Spring 2015 CIS 610

Advanced Geometric Methods in Computer Science
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Homework 3

March 4; Due March 26, 2015

Problem B1 (50). (i) In $\mathbb{R}^3$, the sphere S^2 is the set of points of coordinates (x, y, z) such that $x^2 + y^2 + z^2 = 1$. The point $N = (0, 0, 1)$ is called the *north pole*. The *stereographic projection map* $\sigma_N \colon (S^2 - \{N\}) \to \mathbb{R}^2$ is defined as follows: For every point $M \neq N$ on S^2, the point $\sigma_N(M)$ is the intersection of the line through N and M and the plane of equation $z = 0$. Show that if M has coordinates (x, y, z) (with $x^2 + y^2 + z^2 = 1$), then

$$\sigma_N(M) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

The map σ_N is bijective and its inverse is given by the map $\tau_N \colon \mathbb{R}^2 \to (S^2 - \{N\})$, with

$$(x, y) \mapsto \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}\right).$$

You need not prove these facts!

Using the complex number $u = x + iy$ to represent the point (x, y), the maps $\tau_N \colon \mathbb{R}^2 \to (S^2 - \{N\})$ and $\sigma_N \colon (S^2 - \{N\}) \to \mathbb{R}^2$ can be viewed as maps from $\mathbb{C}$ to $(S^2 - \{N\})$ and from $(S^2 - \{N\})$ to $\mathbb{C}$, defined such that

$$\tau_N(u) = \left(\frac{2u}{|u|^2 + 1}, \frac{|u|^2 - 1}{|u|^2 + 1}\right)$$

and

$$\sigma_N(u, z) = \frac{u}{1 - z}.$$

(ii) Identifying $\mathbb{C}^2$ and $\mathbb{R}^4$, for $z = x + iy$ and $z' = x' + iy'$, we define

$$\|(z, z')\| = \sqrt{x^2 + y^2 + x'^2 + y'^2}.$$

The sphere S^3 is the subset of $\mathbb{C}^2$ (or $\mathbb{R}^4$) consisting of those points (z, z') such that $\|(z, z')\|^2 = 1$.

Prove that $\mathbb{CP}^1$ is in bijection with the quotient of S^3 by the equivalence relation

$$(z_1, z_2) \sim (z_2, z_2') \quad \text{iff} \quad (z_1, z_2) = \lambda(z_2, z_2') \quad \text{for some } \lambda \in \mathbb{C} \text{ with } |\lambda| = 1.$$

If we let $u = z/z'$ (where $z, z' \in \mathbb{C}$) in the map

$$u \mapsto \left(\frac{2u}{|u|^2 + 1}, \frac{|u|^2 - 1}{|u|^2 + 1} \right)$$

and require that $\|(z, z')\|^2 = 1$, show that we get the map $HF\colon S^3 \to S^2$ defined such that

$$HF((z, z')) = (2z\overline{z'},\, |z|^2 - |z'|^2).$$

Prove that $HF\colon S^3 \to S^2$ induces a bijection $\widehat{HF}\colon \mathbb{CP}^1 \to S^2$, and thus that $\mathbb{CP}^1$ is homeomorphic to S^2.

(iii) Prove that

$$HF((z_1, z_1')) = HF((z_2, z_2')) \quad \text{iff} \quad (z_1, z_2) \sim (z_2, z_2'),$$

which means the inverse image $HF^{-1}(s)$ of every point $s \in S^2$ is a circle. Thus S^3 can be viewed as a union of disjoint circles. The map HF is called the *Hopf fibration*.

Problem B2 (80). This problem is from Knapp, *Lie Groups Beyond an Introduction*, Introduction, page 21. Recall that the group $\mathbf{SU}(2)$ consists of all complex matrices of the form

$$A = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \qquad \alpha, \beta \in \mathbb{C}, \qquad \alpha\overline{\alpha} + \beta\overline{\beta} = 1,$$

and the action $\cdot\colon \mathbf{SU}(2) \times (\mathbb{C} \cup \{\infty\}) \to \mathbb{C} \cup \{\infty\}$ is given by

$$A \cdot w = \frac{\alpha w + \beta}{-\overline{\beta} w + \overline{\alpha}}, \quad w \in \mathbb{C} \cup \{\infty\}.$$

This is a transitive action. Using the stereographic projection σ_N from S^2 onto $\mathbb{C} \cup \{\infty\}$ and its inverse σ_N^{-1}, we can define an action of $\mathbf{SU}(2)$ on S^2 by

$$A \cdot (x, y, z) = \sigma_N^{-1}(A \cdot \sigma_N(x, y, z)), \quad (x, y, z) \in S^2,$$

and we denote by $\rho(A)$ the corresponding map from S^2 to S^2.

