# Summer 1, 2013 CIS 610

# Advanced Geometric Methods in Computer Science
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## Homework 4

June 21; Due June 28, 2013

Do the Exercises on pages 4-6 (some properties of $\mathbf{GL}(n, K)$) from the handouts on the web, and the problems below.

**Problem B1 (10).** (a) Find two symmetric matrices, $A$ and $B$, such that $AB$ is not symmetric.

(b) Find two matrices, $A$ and $B$, such that

$$e^A e^B \neq e^{A+B}.$$

Try

$$A = \frac{\pi}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \frac{\pi}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

**Problem B2 (60).** (a) Consider the map, $f\colon \mathbf{GL}(n) \to \mathbb{R}$, given by

$$f(A) = \det(A).$$

Prove that $df(I)(B) = \mathrm{tr}(B)$, the trace of $B$, for any matrix $B$ (here, $I$ is the identity matrix). Then, prove that

$$df(A)(B) = \det(A)\mathrm{tr}(A^{-1}B),$$

where $A \in \mathbf{GL}(n)$.

(b) Use the map $A \mapsto \det(A) - 1$ to prove that $\mathbf{SL}(n)$ is a manifold of dimension $n^2 - 1$.

(c) Let $J$ be the $(n+1) \times (n+1)$ diagonal matrix

$$J = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}.$$

We denote by $\mathbf{SO}(n, 1)$ the group of real $(n+1) \times (n+1)$ matrices

$$\mathbf{SO}(n,1) = \{A \in \mathbf{GL}(n+1) \mid A^{\top} J A = J \quad \text{and} \quad \det(A) = 1\}.$$

Check that $\mathbf{SO}(n,1)$ is indeed a group with the inverse of $A$ given by $A^{-1} = JA^{\top}J$ (this is the *special Lorentz group*.) Consider the function $f\colon \mathbf{GL}^{+}(n+1) \to \mathbf{S}(n+1)$, given by

$$f(A) = A^{\top}JA - J,$$

where $\mathbf{S}(n+1)$ denotes the space of $(n+1)\times(n+1)$ symmetric matrices. Prove that

$$df(A)(H) = A^{\top}JH + H^{\top}JA$$

for any matrix, $H$. Prove that $df(A)$ is surjective for all $A \in \mathbf{SO}(n,1)$ and that $\mathbf{SO}(n,1)$ is a manifold of dimension $\frac{n(n+1)}{2}$.

**Problem B3 (30 + 30 pts).** (a) Given any matrix

$$B = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C}),$$

if $\omega^2 = a^2 + bc$ and $\omega$ is any of the two complex roots of $a^2 + bc$, prove that if $\omega \neq 0$, then

$$e^{B} = \cosh\omega\, I + \frac{\sinh\,\omega}{\omega}\, B,$$

and $e^{B} = I + B$, if $a^2 + bc = 0$. Observe that $\mathrm{tr}(e^{B}) = 2\cosh\,\omega$.

Prove that the exponential map, $\exp\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathbf{SL}(2,\mathbb{C})$, is *not* surjective. For instance, prove that

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

is not the exponential of any matrix in $\mathfrak{sl}(2,\mathbb{C})$.

(b) (**Extra Credit 30 pts.**) Recall that a matrix, $N$, is *nilpotent* iff there is some $m \geq 0$ so that $N^m = 0$. Let $A$ be any $n \times n$ matrix of the form $A = I - N$, where $N$ is nilpotent. Why is $A$ invertible? prove that there is some $B$ so that $e^{B} = I - N$ as follows: Recall that for any $y \in \mathbb{R}$ so that $|y - 1|$ is small enough, we have

$$\log(y) = -(1-y) - \frac{(1-y)^2}{2} - \cdots - \frac{(1-y)^k}{k} - \cdots.$$

As $N$ is nilpotent, we have $N^m = 0$, where $m$ is the smallest integer with this propery. Then, the expression

$$B = \log(I - N) = -N - \frac{N^2}{2} - \cdots - \frac{N^{m-1}}{m-1}$$

is well defined. Use a formal power series argument to show that

$$e^{B} = A.$$

We denote $B$ by $\log(A)$.

**Problem B4 (120 pts).** Recall from Homework 3, Problem B7, the Cayley parametrization of rotation matrices in $\mathbf{SO}(n)$ given by

$$C(B) = (I - B)(I + B)^{-1},$$

where $B$ is any $n \times n$ skew symmetric matrix.

