

(3) Given a vector space E of dimension n, a *flag* in E is a sequence $F = (V_i)$ of subspaces of E such that

$$(0) = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = E,$$

such that $\dim(V_i) = i$. Define $\mathfrak{n}(F)$ by

$$\mathfrak{n}(F) = \{f \in \mathrm{End}(E) \mid f(V_i) \subseteq V_{i-1},\ i = 1, \ldots, n\}.$$

If we pick a basis $(e_1, \ldots, e_n)$ of E such that $e_i \in V_i$, then check that every $f \in \mathfrak{n}(F)$ is represented by a strictly upper triangular matrix (the diagonal entries are 0).

Prove that $\mathfrak{n}(F)$ is a Lie subalgebra of $\mathrm{End}(E)$ and that it is nilpotent.

If $\mathfrak{g}$ is a nilpotent Lie algebra, then prove that ad_x is nilpotent for every $x \in \mathfrak{g}$.

(4) The *derived series* (or *commutator series*) $(D^k\mathfrak{g})$ of $\mathfrak{g}$ is defined as follows:

$$\begin{aligned} D^0\mathfrak{g} &= \mathfrak{g} \\ D^{k+1}\mathfrak{g} &= [D^k\mathfrak{g}, D^k\mathfrak{g}], \quad k \geq 0. \end{aligned}$$

We have a decreasing sequence

$$\mathfrak{g} = D^0\mathfrak{g} \supseteq D^1\mathfrak{g} \supseteq D^2\mathfrak{g} \supseteq \cdots.$$

We say that $\mathfrak{g}$ is *solvable* iff $D^k\mathfrak{g} = (0)$ for some $k \geq 1$.

Recall that a Lie algebra $\mathfrak{g}$ is *abelian* if $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. Check that If $\mathfrak{g}$ is abelian, then $\mathfrak{g}$ is solvable.

Prove that a nonzero solvable Lie algebra has a nonzero abelian ideal.

Prove that the following statements are equivalent:

1. The algebra $\mathfrak{g}$ is solvable.

2. There is a chain of ideals

$$\mathfrak{g} = \mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \cdots \supseteq \mathfrak{a}_n = (0)$$

such that $[\mathfrak{a}_i, \mathfrak{a}_i] \subseteq \mathfrak{a}_{i+1}$ for $i = 0, \ldots, n-1$ $(n \geq 1)$.

Given any flag $F = (V_i)$ in E (where E is a vector space of dimension n), define $\mathfrak{b}(F)$ by

$$\mathfrak{b}(F) = \{f \in \mathrm{End}(E) \mid f(V_i) \subseteq V_i,\ i = 0, \ldots, n\}.$$

If we pick a basis $(e_1, \ldots, e_n)$ of E such that $e_i \in V_i$, then check that every $f \in \mathfrak{b}(F)$ is represented by an upper triangular matrix.

Prove that $\mathfrak{b}(F)$ is a Lie subalgebra of $\mathrm{End}(E)$ and that it is solvable (observe that $D^1(\mathfrak{b}(F)) \subseteq \mathfrak{n}(F)$).

(5) Prove that

$$D^k\mathfrak{g} \subseteq C^k\mathfrak{g} \quad k \geq 0.$$

Deduce that every nilpotent Lie algebra is solvable.

(6) If $\mathfrak{g}$ is a solvable Lie algebra, then prove that every Lie subalgebra of $\mathfrak{g}$ is solvable, and for every ideal $\mathfrak{a}$ of $\mathfrak{g}$, the quotient Lie algebra $\mathfrak{g}/\mathfrak{a}$ is solvable.

Given a Lie algebra $\mathfrak{g}$, if $\mathfrak{a}$ is a solvable ideal and if $\mathfrak{g}/\mathfrak{a}$ is also solvable, then $\mathfrak{g}$ is solvable.

Given any two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of a Lie algebra $\mathfrak{g}$, prove that $(\mathfrak{a} + \mathfrak{b})/\mathfrak{a}$ and $\mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})$ are isomorphic Lie algebras.

Given any two solvable ideals $\mathfrak{a}$ and $\mathfrak{b}$ of a Lie algebra $\mathfrak{g}$, prove that $\mathfrak{a} + \mathfrak{b}$ is solvable. Conclude from this that there is a largest solvable ideal $\mathfrak{r}$ in $\mathfrak{g}$ (called the *radical* of $\mathfrak{g}$).

TOTAL: 420 **points.**