Summary 1, 2009 CIS 610

Advanced Geometric Methods in Computer Science
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Homework 3

June 23, 2009; Due June 30 2009

"B problems" must be turned in.

Problem B1 (30 pts). Let $(v_1, \ldots, v_n)$ be a sequence of n vectors in $\mathbb{R}^d$ and let V be the $d \times n$ matrix whose j-th column is v_j. Prove the equivalence of the following two statements:

(a) There is no nontrivial positive linear dependence among the v_j, which means that there is no nonzero vector, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, with $y_j \geq 0$ for $j = 1, \ldots, n$, so that

$$y_1 v_1 + \cdots + y_n v_n = 0$$

or equivalently, $Vy = 0$.

(b) There is some vector, $c \in \mathbb{R}^d$, so that $c^\top V > 0$, which means that $c^\top v_j > 0$, for $j = 1, \ldots, n$.

Problem B2 (20 pts). Let E be a real vector space of finite dimension, $n \geq 1$. Say that two bases, $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$, of E have the *same orientation* iff $\det(P) > 0$, where P the change of basis matrix from $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$, namely, the matrix whose jth columns consist of the coordinates of v_j over the basis $(u_1, \ldots, u_n)$.

(a) Prove that having the same orientation is an equivalence relation with two equivalence classes.

An *orientation* of a vector space, E, is the choice of any fixed basis, say $(e_1, \ldots, e_n)$, of E. Any other basis, $(v_1, \ldots, v_n)$, has the *same orientation* as $(e_1, \ldots, e_n)$ (and is said to be *positive* or *direct*) iff $\det(P) > 0$, else it said to have the *opposite orientation* of $(e_1, \ldots, e_n)$ (or to be *negative* or *indirect*), where P is the change of basis matrix from $(e_1, \ldots, e_n)$ to $(v_1, \ldots, v_n)$. An *oriented* vector space is a vector space with some chosen orientation (a positive basis).

(b) Let $B_1 = (u_1, \ldots, u_n)$ and $B_2 = (v_1, \ldots, v_n)$ be two orthonormal bases. For any sequence of vectors, $(w_1, \ldots, w_n)$, in E, let $\det_{B_1}(w_1, \ldots, w_n)$ be the determinant of the matrix whose columns are the coordinates of the w_j's over the basis B_1 and similarly for $\det_{B_2}(w_1, \ldots, w_n)$.

Prove that if B_1 and B_2 have the same orientation, then

$$\det_{B_1}(w_1, \ldots, w_n) = \det_{B_2}(w_1, \ldots, w_n).$$

Given any oriented vector space, E, for any sequence of vectors, $(w_1, \ldots, w_n)$, in E, the common value, $\det_B(w_1, \ldots, w_n)$, for all positive orthonormal bases, B, of E is denoted

$$\lambda_E(w_1, \ldots, w_n)$$

and called a *volume form* of $(w_1, \ldots, w_n)$.

(c) Given any Euclidean oriented vector space, E, of dimension n for any $n-1$ vectors, $w_1, \ldots, w_{n-1}$, in E, check that the map

$$x \mapsto \lambda_E(w_1, \ldots, w_{n-1}, x)$$

is a linear form. Then, prove that there is a unique vector, denoted $w_1 \times \cdots \times w_{n-1}$, such that

$$\lambda_E(w_1, \ldots, w_{n-1}, x) = (w_1 \times \cdots \times w_{n-1}) \cdot x,$$

for all $x \in E$. The vector $w_1 \times \cdots \times w_{n-1}$ is called the *cross-product* of $(w_1, \ldots, w_{n-1})$. It is a generalization of the cross-product in $\mathbb{R}^3$ (when $n = 3$).

