Homework III (due October 28), Math 602, Fall 2002. (GJSZ)

BIV (a). Assume that R is a local ring and view R as a module over itself. As such, R is generated by 1, and so, the endomorphism ring of linear maps, $\mathrm{End}_R(R, R)$, is isomorphic to R. Indeed, every R-linear map, $f\colon R \to R$, is completely determined by $f(1) \in R$.

The crucial property of a local ring, R, is that its unique maximal ideal, $\mathfrak{M}$, consists of the nonunits of R (equivalently, $R - \mathfrak{M}$ is the set of units of R). This is because if $a \in R$ is not a unit and $a \notin \mathfrak{M}$, then the principal ideal, (a), generated by a is a proper ideal of R. But every ideal is contained in some maximal ideal (by an application of Zorn's lemma), and since R has a *unique* maximal ideal, $\mathfrak{M}$, the ideal, (a), must be contained in $\mathfrak{M}$, and so, $a \in \mathfrak{M}$, a contradiction. Obviously, $\mathfrak{M}$ can't contain any units (otherwise, we would have $\mathfrak{M} = R$, contradicting the fact that maximal ideals are proper), and so, $\mathfrak{M}$ consists exactly of the nonunits of R.

If $R \cong M_1 \coprod M_2$, where $M_1, M_2 \neq R$, then the natural projections $pr_1\colon R \to M_1$ and $pr_2\colon R \to M_2$ are nontrivial idempotents, which means that $pr_1^2 = pr_1$, $pr_2^2 = pr_2$, and pr_1 and pr_2 are neither the constant map 0 nor the identity. In view of the isomorphism $\mathrm{End}_R(R, R) \cong R$, the projections pr_1 and pr_2 correspond to nontrivial idempotents in R, say p_1 and p_2. We claim that this is impossible.

Indeed, assume that $a^2 = a$ for any element, a, of a local ring, R. Then $a(a - 1) = 0$. If a or $a - 1$ is a unit, we conclude that either $a = 1$ or $a = 0$. We now consider the case where a and $a - 1$ are nonunits. But, in a local ring, R, the unique maximal ideal, $\mathfrak{M}$, is the set of nonunits in R. So, both a and $a - 1$ must belong to $\mathfrak{M}$, and then, $1 \in \mathfrak{M}$. This implies that $\mathfrak{M} = R$, a contradiction. Therefore, any idempotent in a local ring is trivial. From this we conclude that either $M_1 = R$ or $M_2 = R$, and R is indecomposable.

BIV (b) If R is any ring with unity, there is a unique homomorphism, $\mathbb{Z} \longrightarrow R$, given by $n \mapsto n \cdot 1$. The kernel of this homomorphism is a principal ideal, $n\mathbb{Z}$, of $\mathbb{Z}$, for some $n \geq 0$ called the *characteristic* of R. Observe that n is the smallest integer so that $n \cdot a = 0$ for all $a \in R$. From now on, to simplify the notation, we write n for $n \cdot 1$.

Now, assume that R is a local ring, with unique maximal ideal, $\mathfrak{M}$. The case $n = 0$ may arise. For example, every field is a local ring. Next, consider the case $n > 0$. We can factor n into its prime factors as $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$, where the p_j's are distinct primes. If $k = 1$, we are done. Thus, assume that $k \geq 2$. By hypothesis,

$$p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k} = 0.$$

If any $p_i^{m_i}$ is a unit, we get

$$\prod_{j \neq i} p_j^{m_j} = 0,$$

contradicting the fact that $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ is the smallest integer so that $n = 0$ in R. Thus, $p_1^{m_1}, p_2^{m_2}, \ldots, p_k^{m_k}$ are all nonunits, and since R is a local ring, they all belong to $\mathfrak{M}$. Now, a maximal ideal is a prime ideal, and since $p_j^{m_j} \in \mathfrak{M}$ for every j, we have $p_j \in \mathfrak{M}$ for all j. However, the p_j are distinct primes, and so, they are relatively prime. By the Bezout identity, there are integers $h_1, \ldots, h_k$ so that

$$h_1 p_1 + h_2 p_2 + \cdots + h_k p_k = 1,$$

which implies that $1 \in \mathfrak{M}$, a contradiction. Therefore, $k = 1$ and the characteristic of a local ring is either 0 or a prime power, p^m.