(a) Now, consider $n = 3$, i.e., $\mathbf{SO}(3)$. Let $E_1$, $E_2$ and $E_3$ be the rotations about the $x$-axis, $y$-axis, and $z$-axis, respectively, by the angle $\pi$, i.e.,

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove that the four maps

$$\begin{aligned} B &\mapsto C(B) \\ B &\mapsto E_1C(B) \\ B &\mapsto E_2C(B) \\ B &\mapsto E_3C(B) \end{aligned}$$

where $B$ is skew symmetric, are parametrizations of $\mathbf{SO}(3)$ and that the union of the images of $C$, $E_1C$, $E_2C$ and $E_3C$ covers $\mathbf{SO}(3)$, so that $\mathbf{SO}(3)$ is a manifold.

(b) Let $A$ be *any* matrix (not necessarily invertible). Prove that there is some diagonal matrix, $E$, with entries $+1$ or $-1$, so that $EA + I$ is invertible.

(c) Prove that every rotation matrix, $A \in \mathbf{SO}(n)$, is of the form

$$A = E(I - B)(I + B)^{-1},$$

for some skew symmetric matrix, $B$, and some diagonal matrix, $E$, with entries $+1$ and $-1$, and where the number of $-1$ is even. Moreover, prove that every orthogonal matrix $A \in \mathbf{O}(n)$ is of the form

$$A = E(I - B)(I + B)^{-1},$$

for some skew symmetric matrix, $B$, and some diagonal matrix, $E$, with entries $+1$ and $-1$. The above provide parametrizations for $\mathbf{SO}(n)$ (resp. $\mathbf{O}(n)$) that show that $\mathbf{SO}(n)$ and $\mathbf{O}(n)$ are manifolds. However, observe that the number of these charts grows exponentially with $n$.

**Problem B5 (100 + 200 pts).** Consider the affine maps $\rho\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined such that

$$\rho \begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

where $\theta, w_1, w_2, \alpha \in \mathbb{R}$, with $\alpha > 0$. These maps are called (direct) *affine similitudes* (for short, *similitudes*). The number $\alpha > 0$ is the *scale factor* of the similitude. These affine maps are the composition of a rotation of angle $\theta$, a rescaling by $\alpha > 0$, and a translation.

(a) Prove that these maps form a group that we denote by $\mathbf{SIM}(2)$.

Given any map $\rho$ as above, if we let

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{and} \quad W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

then $\rho$ can be represented by the $3 \times 3$ matrix

$$A = \begin{pmatrix} \alpha R & W \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha\cos\theta & -\alpha\sin\theta & w_1 \\ \alpha\sin\theta & \alpha\cos\theta & w_2 \\ 0 & 0 & 1 \end{pmatrix}$$

in the sense that

$$\begin{pmatrix} \rho(X) \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha R & W \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}$$

iff

$$\rho(X) = \alpha RX + W.$$

(b) Consider the set of matrices of the form

$$\begin{pmatrix} \lambda & -\theta & u \\ \theta & \lambda & v \\ 0 & 0 & 0 \end{pmatrix}$$

where $\theta, \lambda, u, v \in \mathbb{R}$. Verify that this set of matrices is a vector space isomorphic to $(\mathbb{R}^4, +)$. This vector space is denoted by $\mathfrak{sim}(2)$.

(c) Given a matrix

$$\Omega = \begin{pmatrix} \lambda & -\theta \\ \theta & \lambda \end{pmatrix},$$

prove that

$$e^{\Omega} = e^{\lambda} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

*Hint*. Write

$$\Omega = \lambda I + \theta J,$$

with

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Observe that $J^2 = -I$, and prove by induction on $k$ that

$$\Omega^k = \frac{1}{2}\left((\lambda + i\theta)^k + (\lambda - i\theta)^k\right) I + \frac{1}{2i}\left((\lambda + i\theta)^k - (\lambda - i\theta)^k\right) J.$$

(d) As in (c), write

$$\Omega = \begin{pmatrix} \lambda & -\theta \\ \theta & \lambda \end{pmatrix},$$

let

$$U = \begin{pmatrix} u \\ v \end{pmatrix},$$

and let

$$B = \begin{pmatrix} \Omega & U \\ 0 & 0 \end{pmatrix}.$$

Prove that

$$B^n = \begin{pmatrix} \Omega^n & \Omega^{n-1}U \\ 0 & 0 \end{pmatrix}$$

where $\Omega^0 = I_2$.