(1) If we write $\alpha = a + ib$ and $\beta = c + id$, prove that $\rho(A)$ is given by the matrix

$$\rho(A) = \begin{pmatrix} a^2 - b^2 - c^2 + d^2 & -2ab - 2cd & -2ac + 2bd \\ 2ab - 2cd & a^2 - b^2 + c^2 - d^2 & -2ad - 2bc \\ 2ac + 2bd & 2ad - 2bc & a^2 + b^2 - c^2 - d^2 \end{pmatrix}.$$

Prove that $\rho(A)$ is indeed a rotation matrix which represents the rotation whose axis is the line determined by the vector $(d, -c, b)$ and whose angle $\theta \in [-\pi, \pi]$ is determined by

$$\cos\frac{\theta}{2} = |a|.$$

Hint. Recall that the axis of a rotation matrix $R \in \mathbf{SO}(3)$ is specified by any eigenvector of 1 for R, and that the angle of rotation θ satisfies the equation

$$\mathrm{tr}(R) = 2\cos\theta + 1.$$

(3) We can compute the derivative $d\rho_I \colon \mathfrak{su}(2) \to \mathfrak{so}(3)$ of ρ at I as follows. Recall that $\mathfrak{su}(2)$ consists of all complex matrices of the form

$$\begin{pmatrix} ib & c+id \\ -c+id & -ib \end{pmatrix}, \quad b, c, d \in \mathbb{R},$$

so pick the following basis for $\mathfrak{su}(2)$,

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and define the curves in $\mathbf{SU}(2)$ through I given by

$$c_1(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}, \quad c_2(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad c_3(t) = \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix}.$$

Prove that $c_i'(0) = X_i$ for $i = 1, 2, 3$, and that

$$d\rho_I(X_1) = 2\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad d\rho_I(X_2) = 2\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad d\rho_I(X_3) = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus, we have

$$d\rho_I(X_1) = 2E_3, \quad d\rho_I(X_2) = -2E_2, \quad d\rho_I(X_3) = 2E_1,$$

where (E_1, E_2, E_3) is the basis of $\mathfrak{so}(3)$ given in Section 2.5. Conclude that $d\rho_I$ is an isomorphism between the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$.

(4) Recall from Proposition 2.37 that we have the commutative diagram

$$\begin{tikzcd} \mathbf{SU}(2) \arrow[r, "\rho"] & \mathbf{SO}(3) \\ \mathfrak{su}(2) \arrow[u, "\exp"] \arrow[r, "d\rho_I"'] & \mathfrak{so}(3) \arrow[u, "\exp"'] \end{tikzcd}.$$

Since $d\rho_I$ is surjective and the exponential map $\exp\colon \mathfrak{so}(3) \to \mathbf{SO}(3)$ is surjective, conclude that ρ is surjective. Prove that $\mathrm{Ker}\,\rho = \{I, -I\}$.

Problem B3 (20). (a) Let A be any invertible (real) $n \times n$ matrix. Prove that for every SVD, $A = VDU^\top$, of A, the product $VU^\top$ is the same (i.e., if $V_1DU_1^\top = V_2DU_2^\top$, then $V_1U_1^\top = V_2U_2^\top$). What does $VU^\top$ have to do with the polar form of A?

(b) Given any invertible (real) $n \times n$ matrix, A, prove that there is a unique orthogonal matrix, $Q \in \mathbf{O}(n)$, such that $\|A - Q\|_F$ is minimal (under the Frobenius norm). In fact, prove that $Q = VU^\top$, where $A = VDU^\top$ is an SVD of A. Moreover, if $\det(A) > 0$, show that $Q \in \mathbf{SO}(n)$.

