(GJSZ)

Homework II (due October 14), Math 602, Fall 2002.

BI (b). The objective of this section is to prove that if G is a finite group acting trivially on $\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}$, then

$$\mathrm{Hom}\,(G, \mathbb{Q}/\mathbb{Z}) \cong H^2(G, \mathbb{Z}).$$

The first step if to define a homomorphism $\theta\colon \mathrm{Hom}\,(G, \mathbb{Q}/\mathbb{Z}) \to H^2(G, \mathbb{Z})$. We proceed as follows: Given any homomorphism $\chi\colon G \to \mathbb{Q}/\mathbb{Z}$, let $\chi'\colon G \to \mathbb{Q}$ be any map representing χ, which means that

$$\chi'(\sigma) = \chi(\sigma) \ (\mathrm{mod}\, \mathbb{Z}), \quad \text{for all } \sigma \in G.$$

Since χ is a homomorphism, $\chi(\sigma\tau) = \chi(\sigma) + \chi(\tau)$. Writing $\chi'(\sigma) = \chi(\sigma) + k_1$, $\chi'(\tau) = \chi(\tau) + k_2$ and $\chi'(\sigma\tau) = \chi(\sigma\tau) + k_3$, with $k_1, k_2, k_3 \in \mathbb{Z}$, we have

$$\begin{aligned} \chi'(\tau) - \chi'(\sigma\tau) + \chi'(\sigma) &= \chi(\tau) + k_2 - (\chi(\sigma\tau) + k_3) + \chi(\sigma) + k_1 \\ &= \chi(\tau) + k_2 - \chi(\sigma) - \chi(\tau) - k_3 + \chi(\sigma) + k_1 \\ &= k_2 - k_3 + k_1, \end{aligned}$$

which shows that $\chi'(\tau) - \chi'(\sigma\tau) + \chi'(\sigma) \in \mathbb{Z}$. Observe that

$$(\delta\chi')(\sigma, \tau) = \chi'(\tau) - \chi'(\sigma\tau) + \chi'(\sigma),$$

where χ' is a 1-cochain in $C^1(G, \mathbb{Q})$, and so, $(\delta\chi')(\sigma, \tau) \in B^2(G, \mathbb{Q})$ and

$$\delta_2(\delta\chi') \equiv 0,$$

which means that $\delta\chi'$ is a 2-cocycle in $Z^2(G, \mathbb{Z})$. Furthermore, if χ'' is any other map representing χ, then $\chi'' - \chi'$ is clearly integer valued, so $\delta\chi'' - \delta\chi'$ is a 2-coboundary in $B^2(G, \mathbb{Z})$. If f is any 2-cocycle in $Z^2(G, \mathbb{Z})$, let $[f]$ denote its image in $H^2(G, \mathbb{Z})$ (its cohomology class). As a consequence of the above, the map $\theta\colon \mathrm{Hom}\,(G, \mathbb{Q}/\mathbb{Z}) \to H^2(G, \mathbb{Z})$ given by

$$\theta(\chi)(\sigma, \tau) = [\chi'(\tau) - \chi'(\sigma\tau) + \chi'(\sigma)]$$

where χ' is any cochain in $C^1(G, \mathbb{Q})$ representing χ, is indeed well-defined. Since the group operation in $\mathrm{Hom}\,(G, \mathbb{Q})$ is addition of functions, it is obvious that θ is a homomorphism.

We prove that $\theta\colon \mathrm{Hom}\,(G, \mathbb{Q}/\mathbb{Z}) \to H^2(G, \mathbb{Z})$ is injective as follows. Assume that $\theta(\chi)(\sigma, \tau) = 0$ in $H^2(G, \mathbb{Z})$, which means that $\chi'(\tau) - \chi'(\sigma\tau) + \chi'(\sigma)$ is a 2-coboundary in $B^2(G, \mathbb{Z})$. So, there is some 1-cochain $g \in C^1(G, \mathbb{Z})$ so that

$$\chi'(\tau) - \chi'(\sigma\tau) + \chi'(\sigma) = g(\tau) - g(\sigma\tau) + g(\sigma)$$

for all σ and τ in G. Since G is finite, if we let $|G| = n$, we can sum the above with respect to σ, and we get

$$\sum_\sigma \chi'(\tau) - \sum_\sigma \chi'(\sigma\tau) + \sum_\sigma \chi'(\sigma) = \sum_\sigma g(\tau) - \sum_\sigma g(\sigma\tau) + \sum_\sigma g(\sigma),$$

that is,

$$n\chi'(\tau) - \sum_{\sigma} \chi'(\sigma) + \sum_{\sigma} \chi'(\sigma) = ng(\tau) - \sum_{\sigma} g(\sigma) + \sum_{\sigma} g(\sigma),$$

so

$$n\chi'(\tau) = ng(\tau), \quad \text{for all } \tau \in G,$$

and since $\chi'(\tau) \in \mathbb{Q}$ and $g(\tau) \in \mathbb{Z}$,

$$\chi'(\tau) = g(\tau), \quad \text{for all } \tau \in G,$$

which means that $\chi \equiv 0$ in $\mathrm{Hom}\,(G, \mathbb{Q}/\mathbb{Z})$. Thus, θ is indeed injective.