Prove that

$$e^B = \begin{pmatrix} e^\Omega & VU \\ 0 & 1 \end{pmatrix},$$

where

$$V = I_2 + \sum_{k\geq 1} \frac{\Omega^k}{(k+1)!}.$$

(e) Use the formula

$$V = I_2 + \sum_{k\geq 1} \frac{\Omega^k}{(k+1)!} = \int_0^1 e^{\Omega t} dt$$

to prove that if $\lambda = \theta = 0$, then

$$V = I_2,$$

else

$$V = \frac{1}{\lambda^2 + \theta^2} \begin{pmatrix} \lambda(e^\lambda \cos\theta - 1) + e^\lambda \theta \sin\theta & -\theta(1 - e^\lambda \cos\theta) - e^\lambda \lambda \sin\theta \\ \theta(1 - e^\lambda \cos\theta) + e^\lambda \lambda \sin\theta & \lambda(e^\lambda \cos\theta - 1) + e^\lambda \theta \sin\theta \end{pmatrix}.$$

Conclude that if $\lambda = \theta = 0$, then

$$e^B = \begin{pmatrix} I & U \\ 0 & 1 \end{pmatrix},$$

else

$$e^B = \begin{pmatrix} e^\Omega & VU \\ 0 & 1 \end{pmatrix},$$

with

$$e^\Omega = e^\lambda \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

and

$$V = \frac{1}{\lambda^2+\theta^2}\begin{pmatrix} \lambda(e^\lambda \cos\theta - 1) + e^\lambda \theta \sin\theta & -\theta(1-e^\lambda\cos\theta) - e^\lambda \lambda \sin\theta \\ \theta(1-e^\lambda\cos\theta) + e^\lambda\lambda\sin\theta & \lambda(e^\lambda\cos\theta - 1) + e^\lambda\theta\sin\theta \end{pmatrix},$$

and that $e^B \in \mathbf{SIM}(2)$, with scale factor $e^\lambda$.

(f) Prove that the exponential map $\exp\colon \mathfrak{sim}(2) \to \mathbf{SIM}(2)$ is surjective.

(g) Similitudes can be used to describe certain deformations (or flows) of a deformable body $\mathcal{B}_t$ in the plane. Given some initial shape $\mathcal{B}$ in the plane (for example, a circle), a deformation of $\mathcal{B}$ is given by a piecewise differentiable curve

$$\mathcal{D}\colon [0,T] \to \mathbf{SIM}(2),$$

where each $\mathcal{D}(t)$ is a similitude (for some $T > 0$). The deformed body $\mathcal{B}_t$ at time $t$ is given by

$$\mathcal{B}_t = \mathcal{D}(t)(\mathcal{B}).$$

The surjectivity of the exponential map $\exp\colon \mathfrak{sim}(2) \to \mathbf{SIM}(2)$ implies that there is a map $\log\colon \mathbf{SIM}(2) \to \mathfrak{sim}(2)$, although it is multivalued. The exponential map and the log "function" allows us to work in the simpler (noncurved) Euclidean space $\mathfrak{sim}(2)$.

For instance, given two similitudes $A_1, A_2 \in \mathbf{SIM}(2)$ specifying the shape of $\mathcal{B}$ at two different times, we can compute $\log(A_1)$ and $\log(A_2)$, which are just elements of the Euclidean space $\mathfrak{sim}(2)$, form the linear interpolant $(1-t)\log(A_1) + t\log(A_2)$, and then apply the exponential map to get an interpolating deformation

$$t \mapsto e^{(1-t)\log(A_1) + t\log(A_2)}, \quad t \in [0,1].$$

Also, given a sequence of "snapshots" of the deformable body $\mathcal{B}$, say $A_0, A_1, \ldots, A_m$, where each is $A_i$ is a similitude, we can try to find an interpolating deformation (a curve in $\mathbf{SIM}(2)$) by finding a simpler curve $t \mapsto C(t)$ in $\mathfrak{sim}(2)$ (say, a $B$-spline) interpolating $\log A_1, \log A_1, \ldots, \log A_m$. Then, the curve $t \mapsto e^{C(t)}$ yields a deformation in $\mathbf{SIM}(2)$ interpolating $A_0, A_1, \ldots, A_m$.

(1) **(75 pts).** Write a program interpolating between two deformations.

(2) **Extra credit (125 pts).** If you konw about cubic spline interpolation, write a program to interpolate a sequence of deformations given by similitudes $A_0, A_1, \ldots, A_m$ by a $C^2$-curve.