What can you say if A is singular (i.e., non-invertible)?

Problem B4 (40 pts). Consider the action of the group $\mathbf{SL}(2, \mathbb{R})$ on the upper half-plane, $H = \{z = x + iy \in \mathbb{C} \mid y > 0\}$, given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

(a) Check that for any $g \in \mathbf{SL}(2, \mathbb{R})$,

$$\Im(g \cdot z) = \frac{\Im(z)}{|cz + d|^2},$$

and conclude that if $z \in H$, then $g \cdot z \in H$, so that the action of $\mathbf{SL}(2, \mathbb{R})$ on H is indeed well-defined (Recall, $\Re(z) = x$ and $\Im(z) = y$, where $z = x + iy$.)

(b) Check that if $c \neq 0$, then

$$\frac{az + b}{cz + d} = \frac{-1}{c^2 z + cd} + \frac{a}{c}.$$

Prove that the group of Möbius transformations induced by $\mathbf{SL}(2, \mathbb{R})$ is generated by Möbius transformations of the form

1. $z \mapsto z + b$,
2. $z \mapsto kz$,
3. $z \mapsto -1/z$,

where $b \in \mathbb{R}$ and $k \in \mathbb{R}$, with $k > 0$. Deduce from the above that the action of $\mathbf{SL}(2, \mathbb{R})$ on H is transitive and that transformations of type (1) and (2) suffice for transitivity.

(c) Now, consider the action of the discrete group $\mathbf{SL}(2, \mathbb{Z})$ on H, where $\mathbf{SL}(2, \mathbb{Z})$ consists of all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{Z}.$$

Why is this action not transitive? Consider the two transformations

$$S\colon z \mapsto -1/z$$

associated with $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and

$$T\colon z \mapsto z+1$$

associated with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Define the subset, D, of H, as the set of points, $z = x + iy$, such that $-1/2 \leq x \leq -1/2$ and $x^2 + y^2 \geq 1$. Observe that D contains the three special points, i, $\rho = e^{2\pi i/3}$ and $-\overline{\rho} = e^{\pi i/3}$.

Draw a picture of this set, known as a *fundamental domain* of the action of $G = \mathbf{SL}(2, \mathbb{Z})$ on H.

Remark: Gauss proved that the group $G = \mathbf{SL}(2, \mathbb{Z})$ is generated by S and T.

Problem B5 (30 pts). Let J be the 2×2 matrix

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and let $\mathbf{SU}(1,1)$ be the set of 2×2 complex matrices

$$\mathbf{SU}(1,1) = \{A \mid A^* J A = J, \quad \det(A) = 1\},$$

where A^* is the conjugate transpose of A.

(a) Prove that $\mathbf{SU}(1,1)$ is the group of matrices of the form

$$A = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}, \quad \text{with} \quad a\overline{a} - b\overline{b} = 1.$$

If

$$g = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

prove that the map from $\mathbf{SL}(2, \mathbb{R})$ to $\mathbf{SU}(1,1)$ given by

$$A \mapsto gAg^{-1}$$

is a group isomorphism.

(b) Prove that the Möbius transformation associated with g,

$$z \mapsto \frac{z-i}{z+i}$$

is a bijection between the upper half-plane, H, and the unit open disk, $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Prove that the map from $\mathbf{SU}(1,1)$ to $S^1 \times D$ given by

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \mapsto (a/|a|, b/a)$$

is a continuous bijection (in fact, a homeomorphism). Conclude that $\mathbf{SU}(1,1)$ is topologically an open solid torus.

(c) Check that $\mathbf{SU}(1,1)$ acts transitively on D by

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \cdot z = \frac{az+b}{\overline{b}z + \overline{a}}.$$

Find the stabilizer of $z = 0$ and conclude that

$$\mathbf{SU}(1,1)/\mathbf{SO}(2) \cong D.$$

Problem B6 (80 pts). Given a finite dimensional Lie algebra $\mathfrak{g}$ (as a vector space over $\mathbb{R}$), we define the function $B \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ by

$$B(X,Y) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)), \quad X, Y \in \mathfrak{g}.$$

(1) Check that B is $\mathbb{R}$-bilinear and symmetric.