To prove surjectivity, we prove a general property of 2-cocycle in $H^2(G, A)$, when G is finite and A is any G-module.

Recall that if f is a 1-cochain, then

$$(\delta_1 f)(\sigma, \tau) = \sigma \cdot f(\tau) - f(\sigma\tau) + f(\sigma),$$

and if f is a 2-cochain, then

$$(\delta_2 f)(\sigma, \tau, \rho) = \sigma \cdot f(\tau, \rho) - f(\sigma\tau, \rho) + f(\sigma, \tau\rho) - f(\sigma, \tau).$$

Assume that G is a finite group, and let $|G| = n$. Then, if f is a 2-cochain, we can define the 1-cochain, g, by

$$g(\sigma) = \sum_{\rho \in G} f(\sigma, \rho).$$

Let us compute $\Sigma_{\rho \in G} (\delta_2 f)(\sigma, \tau, \rho)$. We have

$$\begin{aligned}
\sum_{\rho \in G} (\delta_2 f)(\sigma, \tau, \rho) &= \sum_{\rho \in G} \sigma \cdot f(\tau, \rho) - \sum_{\rho \in G} f(\sigma\tau, \rho) + \sum_{\rho \in G} f(\sigma, \tau\rho) - \sum_{\rho \in G} f(\sigma, \tau) \\
&= \sum_{\rho \in G} \sigma \cdot f(\tau, \rho) - \sum_{\rho \in G} f(\sigma\tau, \rho) + \sum_{\rho \in G} f(\sigma, \rho) - nf(\sigma, \tau) \\
&= (\delta_1 g)(\sigma, \tau) - nf(\sigma, \tau).
\end{aligned}$$

If f is a 2-cocycle, $(\delta_2 f)(\sigma, \tau, \rho) = 0$, and we get

$$nf(\sigma, \tau) = (\delta_1 g)(\sigma, \tau).$$

Now, if $f \in Z^2(G, \mathbb{Z})$ is a 2-cocycle, by the above, there is a 1-cochain, g, in $C^1(G, \mathbb{Z})$ so that

$$nf(\sigma, \tau) = (\delta_1 g)(\sigma, \tau),$$

and thus, $\chi' = g/n$ is a 1-cochain *in* $C^1(G, \mathbb{Q})$ so that

$$f(\sigma, \tau) = (\delta\chi')(\sigma, \tau),$$

Now, since $f(\sigma, \tau) = (\delta\chi')(\sigma, \tau)$ and $f \in Z^2(G, \mathbb{Z})$ we also have

$$(\delta\chi')(\sigma, \tau) = \chi'(\tau) - \chi'(\sigma\tau) + \chi'(\sigma) \in \mathbb{Z},$$

which means that

$$\chi'(\tau) - \chi'(\sigma\tau) + \chi'(\sigma) = 0 \quad \text{in } \mathbb{Q}/\mathbb{Z},$$

i.e., χ' represents a homomorphism $\chi \in \mathrm{Hom}\,(G, \mathbb{Q}/\mathbb{Z})$. Therefore, we have $\theta(\chi) = [f]$, where $\chi \in \mathrm{Hom}\,(G, \mathbb{Q}/\mathbb{Z})$ is the homomorphism represented by χ'. This proves the surjectivity of θ. Finally, we proved that $\theta\colon \mathrm{Hom}\,(G, \mathbb{Q}/\mathbb{Z}) \to H^2(G, \mathbb{Z})$ is an isomorphism.

