Summer 1, 2013 CIS 610

Advanced Geometric Methods in Computer Science
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Homework 3

June 14; Due June 21, 2013

Do Exercise 2.IV.1, 2.IV.4, 3.III.1, and 3.IV.1, from the handouts on the web, and the problems below.

Problem B1 (20 pts). Let $\varphi\colon E \times E \to \mathbb{R}$ be a bilinear form on a real vector space E of finite dimension n. Given any basis $(e_1, \ldots, e_n)$ of E, let $A = (a_{i\,j})$ be the matrix defined such that

$$a_{i\,j} = \varphi(e_i, e_j),$$

$1 \leq i, j \leq n$. We call A *the matrix of* φ *w.r.t. the basis* $(e_1, \ldots, e_n)$.

(a) For any two vectors x and y, if X and Y denote the column vectors of coordinates of x and y w.r.t. the basis $(e_1, \ldots, e_n)$, prove that

$$\varphi(x, y) = X^\top AY.$$

(b) Recall that A is a *symmetric* matrix if $A = A^\top$. Prove that φ is symmetric if A is a symmetric matrix.

(c) If $(f_1, \ldots, f_n)$ is another basis of E and P is the change of basis matrix from $(e_1, \ldots, e_n)$ to $(f_1, \ldots, f_n)$, prove that the matrix of φ w.r.t. the basis $(f_1, \ldots, f_n)$ is

$$P^\top AP.$$

The common rank of all matrices representing φ is called the *rank* of φ.

Problem B2 (40 pts). Let $\varphi\colon E \times E \to \mathbb{R}$ be a symmetric bilinear form on a real vector space E of finite dimension n. Two vectors x and y are said to be *conjugate or orthogonal w.r.t.* φ if $\varphi(x, y) = 0$. The main purpose of this problem is to prove that there is a basis of vectors that are pairwise conjugate w.r.t. φ.

(a) Prove that if $\varphi(x, x) = 0$ for all $x \in E$, then φ is identically null on E.

Otherwise, we can assume that there is some vector $x \in E$ such that $\varphi(x, x) \neq 0$. Use induction to prove that there is a basis of vectors that are pairwise conjugate w.r.t. φ.

For the induction step, proceed as follows. Let $(e_1, e_2, \ldots, e_n)$ be a basis of E, with $\varphi(e_1, e_1) \neq 0$. Prove that there are scalars $\lambda_2, \ldots, \lambda_n$ such that each of the vectors

$$v_i = e_i + \lambda_i e_1$$

is conjugate to e_1 w.r.t. φ, where $2 \leq i \leq n$, and that $(e_1, v_2, \ldots, v_n)$ is a basis.

(b) Let $(e_1, \ldots, e_n)$ be a basis of vectors that are pairwise conjugate w.r.t. φ, and assume that they are ordered such that

$$\varphi(e_i, e_i) = \begin{cases} \theta_i \neq 0 & \text{if } 1 \leq i \leq r, \\ 0 & \text{if } r+1 \leq i \leq n, \end{cases}$$

where r is the rank of φ. Show that the matrix of φ w.r.t. $(e_1, \ldots, e_n)$ is a diagonal matrix, and that

$$\varphi(x, y) = \sum_{i=1}^{r} \theta_i x_i y_i,$$

where $x = \sum_{i=1}^{n} x_i e_i$ and $y = \sum_{i=1}^{n} y_i e_i$.

Prove that for every symmetric matrix A, there is an invertible matrix P such that

$$P^{\top} A P = D,$$

where D is a diagonal matrix.

(c) Prove that there is an integer p, $0 \leq p \leq r$ (where r is the rank of φ), such that $\varphi(u_i, u_i) > 0$ for exactly p vectors of every basis $(u_1, \ldots, u_n)$ of vectors that are pairwise conjugate w.r.t. φ (*Sylvester's inertia theorem*).

Proceed as follows. Assume that in the basis $(u_1, \ldots, u_n)$, for any $x \in E$, we have

$$\varphi(x, x) = \alpha_1 x_1^2 + \cdots + \alpha_p x_p^2 - \alpha_{p+1} x_{p+1}^2 - \cdots - \alpha_r x_r^2,$$

where $x = \sum_{i=1}^{n} x_i u_i$, and that in the basis $(v_1, \ldots, v_n)$, for any $x \in E$, we have

$$\varphi(x, x) = \beta_1 y_1^2 + \cdots + \beta_q y_q^2 - \beta_{q+1} y_{q+1}^2 - \cdots - \beta_r y_r^2,$$

where $x = \sum_{i=1}^{n} y_i v_i$, with $\alpha_i > 0$, $\beta_i > 0$, $1 \leq i \leq r$.

Assume that $p > q$ and derive a contradiction. First, consider x in the subspace F spanned by

$$(u_1, \ldots, u_p, u_{r+1}, \ldots, u_n),$$

and observe that $\varphi(x, x) \geq 0$ if $x \neq 0$. Next, consider x in the subspace G spanned by

$$(v_{q+1}, \ldots, v_r),$$

and observe that $\varphi(x, x) < 0$ if $x \neq 0$. Prove that $F \cap G$ is nontrivial (i.e., contains some nonnull vector), and derive a contradiction. This implies that $p \leq q$. Finish the proof.