(2) Let $\mathfrak{g} = \mathfrak{gl}(2,\mathbb{R}) = \mathrm{M}_2(\mathbb{R})$. Given any matrix $A \in \mathrm{M}_2(\mathbb{R})$ with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

show that in the basis $(E_{12}, E_{11}, E_{22}, E_{21})$, the matrix of $\mathrm{ad}(A)$ is given by

$$\begin{pmatrix} a-d & -b & b & 0 \\ -c & 0 & 0 & b \\ c & 0 & 0 & -b \\ 0 & c & -c & d-a \end{pmatrix}.$$

Show that

$$\det(xI - \mathrm{ad}(A)) = x^2(x^2 - ((a-d)^2 + 4bc)).$$

(3) Given $A, A' \in \mathrm{M}_2(\mathbb{R})$ with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

prove that

$$B(A, A') = 2(d - a)(d' - a') + 4bc' + 4cb' = 4\mathrm{tr}(AA') - 2\mathrm{tr}(A)\mathrm{tr}(A').$$

(4) Next, let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. Check that the following three matrices form a basis of $\mathfrak{sl}(2, \mathbb{R})$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Prove that in the basis (H, X, Y), for any

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R}),$$

the matrix of $\mathrm{ad}(A)$ is

$$\begin{pmatrix} 0 & -c & b \\ -2b & 2a & 0 \\ 2c & 0 & -2a \end{pmatrix}.$$

Prove that

$$\det(xI - \mathrm{ad}(A)) = x(x^2 - 4(a^2 + bc)).$$

(5) Given $A, A' \in \mathfrak{sl}(2, \mathbb{R})$ with

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad A' = \begin{pmatrix} a' & b' \\ c' & -a'' \end{pmatrix},$$

prove that

$$B(A, A') = 8aa' + 4bc' + 4cb' = 4\mathrm{tr}(AA').$$

(6) Let $\mathfrak{g} = \mathfrak{so}(3)$. For any $A \in \mathfrak{so}(3)$, with

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

we know from Proposition 2.39 that in the basis (E_1, E_2, E_3), the matrix of $\mathrm{ad}(A)$ is A itself. Prove that

$$B(A, A') = -2(aa' + bb' + cc') = \mathrm{tr}(AA').$$

(7) Recall that a symmetric bilinear form B is *nondegenerate* if for every X, if $B(X, Y) = 0$ for all Y, then $X = 0$.

Prove that B on $\mathfrak{gl}(2, \mathbb{R}) = \mathrm{M}_2(\mathbb{R})$ is degenerate; B on $\mathfrak{sl}(2, \mathbb{R})$ is nondegenerate but neither positive definite nor negative definite; B on $\mathfrak{so}(3)$ is nondegenerate negative definite.

(8) **Extra Credit (45) points.** Recall that a subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a *subalgebra* of $\mathfrak{g}$ if $[x, y] \in \mathfrak{h}$ for all $x, y \in \mathfrak{h}$, and an *ideal* if $[h, x] \in \mathfrak{h}$ for all $h \in \mathfrak{h}$ and all $x \in \mathfrak{g}$. Check that $\mathfrak{sl}(n, \mathbb{R})$ is an ideal in $\mathfrak{gl}(n, \mathbb{R})$, and that $\mathfrak{so}(n)$ is a subalgebra of $\mathfrak{sl}(n, \mathbb{R})$, but not an ideal. Prove that if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, then the bilinear form B on $\mathfrak{h}$ is equal to the restriction of the bilinear form B on $\mathfrak{g}$ to $\mathfrak{h}$.