BI (c) We have the G-pairing $\mathbb{Z} \prod A \longrightarrow A$, given by $(n, a) \mapsto na$. By part (a) with $p = 2$ and $q = 0$, we obtain a pairing

$$H^2(G, \mathbb{Z}) \prod H^0(G, A) \longrightarrow H^2(G, A)$$

given by the cup-product. We know from class that $H^0(G, A) \cong A^G = \{a \in A \mid \sigma \cdot a = a\}$ for all $\sigma \in G$. Thus, in view of part (b), we have a pairing

$$\mathrm{Hom}\,(G, \mathbb{Q}/\mathbb{Z}) \prod A^G \longrightarrow H^2(G, A).$$

For any $\chi \in \mathrm{Hom}\,(G, \mathbb{Q}/\mathbb{Z})$ and any $a \in A^G$, using the definition of part (a), the isomorphism $\mathrm{Hom}\,(G, \mathbb{Q}/\mathbb{Z}) \cong H^2(G, \mathbb{Z})$, and the cup-product

$$H^2(G, \mathbb{Z}) \prod H^0(G, A) \longrightarrow H^2(G, A),$$

we have

$$(\delta\chi \cup a)(\sigma, \tau) = [\delta\chi'(\sigma, \tau)(\sigma\tau \cdot a)]$$

where χ' is any map representing χ. Since $a \in A^G$, we have $\sigma\tau \cdot a = a$, and so,

$$(\delta\chi \cup a)(\sigma, \tau) = [\delta\chi'(\sigma, \tau)a] = [(\chi'(\tau) - \chi'(\sigma\tau) + \chi'(\sigma))a],$$

for any χ' representing χ. For any $\alpha \in A$, if $\xi = \mathcal{N}\alpha = \sum_{\sigma \in G} \sigma \cdot \alpha$, we get

$$(\delta\chi \cup \xi)(\tau, \rho) = \left[\delta\chi'(\tau, \rho) \sum_{\sigma \in G} \sigma \cdot \alpha\right] = \left[\sum_{\sigma \in G} \sigma \cdot \delta\chi'(\tau, \rho)\alpha\right].$$

Observe that $\delta\chi'(\tau, \rho)\alpha$ is a 2-cocycle in $Z^2(G, A)$. Thus, if we prove that for every cocycle $f \in Z^2(G, A)$, the 2-cocycle $\mathcal{N}f(\tau, \rho) = \sum_\sigma \sigma \cdot f(\tau, \rho)$ is a 2-coboundary, we will have shown that

$$(\delta\chi \cup \mathcal{N}\alpha)(\sigma, \tau) = 0.$$

Let $u_f(\rho) = \sum_\sigma f(\sigma, \rho)$. Since f is a 2-cocycle, we have

$$0 = \sigma \cdot f(\tau, \rho) - f(\sigma\tau, \rho) + f(\sigma, \tau\rho) - f(\sigma, \tau).$$

If we sum the above over σ, we get

$$\begin{aligned} 0 &= \sum_{\sigma\in G}\sigma\cdot f(\tau,\rho) - \sum_{\sigma\in G} f(\sigma\tau,\rho) + \sum_{\sigma\in G} f(\sigma,\tau\rho) - \sum_{\sigma\in G} f(\sigma,\tau) \\ &= \mathcal{N}f(\tau,\rho) - \sum_{\sigma\in G} f(\sigma,\rho) + \sum_{\sigma\in G} f(\sigma,\tau\rho) - \sum_{\sigma\in G} f(\sigma,\tau) \\ &= \mathcal{N}f(\tau,\rho) - u_f(\rho) + u_f(\tau\rho) - u_f(\tau). \end{aligned}$$

Therefore,

$$\mathcal{N}f(\tau,\rho) = u_f(\tau) + u_f(\rho) - u_f(\tau\rho),$$

which proves that $\mathcal{N}f(\tau,\rho)$ is a 2-coboundary. Thus, for any $\alpha\in A$.

$$(\delta\chi\cup\mathcal{N}\alpha)(\sigma,\tau) = 0.$$

However, observe that $\mathcal{N}A\subseteq A^G$, since if $\xi = \sum_{\sigma\in G}\sigma\cdot\alpha$, then for any $\tau\in G$,

$$\tau\cdot\xi = \tau\cdot\left(\sum_{\sigma\in G}\sigma\cdot\alpha\right) = \sum_{\sigma\in G}\tau\sigma\cdot\alpha = \sum_{\sigma\in G}\sigma\cdot\alpha = \xi.$$

Consequently, we have the pairing

$$\mathrm{Hom}\,(G,\mathbb{Q}/\mathbb{Z})\coprod(A^G/\mathcal{N}A) \longrightarrow H^2(G,A).$$