**Problem B6 (100 pts).** (a) For any matrix

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

if we let $\theta = \sqrt{a^2 + b^2 + c^2}$ and

$$B = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix},$$

prove that

$$\begin{aligned} A^2 &= -\theta^2 I + B, \\ AB &= BA = 0. \end{aligned}$$

From the above, deduce that

$$A^3 = -\theta^2 A.$$

(b) Prove that the exponential map $\exp\colon \mathfrak{so}(3) \to \mathbf{SO}(3)$ is given by

$$\exp A = e^A = \cos\theta\, I_3 + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} B,$$

or, equivalently, by

$$e^A = I_3 + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} A^2,$$

if $\theta \neq 0$, with $\exp(0_3) = I_3$.

(c) Prove that $e^A$ is an orthogonal matrix of determinant $+1$, i.e., a rotation matrix.

(d) Prove that the exponential map $\exp\colon \mathfrak{so}(3) \to \mathbf{SO}(3)$ is surjective. For this, proceed as follows: Pick any rotation matrix $R \in \mathbf{SO}(3)$;

(1) The case $R = I$ is trivial.

(2) If $R \neq I$ and $\mathrm{tr}(R) \neq -1$, then

$$\exp^{-1}(R) = \left\{ \frac{\theta}{2\sin\theta}(R - R^T) \;\middle|\; 1 + 2\cos\theta = \mathrm{tr}(R) \right\}.$$

(Recall that $\mathrm{tr}(R) = r_{1\,1} + r_{2\,2} + r_{3\,3}$, the *trace* of the matrix $R$). Note that both $\theta$ and $2\pi - \theta$ yield the same matrix $\exp(R)$.

(3) If $R \neq I$ and $\mathrm{tr}(R) = -1$, then prove that the eigenvalues of $R$ are $1, -1, -1$, that $R = R^\top$, and that $R^2 = I$. Prove that the matrix

$$S = \frac{1}{2}(R - I)$$

is a symmetric matrix whose eigenvalues are $-1, -1, 0$. Thus, $S$ can be diagonalized with respect to an orthogonal matrix $Q$ as

$$S = Q \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^\top.$$

Prove that there exists a skew symmetric matrix

$$U = \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix}$$

so that

$$U^2 = S = \frac{1}{2}(R - I).$$

Observe that

$$U^2 = \begin{pmatrix} -(c^2 + d^2) & bc & bd \\ bc & -(b^2 + d^2) & cd \\ bd & cd & -(b^2 + c^2) \end{pmatrix}.$$

and use this to conclude that if $U^2 = S$, then $b^2 + c^2 + d^2 = 1$. Then, show that

$$\exp^{-1}(R) = \left\{ (2k+1)\pi \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix},\ k \in \mathbb{Z} \right\},$$

where $(b, c, d)$ is any unit vector such that for the corresponding skew symmetric matrix $U$, we have $U^2 = S$.

(e) To find a skew symmetric matrix $U$ so that $U^2 = S = \frac{1}{2}(R - I)$ as in (d), we can solve the system

$$\begin{pmatrix} b^2 - 1 & bc & bd \\ bc & c^2 - 1 & cd \\ bd & cd & d^2 - 1 \end{pmatrix} = S.$$

We immediately get $b^2, c^2, d^2$, and then, since one of $b, c, d$ is nonzero, say $b$, if we choose the positive square root of $b^2$, we can determine $c$ and $d$ from $bc$ and $bd$.

Implement a computer program to solve the above system.

**Problem B7 (60 pts).** (1) Implement the Gram-Schmidt orthonormalization procedure and the modified Gram-Schmidt procedure. You may use the pseudo-code showed in (2).

(2) Implement the method to compute the QR decomposition of an invertible matrix. You may use the following pseudo-code:

**function** $qr(A$: matrix): $[Q, R]$ pair of matrices
  **begin**
    $n = \dim(A)$;
    $R = 0$; (the zero matrix)
    $Q1(:, 1) = A(:, 1)$;

```
    R(1,1) = sqrt(Q1(:,1)^T · Q1(:,1));
    Q(:,1) = Q1(:,1)/R(1,1);
    for k := 1 to n − 1 do
      w = A(:,k+1);
      for i := 1 to k do
        R(i,k+1) = A(:,k+1)^T · Q(:,i);
        w = w − R(i,k+1)Q(:,i);
      endfor;
    Q1(:,k+1) = w;
    R(k+1,k+1) = sqrt(Q1(:,k+1)^T · Q1(:,k+1));
    Q(:,k+1) = Q1(:,k+1)/R(k+1,k+1);
    endfor;
  end
```

Test it on various matrices, including those involved in Project 1.