Prove the following facts: for all $n \geq 2$:

$$\begin{aligned}
\mathfrak{gl}(n, \mathbb{R})\colon &\qquad B(X, Y) = 2n\mathrm{tr}(XY) - 2\mathrm{tr}(X)\mathrm{tr}(Y) \\
\mathfrak{sl}(n, \mathbb{R})\colon &\qquad B(X, Y) = 2n\mathrm{tr}(XY) \\
\mathfrak{so}(n)\colon &\qquad B(X, Y) = (n-2)\mathrm{tr}(XY).
\end{aligned}$$

Problem B7 (100 pts). As in Problem B6, consider a finite dimensional Lie algebra $\mathfrak{g}$, but this time a vector space over $\mathbb{C}$, and define the function $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ by

$$B(x, y) = \mathrm{tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y)), \quad x, y \in \mathfrak{g}.$$

The bilinear form B is called the *Killing form* of $\mathfrak{g}$. Recall that a *homomorphism* $\varphi\colon \mathfrak{g} \to \mathfrak{g}$ is a linear map such that $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in \mathfrak{g}$, or equivalently such that

$$\varphi \circ \mathrm{ad}(x) = \mathrm{ad}(\varphi(x)) \circ \varphi, \quad \text{for all } x \in \mathfrak{g},$$

and that an *automorphism* of $\mathfrak{g}$ is a homomorphism of $\mathfrak{g}$ that has an inverse which is also a homomorphism of $\mathfrak{g}$.

(1) Prove that for every automorphism $\varphi\colon \mathfrak{g} \to \mathfrak{g}$, we have

$$B(\varphi(x), \varphi(y)) = B(x, y), \quad \text{for all } x, y \in \mathfrak{g}.$$

Prove that for all $x, y, z \in \mathfrak{g}$, we have

$$B(\mathrm{ad}(x)(y), z) = -B(y, \mathrm{ad}(x)(z)),$$

or equivalently

$$B([y, x], z) = B(y, [x, z]).$$

(2) Review the primary decomposition theorem, Section 17.3 of my notes *Fundamentals of Linear Algebra and Optimization* (linalg.pdf), especially Theorem 17.16. For any $x \in \mathfrak{g}$, we can apply the primary decomposition theorem to the linear map $\mathrm{ad}(x)$. Write

$$m(X) = (X - \lambda_1)^{r_1} \cdots (X - \lambda_k)^{r_k}$$

for the minimal polynomial of $\mathrm{ad}(x)$, where $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of $\mathrm{ad}(x)$, and let

$$\mathfrak{g}_x^{\lambda_i} = \mathrm{Ker}\,(\lambda_i I - \mathrm{ad}(x))^{r_i}, \quad i = 1, \ldots, k.$$

We know that 0 is an eigenvalue of $\mathrm{ad}(x)$, and we agree that $\lambda_0 = 0$. Then, we have a direct sum

$$\mathfrak{g} = \bigoplus_{\lambda_i} \mathfrak{g}_x^{\lambda_i}.$$

It is convenient to define $\mathfrak{g}_x^{\lambda}$ when λ is not an eigenvalue of $\mathrm{ad}(x)$ as

$$\mathfrak{g}_x^{\lambda} = (0).$$

Prove that

$$[\mathfrak{g}_x^{\lambda}, \mathfrak{g}_x^{\mu}] \subseteq \mathfrak{g}_x^{\lambda+\mu}, \quad \text{for all } \lambda, \mu \in \mathbb{C}.$$

Hint. First, show that

$$((\lambda + \mu)I - \mathrm{ad}(x))[y, z] = [(\lambda I - \mathrm{ad}(x))(y), z] + [y, (\mu I - \mathrm{ad}(x))(z)],$$

for all $x, y, z \in \mathfrak{g}$, and then that

$$((\lambda + \mu)I - \mathrm{ad}(x))^n[y, z] = \sum_{p=0}^{n} \binom{n}{p} [(\lambda I - \mathrm{ad}(x))^p(y), (\mu I - \mathrm{ad}(x))^{n-p}(z)],$$

by induction on n.

Prove that $\mathfrak{g}_x^0$ is a Lie subalgebra of $\mathfrak{g}$.

(3) Prove that if $\lambda + \mu \neq 0$, then $\mathfrak{g}_x^{\lambda}$ and $\mathfrak{g}_x^{\mu}$ are orthogonal with respect to B (which means that $B(X, Y) = 0$ for all $X \in \mathfrak{g}_x^{\lambda}$ and all $Y \in \mathfrak{g}_x^{\mu}$).