We still need to prove that $u_f(\tau)\in A^G$, since this is needed in part (d). For this, we sum the cocycle condition

$$0 = \sigma\cdot f(\tau,\rho) - f(\sigma\tau,\rho) + f(\sigma,\tau\rho) - f(\sigma,\tau)$$

over τ. We get

$$\begin{aligned} 0 &= \sum_{\tau\in G}\sigma\cdot f(\tau,\rho) - \sum_{\tau\in G} f(\sigma\tau,\rho) + \sum_{\tau\in G} f(\sigma,\tau\rho) - \sum_{\tau\in G} f(\sigma,\tau) \\ &= \sigma\cdot\left(\sum_{\tau\in G} f(\tau,\rho)\right) - \sum_{\tau\in G} f(\tau,\rho) + \sum_{\tau\in G} f(\sigma,\tau) - \sum_{\tau\in G} f(\sigma,\tau) \\ &= \sigma\cdot u_f(\rho) - u_f(\rho), \end{aligned}$$

and so, $\sigma\cdot u_f(\rho) = u_f(\rho)$, for all $\sigma\in G$, as desired.

BI (d) Now, we assume that $G = \{1,\sigma_0,\sigma_0^2,\ldots,\sigma_0^{n-1}\}$ is a finite cyclic group. Let $\chi_0\colon G\to\mathbb{Q}/\mathbb{Z}$ be the homomorphism defined by $\chi_0(\sigma_0) = 1/n \pmod{\mathbb{Z}}$ and let χ_0' be the 1-cocycle in $C^1(G,\mathbb{Q})$ representing χ_0 defined by $\chi_0'(\sigma_0^k) = k/n$, for $k = 0,\ldots,n-1$. We have the homomorphism $\theta\colon A^G/\mathcal{N}A \longrightarrow H^2(G,A)$ defined by $\theta(\alpha) = \delta\chi_0\cup\alpha$. First, we prove injectivity. If $\theta(a) = 0$, this means that there is some 1-cochain $g\in C^1(G,A)$ so that

$$(\delta\chi_0\cup\alpha)(\sigma,\tau) = (\chi_0'(\tau) - \chi_0'(\sigma\tau) + \chi_0'(\tau))\alpha = \sigma\cdot g(\tau) - g(\sigma\tau) + g(\sigma)$$

for all $\sigma, \tau \in G$. If we sum the above over $\sigma \in G$, we get

$$\left(\sum_{\sigma} \chi_0'(\tau) - \sum_{\sigma} \chi_0'(\sigma\tau) + \sum_{\sigma} \chi_0'(\sigma)\right) \alpha = \sum_{\sigma} \sigma \cdot g(\tau) - \sum_{\sigma} g(\sigma\tau) + \sum_{\sigma} g(\sigma),$$

that is,

$$\left(n\chi_0'(\tau) - \sum_{\sigma} \chi_0'(\sigma) + \sum_{\sigma} \chi_0'(\sigma)\right) \alpha = \sum_{\sigma} \sigma \cdot g(\tau) - \sum_{\sigma} g(\sigma) + \sum_{\sigma} g(\sigma),$$

which yields

$$n\chi_0'(\tau)\alpha = \sum_{\sigma} \sigma \cdot g(\tau).$$

If we set $\tau = \sigma_0$, since $\chi_0'(\sigma_0) = 1/n$, we get

$$\alpha = \sum_{\sigma} \sigma \cdot g(\sigma_0),$$

and this shows that $\alpha \in \mathcal{N}A$, as desired.

Now, we need to prove the surjectivity of θ. Unfortunately, this has eluded all our attemps! All we can do is this: For any 2-cocycle $f \in Z^2(G, A)$, if there is some $a \in A^G$ so that

$$(\delta\chi_0 \cup \alpha)(\sigma, \tau) = [(\chi_0'(\tau) - \chi_0'(\sigma\tau) + \chi_0'(\tau))a] = [f(\sigma, \tau)],$$

then by summing over $\sigma \in G$ as above, we get

$$[\chi_0'(\tau)a] = \left[\sum_{\sigma} f(\sigma, \tau)\right],$$

that is,

$$[\chi_0'(\tau)a] = [u_f(\tau)].$$

Since $\chi_0'(\sigma_0^k) = k/n$, this yields

$$u_f(\sigma_0^k) = ka, \quad k = 0, \ldots, n-1,$$

and in particular, we must have $a = u_f(\sigma_0)$. We also observed that

$$\delta\chi_0(\sigma_0^h, \sigma_0^k) = \chi_0'(\sigma_0^h) - \chi_0'(\sigma_0^{h+k}) + \chi_0'(\sigma_0^k) = \begin{cases} 0 & \text{if } 0 \leq h+k \leq n-1 \\ 1 & \text{if } n \leq h+k \leq 2n-2. \end{cases}$$