Consider the ring $R = \mathbb{Z}/p^m\mathbb{Z}$, where p is a prime. We claim that this is a local ring, and obviously, its characteristic is p^m. Indeed, we argue that $\mathfrak{M} = p\mathbb{Z}/p^m\mathbb{Z}$ is the unique maximal ideal of R. Since $p\mathbb{Z}$ is an ideal in $\mathbb{Z}$, it is clear that $p\mathbb{Z}/p^m\mathbb{Z}$ is an ideal in R. Let us show that any $n \in R - \mathfrak{M}$ is a unit. We may assume that $1 \leq n \leq p^m - 1$, and the fact that $n \notin \mathfrak{M}$ means that n is not a multiple of p, and so, $(n, p^m) = 1$. Again, by Bezout, there are integers, u, v, so that $un + vp^m = 1$, and when we reduce modulo p^m, we find that n is invertible in $\mathbb{Z}/p^m\mathbb{Z}$.

BIV (c) We first consider holomorphic functions defined in some open set, Ω, containing 0. Let us denote the germ of a holomorphic function, f, defined near 0, by $[f]$. The set of germs of holomorphic functions defined near 0 can be made into a ring by defining addition and multiplication of germs as follows:

$$\begin{aligned} [f] + [g] &= [f + g] \\ [f][g] &= [fg], \end{aligned}$$

where $(f + g)(z) = f(z) + g(z)$ and $(fg)(z) = f(z)g(z)$, near 0. I will not check that this gives a ring! Let us denote this ring $\mathcal{O}_0$.

Define $\mathfrak{M}$ by

$$\mathfrak{M} = \{[f] \in \mathcal{O}_0 \mid f(0) = 0\}.$$

This is clearly an ideal. Note that $[f] \in \mathcal{O}_0 - \mathfrak{M}$ iff $g(0) \neq 0$ for every $g \in [f]$ (i.e., $f \sim g$, where $\sim$ is the equivalence relation defining germs). Now, it is known from complex analysis that if a holomorphic function, f, is nonzero at some point $z = z_0$, then, it is nonzero in some open set containing z_0, and $1/f$ is holomorphic in some open set containing z_0. This shows that every $[f] \in \mathcal{O}_0 - \mathfrak{M}$ is a unit in $\mathcal{O}_0$, and thus, $\mathfrak{M}$ is the unique maximal ideal of $\mathcal{O}_0$, and $\mathcal{O}_0$ is a local ring.

In order to show that $\mathcal{O}_0$ is a good local ring, first we are going to show that $\mathfrak{M} = ([z])$, the principal ideal generated by the germ of z, and that the ideals of $\mathcal{O}_0$ are exactly the ideals $\mathfrak{M}^k = ([z^k])$, thereby proving that $\mathcal{O}_0$ is a principal ring.

From complex analysis, we know that if f is a homomorphic function defined on an open $\Omega \subseteq \mathbb{C}$ and f is not the zero function, then for every $p \in \Omega$, there is a uniquely defined natural number, $n = \mathrm{ord}_p(f)$, so that, near p,

$$f(z) = (z - p)^n \widetilde{f}(z),$$

where $\widetilde{f}$ is a holomorphic function so that $\widetilde{f}(z) \neq 0$ in an open subset around p. Observe that $\mathrm{ord}_p(f)$ is the same for all the holomorphic functions in the germ of f at p, and so, $\mathrm{ord}_p([f])$ is well-defined.

Let us prove that $\mathcal{O}_0$ is an integral domain. Let $[f], [g] \in \mathcal{O}_0$, with $[f], [g] \neq 0$. Then, near 0, we can write

$$f(z) = z^{n_f}\widetilde{f}(z) \quad \text{and} \quad g(z) = z^{n_g}\widetilde{g}(z),$$

where $\widetilde{f}$ and $\widetilde{g}$ are holomorphic and nonzero in some open subset around 0. Then, for all z near 0, we have

$$(fg)(z) = z^{n_f+n_g}(\widetilde{f}\widetilde{g})(z)$$

where $(\widetilde{f}\widetilde{g})(z) \neq 0$ for all z in some open around 0. Thus, $[fg] \neq 0$, and $\mathcal{O}_0$ is an integral domain.

Let $\mathfrak{I} \subseteq \mathcal{O}_0$ be any ideal in $\mathcal{O}_0$. Let

$$N = \min \{\mathrm{ord}_0([f]) \mid [f] \in \mathfrak{I} - \{0\}\}.$$

There is some holomorphic function, f, so that $[f] \in \mathfrak{I} - \{0\}$ and $\mathrm{ord}_0(f) = N$, and we have

$$f(z) = z^N\widetilde{f}(z),$$

where $\widetilde{f}$ is holomorphic and nonzero in some open subset around 0. Furthermore, $1/\widetilde{f}$ is holomorphic near 0. This implies that