Problem B3 (30 pts). Given p vectors $(u_1, \ldots, u_p)$ in a Euclidean space E of dimension $n \geq p$, the *Gram determinant (or Gramian)* of the vectors $(u_1, \ldots, u_p)$ is the determinant

$$\mathrm{Gram}(u_1, \ldots, u_p) = \begin{vmatrix} \|u_1\|^2 & \langle u_1, u_2\rangle & \ldots & \langle u_1, u_p\rangle \\ \langle u_2, u_1\rangle & \|u_2\|^2 & \ldots & \langle u_2, u_p\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_p, u_1\rangle & \langle u_p, u_2\rangle & \ldots & \|u_p\|^2 \end{vmatrix}.$$

(1) Prove that

$$\mathrm{Gram}(u_1, \ldots, u_n) = \lambda_E(u_1, \ldots, u_n)^2.$$

Hint. If $(e_1, \ldots, e_n)$ is an orthonormal basis and A is the matrix of the vectors $(u_1, \ldots, u_n)$ over this basis,

$$\det(A)^2 = \det(A^\top A) = \det(A_i \cdot A_j),$$

where A_i denotes the ith column of the matrix A, and $(A_i \cdot A_j)$ denotes the $n \times n$ matrix with entries $A_i \cdot A_j$.

(2) Prove that

$$\|u_1 \times \cdots \times u_{n-1}\|^2 = \mathrm{Gram}(u_1, \ldots, u_{n-1}).$$

Hint. Letting $w = u_1 \times \cdots \times u_{n-1}$, observe that

$$\lambda_E(u_1, \ldots, u_{n-1}, w) = \langle w, w \rangle = \|w\|^2,$$

and show that

$$\begin{aligned}\|w\|^4 &= \lambda_E(u_1, \ldots, u_{n-1}, w)^2 = \mathrm{Gram}(u_1, \ldots, u_{n-1}, w) \\ &= \mathrm{Gram}(u_1, \ldots, u_{n-1})\|w\|^2.\end{aligned}$$

Problem B4 (50 pts). Given a Euclidean space E, let U be a nonempty affine subspace of E, and let a be any point in E. We define the *distance* $d(a, U)$ of a to U as

$$d(a, U) = \inf\{\|\mathbf{ab}\| \mid b \in U\}.$$

(a) Prove that the affine subspace $U_a^\perp$ defined such that

$$U_a^\perp = a + \overrightarrow{U}^\perp$$

intersects U in a single point b such that $d(a, U) = \|\mathbf{ab}\|$.
Hint. Recall the discussion after Lemma 2.11.2.

(b) Let $(a_0, \ldots, a_p)$ be a frame for U (not necessarily orthonormal). Prove that

$$d(a, U)^2 = \frac{\mathrm{Gram}(\mathbf{a_0a}, \mathbf{a_0a_1}, \ldots, \mathbf{a_0a_p})}{\mathrm{Gram}(\mathbf{a_0a_1}, \ldots, \mathbf{a_0a_p})}.$$

Hint. Gram is unchanged when a linear combination of other vectors is added to one of the vectors, and thus

$$\mathrm{Gram}(\mathbf{a_0a}, \mathbf{a_0a_1}, \ldots, \mathbf{a_0a_p}) = \mathrm{Gram}(\mathbf{ba}, \mathbf{a_0a_1}, \ldots, \mathbf{a_0a_p}),$$

where b is the unique point defined in question (a).

(c) If D and D' are two lines in E that are not coplanar, $a, b \in D$ are distinct points on D, and $a', b' \in D'$ are distinct points on D', prove that if $d(D, D')$ is the shortest distance between D and D' (why does it exist?), then

$$d(D, D')^2 = \frac{\mathrm{Gram}(\mathbf{aa'}, \mathbf{ab}, \mathbf{a'b'})}{\mathrm{Gram}(\mathbf{ab}, \mathbf{a'b'})}.$$

Problem B5 (30 pts). (1) If an upper triangular $n \times n$ matrix R is invertible, prove that its inverse is also upper triangular.

(2) If an upper triangular matrix is orthogonal, prove that it must be a diagonal matrix.

If A is an invertible $n \times n$ matrix and if $A = Q_1R_1 = Q_2R_2$, where R_1 and R_2 are upper triangular with positive diagonal entries and Q_1, Q_2 are orthogonal, prove that $Q_1 = Q_2$ and $R_1 = R_2$.

TOTAL: 160 points.