The pair $(p, r-p)$ is called the *signature* of φ.

(d) A symmetric bilinear form φ is *definite* if for every $x \in E$, if $\varphi(x,x) = 0$, then $x = 0$.

Prove that a symmetric bilinear form is definite iff its signature is either $(n, 0)$ or $(0, n)$. In other words, a symmetric definite bilinear form has rank n and is either positive or negative.

Problem B3 (30 pts). Let $\| \; \|$ be any matrix norm. Given an invertible $n \times n$ matrix A, if $c = 1/(2\,\|A^{-1}\|)$, then for every $n \times n$ matrix H, if $\|H\| \leq c$, then $A + H$ is invertible. Furthermore, if $\| \; \|$ is an operator norm and if $\|H\| \leq c$, then $\|(A+H)^{-1}\| \leq 1/c$.

Problem B4 (20 pts). Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Compute the directional derivative $\mathrm{D}_u f(0,0)$ of f at $(0,0)$ for every vector $u = (u_1, u_2) \neq 0$.

(b) Prove that the derivative $\mathrm{D}f(0,0)$ does not exist. What is the behavior of the function f on the parabola $y = x^2$ near the origin $(0,0)$?

Problem B5 (40 pts). (a) Let $f \colon \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$ be the function defined on $n \times n$ matrices by

$$f(A) = A^2.$$

Prove that

$$\mathrm{D}f_A(H) = AH + HA,$$

for all $A, H \in \mathrm{M}_n(\mathbb{R})$.

(b) Let $f \colon \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$ be the function defined on $n \times n$ matrices by

$$f(A) = A^3.$$

Prove that

$$\mathrm{D}f_A(H) = A^2 H + AHA + HA^2,$$

for all $A, H \in \mathrm{M}_n(\mathbb{R})$.

(c) Let $f \colon \mathrm{GL}(n, \mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$ be the function defined on invertible $n \times n$ matrices by

$$f(A) = A^{-1}.$$

Prove that

$$\mathrm{D}f_A(H) = -A^{-1} H A^{-1},$$

for all $A \in \mathrm{GL}(n, \mathbb{R})$ and for all $H \in \mathrm{M}_n(\mathbb{R})$.

Problem B6 (60 pts). Let H be a symmetric positive definite matrix and let K be any symmetric matrix.

(1) Prove that HK is diagonalizable, with real eigenvalues.

(2) If K is also positive definite, then prove that the eigenvalues of HK are positive.

(3) If K is any symmetric matrix, prove that the number of positive (resp. negative) eigenvalues of HK is equal to the number of positive (resp. negative) eigenvalues of K.

Problem B7 (80 pts). Recall that a matrix $B \in \mathrm{M}_n(\mathbb{R})$ is skew-symmetric if

$$B^\top = -B.$$

Check that the set $\mathfrak{so}(n)$ of skew-symmetric matrices is a vector space of dimension $n(n-1)/2$, and thus is isomorphic to $\mathbb{R}^{n(n-1)/2}$.

(a) Given a rotation matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $0 < \theta < \pi$, prove that there is a skew symmetric matrix B such that

$$R = (I - B)(I + B)^{-1}.$$

(b) Prove that the eigenvalues of a skew-symmetric matrix are either 0 or pure imaginary (that is, of the form $i\mu$ for $\mu \in \mathbb{R}$.).

Let $C \colon \mathfrak{so}(n) \to \mathrm{M}_n(\mathbb{R})$ be the function given by

$$C(B) = (I - B)(I + B)^{-1}.$$

Prove that if B is skew-symmetric, then $I - B$ and $I + B$ are invertible, and so C is well-defined. Prove that

$$(I + B)(I - B) = (I - B)(I + B),$$

and that

$$(I + B)(I - B)^{-1} = (I - B)^{-1}(I + B).$$

Prove that

$$(C(B))^\top C(B) = I$$

and that

$$\det C(B) = +1,$$

so that $C(B)$ is a rotation matrix. Furthermore, show that $C(B)$ does not admit -1 as an eigenvalue.

(c) Let $\mathbf{SO}(n)$ be the group of $n \times n$ rotation matrices. Prove that the map

$$C\colon \mathfrak{so}(n) \to \mathbf{SO}(n)$$

is bijective onto the subset of rotation matrices that do not admit -1 as an eigenvalue. Show that the inverse of this map is given by

$$B = (I + R)^{-1}(I - R) = (I - R)(I + R)^{-1},$$

where $R \in \mathbf{SO}(n)$ does not admit -1 as an eigenvalue. Check that C is a homeomorphism between $\mathfrak{so}(n)$ and $C(\mathfrak{so}(n))$.

(d) If $f\colon \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$ and $g\colon \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$ are differentiable matrix functions, prove that

$$d(fg)_A(B) = df_A(B)g(A) + f(A)dg_A(B),$$

for all $A, B \in \mathrm{M}_n(\mathbb{R})$.

(e) Prove that

$$dC(B)(A) = -[I + (I - B)(I + B)^{-1}]A(I + B)^{-1}.$$

Prove that $dC(B)$ is injective, for every skew-symmetric matrix B. Prove that C a parametrization of $\mathbf{SO}(n)$.