$$z^N = f(z)/\widetilde{f}(z)$$

near 0, and since $\mathfrak{I}$ is an ideal and $[f] \in \mathfrak{I}$, we get $[z^N] \in \mathfrak{I}$. For any $[g] \in \mathfrak{I}$, as for $[f]$, we have

$$g(z) = z^{n_g}\widetilde{g}(z),$$

where $\widetilde{g}$ is holomorphic and nonzero near 0, and since by hypothesis, $N \leq n_g$, we can write

$$g(z) = z^N z^{n_g - N}\widetilde{g}(z).$$

Since $\mathfrak{I}$ is an ideal, we get $g \in ([z^N])$, the principal ideal generated by $[z^N]$. Thus, $\mathfrak{I} \subseteq ([z^N])$, and since $[z^N] \in \mathfrak{I}$, we have $\mathfrak{I} = [z^N]$.

Therefore, we proved that the maximal ideal of $\mathcal{O}_0$ is the principal ideal $\mathfrak{M} = [z]$, and that every ideal of $\mathcal{O}_0$ is also principal and of the form $\mathfrak{M}^k = [z^k]$. Putting together what we did, we obtain the fact that $\mathcal{O}_0$ is a local ring which is a PID.

Now, for any $[f] \in \mathcal{O}_0$ with $[f] \neq 0$, if we let $n = \mathrm{ord}_0([f])$, we showed that $[f] \in \mathfrak{M}^n$, but $[f] \notin \mathfrak{M}^{n+1}$, and this implies that

$$\bigcap_{k \geq 0} \mathfrak{M}^k = (0).$$

Thus, $\mathcal{O}_0$ is a good local ring.

We now turn to the case of real, C^k-functions, defined near 0. The situation is quite different. As in the holomorphic case, we obtain a ring, $\mathcal{O}_0^k$, and we let

$$\mathfrak{M} = \{[f] \in \mathcal{O}_0^k \mid f(0) = 0\}.$$

It is an ideal, in fact, the unique maximal ideal of $\mathcal{O}_0^k$. This time, to show that every $[f] \in \mathcal{O}_0^k - \mathfrak{M}$ is a unit, we use the fact that if $f(0) \neq 0$, by continuity, $f(x) \neq 0$ in some open containing 0, and so, $1/f$ is C^k in this open subset.

However, $\mathcal{O}_0^k$ is not a good local ring, nor a domain. To show this, consider the function, f, defined by

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Observe that $f(x) \neq 0$ for all $x > 0$, and yet, $f(x)f(-x) \equiv 0$. So, if we show that f is in C^∞, it will also be the case that $f(-x) \in C^\infty$, and so, $[f]$ is a zero-divisor in $\mathcal{O}_0^\infty$, and thus, also in $\mathcal{O}_0^k$ (for k finite).

Consider the case $k = \infty$. If we let $f_n(x) = f(nx)$, we see that $f_n \in C^\infty$ and $f_n(0) = f(0) = 0$, so that $f_n \in \mathfrak{M}$. Moreover, since

$$\left(e^{-1/nx}\right)^n = e^{-1/x},$$

we have $f_n^n = f$ for all $n \geq 0$. But then, $[f] \in \mathfrak{M}^n$ for all $n \geq 0$, and so, $[f] \in \bigcap_{k \geq 0} \mathfrak{M}^k$, and

$$\bigcap_{k \geq 0} \mathfrak{M}^k \neq (0).$$

Thus, $\mathcal{O}_0^\infty$ is not a good local ring. Now, since $f \in C^\infty$, it also belongs to every C^k, and so, $\mathcal{O}_0^k$ is not a good local ring for any finite k.

It remains to prove that f is in C^∞, which is done by computing derivatives by induction. We claim that $f^{(n)}$ is differentiable and is a linear combination of functions of the form

$$h(x) = \begin{cases} \dfrac{e^{-1/x}}{x^k} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0, \end{cases}$$

where $k \geq 0$. The base case is trivial. For the induction step, observe that

$$g'(x) = \frac{-ke^{-1/x}}{x^{k+1}} + \frac{e^{-1/x}}{x^{k+2}}, \quad \text{if } x > 0$$

and $g'(x) = 0$ if $x < 0$. We still need to show that $g'(0) = 0$. Since $g(x) = 0$ for $x \leq 0$, it suffices to show that

$$\lim_{x \to 0_+} \frac{g(x)}{x} = \lim_{x \to 0_+} \frac{e^{-1/x}}{x^{k+1}} = 0.$$

However, (setting $t = 1/x$) this is equivalent to proving that

$$\lim_{t \to +\infty} \frac{t^{k+1}}{e^t} = 0,$$

a well-known property of the exponential.