Hint. For any $X \in \mathfrak{g}_x^{\lambda}$ and any $Y \in \mathfrak{g}_x^{\mu}$, prove that $\mathrm{ad}(X) \circ \mathrm{ad}(Y)$ is nilpotent. Note that for any ν and any $Z \in \mathfrak{g}_x^{\nu}$,

$$(\mathrm{ad}(X) \circ \mathrm{ad}(Y))(Z) = [X, [Y, Z]],$$

so by (2),

$$[\mathfrak{g}_x^{\lambda}, [\mathfrak{g}_x^{\mu}, \mathfrak{g}_x^{\nu}]] \subseteq \mathfrak{g}_x^{\lambda+\mu+\nu}.$$

Conclude that we have an orthogonal direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_x^0 \oplus \bigoplus_{\lambda \neq 0} (\mathfrak{g}_x^{\lambda} \oplus \mathfrak{g}_x^{-\lambda}).$$

Prove that if B is nondegenerate, then B is nondegenerate on each of the summands.

Problem B8 (40 pts). We know that the Lie algebra $\mathfrak{se}(3)$ of $\mathbf{SE}(3)$ consists of all 4×4 matrices of the form

$$\begin{pmatrix} B & u \\ 0 & 0 \end{pmatrix},$$

where $B \in \mathfrak{so}(3)$ is a skew symmetric matrix and $u \in \mathbb{R}^3$. The following 6 matrices form a basis of $\mathfrak{se}(3)$:

$$X_1 = \begin{pmatrix} E_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} E_2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} E_3 & 0 \\ 0 & 0 \end{pmatrix},$$
$$X_4 = \begin{pmatrix} 0 & e_1^3 \\ 0 & 0 \end{pmatrix}, \qquad X_5 = \begin{pmatrix} 0 & e_2^3 \\ 0 & 0 \end{pmatrix}, \qquad X_6 = \begin{pmatrix} 0 & e_3^3 \\ 0 & 0 \end{pmatrix},$$

with

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$e_1^3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad e_2^3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad e_3^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Also recall the isomorphism between $(\mathbb{R}^3, \times)$ and $\mathfrak{so}(3)$ given by

$$u = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto u_\times = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}.$$

We define the bijection $\psi\colon \mathbb{R}^6 \to \mathfrak{se}(3)$ by

$$\psi(e_i^6) = X_i, \quad i = 1, \dots, 6,$$

where $(e_1^6, \dots, e_6^6)$ is the canonical basis of $\mathbb{R}^6$. If we split a vector in $\mathbb{R}^6$ as two vectors $\omega, u \in \mathbb{R}^3$ and write

$$\begin{pmatrix} \omega \\ u \end{pmatrix}$$

for such a vector in $\mathbb{R}^6$, then ψ is given by

$$\psi \begin{pmatrix} \omega \\ u \end{pmatrix} = \begin{pmatrix} \omega_\times & u \\ 0 & 0 \end{pmatrix}.$$

We define a bracket structure on $\mathbb{R}^6$ by

$$\left[\begin{pmatrix} \omega \\ u \end{pmatrix}, \begin{pmatrix} \theta \\ v \end{pmatrix} \right] = \begin{pmatrix} \omega \times \theta \\ u \times \theta + \omega \times v \end{pmatrix}.$$

(1) Check that $\psi\colon (\mathbb{R}^6, [-,-]) \to \mathfrak{se}(3)$ is a Lie algebra isomorphism.

Hint. Use proposition 2.39 to prove that

$$[\omega_\times, \theta_\times] = \omega_\times \theta_\times - \theta_\times \omega_\times = (\omega \times \theta)_\times.$$

(2) For any

$$X = \begin{pmatrix} B & u \\ 0 & 0 \end{pmatrix} \in \mathfrak{se}(3)$$

and any

$$\begin{pmatrix} \theta \\ v \end{pmatrix} \in \mathbb{R}^6,$$

prove that

$$\psi^{-1} \circ \mathrm{ad}(X) \circ \psi \begin{pmatrix} \theta \\ v \end{pmatrix} = \begin{pmatrix} B & 0 \\ u_\times & B \end{pmatrix} \begin{pmatrix} \theta \\ v \end{pmatrix}.$$