(3) Given any invertible matrix $A$, define the sequences $A_k$, $Q_k, R_k$ as follows:

$$\begin{aligned} A_1 &= A \\ Q_k R_k &= A_k \\ A_{k+1} &= R_k Q_k \end{aligned}$$

for all $k \geq 1$, where in the second equation, $Q_k R_k$ is the QR decomposition of $A_k$ given by part (2).

Run the above procedure for various values of $k$ (50, 100, ...) and various real matrices $A$, in particular some symmetric matrices; also run the `Matlab` command `eig` on $A_k$, and compare the diagonal entries of $A_k$ with the eigenvalues given by `eig`$(A_k)$.

What do you observe? How do you explain this behavior?

**Problem B8 (Extra Credit 100 pts).** (a) Consider the set of affine maps $\rho$ of $\mathbb{R}^3$ defined such that

$$\rho(X) = RX + W,$$

where $R$ is a rotation matrix (an orthogonal matrix of determinant +1) and $W$ is some vector in $\mathbb{R}^3$. Every such a map can be represented by the $4 \times 4$ matrix

$$\begin{pmatrix} R & W \\ 0 & 1 \end{pmatrix}$$

in the sense that

$$\begin{pmatrix} \rho(X) \\ 1 \end{pmatrix} = \begin{pmatrix} R & W \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ 1 \end{pmatrix}$$

iff

$$\rho(X) = RX + W.$$

Prove that these maps are affine bijections and that they form a group, denoted by $\mathbf{SE}(3)$ (the *direct affine isometries, or rigid motions*, of $\mathbb{R}^3$).

(b) Let us now consider the set of $4 \times 4$ matrices of the form

$$B = \begin{pmatrix} \Omega & W \\ 0 & 0 \end{pmatrix},$$

where $\Omega$ is a skew-symmetric matrix

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

and $W$ is a vector in $\mathbb{R}^3$.

Verify that this set of matrices is a vector space isomorphic to $(\mathbb{R}^6, +)$. This vector space is denoted by $\mathfrak{se}(3)$. Show that in general, $BC \neq CB$.

(c) Given a matrix

$$B = \begin{pmatrix} \Omega & W \\ 0 & 0 \end{pmatrix}$$

as in (b), prove that

$$B^n = \begin{pmatrix} \Omega^n & \Omega^{n-1}W \\ 0 & 0 \end{pmatrix}$$

where $\Omega^0 = I_3$. Given

$$\Omega = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

let $\theta = \sqrt{a^2 + b^2 + c^2}$. Prove that if $\theta = 0$, then

$$e^B = \begin{pmatrix} I_3 & W \\ 0 & 1 \end{pmatrix},$$

and that if $\theta \neq 0$, then

$$e^B = \begin{pmatrix} e^\Omega & VW \\ 0 & 1 \end{pmatrix},$$

where

$$V = I_3 + \sum_{k \geq 1} \frac{\Omega^k}{(k+1)!}.$$

(d) Prove that

$$e^\Omega = I_3 + \frac{\sin\theta}{\theta}\Omega + \frac{(1-\cos\theta)}{\theta^2}\Omega^2$$

and

$$V = I_3 + \frac{(1-\cos\theta)}{\theta^2}\Omega + \frac{(\theta - \sin\theta)}{\theta^3}\Omega^2.$$

*Hint*. Use the fact that $\Omega^3 = -\theta^2\Omega$.

(e) Prove that $e^B$ is a direct affine isometry in $\mathbf{SE}(3)$. Prove that $V$ is invertible and that

$$V^{-1} = I - \frac{1}{2}\Omega + \frac{1}{\theta^2}\left(1 - \frac{\theta\sin\theta}{2(1-\cos\theta)}\right)\Omega^2,$$

for $\theta \neq 0$.

*Hint*. Assume that the inverse of $V$ is of the form

$$Z = I_3 + a\Omega + b\Omega^2,$$

and show that $a, b$, are given by a system of linear equations that always has a unique solution.

Prove that the exponential map $\exp\colon \mathfrak{se}(3) \to \mathbf{SE}(3)$ is surjective.

**Remark:** As in the case of the plane, rigid motions in $\mathbf{SE}(3)$ can be used to describe certain deformations of bodies in $\mathbb{R}^3$.