(3) For any

$$g = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \in \mathbf{SE}(3),$$

where $R \in \mathbf{SO}(3)$ and $t \in \mathbb{R}^3$ and for any

$$\begin{pmatrix} \theta \\ v \end{pmatrix} \in \mathbb{R}^6,$$

prove that

$$\psi^{-1}\mathrm{Ad}(g) \circ \psi \begin{pmatrix} \theta \\ v \end{pmatrix} = \begin{pmatrix} R & 0 \\ t_\times R & R \end{pmatrix} \begin{pmatrix} \theta \\ v \end{pmatrix}.$$

Problem B9 (60 pts). We can let the group $\mathbf{SO}(3)$ act on itself by conjugation, so that

$$R \cdot S = RSR^{-1} = RSR^\top.$$

The orbits of this action are the *conjugacy classes* of $\mathbf{SO}(3)$.

(1) Prove that the conjugacy classes of $\mathbf{SO}(3)$ are in bijection with the following sets:

1. $\mathcal{C}_0 = \{(0, 0, 0)\}$, the sphere of radius 0.
2. $\mathcal{C}_\theta$, with $0 < \theta < \pi$ and
$$\mathcal{C}_\theta = \{u \in \mathbb{R}^3 \mid \|u\| = \theta\},$$
the sphere of radius θ.
3. $\mathcal{C}_\pi = \mathbb{RP}^2$, viewed as the quotient of the sphere of radius π by the equivalence relation of being antipodal.

(2) Give $\mathrm{M}_3(\mathbb{R})$ the Euclidean structure where

$$\langle A, B \rangle = \frac{1}{2}\mathrm{tr}(A^\top B).$$

Consider the following three curves in $\mathbf{SO}(3)$:

$$c(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

for $0 \leq t \leq 2\pi$,

$$\alpha(\theta) = \begin{pmatrix} -\cos 2\theta & 0 & \sin 2\theta \\ 0 & -1 & 0 \\ \sin 2\theta & 0 & \cos 2\theta \end{pmatrix},$$

for $-\pi/2 \leq \theta \leq \pi/2$, and

$$\beta(\theta) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\cos 2\theta & \sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix},$$

for $-\pi/2 \leq \theta \leq \pi/2$.

Check that $c(t)$ is a rotation of angle t and axis $(0, 0, 1)$, that $\alpha(\theta)$ is a rotation of angle π whose axis is in the (x, z)-plane, and that $\beta(\theta)$ is a rotation of angle π whose axis is in the (y, z)-plane. Show that a log of $\alpha(\theta)$ is

$$B_\alpha = \pi \begin{pmatrix} 0 & -\cos\theta & 0 \\ \cos\theta & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix},$$

and that a log of $\beta(\theta)$ is

$$B_\beta = \pi \begin{pmatrix} 0 & -\cos\theta & \sin\theta \\ \cos\theta & 0 & 0 \\ -\sin\theta & 0 & 0 \end{pmatrix}.$$

(3) The curve $c(t)$ is a closed curve starting and ending at I that intersects $\mathcal{C}_\pi$ for $t = \pi$, and α, β are contained in $\mathcal{C}_\pi$ and coincide with $c(\pi)$ for $\theta = 0$. Compute the derivative $c'(\pi)$ of $c(t)$ at $t = \pi$, and the derivatives $\alpha'(0)$ and $\beta'(0)$, and prove that they are pairwise orthogonal (under the inner product $\langle -, - \rangle$).

Conclude that $c(t)$ intersects $\mathcal{C}_\pi$ transversally in $\mathbf{SO}(3)$, which means that

$$T_{c(\pi)}\, c + T_{c(\pi)}\, \mathcal{C}_\pi = T_{c(\pi)}\, \mathbf{SO}(3).$$

This fact can be used to prove that all closed curves smoothly homotopic to $c(t)$ must intersect $\mathcal{C}_\pi$ transversally, and consequently $c(t)$ is not (smoothly) homotopic to a point. This implies that $\mathbf{SO}(3)$ is not simply connected, but this will have to wait for another homework!

TOTAL: $500 + 45$ **points.**